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REFLECTION AND FOCUSING OF SONIC BOOMS BY TWO-DIMENSIONAL CURVED SURFACES

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Jack Werner

April, 1970

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New York University
School of Engineering and Science
University Heights, New York, N.Y. 10453

Research Division
Department of Aeronautics & Astronautics

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FOREWORD

This report was prepared by Dr. Jack Werner, Associate Professor of Aeronautics and Astronautics.

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LIST OF SYMBOLS

a, b, c	values of x at intersections of WI and ΓI
c_o	speed of sound
$H(\zeta)$	heaviside step function
$p(\xi,\eta,\zeta)$	pressure at pt (ξ,η,ζ)
p_∞	pressure in the incident plane wave
p_ω	pressure on the reflecting surface
$p_H(\xi\eta\zeta)$	$p(\xi\eta\zeta) - H(\zeta+\xi)$
r	$\sqrt{(x-\xi)^2 + (y-\eta)^2}$
s	displacement normal to plane wave front
S_ω	reflecting surface in xyz space
S_i	initial front of incident plane wave in xyz space
t	time
WI	intersection of S_ω and S_I
x,y	space coordinates
z	$c_o t$
x', y', z'	$(x-\xi)$, $(y-\eta)$, $(z-\zeta)$
Γ	characteristic cone through ξ,η,ζ
ΓI	intersection of S_i and Γ
ξ,η,ζ	point in xyz space
ζ_o	"duration" of N wave

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ABSTRACT

An integral relation has been derived describing the pressure due to the reflection of a plane acoustic wave of arbitrary wave form incident on a two dimensional curved surface with plane asymptotes. The results were applied to the problem of sonic boom incident over a reflecting hyperbolic surface. It was found that singularities or "focal points" occur at which the pressure becomes infinite according to linear theory. A criterion for these focal points to occur was developed and the locus of singularities was determined. The pressure disturbances in the neighborhood of these focal points was investigated for incident step function, linear and N-wave forms. It was found that in the case of the N-wave the major contribution to the disturbance near the focal points comes from the reflection of the discontinuities at the leading and trailing edges of the incident wave.

I. Introduction

An assessment of many of the effects of sonic boom involves a basic knowledge of the manner in which pressure pulses are reflected from solid surfaces. Normally the disturbances occasioned by sonic boom are of acoustic proportions, that is to say they are small enough to be governed locally by linearized theory. However the overall effects of these disturbances may become large owing to two factors both of a cumulative nature. Thus a disturbance incident on large areas such as the side of a glass walled building may develop total integrated forces which are large enough to do damage. On the other hand, reflections from a large curved surface may be focused into a small region so that the cumulative effect is one of high concentration of acoustic energy with very high local pressure. The reflection by a plane surface with a single sharp bend raises the local pressure by a factor of two.[1] In comparison the focusing of disturbances by a suitably curved surface concave to the incident wave may give rise to pressures which are locally ten to one hundred times the disturbance of the incident wave. It is this latter class of phenomena that are the subject of this paper.

Figure (1) depicts a typical wave pattern generated by an aircraft in supersonic flight over a valley or depression in the ground. The usual reflected pattern which would be present over a plane surface is omitted in this case since it is assumed that the presence of curvature causes continuous pressure variations behind the incident wave rather than the reflected discontinuities found over plane surfaces. To facilitate an analysis of the situation depicted in Figure (1) a mathematical model is adopted which will retain the principal physical features of the reflection. The model consists of a distribution of plane waves incident on a two dimensional curved surface with plane asymptotes. A simple example of such a surface is the hyperbola $xy = 1$ to which the general analysis developed here will be applied. This latter case is illustrated in Figure (2).

Typical of incident sonic boom waves are the "N" waves illustrated in Figure (1) having a discontinuity at the leading and trailing edges. Since the magnitude of these disturbances are usually acoustic in level the discontinuities are in the nature of acoustic shock fronts whose propagations are governed by the relations of geometrical acoustics.[2] As briefly mentioned above, owing to the continuous curvature of the reflecting surface out to infinity the reflection of a discontinuous wave front shows up, not as a discontinuity but rather as a region of rapidly varying but continuous pressure rise as indicated in Figure (3). Far from the wall the variation becomes increasingly rapid approaching a discontinuous jump as we go to infinity losing the details of the local curvature.

II. Basic Equations

The continuous pressure perturbation field behind the incident wave front is governed by the acoustic wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{C_o^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

For convenience we introduce the time-like variable $z = C_o t$ and write Equation (1) in x,y,z space

$$L(p) = \nabla^2 p - \frac{\partial^2 p}{\partial z^2} = 0 \tag{2}$$

Volterra has formulated the wave equation as an integral equation.[3] Briefly, his method was to apply Green's theorem to a volume V in x,y,z space, indicated in Figure (4), bounded by the data support s, the characteristic cone Γ, and a cylinder C of infinitesimal radius parallel to the z axis. The cylinder C is introduced to exclude singularities from the interior of the volume. Since the wave equation (2) is self-adjoint, Green's theorem reduces to:[3]

$$\int [vL(p)-pL(v)]dV = \int_{\sigma=s+c+\Gamma} \left(p\frac{\partial v}{\partial \nu} - v\frac{\partial p}{\partial \nu}\right) d\sigma \tag{3}$$

The "conormal" ν for the wave equation is the reflection of the inner normal in the plane z = const. The characteristic cone Γ is represented by

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2} = -(z-\zeta) \tag{4}$$

and we note that on Γ the conormal ν lies in the characteristic cone. Finally, the adjoint function v is chosen to satisfy

$$L(v) = 0, \quad v = 0 \text{ on } \Gamma \tag{5}$$

A function satisfying relations (5) was determined by Volterra

$$v = \ln\left[-\frac{z'}{r} + \sqrt{\left(\frac{z'}{r}\right)^2 - 1}\right] \tag{6}$$

$$z' = z-\zeta, \qquad r^2 = (x')^2 + (y')^2$$

$$x' = x-\xi, \qquad y' = y-\eta$$

This result for v is substituted into the integral relation (3) and the limiting process of allowing the radius of C to tend to zero is performed. In the latter operation two types of data support must be distinguished, those which intersect the cylinder C in a closed curve c_s, Figure (4) and those having a portion of their surface parallel to the z axis as in Figure (5). In the second case when pt P lies on the vertical portion of the data support only a segment of the cylinder C lies in the interior of V. Let this segment be comprised of the circular segment C_{sn} of angle $N\pi$. If the point P does not lie on the data support the case reverts to that of Figure (4). The introduction of v into Equation (3) then yields:

$$N\pi \int_{z_s}^{\zeta} p dz = \int_{s} (p \frac{\partial v}{\partial \nu} - v \frac{\partial p}{\partial \nu}) ds \tag{7}$$

This last result is actually equivalent to a representation of the impulse. Differentiating to obtain an explicit relation for the pressure:

$$p(\xi\eta\zeta) = \frac{1}{N\pi} \frac{\partial}{\partial \zeta} \int (p \frac{\partial v}{\partial \nu} - v \frac{\partial p}{\partial \nu}) ds \tag{8}$$

$$N = \begin{cases} 1, & \xi = \xi_s, \quad \eta = \eta_s \quad \mathrm{Up}(\zeta\eta) \text{ on } s \\ 2, & p(\xi,\eta) \text{ not on } s \end{cases}$$

Figure (6) represents the data support and characteristic surface Γ in x,y,z space corresponding to the problem of a distribution of plane waves incident on a hyperbolic cylindrical surface. The data support consists of the surfaces S_ω, S_i and S_∞. S_ω represents the hyperbolic wall in space-time and on S_ω the conormal ν_ω and normal n_ω coincide. S_i represents the locus in space-time of the initial wave front and is a plane surface with normal n_i at 45° to the z axis. The conormal ν_i lies in the surface S_i. Finally S_∞ is a surface perpendicular to S_i. Its presence is required to close the surface bounding V since the bicharacteristics or generators of Γ also make an angle of 45° with the z axis. The intersection of S_i and Γ is thus parabolic a consequence of the assumption that the incident disturbance is acoustic in level. The distribution of p on S_∞ specifies the pressure distribution of the incident plane waves and is considered to be given. It is evident that ν_∞ lies in S_∞ as indicated in Figure (6).

III. Boundary Conditions

A condition on the pressure at the surface S_i is provided by the theory of geometrical acoustics applied to the initial plane discontinuity. Thus the pressure jump across S_i is found to be constant. Let this pressure be P_o. Since the region ahead of the initial wave front (or below it in x,y,z space) is undisturbed we have

$$p = P_o \text{ on } S_i \tag{9}$$

On the wall the boundary condition on p may be determined from the normal component of the momentum equation for an acoustic disturbance under the restriction the velocity normal to the wall is zero. The resulting boundary condition is:

$$\frac{\partial p}{\partial n} = \frac{\partial p}{\partial \nu} = 0 \quad \text{on } S_\omega \tag{10}$$

An additional boundary condition on the pressure at the wall immediately behind the initial wave front is provided by the requirement that the flow at this point be tangent to the wall. Thus, as the front is approached along the wall the pressure disturbance approaches the value of $2P_o$, ie:

$$p(x,y,-x) = 2p_o \quad \text{at intersection of } S_\omega \text{ and } S_i \tag{11}$$

Finally, letting s be a coordinate measured in the direction of $-\nu_\infty$ on S_∞ as shown in Figure (6) the function $p_\infty(s)$ on S_∞ is given. This represents the given pressure distribution of the incident plane disturbance, i.e.:

$$p = p_\infty(s) \quad \text{on } S_\infty \tag{12}$$

IV. The Integral Pressure Relation

The boundary conditions provided by relations (9), (10) and (12) are substituted into the integral equation (8).

$$p(\xi,\zeta,) = \frac{1}{N\pi}\frac{\partial}{\partial\zeta}\left\{\int_{S_\omega} p(x,y,z)\,\frac{\partial v}{\partial n}\,dS + p_o\int_{S_i}\frac{\partial v}{\partial \nu_i}\,dS - \int_{S_\infty}\left(p_\infty(s)\,\frac{\partial v}{\partial s} - v\,\frac{\partial p_\infty}{\partial s}\right)dS\right\} \tag{13}$$

As yet a specific set of coordinate axes has not been chosen. For convenience in evaluating the integrals over S_i and S_∞ the x axis is chosen parallel to the direction of propagation of the incoming wave in physical space. The origin of the system is chosen so that in the absence of a wall the incident wave front would pass thru the origin at t=0. Then, noting that v=0 on Γ

$$\int_{S_i}\frac{\partial v}{\partial \nu_i}\,dS = \int_{S_i}\int\frac{\partial v}{\partial \nu_i}\,d\nu_i\,dy = \int_{y_3}^{y_4} v_{i\infty}\,dy - \int_{y_1}^{y_2} v_{\omega i}\,dy$$

where $(\ )_{\omega i}$, $(\ )_{i\infty}$ denote evaluation along the intersections of S_i, S_ω and S_i, S_∞ respectively. Referring to Figure (6), $(\ )_1$, $(\ )_2$

represent quantities evaluated at the intersections of S_i, S_ω, Γ while $(\ )_3$, $(\ )_4$ represents evaluation at the intersections of S_i, S_∞, Γ.

Turning to the integral over S_∞ we first consider

$$\int_{S_\infty} v \frac{\partial p_\infty}{\partial s} dS = \int_{y_3}^{y_4} \left[\int_o^{s(\Gamma)} v \frac{\partial p}{\partial s} ds \right] dy$$

Integrating by parts and noting that $p_\infty(o) = P_o$

$$\int_{S_\infty} v \frac{\partial p_\infty}{\partial s} dS = -P_o \int_{y_3}^{y_4} v_{i\infty} dy - \int_{y_3}^{y_4} \int_o^{s(\Gamma)} p_\infty \frac{\partial v}{\partial s} ds\, dy$$

With the incorporation of these results Equation (13) becomes:

$$p(\xi,\eta,\zeta) = \frac{1}{N\pi} \frac{\partial}{\partial \zeta} \left\{ \int_{S_\omega} p(x,y,z) \frac{\partial v}{\partial n} dS - P_o \int_{y_1}^{y_2} v_{\omega i}\, dy - 2 \int_{S_\infty} p_\infty(s) \frac{\partial v}{\partial s} ds \right\} \tag{14}$$

Since the incident pressure wave $P_\infty(s)$ is assumed independent of y the integral over S_∞ may be further simplified:

$$\int_{S_\infty} p_\infty(s) \frac{\partial v}{\partial s} dS = \int_{s=o}^{s(\Gamma)} p_\infty(s) \left[\int_{\eta-a(s)}^{\eta+a(s)} \frac{\partial v}{\partial s} dy \right] ds \tag{15}$$

where $a(s) = \sqrt{z'(s)^2 - x'(s)^2}$ and $\partial v/\partial s$ evaluated an S_∞ is:

$$\frac{\partial v}{\partial s} = \frac{1}{\sqrt{2}} \frac{\frac{z'x'}{r^2} - 1}{\sqrt{(z')^2 - r^2}}$$

Performing the integration with respect to y or equivalently with respect to y' and letting x' go to infinity:

$$\lim_{x' \to \infty} \left\{ \frac{1}{\sqrt{2}} \int_{y_1'}^{\sqrt{(z')^2-(x')^2}} \frac{1}{\sqrt{(z')^2-(x')^2-(y')^2}} \left[\frac{z'x'}{(x')^2+(y')^2} - 1 \right] dy' \right\}$$

$$= \begin{cases} -\pi\sqrt{2}\ , & y_1' = -\sqrt{(z')^2-(x')^2} \\ -\pi/\sqrt{2}\ , & y_1' = \text{const.} \end{cases}$$

In which we have noted that $z'/\sqrt{(z')^2} = -1$. Equation (15) then becomes

$$-\frac{\partial}{\partial \zeta} \int_{S_\infty} p_\infty \frac{\partial v}{\partial s}\, dS = \frac{\pi\sqrt{2}}{M} \frac{\partial}{\partial \zeta} \int_0^{(\zeta+\xi)/\sqrt{2}} p_\infty(s)\, ds = \frac{\pi}{M}\, p_\infty(\zeta+\xi) \qquad (16)$$

where

$$M = \begin{cases} 1\ , & y_1' = -\sqrt{(z')^2-(x')^2} \\ 2\ , & y_1' = \text{const.} \end{cases}$$

and $(\zeta+\xi)/C_o$ is the time measured from the arrival of the initial wave front at (ξ,η).

Considering the integral over S_ω in Equation (14) we note that for any cylindrical surface $y = y(x)$ we have

$$\frac{\partial v}{\partial n}\, dS = \frac{\partial v}{\partial y}\, dxdz - \frac{\partial v}{\partial x}\, dydz$$

so that

$$\frac{\partial v}{\partial n}\, dS = \frac{z'/r^2}{\sqrt{(z')^2-r^2}}\,[(y-\eta)dx-(x-\xi)dy]dz \tag{17}$$

Finally, noting that $v_{\omega i} = 0$ at y_1 and y_2

$$\frac{\partial}{\partial\zeta}\int_{y_1}^{y_2} v_{\omega i}\, dy = \int_{y_1}^{y_2} \frac{dy}{\sqrt{(x+\zeta)^2-r_{\omega i}^2}} \tag{18}$$

Substituting the results of Equations (16), (17) and (18) into Equation (14) for $p(\xi,\eta,\zeta)$ an expression is obtained for the pressure field resulting from an incident plane wave reflected by a two dimensional surface of the form represented in Figure (6).

$$p(\xi,\eta,\zeta) = \frac{2}{NM} p_\infty(\zeta+\xi) - \frac{p_o}{N\pi}\int_{y_1}^{y_2} \frac{dy}{\sqrt{(x+\zeta)^2-r_{\omega i}^2}}$$

$$+ \frac{1}{N\pi}\frac{\partial}{\partial\zeta}\int\int_{S_\omega} \frac{z' p(x,y,z)}{r^2\sqrt{(z')^2-r^2}} \left[(y-\eta)dx-(x-\xi)dy\right]dz \tag{19}$$

The first term on the right in Equation (19) represents the pressure of the incident disturbance as it propagates over the point (ξ,η). The remaining terms represent the contributions from the region of the wall within the zone of dependence of (ξ,η,ζ).

As it stands Equation (19) is not yet a completely explicit relation for the pressure but rather a representation of the original partial differential equation in a more convenient integral form. This is because the integral over S_ω cannot be evaluated until the pressure on the wall is known. To determine p_ω on the wall we note that when ξ,η take on values at the wall Equation (19) becomes an integral equation for p_ω the solution of which allows the determination of the last integral in Equation (19) for arbitrary ξ,η,ζ thus yielding an explicit relation for $p(\xi,\eta,\zeta)$.

V. Reflection from the Hyperbola xy = 1

Equation (19) will now be employed in treating the specific case of a plane wave propagating along the x axis with a step function pressure profile

$$p_\infty = P_o H(x+z)$$

incident on the surface xy = 1 in the first quadrant. A representation of the problem in x,y,z space is illustrated in Figure (7). Comparing this with Figure (6) it becomes evident that there will always be a region of S_ω extending to infinitely along the x axis, and $y_2' = -\eta$ so that the integer M in Equation (19) is set equal to 2. (See Equation (16)).

To determine p_ω let ξ,η take positions on the wall, i.e. $\xi = 1/\eta$. Similarly since the surface integrals are carried out along the wall we must set $y = 1/x$. Then, Equation (19) with M=2, N=1 becomes:

$$p_\omega(\xi,1/\xi,\zeta) = P_o - \frac{P_o}{\pi}\int_{wi} \frac{dx}{x^2\sqrt{(x+\zeta)^2-r_{wi}{}^2}}$$

$$-\frac{1}{\pi}\frac{\partial}{\partial\zeta}\int\int_{S_w} \frac{z'(x-\xi)^2}{\xi x^2 r^2} \cdot \frac{P_\omega(x,1/x,z)}{\sqrt{(z')^2-r^2}}\, dxdz \qquad (20)$$

where $\int_{wi}$ denotes integration along those portions of the intersection of S_ω and S_i within the characteristic cone Γ or "zone of influence". Integrations are to be carried out in the sense of increasing x.

Once p_ω is determined by Equation (20) the pressure at any arbitrary point in space-time is obtained by letting ξ,η,ζ take independent values while the variables of integrations are, of course, still related by xy = 1. Thus, with M=2, N=2;

$$p(\xi,\eta,\zeta) = \frac{P_o}{2} - \frac{P_o}{2\pi}\int_{wi} \frac{dx/x^2}{\sqrt{(z')^2-r_{wi}^2}}$$

$$+ \frac{1}{2\pi}\frac{\partial}{\partial\xi}\int\int_{S_w} \frac{z'p_w(x,1/x,z)\ [(\frac{1}{x} - \eta) + \frac{x-\xi}{x^2}]\ dxdz}{r^2\sqrt{(z')^2-r^2}} \tag{21}$$

Equation (20) for p_ω is sufficiently complex to discourage attempts at an analytic solution at this stage although a numerical solution with the aid computers should prove reasonably straightforward. Nevertheless a considerable amount of useful information may be derived from Equation (21) if we introduce some reasonable approximations for p_ω. Specifically we shall consider values of ξ,η,ζ for which the radical $\sqrt{(z')^2-r^2}$ occurring in the integral terms remains small over most of the range of integration. Under these circumstances the contributions from the integrals should be large and in addition we are able to use the simplifying approximation $x+\zeta \approx r$. To derive the conditions under which this approximation is valid we note that on the characteristic cone $\sqrt{(z')^2-r^2} = 0$. Therefore this radical will be small on those portions of S_ω which lie near the surface Γ. From Figure (7) it is evident that when the intersections WI and ΓI lie close to each other the value of $\sqrt{(z')^2-r^2}$ will be small over the region S_w. Furthermore the range of z in the integral over S_w is restricted to values near $z = -x$, and it is reasonable to assume that in this instance for $c<x<b$

$$p_\omega(x,1/x,z) \approx p_\omega(x,1/x,-x) = 2p_o \tag{22}$$

Returning to Figure (7) and considering the region S_{w2} we see that over a large part of this region the range of z is large and in fact becomes infinite with x. However, over this region the slope of the wall is vanishingly small as x becomes infinite, allowing the introduction of a convenient approximation to the disturbance pressure on S_{w2}. Referring to Figure (8) let us assume that for $x>x_p$ the slope of the wall is so small as to cause negligible disturbance. The curve γ represents the limit of the range of incluence of the wall section between x_z and x_p. Outside this range a given point in space will be influenced only by the section of the wall for which

$x>x_p$ and since this influence is assumed to be negligible the pressure disturbance beyond γ will be P_o. The pressure distribution along the wall is also illustrated in Figure (8). Between x_t and $x_\alpha \cos\theta$ the distribution along the wall is approximately $2P_o$. Beyond x_α the disturbance level falls to P_o. As both x_t and x_p move out to infinity along the wall the region between x_t and x_γ becomes vanishingly small while the disturbance beyond x_γ drops rapidly to P_o. Thus, over those sections of S_{w2} over which the slope of the wall is small the disturbance is approximately P_o except for a small region near the wave front where the pressure is approximately $2P_o$. Translated into x,y,z space this means that in the field described by Figure (7) we may adopt the following approximations.

$$
\begin{aligned}
p_\omega &= 2P_o \quad \text{on } S_{w1} \quad c \le x \le b \\
p_\omega &= 2P_o \quad \text{on } S_{w2} \quad a \le x \le x_a \qquad (23)\\
p_\omega &= P_o \quad \text{on } S_{w2} \quad x_a \le x < \infty
\end{aligned}
$$

In keeping with the specification, made in the introduction, that the reflecting surface has plane asymptotes we may assume that for $x>x_a$ the wall is sufficiently near the x axis to make the approximation $(y-\eta)dx - (x-\xi)dy \approx -\eta dx$. Introducing these approximations into Equation (21) we are now able to perform the integrations with respect to z. After carrying out the differentation of the last term with respect to ζ Equation (21) becomes:

$$p(\xi,\eta,\zeta) = \frac{P_o}{2} - \frac{1}{2\pi}\left\{\int_{c(\zeta)}^{b(\zeta)} + \int_{a(\zeta)}^{x_a(\zeta)} + \int_{x_a(\zeta)}^{\infty} \left(\frac{P_o dx}{x^2\sqrt{(x+\zeta)^2-r^2}}\right)\right.$$

$$+ \int_{c(\zeta)}^{b(\zeta)} + \int_{a(\zeta)}^{x_a(\zeta)} \left(\frac{2P_o(x+\zeta)[(1-\eta x)x+(x-\xi)]}{r^2x^2\sqrt{(x+\zeta)^2-r^2}}\, dx\right) + \int_{x_a(\zeta)}^{\infty} \frac{P_o(x+\zeta)(-\eta)dx}{r^2\sqrt{(x+\zeta)^2-r^2}}$$

$$\left. + P_o \frac{\partial x_a}{\partial \zeta}\sqrt{(x_a+\zeta)^2-(x_a-\xi)^2-(\frac{1}{x_a}-\eta)^2}\left[\frac{(1-\eta x_a)x_a+(x_a-\xi)}{x_a^2[(x_a-\xi)^2+(\frac{1}{x_a}-\eta)^2]}\right]\right\} \tag{24}$$

in which we recall that the limits a,b,c are the values of x at which the curves WI and ΓI intersect and as such they are functions of ξ,η,ζ. We also note that $\sqrt{(z')^2-r^2} = 0$ at these points since they lie on Γ. x_a is the value of x at which ΓI intersects the x axis, thus we find that:

$$x_a = \frac{\eta^2}{2(\zeta+\xi)} - \frac{\zeta-\xi}{2} \tag{25}$$

It is assumed that beyond the point $x=x_a$ WI is approximated by the x axis as a result of which we may set

$$\left.\begin{aligned} r^2 &\approx (x-\xi)^2+\eta^2 \\ \sqrt{(x+\zeta)^2-r^2} &\approx \sqrt{2(\zeta+\xi)(x-x_a)} \end{aligned}\right\} x>x_a$$

The last integral appearing in Equation (24) may, after a good deal of algebra, be directly evaluated upon writing in the form

$$\frac{P_o}{\sqrt{2(\zeta+\xi)}} \int_{x_a}^{\infty} \frac{(x+\zeta)(-\eta)dx}{[(x-\xi)^2+\eta^2]\sqrt{(x-x_a)}} = -\pi P_o \tag{26}$$

In the same manner we evaluate: the third integral in Equation (24):

$$\frac{P_o}{\sqrt{2(\zeta+\xi)}} \int_{x_a}^{\infty} \frac{dx}{x^2\sqrt{(x-x_a)}} = \frac{P_o\pi}{2x_a^{3/2}\sqrt{2(\zeta+\xi)}} \tag{27}$$

Over the range of computations considered in this paper this last term will be found to make negligible contribution to the pressure. Basically this is because we have:

$$\frac{dx}{x^2\sqrt{(x-x_a)}} = \frac{dy}{\sqrt{(x-x_a)}} \quad \text{on the wall.}$$

When the assumption that WI may be replaced by the x axis is valid we have $dy \approx 0$ and the integral becomes small. The term in $\partial x_a/\partial\zeta$ is also found to be small over the range of computations considered here. The remaining integrals are evaluated over ranges for which $\sqrt{(x+\zeta)^2-r^2}$ is small. Consequently:

$$r \approx x+\zeta, \qquad \frac{x+\zeta}{r^2} \approx \frac{1}{x+\zeta}$$

as they appear in the remaining integrals. On the hyperbola $y = 1/x$ the quantity $(x+\zeta)^2 - r^2$ takes the form $k\,\underline{\overline{X}}^3(x)/x^2$ where $\underline{\overline{X}}^3$ represents a third degree polynomial. The zeros of this polynomial are those points at which ΓI intersects WI, ie. at the intersection of S_w, Γ and S_I. These are the points $x = a,b,c$ (assuming that ξ,η,ζ are in the range for which three intersections occur); consequently we may write

$$\underline{\overline{X}}^3 = (x-a)(x-b)(x-c)$$

or more explicitly:

$$\sqrt{(x+\zeta)^2 - r^2} = \begin{cases} \dfrac{1}{x}\sqrt{2(\zeta+\xi)(a-x)(b-x)(x-c)}, & c<x\leq b \\ \\ \dfrac{1}{x}\sqrt{2(\zeta+\xi)(x-a)(x-b)(x-c)}, & a<x\leq x_a \end{cases}$$

Finally, assuming $\zeta \neq 0$ and splitting the factor $1/x(x+\zeta)$ in Equation (24) into partial fractions we obtain

$$p(\xi,\eta,\zeta) = P_o\left\{1 + \frac{H_1 - (1+\eta\zeta)H_2}{\pi\zeta}\right\}, \quad \zeta \neq 0 \tag{28}$$

where

$$H_1 = \frac{1}{\sqrt{2(\zeta+\xi)}} \int_c^b + \int_a^{x_a} \frac{(\xi-\zeta-x)dx}{x\sqrt{(x-a)(x-b)(x-c)}}$$

$$H_2 = \frac{1}{\sqrt{2(\zeta+\xi)}} \int_c^b + \int_a^{x_a} \frac{(\frac{\zeta+\xi}{1+\eta} - x)dx}{(x+\zeta)\sqrt{(x-a)(x-b)(x-c)}}$$

H_1 and H_2 may be evaluated in terms of elliptic integrals[4] for which tables are available.[5] After some algebraic manipulations H_1 and H_2 may be reduced to the following forms:

$$H_1 = \sqrt{\frac{2}{(\zeta+\xi)(a-c)}} \left\{ \frac{\zeta-\xi}{c} \left[F(\phi,k)+F(\pi/2,k)-\pi(\phi, \frac{-c}{a-c}, k) \right.\right.$$

$$\left. -\pi(\pi/2, \frac{-c}{a-c}, k) + \frac{c(a-c)}{ab}\left(\frac{\pi}{2} + \tan^{-1}\sqrt{\frac{ac}{b}\frac{x_a-a}{(x_a-b)(x_a-c)}}\right)\right]$$

$$\left. -F(\phi,k) - F(\pi/2,k) \right\} \tag{29}$$

$$H_2 = \sqrt{\frac{2}{(\zeta+\xi)(a-c)}} \left\{ \frac{\frac{\zeta+\xi}{1+\eta\zeta} + \zeta}{c+\zeta} \left[\pi\left(\phi, -\frac{b-c}{c+\zeta}, k\right) + \pi\left(\pi/2, -\frac{b-c}{c+\zeta}, k\right) \right.\right.$$

$$\left. + \sqrt{\frac{(c+\zeta)(a-c)}{(a+\zeta)(b+\zeta)}} \quad \tan^{-1}\sqrt{\frac{(c+\zeta)(a+\zeta)(x_a-a)}{(b+\zeta)(x_a-b)(x_a-c)}} - \tan^{-1}\sqrt{\frac{(a+\zeta)(b+\zeta)(x_a-a)(x_a-b)}{(c+\zeta)(x_a-c)(a-b)^2}} \right]$$

$$\left. -F(\phi,k) - F(\pi/2,k) \right\} \tag{30}$$

where $F(\phi,k)$, $\pi(\phi,\alpha^2,k)$ are the elliptic integrals:[4]

$$f(\phi,k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{31}$$

$$\pi(\phi,\alpha^2,k) = \int_0^\phi \frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{32}$$

with $\phi = \sin^{-1}\sqrt{\frac{x_a-a}{x_a-b}}$, $\quad k^2 = \frac{b-c}{a-c}$

To obtain the above expressions for H_1 and H_2 use has been made of addition formulas relating elliptic integrals with different arguments. These have been introduced in order to obtain ranges of α^2 and k for which tables are available.

The solution given in Equation (28) is valid as long as $\zeta \neq 0$ a restriction introduced when $1/x(x+\zeta)$ was split into partial fractions. When ζ is near zero, Equation (28) becomes inconvenient for computation and a modified form of the solution must be used. Thus, assuming ζ may be neglected compared to x and proceeding from Equation (24) as before we obtain for $\zeta \gtrsim 0$:

$$p(\xi,\eta,\zeta) \quad P_o \left\{ 1 + \frac{\eta J_1 + \xi J_2}{\pi} \right\}, \qquad \zeta \gtrsim 0 \tag{33}$$

where

$$J_1 = \frac{1}{\sqrt{2(\zeta+\xi)}} \left\{ \int_e^b + \int_a^{x_a} \frac{(x - \frac{5}{2\eta})dx}{x\sqrt{(x-a)(x-b)(x-c)}} \right\}$$

$$J_2 = \frac{1}{\sqrt{2(\zeta+\xi)}} \left\{ \int_c^b + \int_a^{x_a} \frac{dx}{x^2\sqrt{(x-a)(x-b)(x-c)}} \right\}$$

which may again be evaluated in terms of elliptic integrals

$$J_1 = -\sqrt{\frac{2}{(\zeta+\xi)(a-c)}}\left\{\frac{1}{\eta c}\left[F(\phi,k) + F(\pi/2,k) - \pi(\phi, \frac{-c}{a-c}, k)\right.\right.$$

$$\left. -\pi(\pi/2, \frac{-c}{a-c}, k) + \sqrt{\frac{c(a-c)}{ab}}\left(\frac{\pi}{2} + \tan^{-1}\sqrt{\frac{ac}{b}\frac{(x_a-a)}{(x_a-b)(x_a-c)}}\right)\right]$$

$$\left. -F(\phi,k) - F(\pi/2,k)\right\}$$

$$J_2 = \sqrt{\frac{2}{(\zeta+\xi)(a-c)}}\left\{\frac{1}{c^2}\left(1 + \frac{(b-c)(a-c)}{2ab}\right.\right. \pi(\pi/2, \frac{-(b-c)}{c}, k)$$

$$\frac{-(a-c)}{2abC} E(\pi/2,k) - \frac{1}{2bC} F(\pi/2,k) + \frac{1}{ac} F(\phi,k)$$

$$\left. -\frac{1}{a}\pi(\phi, \frac{-c}{a-c}, k) + \frac{1}{a}\sqrt{\frac{a-c}{ab^c}}\tan^{-1}\sqrt{\frac{ac}{b}\frac{(x_a-a)}{(x_a-b)}\frac{1}{(x_a-c)}}\right\}$$

with

$$E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\, d\phi$$

and F,π,ϕ,k defined as before in Equations (31) and (32).

VI. Focusing of Pressure Disturbances

Before turning to the computation of the pressure predicted by Equations (28) and (33) we shall investigate one of the more important properties of these solutions, namely, the focusing effect of the curved surface on the reflected pressure distribution. This property is revealed upon examination of the integrals in H_1, H_2, J_1, J_2. Thus, when the points a and b coincide we have

$$\sqrt{(x-a)(x-b)(x-c)} = |a-x|\sqrt{x-c}$$

and the integrals become singular, yielding infinite pressure for the corresponding values of ξ,η,ζ . The physical significance of this singularity may be appreciated with the aid of Figure (9) which illustrates the projection of WI and ΓI on the xy plane for the case $\xi=1$, $\eta=2$ and $\zeta<1$. The coincidence of a and b means that WI and ΓI are tangent at the point of coincidence. Now in the general case a,b,c represent three discrete points from which a reflected pressure disturbance will arrive at ξ,η simultaneously at the instant ζ. However when a and b coincide a disturbance arrives at $\xi\eta$ simultaneously from an infinity of points in the neighborhood of the point of tangency. We must note that the order of contact of WI and ΓI is a factor of prime importance. This is illustrated by the fact that when b=c no singularity occurs although WI is again tangent to ΓI. This comes about because the range of integration of the first integrals appearing in H and J extends from c to b and hence shrinks to zero while the integrals from a to x_a do not contain the singularity of the integrand at x=b.

It is of interest to determine the locus of all points (ξ,η,ζ) at which singularities of the pressure occur. Although a linear theory is no longer valid in the neighborhood of these points the locus would provide a useful indication of where and when reflected pressure disturbances are large enough to give cause for concern. Specifically we seek values of ξ,η,ζ for which a=b. Substituting $y = 1/x$ into the expression for $(x+\zeta)^2-r^2$ we obtain

$$(x+\zeta)^2-r^2 = \frac{2(\zeta+\xi)}{x^2}\,\underline{\overline{X}}^3 \tag{34}$$

$$\underline{\overline{X}}^3 = x^3 + \left(\frac{\zeta-\xi}{2} - \frac{\eta^2}{2(\zeta+\xi)}\right)x^2 + \frac{\eta}{2(\zeta+\xi)}\,x - \frac{1}{2(\zeta+\xi)}$$

The polynomial $\underline{\overline{X}}^3$ shall be represented simply as:

$$\underline{\overline{X}}^3 = x^3+dx^2+\ell x+f \tag{35}$$

Since a, b and c are solutions of $\underline{\overline{X}}^3 = 0$ we seek value of ξ,η,ζ for which there are three real roots of $\underline{\overline{X}}^3$, two of which are equal. From the algebraic theory of cubic equations we know that for this condition to obtain we must have for the discriminant

$$\frac{G^2}{4} + \frac{K^2}{27} = 0 \tag{36}$$

where

$$G = \frac{3\ell - d^2}{3}, \qquad K = \frac{2d^3 - 9\ell d + 27f}{27}$$

Before carrying out the substitutions indicated in Equations (34) to (36) it is much more convenient to introduce the variables;

$$\begin{aligned} R &= -1/2(\zeta+\xi) \\ S &= (\zeta-\xi)/2 \end{aligned} \tag{37}$$

R = constant represents lines parallel to S_I while S = constant represents lines normal to S_I. Maintaining R constant while varying S translates the parabolic curve ΓI without changing its shape. In terms of these new variables we have;

$$\begin{aligned} d &= S + \eta^2 R \\ \ell &= -2\eta R \\ f &= R \end{aligned}$$

and Equation (36) becomes:

$$R^2(\eta^4 S + \eta^3) + R(2\eta^2 S^2 + 9\eta S + \frac{27}{4}) + S^3 = 0 \tag{38}$$

This last equation is most conveniently solved for R as a function of η and S since it is only quadratic in R. The two solutions obtained resulted from the fact that xy = 1 has a second branch in the third quadrant while the characteristic cone $z'^2 - r^2 = 0$ has two sections given by $z' = \pm r$. Choosing the solution for R corresponding to the tangency of ΓI and WI in the first quadrant:

$$R = -\frac{Q(\eta s) + \sqrt{Q^2(\eta s) - 4(\eta s)^3(1+\eta s)}}{2\eta^3(1+\eta s)} \tag{39}$$

$$Q(\eta s) = 2(\eta s)^2 + 9(\eta s) + 27/4$$

By eliminating R between Equations (37) and (39) ζ and ξ may be written in terms of η and the parameter ηs.

$$\begin{aligned} \zeta &= (\eta s)/\eta - \eta^3 \mathcal{F}(\eta s) \\ \xi &= -(\eta s)/\eta - \eta^3 \mathcal{F}(\eta s) \end{aligned} \tag{40}$$

where

$$\mathcal{F}(\eta s) = -\frac{(1+\eta s/2}{Q + \sqrt{Q^2 - 4(\eta s)^3(1+\eta s)}}$$

For a fixed value of η we assume values of the parameter ηs and compute the corresponding values of ζ and ξ. The results yield a curve in the plane η = constant along which the pressure predicted by linear theory becomes infinite, i.e. along which focusing occurs. For convenience the function $\mathcal{J}(\eta s)$ is plotted in Figure (10) and curves of ζ vs. ξ for focusing are shown in Figure (11) for $1 \leq \eta \leq 3$. To the left these curves terminate on the reflecting hyperbola itself. Beyond the right hand limit tangency with a and b coincident is no longer possible.

VII. Time Dependence of the Pressure

From Figure (11) we see that the locus of singularities form a surface in ξ,η,ζ space. It is instructive to choose a point below this surface i.e.: before the pressure singularity occurs, and to calculate the pressure as a function of time as the singularity is approached. For this purpose we choose the point $\xi,\eta = (1,2)$. At this location the singularity occurs at $\zeta = 0$. The projection of ΓI for this case was illustrated in Figure (8). Thus at $\zeta = -.08$ we obtain curve A indicating that at this instant the point (1,2) receives the initial reflection from the region of high curvature of the hyperbola. Beyond this value of ζ we may expect a rapid increase in pressure to occur since a small increase in ζ brings a large section of WI within the zone of dependence of (ξ,η,ζ). The reflected component $p_H = p(\xi\eta\zeta) - H(\zeta + \xi)$ due to a unit step function disturbance ($P_o = 1$) has been computed from Equation (33) and plotted in Figure (12).

VIII. The Linear Pressure Pulse

In the last section the reflection of a step function pulse was considered. In this section we consider the reflection of a second basic pulse specified by:

$$p_\infty = (\zeta + \xi)H(\zeta + \xi) \tag{42}$$

i.e. a disturbance which varies linearly with unit slope behind the initial wave front. In this case the initial pressure P_o on SI is zero. In keeping with the assumptions which lead to the approximations for p_ω given in Equation (23) we obtain

$$\begin{aligned} p_\omega &= 2(z+x) \quad \text{on } S_{\omega 1} \quad c \le x \le b \\ p_\omega &= 2(z+x) \quad \text{on } S_{\omega 2} \quad a \le x \le X_a \\ p_\omega &= (z+x) \quad \text{on } S_{\omega 2} \quad X_a < x < \infty \end{aligned}$$

Substituting these values of P_o and p_ω into Equation (19) we again obtain a result valid for the range ξ, η, ζ for which $z'^2 - r^2$ is small.

$$p(\xi,\eta,\zeta) = \frac{\zeta+\xi}{2} - \frac{1}{\pi}\left\{ 2\int_c^b + 2\int_a^{X_a} + \int_{X_a}^{\infty} \frac{\sqrt{(x+\zeta)^2 - r^2}}{r^2} [y-\eta)dx - (x-\xi)dy] \right\} \tag{44}$$

Over the last interval $X_a \le x < \infty$ the same approximations used in obtaining Equation (26) are again introduced

$$\int_{X_a}^{\infty} \frac{\sqrt{(x+\zeta)^2 - r^2}}{r^2} \; [(y-\eta)dx - (x-\xi)dy$$

$$-\eta\sqrt{2(\zeta+\xi)} \int_{X_a}^{\infty} \frac{\sqrt{X-X_a}}{(x-\xi)^2+\eta^2}\, dx = -\frac{\pi(\zeta+\xi)}{2} \qquad (45)$$

Integration over the remaining intervals in Equation (44) may be performed with the aid of a further approximation. From Figure (13) it is evident that for values of ξ,η,ζ of interest here the sections of the wall included in the intervals $c \le x \le b$, $a \le x \le X_a$ may be approximated by two linear segments passing through the end points X_a,A and b,c. These segments are represented by:

$$\begin{aligned} y &= -x/bc + (b+c) \qquad c \le x \le b \\ y &= -x/aX_a + (a+X_a) \qquad a \le x \le X_a \end{aligned} \qquad (46)$$

Again letting $r^2 \approx (x+\zeta)^2$ in these intervals and introducing the results of Equation (45) we obtain

$$p(\xi,\eta,\zeta) = (\zeta+\xi) - \left(\frac{1}{b} + \frac{1}{c} - \frac{\xi}{bc} - \eta\right) \int_c^b \frac{\sqrt{(b-x)(x-c)}}{(x+\zeta)^2}\,dx$$

$$- \left(\frac{1}{X_a} + \frac{1}{a} - \frac{\xi}{aX_a} - \eta\right) \int_a^{X_a} \frac{\sqrt{(X_a-x)(x-a)}}{(x+\zeta)^2}\,dx \tag{47}$$

Evaluation of the integrals in Equation (47) then yields an explicit expression for the pressure distribution due to the reflection of a linear pulse of unit slope over the range of ξ,η,ζ such that z'^2-r^2 is small on $S_{\omega 1}$ for $c\le x\le b$, $a\le x\le X_a$ and $y-\eta\approx-\eta$ for $X_a\le x<\infty$.

$$p(\xi,\eta,\zeta) = (\zeta+\xi) + \frac{\xi+\eta bC-(b+c)}{2(bC)^2}\left[\sqrt{\frac{b+\zeta}{c+\zeta}} - 1\right]\left[1 - \sqrt{\frac{c+\zeta}{b+\zeta}}\right]$$

$$+ \frac{\xi+\eta aX_a-(a+X_a)}{2(aX_a)^2}\left[\sqrt{\frac{X_a+\zeta}{a+\zeta}} - 1\right]\left[1 - \sqrt{\frac{a+\zeta}{X_a+\zeta}}\right] \tag{48}$$

The above pressure is again computed for the same region of ξ,η,ζ as in the previous case i.e. $(\xi,\eta) = (1,2)$ and ζ near zero. Figure (14) presents the contribution of the reflected pressure disturbance i.e. $P_L = p(\xi,\eta,\zeta) - (\zeta+\xi)H(\zeta+\xi)$ to the pressure field. From Equation (48) we see that there is no singularity at $\zeta = 0$ as in the previous instance although there is a magnification of the pressure due to the focusing effect of the hyperbola.

IX. Reflection of an N Wave

The results of sections (7) and (8) may be combined to study the reflection of the N wave typically encountered in sonic boom phenomena. Thus, an incident N wave of "duration" ζ_o may be constructed of unit step functions and linear pulses:

$$p_\infty(\xi,\eta,\zeta) = P_o[1 - \frac{2(\zeta+\xi)}{\zeta_o}] H(\zeta+\xi) + P_o[1 + \frac{2(\zeta+\xi+\zeta_o)}{\zeta_o}] H(\zeta+\xi+\zeta_o) \tag{50}$$

A sample calculation has been carried out for $\zeta_o = .02$, $(\xi,\eta) = (1,2)$ and $-.1< \zeta <0$. This represents a scale typical of the reflection of a sonic boom disturbance from a valley floor or from the side of a mountain. The result is given in Figure (12). From an examination of these results in the light of those presented in Figure (14) it is seen that the pressure field near the "focal point" is dominated by the reflection of the step function pulses $P_o(H(\zeta+\xi) + H(\zeta+\xi+\zeta_o)]$ at the beginning and end of the incident N wave. The linear component representing the structural details of wave contributes relatively little to the pressure in the neighborhood of the focal point.

X. Conclusion

An integral relation has been derived describing the pressure due to a plane wave of arbitrary wave form incident on a two dimensional curved surface with plane asymptotes. Pressure is given in terms of integrals involving the pressure distribution over the surface. In principle this relationship represents an integral equation for the pressure on the wall. Its form is such that this integral equation may be solved by numerical techniques. The integral relation would then become a direct expression for the pressure at any point in time and space.

From the form of the integral equation in the instance of reflection from a hyperbola it was found that singularities in the pressure field occur when the characteristic cone in xyz space is tangent to the intersection WI in such a way that WI lies inside the cone near the tangency point. These singularities represent focal points inasmuch as they are locations which are simultaneously influenced by reflection from a continuum of points on the reflector.

Depending on the magnitude of the incident acoustic wave the linear theory of course cannot be extended into the immediate neighborhood of the "focal points". However, the location of these singularities does serve to indicate the region in which pressure pulses are likely to be amplified beyond what would be ordinarily expected. In studying the reflection from a hyperbola it was found that singularities may occur along a surface in xyz space rather than at a single point.

Applying the results of the theory to reflection of an N wave from a hyperbola it was found that in the neighborhood of the focal points the principal contribution to pressure came from the reflection of the discontinuities at the leading and trailing edges of the wave. The continuous component representing the structural details of the incident wave was found to contribute little to the pressure near the focal points in xyz space.

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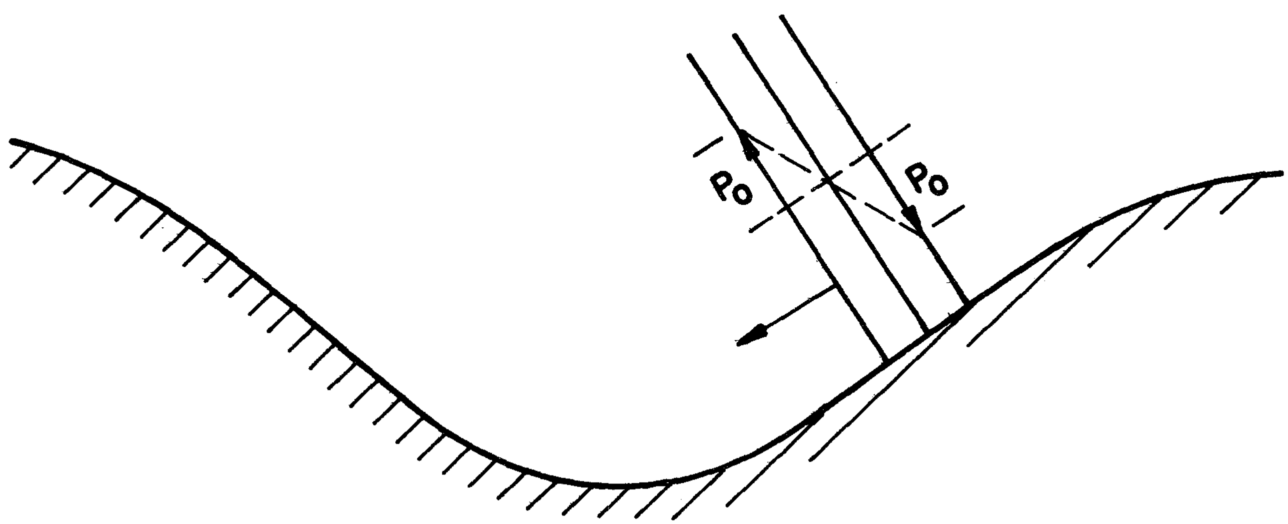


Fig. 1 "Sonic Boom" Pressure Wave Propagating Over a Valley or Depression

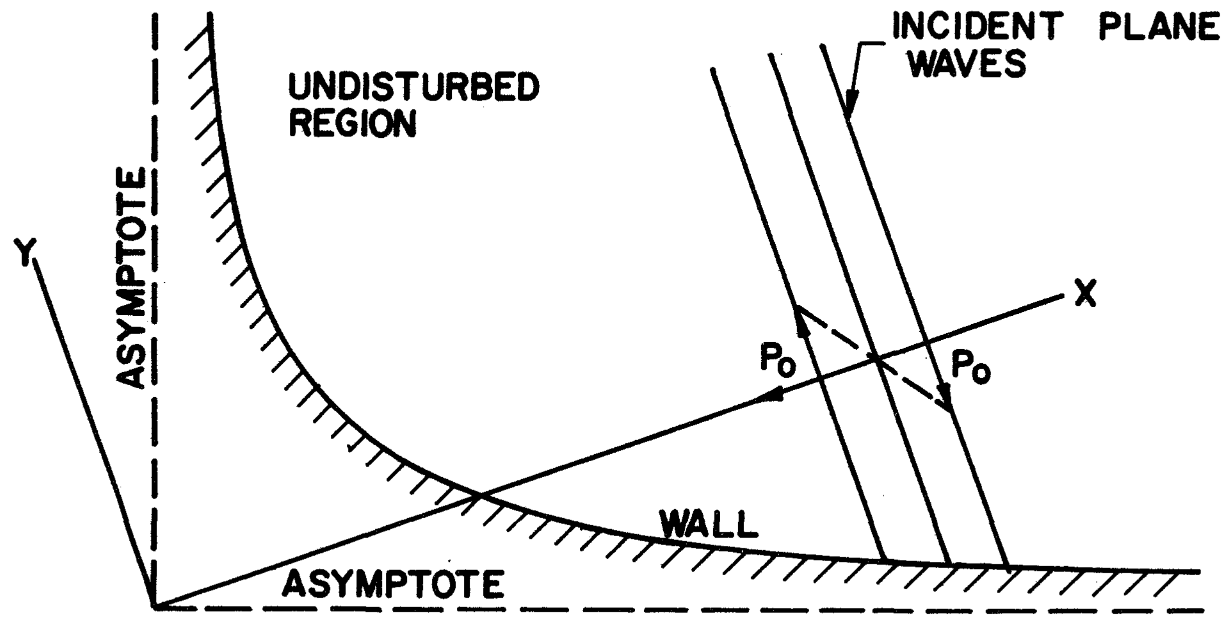
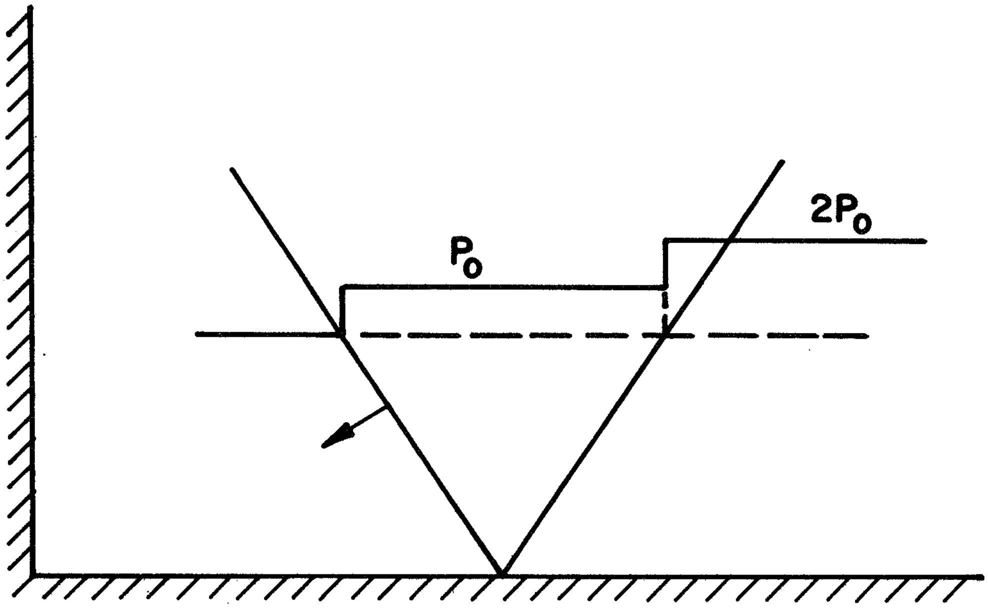
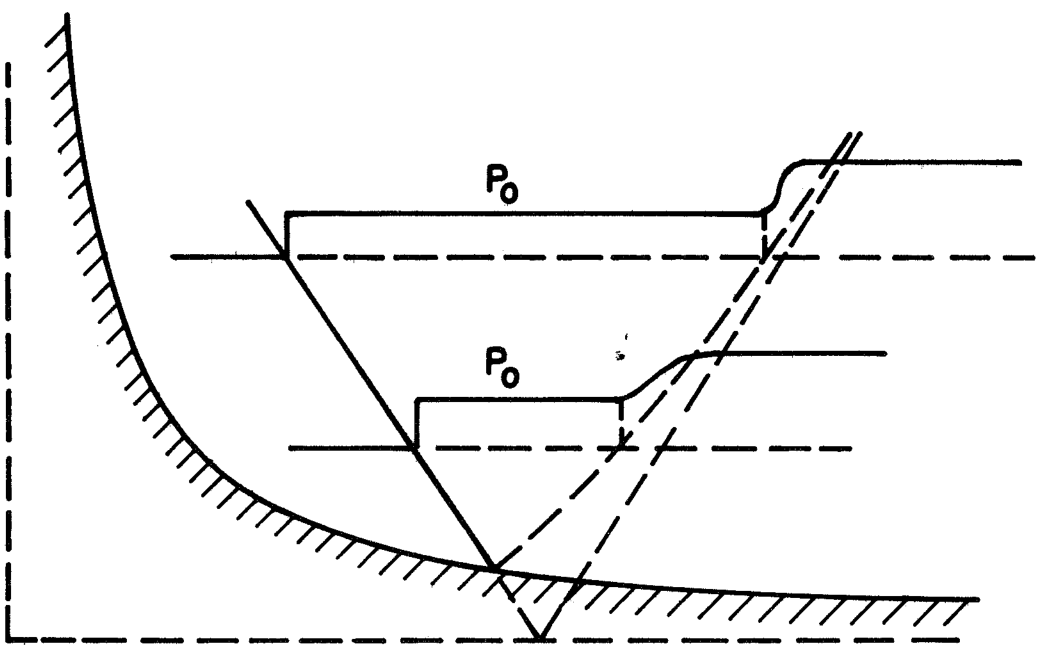


Fig. 2 Analytical Model of Sonic Boom Reflection from 2-Dim. Wall with Plane Asymptotes



REFLECTION FROM PLANE WALL



REFLECTION FROM CURVED SURFACE
WITH PLANE ASYMPTOTES

Fig. 3 Reflection of Plane Wave

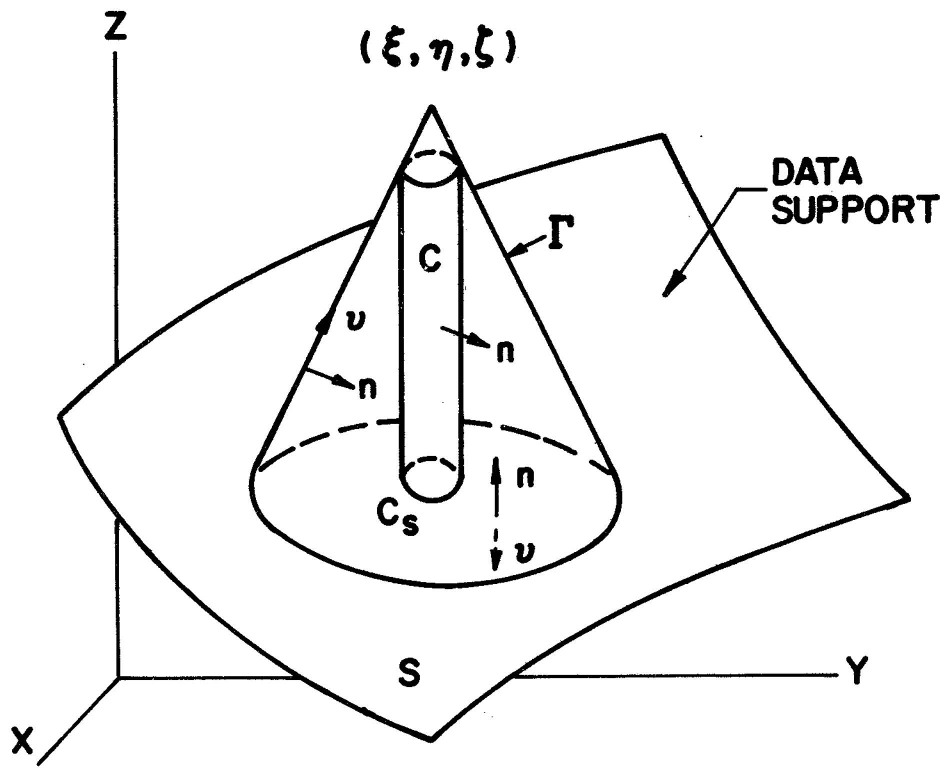


Fig. 4 Characteristic Cone and Data Support

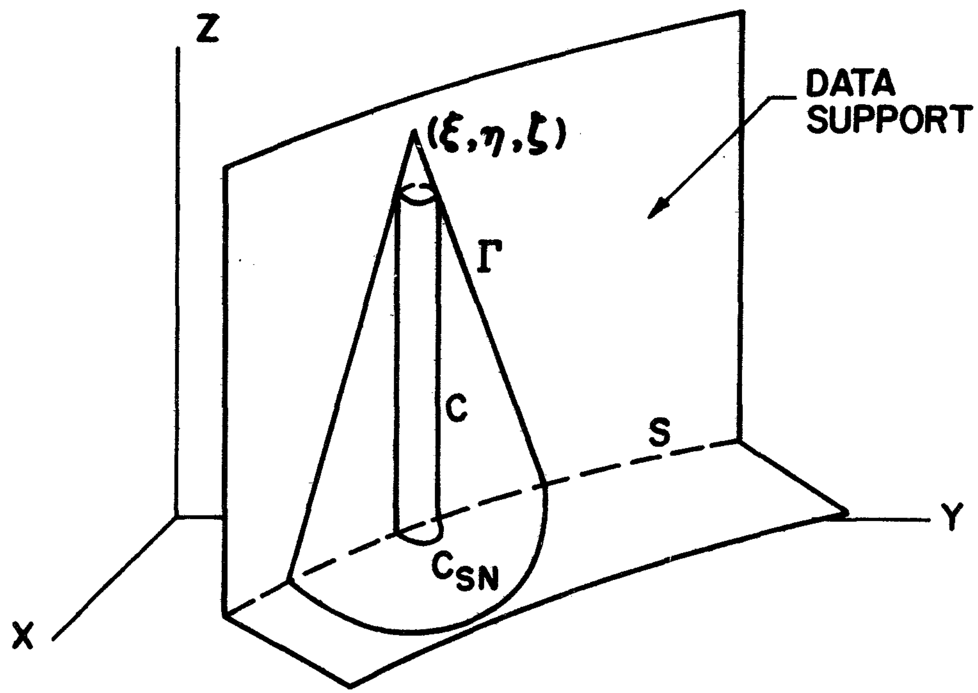


Fig. 5 Data Support Parallel to z axis

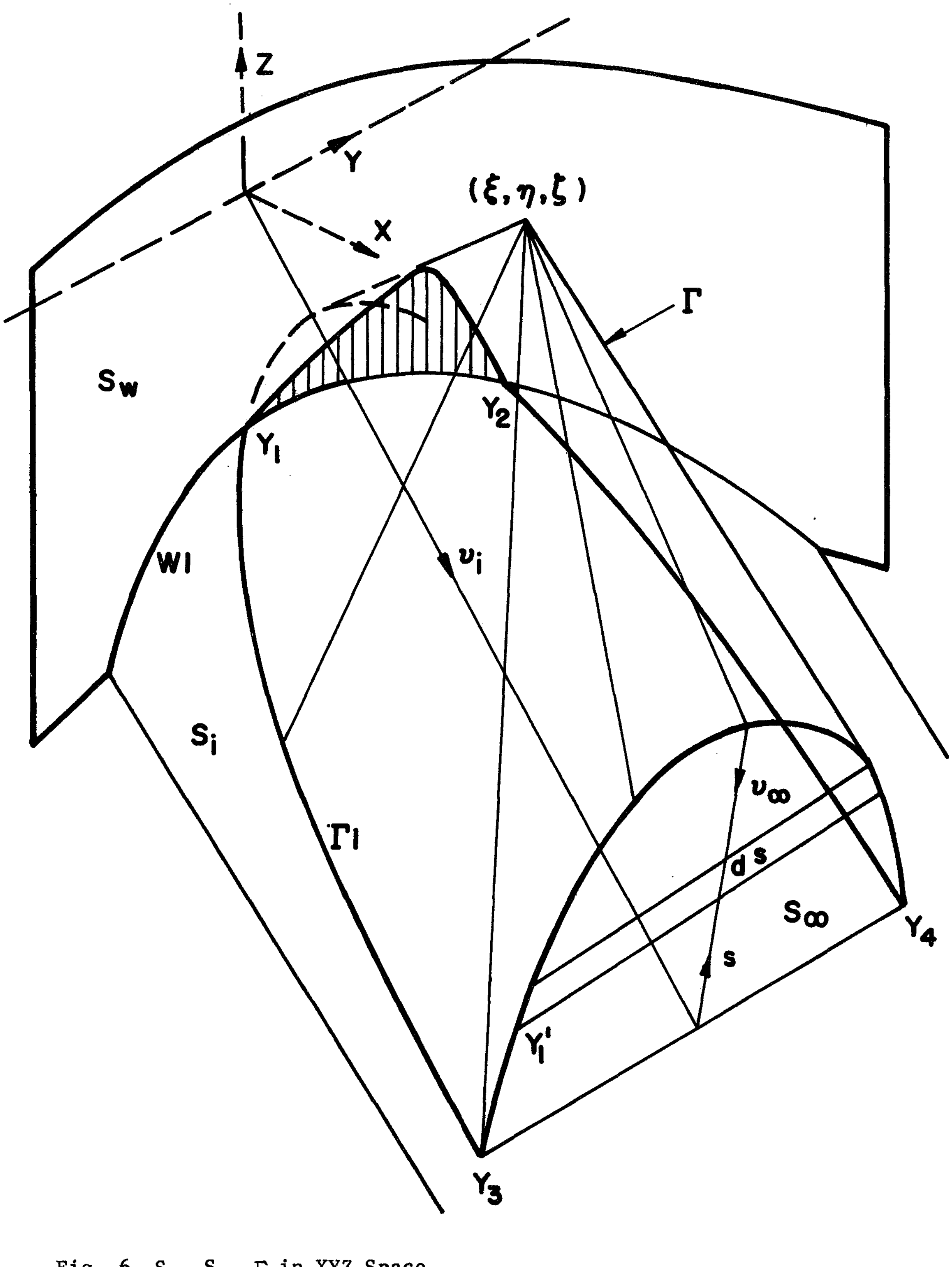


Fig. 6 S_ω, S_i, Γ in XYZ Space

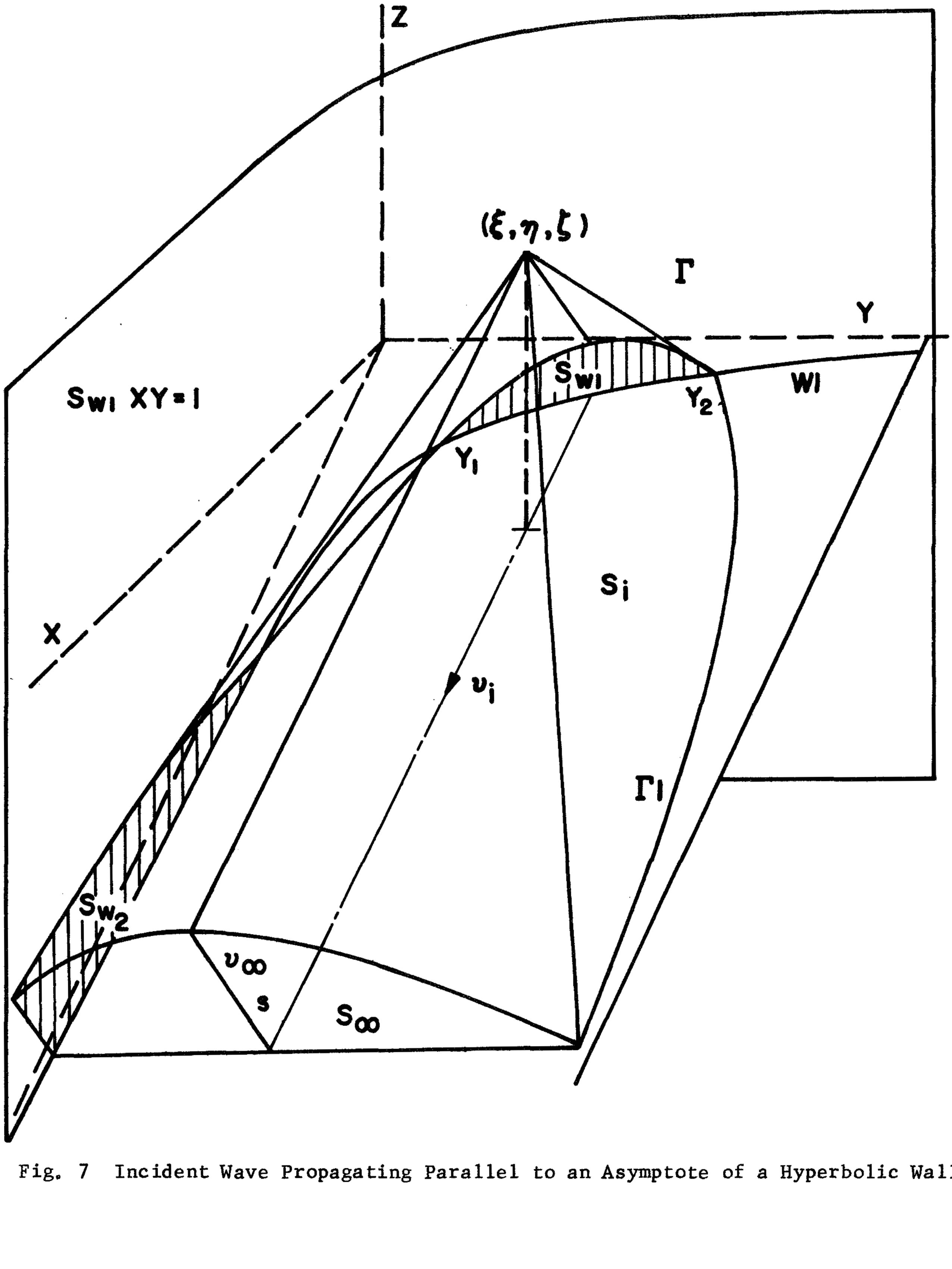


Fig. 7 Incident Wave Propagating Parallel to an Asymptote of a Hyperbolic Wall

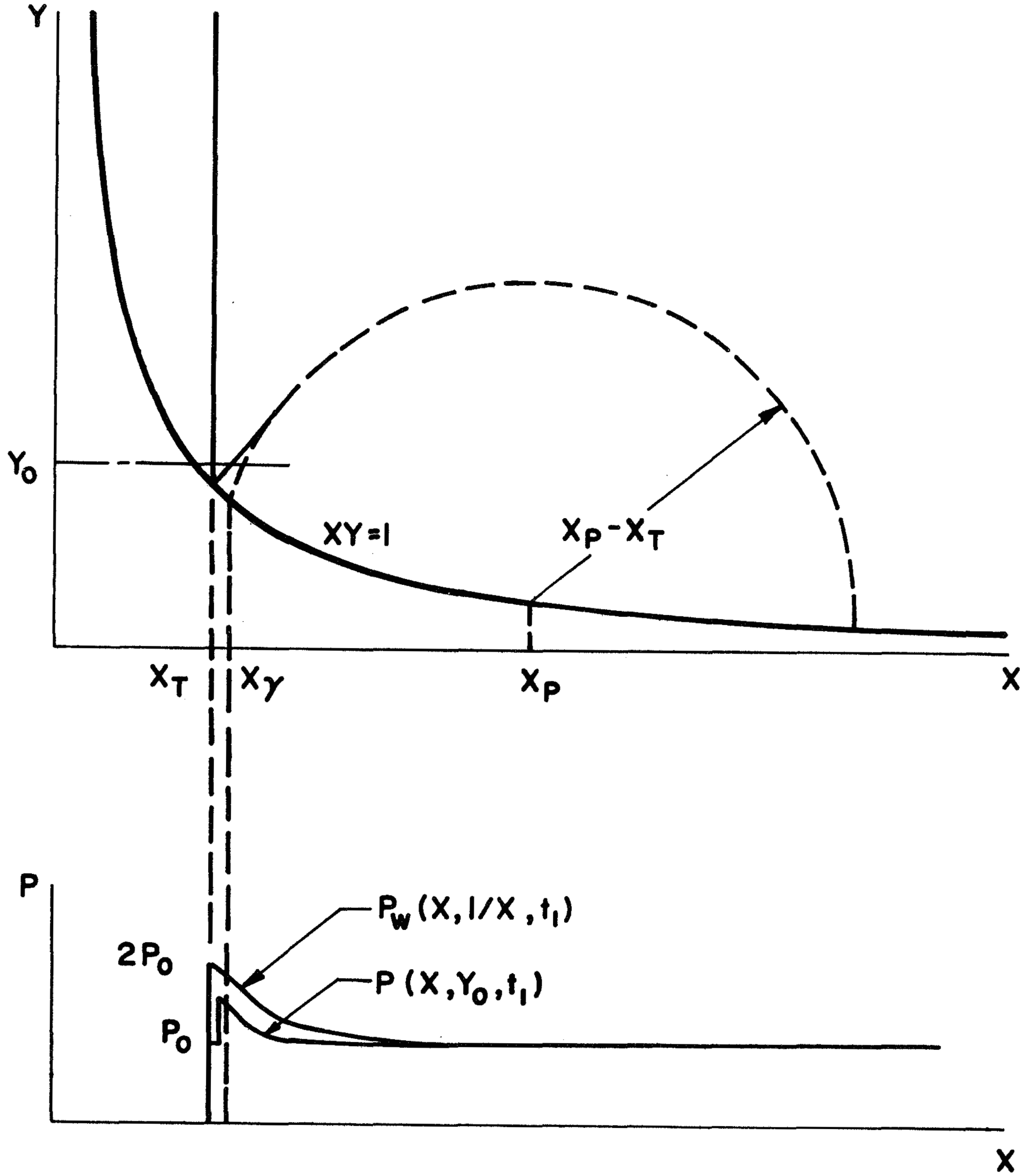


Fig. 8 Wave Pattern and Pressure Distribution

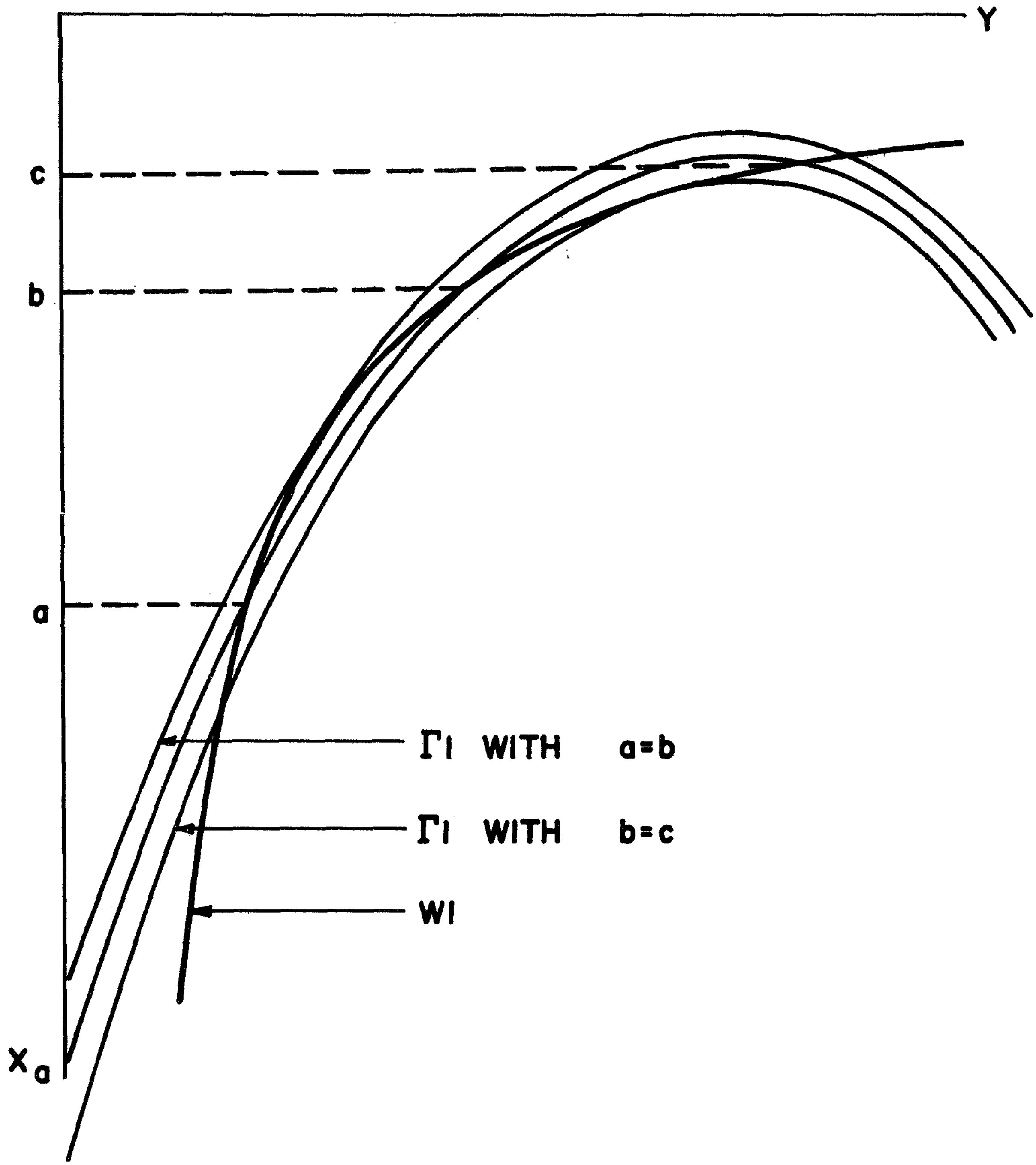


Fig. 9 Projection of ΓI and WI on XY Plane

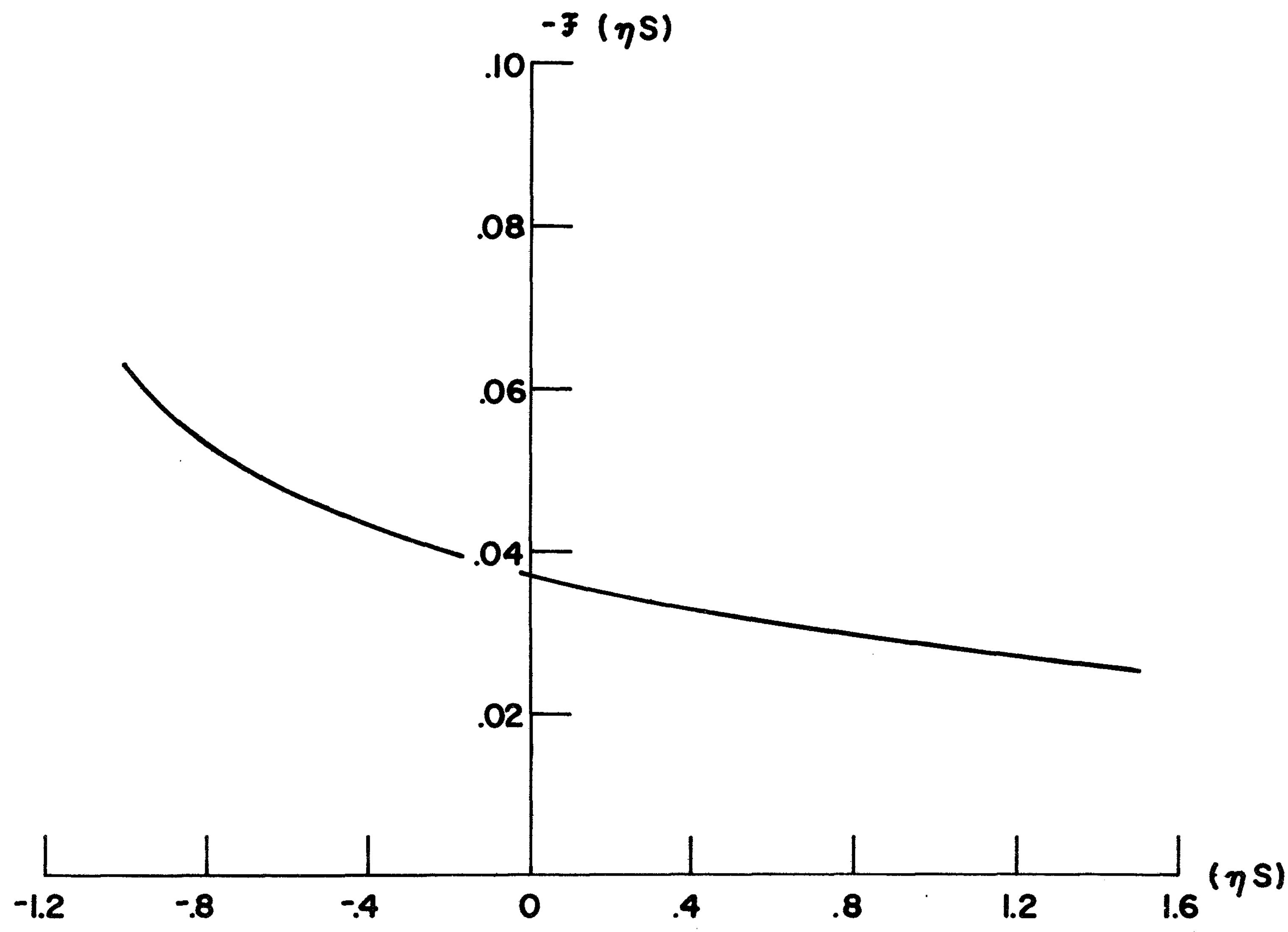


Fig. 10 $\mathcal{F}(\eta s)$ vs. (ηs)

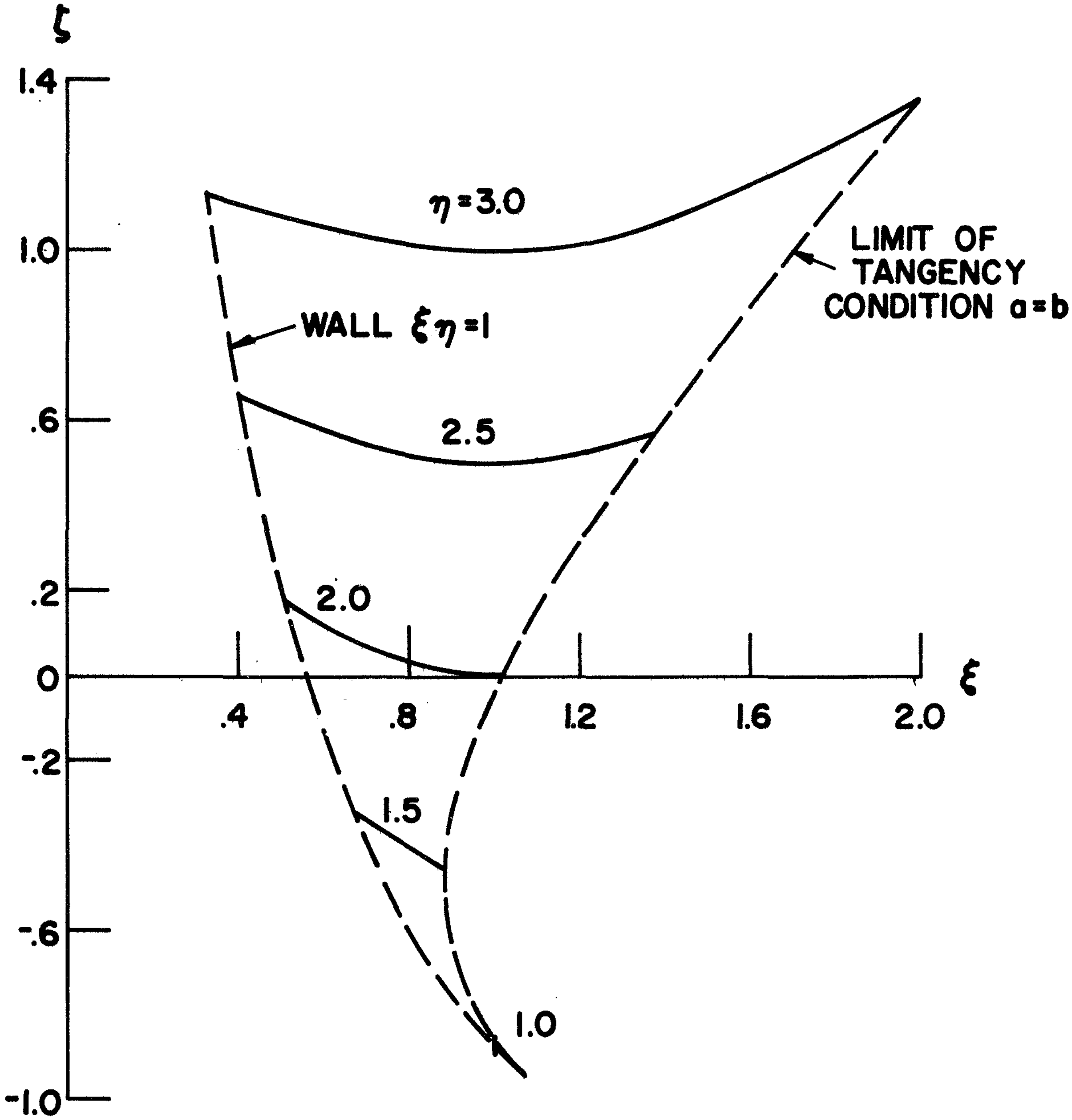


Fig. 11 Locus of Singular Points

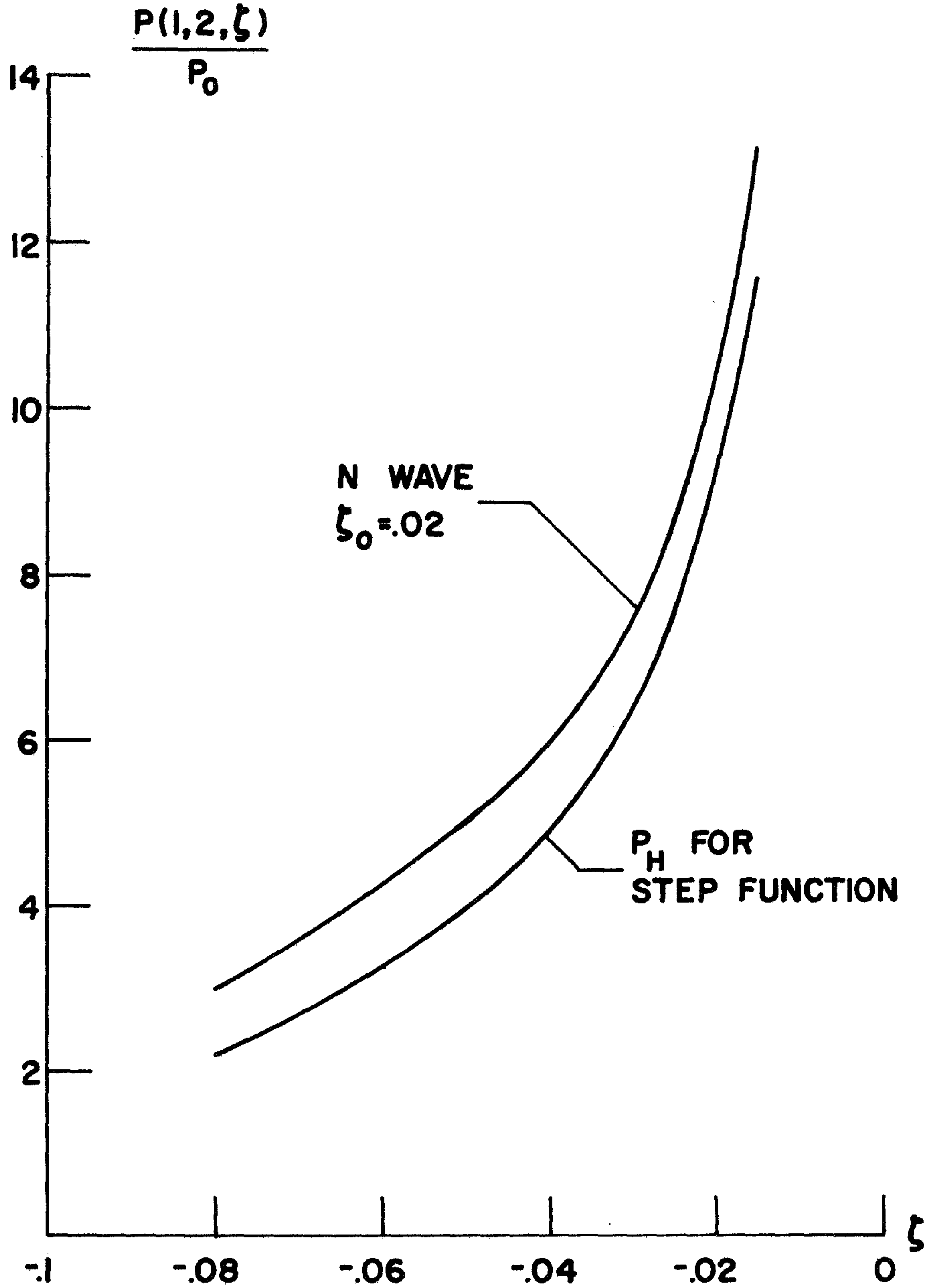


Fig. 12 Reflection of Step Function Pulse and N Wave from Hyperbolic Surface xy = 1

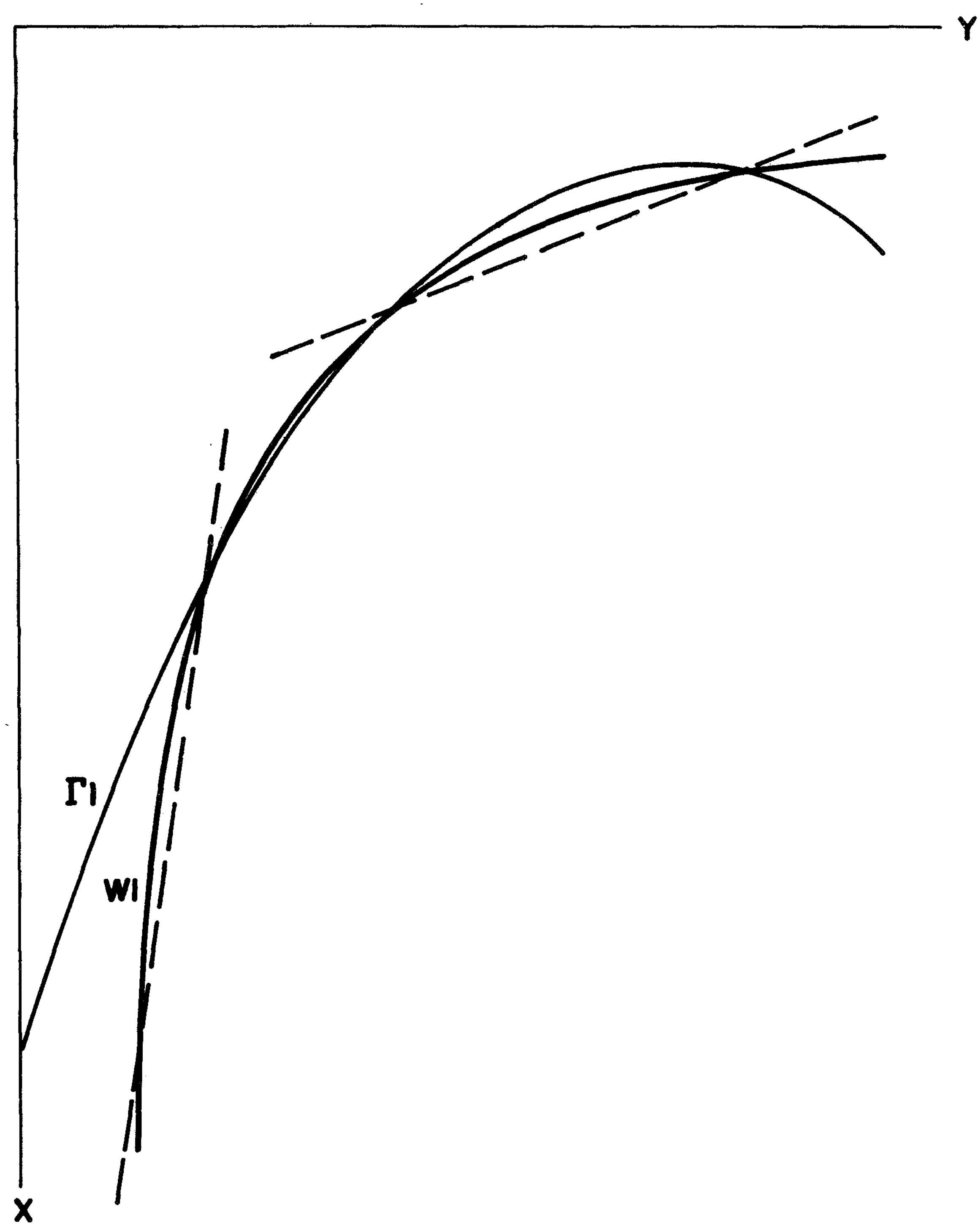


Fig. 13 Linear Wall Approximation

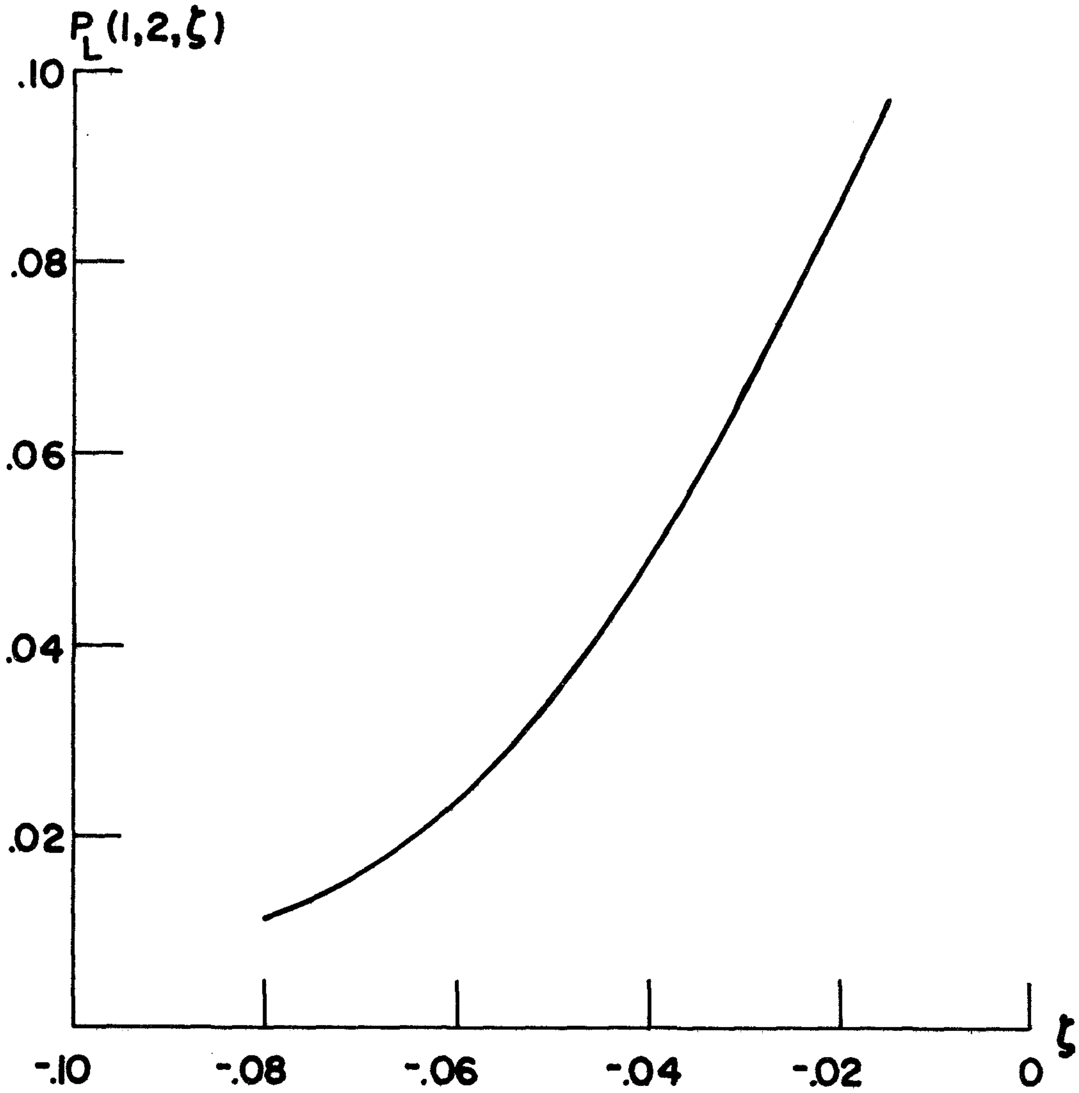


Fig. 14 Reflection of Linear Pulse $(\zeta+\xi)H(\zeta+\xi)$ from Hyperbolic Surface $xy = 1$