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THE CALCULATION OF N-DIMENSIONAL PROBABILITY ELLIPSOIDS

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ABSTRACT

The problem is to find a real number c such that the probability is p that a normally distributed random variable x will lie in an ellipsoid defined $x'Ax = c$, where A is the inverse of the covariance matrix of x. This problem arises, for example, in orbit determination when it is desired to know the dimensions of an ellipsoid centered about the nominal value of the state vector at a given time such that the probability is p that the state error vector will be in the ellipsoid.

THE CALCULATION OF N-DIMENSIONAL PROBABILITY ELLIPSOIDS

INTRODUCTION

If y is a real normally distributed random vector, the probability that y is in a region S is

$$p = \int_S f(x)\,d\omega \tag{1}$$

where

$$f(x) = k\,e^{-\frac{1}{2}x' A x} \tag{2}$$

$$d\omega = dx_1 dx_2 \ldots dx_n, \tag{3}$$

and

$$k = |A|^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}n}. \tag{4}$$

The real vectors x and y have n components, x' is the transpose of x, and A is an n-by-n symmetric positive-definite matrix of real numbers with determinant $|A|$.

The equation

$$x' A x = c, \tag{5}$$

where c is real, defines an ellipsoid in n-space with center at $x = 0$. The problem is to find c, given that p is known and that S is defined by (5). The matrix A is the inverse of the co-variance matrix of y. The mean of y has been taken to be 0.

This problem arises, for example, in orbit determination when it is desired to know the dimensions of an ellipsoid centered about the nominal value of the state vector at a given time such that the probability is p that the state error vector will be in the ellipsoid. Instead of the full state error vector y we can have the error vector for a subspace of the entire state space, i.e., a vector of dimension less than n whose components form a subset of the full error vector. This is true since the marginal distribution of any set of components of y is normal with mean 0 and variances and covariances obtained by taking the proper components of the inverse of A.

INTEGRALS OF QUADRATIC FORMS

As a step toward solving the problem described in the introduction, we consider integrals of the form

$$g(A) = \int_S f(x' A x) d\omega, \tag{6}$$

where A and $d\omega$ are defined in the introduction and S is the ellipsoid of (5). Since A is positive definite, there exists a nonsingular matrix C such that

$$C' A C = I, \tag{7}$$

where I is the identity matrix and C' is the transpose of C.

Let

$$x = C z. \tag{8}$$

Then $\quad x' A x = z' C' A C z = z'z.$

The Jacobian of the transformation (8) is $\overline{C}$, the absolute value of the determinant of C. Thus (6) becomes

$$g(A) = \bar{C} \int_R f(z'z)d\sigma \tag{9}$$

where $d\sigma = dz_1\ dz_2 \ldots dz_n$, and R is the sphere with center at $z = 0$ defined by

$$z'z = c. \tag{10}$$

We now transform to generalized spherical coordinates by

$$z_1 = r \cos \varphi_1$$

$$z_i = r \left(\prod_{j=1}^{i-1} \sin \varphi_j \right) \cos \varphi_i;\ i = 2, 3, \ldots, n-2$$

$$z_{n-1} = r \left(\prod_{j=1}^{n-2} \sin \varphi_j \right) \cos \theta$$

$$z_n = r \left(\prod_{j=1}^{n-2} \sin \varphi_j \right) \sin \theta$$

where $0 \leq \varphi_i \leq \pi$; $0 \leq r < \infty$, $n \geq 3$. If $n=2$, $z_1 = r \cos \theta$, $z_2 = r \sin \theta$. The Jacobian of this transformation is

$$J = r^{n-1} \prod_{j=1}^{n-2} \sin^{n-1-j} \varphi_j,\ n \geq 3.$$

Also $$z'z = r^2.$$

If $n=2$, $J=r$.

Thus (8) becomes

$$g(A) = \bar{C} \int_0^c r^{n-1} f(r^2)dr \int_0^{2\pi} d\theta \prod_{j=1}^{n-2} \int_0^{\pi} \sin^{n-1-j} \varphi_j\ d\varphi_j$$

From (7) $$\bar{C} = 1/|A|^{\frac{1}{2}}.$$

$$\prod_{j=1}^{n-2} \int_0^{\pi} \sin^{n-1-j} \varphi_j \, d\varphi_j = \Gamma^{n-2}(\tfrac{1}{2})/\Gamma(n/2),$$

where Γ is the gamma function. Thus we have

$$g(A) = \bar{C} K \int_0^c r^{n-1} f(r^2) dr, \tag{11}$$

where $$K = 2\pi \, \Gamma^{n-2}(\tfrac{1}{2})/\Gamma(n/2),$$

i.e., $$K = 2\pi^{k}/(k-1)! \quad \text{if} \quad n = 2k,$$

$$K = 2^{k+1} \pi^{k}/[1\cdot3\cdot5\cdot7\cdots(2k-1)] \ \text{if}\ n = 2k + 1,$$

where k is an integer ≥ 1.

THE NORMAL PROBABILITY ELLIPSOID

Applying (11) to (1) gives

$$p = \frac{1}{2^{k-1}(k-1)!} \int_0^c r^{2k-1} e^{-\frac{1}{2} r^2} dr, \qquad n = 2k, \tag{12}$$

$$p = \frac{2/\sqrt{\pi}}{1\cdot3\cdot5\cdots(2k-1)} \int_0^c r^{2k} e^{-\frac{1}{2} r^2} dr, \quad n = 2k + 1. \tag{13}$$

These integrals can be evaluated by formula 313.3 of reference 1.

The function $p(c)$ is monotone increasing on the interval $(0, \infty)$ and has a single inflection point in the interval. Thus the Newton-Raphson method will find the value of c for a given value of p if we use the value of c corresponding to the inflection point as the starting value.

REFERENCES

1. Gröbner, W., and Hofreiter, N., "Integraltafel, Erster Teil," Springer-Verlag, Wien, 1957.