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A LINEARIZED APPROACH TO WAVES IN A DILUTE, CHEMICALLY REACTING ATMOSPHERE

PART I

ANALYTICAL TREATMENT

KENNETH D. SHERE
IGOR J. EBERSTEIN

APRIL 1970

GODDARD SPACE FLIGHT CENTER
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ABSTRACT

The major chemical systems of the Earth's atmosphere are tied to absorption of radiation from the sun, and thus have a dawn and dusk dependence. Also, total solar eclipses may cause sharp local variations.

A study is made of the behavior of chemically generated waves in a simplified atmosphere. The atmosphere is assumed unbounded, isothermal, one-space-dimensional and initially quiescent. At an initial time a dissociation reaction commences and drives the subsequent wave motion. The fraction of reactant in the atmosphere, X, is assumed to be small. The system of governing equations is then expanded in terms of the small parameter, X, and an integral solution asymptotic to $X \to 0$ is obtained. A series solution in terms of Young functions is also obtained. These are a subspecies of Lommel functions.

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*Now at the Division of Applied Mathematics, U. S. Naval Ordnance Laboratory.

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PART I
ANALYTICAL TREATMENT

INTRODUCTION

Waves in a stratified fluid under the influence of gravity appear to have been initially discussed by Burnside (1889) and Love (1891). Both authors treated an incompressible fluid. Görtler (1943) used schlierin photography to show experimentally that disturbances in an incompressible stratified medium under the influence of gravity propagate along characteristic rays. Lamb (1908) treated a compressible, adiabatic, ideal gas whose density is stratified by gravity. Since the original contributions by Burnside, Love and Lamb there have been many theoretical papers on various types of gravity waves. Reasonably up-to-date treatments of the subject may be found in Eckart (1960) and Yih (1965). However, gravity wave theory is currently undergoing an active phase of development and contributions are being added to the literature at a rapid rate.

Recent experimental data from the earth's atmosphere and oceans have greatly stimulated interest in gravity-acoustic waves. Gossard (1962) has observed gravity waves in the troposphere. Hines (1960) has shown that internal atmospheric gravity waves may account for many of the phenomena observed in the lower ionosphere, and gravity waves have been observed in the thermosphere by Newton et. al., (1969) and Harris et. al., (1969).

It seems to be generally agreed that gravity waves are generated in the troposphere, then propagate upwards. Thus Gossard (1962) has observed gravity waves near the earth's surface and Eberstein (1970) has illustrated the development of gravity waves between 30 km and 120 km.

The simpler forms of gravity wave theory treat propagation through an inviscid, non-conducting gas, while more sophisticated theories include effects of heat transfer, viscosity, and high altitude phenomena such as ion drag. However, the effect of chemical reactions does not seem to have been adequately considered to date.

Between the troposphere and the thermosphere there are several regions where important chemical reactions take place.

First, there is the ozonosphere between approximately 15 km and 45 km, with peak ozone concentration at about 35 km. Ozone mole fraction is in the order of parts per million (Mitra, 1952).

Between 60 km and 96 km there is Nitric Oxide (Pearce, 1969). The mixing ratio for Nitric Oxide is also in ppm.

Between 90 km and 120 km molecular oxygen dissociates into atomic oxygen. The dissociation ratio, α being 3×10^{-8} at 90 km and 0.998 at 120 km (Mitra, 1952). The oxygen is no longer a truly dilute reactant since its mixing ratio is 20%.

The major chemical systems are tied to the absorption of ultraviolet radiation from the sun, and are thus have a dawn and dusk dependence.

In what follows, an initial study is made of atmosphere. Simplifying assumptions are made to make the mathematics more tractable.

2. Governing Equations

The continuity and momentum equations (cf. Shere and Bowhill, 1969) are not affected by the chemical reaction provided that the state variables are given the extended definition:

$$P \equiv P(\rho, T, \alpha)$$

where P, ρ, T and α denote pressure, density, temperature and degree of advancement of the reaction, respectively. These equations are in tensor notation:

$$\frac{D\rho}{DT} + \rho \frac{\partial u_i}{\partial x_i} = 0 \tag{2.1}$$

and

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial P}{\partial x_i} + \frac{\partial \sigma_{ji}}{\partial x_j} \quad (i = 1, 2) \tag{2.2}$$

where

$$\sigma_{ji} \equiv \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]$$

We denote time, space variable (x_2 vertical), velocity component, gravitation vector component, viscous stress tensor, dynamic viscosity and the Kronecker delta by t, x_i, u_i, g_i, σ_{ji}, μ and δ_{ij} respectively. We also take $g_i = -g\delta_{i2}$ and we define D/Dt to be the total or Eulerian derivative.

In order to develop the energy equation, it is necessary to consider the thermodynamics of the system. The derivation of the equation of state parallels the work of Liepmann and Roshko (1957, p. 29) and the derivation of the reaction rate equation parallels the work of Eberstein (1966). The following discussion applies to the reaction

$$J + AB \rightleftarrows A + B + J$$

where J is an inert element. We assume that each component of the mixture is a perfect gas. The mixture, however, is not a perfect gas. The partial pressure of the ith constituent is

$$P_i = m_i \rho R_i T \tag{2.3}$$

where $R_i \equiv R/W_i$ with R the universal gas constant, W_i the molecular weight, and m_i the mass fraction of ith-component. Letting α be the fraction of AB dissociated, n_i the number of moles of ith component, and n_0 the total number of moles when $\alpha = 0$,

$$\begin{aligned} n_{AB} &= n_0 X_0 (1 - \alpha) \\ n_A &= n_0 X_0 \alpha \\ n_B &= n_0 X_0 \alpha \\ n_J &= (1 - X_0) n_0 \end{aligned} \tag{2.4}$$

where X_0 is the mole fraction of AB when $\alpha = 0$. Summing the partial pressures yields

$$P = R(\alpha)\rho T \qquad (2.5)$$

where

$$R(\alpha) = R(0)\ (1 + X_0\,\alpha) \qquad (2.6)$$

and

$$R(0) = R/\left[W_J + X_0\ (W_{AB} - W_J)\right]$$

The reaction equation is as follows:

$$DN_{AB}/Dt = -\left\{k_F(T)\left[\frac{n_{AB}}{V}\right] - k_B\,(T)\left[\frac{n_A}{V}\right]\left[\frac{n_B}{V}\right]\right\}N \qquad (2.7)$$

where

$$N = \sum n_i$$

and k is a rate constant. In determining (2.7) it has been implicity assumed that all molecules present may act as an inert element with equal collision effectiveness.

Substituting (2.4) into (2.7) yields

$$\frac{D\alpha}{Dt} = \left[\rho k_F(T)/M\right]\left\{1 - \alpha - [K(T)\rho/M]\left(\frac{X_0\alpha^2}{[1+X_0\alpha]}\right)\right\} \tag{2.8}$$

where

$$K(T) \equiv \frac{k_B(T)}{k_F(T)}$$

and M is the mean molecular weight

$$M = W_J + X_0(W_{AB} - W_J)$$

For chemical equilibrium, the rate equation reduces to

$$\frac{1-\alpha}{\alpha^2} = \frac{P}{K_P}\left(\frac{X_0}{1+\alpha X_0}\right)$$

where use was made of the equality

$$K = K_C = \frac{K_P}{RT}$$

In atmospheric problems the reactant fraction is generally small, and it will be assumed that $X_0 << 1$. It follows that $\alpha X_0 << 1$. The rate equation may thus be simplified to give:

$$\frac{D\alpha}{Dt} = \left[\rho k_F(T)/M\right]\left\{1 - \alpha - \left[K(T)\rho/M\right] X_0\,\alpha^2\right\} \tag{2.9}$$

The energy equation is now developed in the usual manner (cf. Liepmann and Roshko, 1957, p. 185+). Summing the rate of change of the internal, kinetic and potential energy yields

$$\int_V \frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho u_i u_i + \rho g x_2\right) dV$$

$$+ \int_A \left(\rho e + \frac{1}{2}\rho u_i u_i + \rho g x_2\right) u_j n_j \, dA$$

$$= - \int_A P n_i u_i \, dA + \int_A u_i \sigma_{ij} \, dA + \int_A k \frac{\partial T}{\partial x_i} n_i \, dA \qquad (2.10)$$

where n_i is the ith component of the unit normal of the closed surface ϕ, V is the volume of ϕ and A the area; e is the internal energy per unit mass and k is the coefficient of thermal conductivity.

By applying Gauss' theorem and substituting the continuity and momentum equations into (2.10), we obtain

$$\rho \frac{De}{Dt} = -P \frac{\partial u_i}{\partial x_i} + \sigma_{ij} \frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) \qquad (2.11)$$

We now use the first law of thermodynamics, $h = e + P/\rho$, and the continuity equation to obtain

$$\rho \frac{Dh}{Dt} = \frac{DP}{Dt} + \sigma_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) \qquad (2.12)$$

In a reacting gas mixture the enthalpy depends on the temperature and the degree of reaction, i.e.

$$h = h(T, \alpha)$$

The rate of change of enthalpy per unit mass is then given by:

$$\frac{Dh}{Dt} = C_p \frac{DT}{Dt} + X_0 B \frac{D\alpha}{Dt} \tag{2.13}$$

where B is the enthalpy of reaction and C_p is heat capacity. Both the enthalpy of reaction and the heat capacity have a weak temperature dependence, and an even weaker pressure dependence. For simplicity both C_p and B will be assumed constant throughout this paper.

Eliminating h between (2.12) and (2.13) one obtains the enthalpy equation in the form:

$$\rho C_p \frac{DT}{Dt} + X_0 B \frac{D\alpha}{Dt} = \frac{DP}{Dt} + \sigma_{ik} \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) \tag{2.14}$$

To summarize, we have found the system

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \tag{2.1}$$

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial P}{\partial x_i} + \frac{\partial \sigma_{ji}}{\partial x_i} \tag{2.2}$$

$$P = \rho R(\alpha) T \tag{2.5}$$

$$\frac{D\alpha}{Dt} = \rho\frac{k_F(T)}{M}\left[1 - \alpha - \rho\frac{K(T)X_0\,\alpha^2}{M}\right] \tag{2.9}$$

$$\rho C_p\frac{DT}{Dt} + X_0\,\rho B\frac{D\alpha}{Dt} = \frac{DP}{Dt} + \sigma_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) \tag{2.11}$$

The dependent variables are P, ρ, T, u_i, and α; the independent variables are t, x_i. k_F, K, c_p and B are known parameters of the system, X_0 is a small (known) parameter.

For the remainder of this paper we assume that the atmosphere is inviscid and nonconducting. Hence, the governing equations are

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0 \tag{2.15}$$

$$\rho\frac{Du_i}{Dt} = \rho g_i - \frac{\partial P}{\partial x_i} \tag{2.16}$$

$$P = \rho RT \tag{2.17}$$

$$\frac{D\alpha}{Dt} = \rho\left(\frac{k_F}{M}\right)\left[1 - \alpha - \rho\left(\frac{K}{M}\right)X_0\,\alpha^2\right] \tag{2.18}$$

$$\rho C_p\frac{DT}{Dt} + X_0\,B\rho\frac{D\alpha}{Dt} = \frac{DP}{Dt} \tag{2.19}$$

3. Nondimensionalization of the Governing Equations

Since mathematical operations are performed on pure numbers it is proper to non-dimensionalize the governing equations.

Some of the non-dimensionalization is almost trivial, as shown immediately below:

$$R' = R/R(0)$$

$$C_P' = C_P/R(0)$$

$$\rho' = \rho/\rho_*$$

$$M' = M/M = 1$$

$$k_F' = k_F\rho_*/M\omega_B$$

$$K' = K\rho_*/M\omega_B$$

where R(0) is the gas constant for $\alpha = 0$.

ρ_* is a reference density.

Other non-dimensionalizations have a more direct physical or kinematic meaning. Thus, distance is defined in terms of scale height and time is related to the Brunt-Vaisala frequency, i. e.

$$x_i' = x_i/H$$

$$t' = t\omega_B$$

Further non-dimensionalizations are:

$$T' = R(0)\,T/\omega_B^2 H^2$$

$$B' = B/\omega_B^2 H^2$$

$$g' = g/H\omega_B^2$$

$$P' = P/\rho_*\,\omega_B^2 H^2$$

It is seen that in the primed system we have

$$P' = \rho' R' T'$$

In general, the non-dimensional system of governing equations looks very similar to the dimensional system.

The quantity $\omega_B{}^2 H^2$ is proportional to the energy contained in a wave oscillating with the Brunt frequency, and having amplitude equal to a pressure scale height. The quantity $H\omega_B{}^2$ is a measure of the acceleration experienced in such a wave. Since such acceleration must be less than that due to gravity we would expect

$$g' > 1$$

One may estimate the magnitudes of the non-dimensional parameters.

Taking a scale height of 6 km and a Brunt period of 5 minutes, one obtains

$$(\omega_B H)^2 \sim 0.1 \text{ calories/gram}$$

and

$$\omega_B{}^2 H \sim 5 \text{ cm/sec}^2, \text{ giving } \begin{array}{l} g' \sim 200 \\ T' \sim 200 \end{array}$$

Further, taking $B = 10$ kcal/gmole one obtains $B' \sim 10^5$.

4. Asymptotic Development

The non-dimensionalized system of equations will be expanded about the parameter X_0, i.e., we expand each dependent variable in a possible series of the form:

$$f(t, x, z) = \sum_{N=0}^{\infty} f^{(N)}(t, x, z)\, X_0^{N} \qquad (4.1)$$

where higher order terms may be dropped as $X_0 \to 0$.

Assume that the atmosphere is initially stratified, quiescent, and bounded below by the earth.

Hence we have defined an initial value problem with initial conditions:

$$\begin{aligned} T(0, x, z) &= T_0 = T_* \\ u_i(0, x, z) &= 0 \qquad (i = 1,\ 2) \\ \rho(0, x, z) &= e^{-z} \\ \alpha(0, x, z) &= 0 \end{aligned} \qquad (4.2)$$

and boundary condition:

$$u_2(t, x, 0) = 0$$

Using the equation of state to eliminate P from (2.18) - (2.19) and substituting (4.1) into the resultant system yields the zero-order solution

$$\begin{aligned} T^{(0)}(t, x, z) &= T_0 = T_* \\ u_i^{(0)}(t, x, z) &= 0 \\ \rho^{(0)}(t, x, z) &= e^{-z} \\ \alpha^{(0)}(t, x, z) &= 1 - \exp\left[-k_F\, \rho^{(0)}\, t\right] \end{aligned} \qquad (4.3)$$

Defining $T^{(1)} = T^{(0)}\, T_{(1)}$; $\rho^{(1)} = \rho^{(0)}\, \rho_{(1)}$ and $\alpha^{(1)} = \alpha^{(0)}\, \alpha_{(1)}$, we get the first order system

$$\frac{\partial u_i^{(1)}}{\partial t} + T_0 \frac{\partial \rho_{(1)}}{\partial x} + T_0 \frac{\partial T_{(1)}}{\partial x} = 0 \tag{4.4}$$

$$\frac{\partial u_2^{(1)}}{\partial t} + T_0 \frac{\partial \rho_{(1)}}{\partial z} - T_{(0)}\, T_{(1)} + T_0 \frac{\partial T_{(1)}}{\partial z} = T_0\, \alpha^{(0)} \tag{4.5}$$

$$\frac{\partial \rho_{(1)}}{\partial t} + \frac{\partial u_1^{(1)}}{\partial x} + \frac{\partial u_2^{(1)}}{\partial z} - u_2^{(1)} = 0\,. \tag{4.6}$$

$$- \frac{\partial \alpha_{(1)}}{\partial t} + \frac{1}{\gamma - 1} \frac{\partial T_{(1)}}{\partial t} + u_2^{(1)} = -\mathbb{B}\, \frac{\partial \alpha^{(0)}}{\partial t} \tag{4.7}$$

$$\frac{\partial \alpha_{(1)}}{\partial t}\, \alpha^{(0)} + \alpha_{(1)} \frac{\partial \alpha^{(0)}}{\partial t} = \Gamma^{(1)} \left(\rho^{(0)}, \alpha^{(0)}\right) \tag{4.8}$$

where

$$\mathbb{B} = \left(\frac{B}{T_0} - 1\right)$$

The above forms a linear system of four partial differential equations in four unknown functions. The system is inhomogeneous. The solution of (4.8) for $\alpha_{(1)}$ is needed only for the calculation of second order terms. In this paper we determine only the first order terms.

Dropping the (1) subscript or superscript and eliminating we obtain the equation

$$\left\{\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial t^2} - C^2 \frac{\partial^2}{\partial x^2} - C^2 \frac{\partial^2}{\partial z^2} + C^2 \frac{\partial}{\partial z}\right] - \frac{\gamma - 1}{\gamma^2}\, C^4 \frac{\partial^2}{\partial x^2}\right\} T = f\left(\alpha^{(0)}\right) \tag{4.9}$$

where

$$f(\alpha^{(0)}) \equiv -(\gamma-1)\,\mathbb{B}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial t^2} - T_0\frac{\partial^2}{\partial x^2} - T_0\frac{\partial^2}{\partial z^2}\right]\alpha^{(0)}$$

$$-(\gamma-1)\,\mathbb{B}\frac{\partial^3\alpha^{(0)}}{\partial t^2\partial z} + (\gamma-1)\,T_0{}^2\frac{\partial^2\alpha^{(0)}}{\partial x^2} \tag{4.10}$$

and

$$C^2 \equiv \gamma T_0$$

Using the results of Garding (1950), A. Lax (1956), and Courant and A. Lax (1955) it can be shown that (4.9) is a well-posed hyperbolic equation.

After (4.9) is solved the other dependent variables are determined by the equation

$$\left[\frac{\partial^2}{\partial t^2} + T_0\frac{\partial}{\partial z}\right]\rho = \frac{1}{\gamma-1}\frac{\partial^2 T}{\partial t^2} - T_0\frac{\partial T}{\partial z} + T_0\,T + \mathbb{B}\frac{\partial^2\alpha^{(0)}}{\partial t^2} + T_0\,\alpha^{(0)} \tag{4.11}$$

$$\frac{\partial u_1}{\partial x} = -\frac{\partial^2\rho}{\partial t\,\partial z} + \frac{1}{\gamma-1}\frac{\partial^2 T}{\partial t\,\partial z} - \frac{1}{\gamma-1}\frac{\partial T}{\partial t} + \mathbb{B}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z} - 1\right)\alpha^{(0)} \tag{4.12}$$

and

$$u_2 = \frac{\partial\rho}{\partial t} - \frac{1}{\gamma-1}\frac{\partial T}{\partial t} - \mathbb{B}\frac{\partial\alpha^{(0)}}{\partial t} \tag{4.13}$$

Now define

$$\theta(t, x, z) \equiv e^{-z/2}\,T(t, x, z) \tag{4.14}$$

The above definition allows the temperature perturbation to grow exponentially with altitude without a corresponding growth in θ.

Substituting the definition of θ into (4.9) yields:

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial t^2} - C^2\nabla^2 + \frac{C^2}{4}\right]\theta - \frac{C^2(\gamma-1)}{\gamma^2}\frac{\partial^2\theta}{\partial x^2} = e^{-z/2} f(\alpha) \qquad \text{(4.15}$$

$f\left(\alpha^{(0)}\right)$ was defined in Equation (4.10).

The dispersion equation of (4.15) is:

$$\lambda_t^4 - C^2\left(\lambda_x^2 + \lambda_z^2 + \frac{1}{4}\right)\lambda_t^2 + \left[(\gamma-1)\, C^4/\gamma\right]\lambda_x^2 = 0 \qquad (4.16)$$

where λ_i is the wave number of i.

For a discussion of dispersion relations and how these are obtained for partial differential equations reference is made to Courant and Hilbert, especially p. 588.

5. Solution of the First-Order System

In this section we first demonstrate that the governing temperature Equation (4.16) is reducible to an inhomogeneous telegraph equation with homogeneous initial conditions. For a nonreacting atmosphere which is initially perturbed, Lamb [1909] obtained a homogeneous telegraph equation with inhomogeneous initial conditions.

We take for initial conditions of $\theta(t, x, z)$,

$$\theta(0, x, z) = \theta_t(0, x, z) = 0 \qquad (5.1a)$$

$$\theta_{tt}(0, x, z) = -(\gamma-1)\,\mathbb{B}\,k_F^2 \exp(-5z/2) \qquad (5.1b)$$

$$\theta_{ttt}(0, x, z) = (\gamma-1)\left[\mathbb{B}\,k_F^2\,{\rho^{(0)}}^2 + T_0\right]\exp(-3z/2) \qquad (5.1c)$$

Conditions (5.1b) and (5.1c) are determined by the requirement that $\theta(t, x, z)$ remain bounded as $t \to \infty$. The negativeness of θ_{tt} implies that the atmosphere initially cools. This is in correspondence with the endothermic reaction chosen.

Since the inhomogeneous parts of (4.15) - (5.1) do not depend upon the horizontal space variable, neither does the solution; this is obvious from a consideration of the Fourier representation of $\theta(t, x, z)$. Thus (4.15) reduces to

$$\frac{\partial^2}{\partial t^2} M[\theta] = e^{-z/2} f(\alpha) \tag{5.2}$$

where $M[\cdot]$ is the telegraph operator

$$M[\cdot] \equiv \partial^2/\partial t^2 - C^2 \partial^2/\partial z^2 + C^2/4 . \tag{5.3}$$

It is worth noting that the initial conditions and the alpha dependence chosen are such as to imply the relevance of only one space dimension. Other initial or boundary conditions would require that the second space dimension be kept. Integrating (5.2) with respect to time twice,

$$M[\theta] = w(t, z) = k_F(\gamma - 1) \Big[(1 - T_0 t^2) \mathbb{B} k_f e^{-z} + (2\mathbb{B} + 1)(C^2/\gamma)t\Big] e^{-3z/2 - k_F e^{-z} t} \tag{5.4}$$

subject to initial conditions (5.1a).

The inhomogeneous part of (5.4) is the effect of the chemical reaction. For a realistic approximation to the atmosphere, the inhomogeneous part of (5.4) must be multiplied by a suitable weighting function, since the chemical contribution only extends over a finite altitude regime. We also note that at each fixed altitude the solution must tend asymptotically to zero in the Poincaré sense as $t \to \infty$.

The assumptions used in this paper are also suitable for a multilayer approximation of the atmosphere. In this case we would consider either an initial-boundary or pure boundary value problem for (4.15). The x-dependence may not generally be eliminated for this case.

The solution of (5.4) - (5.1a) is well-known [cf. Courant and Hilbert, II, pp. 695, 202] and is given by

$$\theta(t, z) = \int_0^{tc} W(t, z; \tau)\, d\tau \tag{5.5}$$

where

$$W(t, z; \tau) \equiv e^{-(tc-\tau)/2} \int_0^{tc-\tau} \left[I_0\left(\frac{1}{2} \sqrt{(tc-\tau)^2 - \eta^2} \right) \right] \cdot Q(z, \eta)\, d\eta \tag{5.6}$$

and

$$Q(z, \eta) \equiv \left[w(t, z + \eta) - w(\tau, z - \eta) \right] / C^2 \tag{5.7}$$

Although (5.5) - (5.7) provides an exact solution of (5.4) - (5.1a), this solution is complicated and does not easily yield qualitative information.

The first order terms are from (4.12) - (4.13):

$$T(t, z) = e^{z/2}\, \theta(t, z)$$

$$u_2(t, z) = -\int_{-z_s}^{z} \frac{\partial}{\partial t} \left[\frac{T}{\gamma - 1} + \mathcal{B}\alpha \right] dz$$

where z_s is at the earth surface and

$$\rho(t, z) = \int_0^t \left[u_2 - \frac{\partial u_2}{\partial z} \right] dt \, .$$

To obtain qualitative information we consider the Laplace transform of (5.4):

$$-W_{zz}(z,\sigma) + (\sigma^2 + 1/4)W = (\gamma - 1)\,(2\mathcal{B} + 1)\,(k_F/\gamma C) \times \frac{e^{-3z/2}}{\left[\sigma + (k_f/C)e^{-z}\right]^2} + \frac{\mathcal{B}(k_F/C)^2\,(\gamma - 1)e^{-5z/2}}{\left[\sigma + (k_F/C)e^{-z}\right]} + \frac{\mathcal{B}(k_F/C)^2\,\left[(\gamma - 1)/\gamma\right]e^{-5z/2}}{\left[\sigma + (k_F/C)\,e^{-z}\right]^3} \qquad (5.8)$$

where $W(z,\sigma) = \mathcal{L}\left\{\theta(z\,,\,\tau)\right\}$ and $\tau = tc$.

Equation (5.8) is most readily solved by superposition. We obtain

$$W(z,\sigma) = \sum_{N=0}^{\infty} \mathcal{B}(k_F/C)^2\,(\gamma - 1)\left\{\frac{(-1)^N\,(k_F/C)^N}{\sigma^{N+1}\left[\sigma^2 - (N+3)\,(N+2)\right]}\right\}e^{-(5/2+N)z}$$

$$+ \sum_{n=0}^{\infty} (\gamma - 1)\,(2\mathcal{B} + 1)(k_F/\gamma C)\left\{\frac{(n+1)\,(-k_F/C)^n}{\sigma^{n+2}\left[\sigma^2 - (n+1)\,(n+2)\right]}\right\}e^{-(3/2+n)z}$$

$$+ \sum_{n=0}^{\infty} \frac{\mathcal{B}(k_F/C)^2\,(\gamma - 1)}{2\gamma}\left\{\frac{(n+1)\,(n+2)\,(-k_F/C\sigma)^n}{\sigma^{n+3}\left[\sigma^2 - (n+3)\,(n+2)\right]}\right\}e^{-(5/2+n)z} \qquad (5.9)$$

Using Bateman's formula (10) [Erdelyi, et. al. I, pg. 238, 1954] we take the inverse Laplace transform:

$$\theta(t, z) = \sum_{n=0}^{\infty} \left\{ \frac{(\gamma - 1)\mathbb{B}(-k_F t)^{n+2}}{\Gamma(n+3)} \; {}_1F_2\left(1; \frac{n+3}{2}, \frac{n+4}{2}; \frac{(n+2)(n+3)C^2 t^2}{4}\right) \right.$$

$$\left. + \frac{(\gamma - 1)\mathbb{B}C^2(-k_F t)^{n+4}}{2\gamma k_F^2 \, \Gamma(n+5)} \; {}_1F_2\left(1; \frac{n+5}{2}, \frac{n+6}{2}; \frac{(n+2)(n+3)C^2 t^2}{4}\right) \right\} e^{-(5/2+n)z}$$

$$- \sum_{n=0}^{\infty} \frac{(\gamma - 1)(2\mathbb{B} + 1)C^2(-k_F t)^{n+3}}{\gamma k_F^2 \, \Gamma(n+4)} \; {}_1F_2\left(1; \frac{n+4}{2}, \frac{n+5}{2}; \frac{(n+1)(n+2)C^2 t^2}{4}\right) e^{-(3/2+n)z} \tag{5.10}$$

We identify the generalized hypergeometric functions appearing in (5.10) as Young's functions [Young, 1912], a special case of Lommel functions.

It had been noted earlier that $\theta(t, z)$ must be asymptotic to zero as $t \to \infty$. This behavior in time can be obtained by applying Luke's formulae 5.11.2 (T) and 5.11.2 (9) [I, 1969, pg. 199]. The series (5.10) is rapidly covergent.

The results of some calculations on the chemical effect are given in Part II.

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