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RELIABILITY TESTING AND DEMONSTRATION SESSION IIIB

by Vincent R. Lalli
Lewis Research Center
Cleveland, Ohio

Lecture Notes for Seventh Annual Reliability Engineering and Management Institute sponsored by the University of Arizona
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INTRODUCTION

The outline shown below describes the material that will be covered in the Reliability Testing and Demonstration Lecture III B:

III B. Application of Statistical Methods

1. Testing with normal units
2. Determination of confidence limits
3. Testing with lognormal units
4. Determination of confidence limits
5. Testing with bionomial and Poisson events
6. Determination of confidence limits

A great deal of work has been done by various researchers to develop mathematical concepts suitable for reliability studies. The interested reader should consult References 1 through 4 for additional details pertaining to statistical methods for discrete and continuous random variables.

In these notes effort will be concentrated on four functions: (1) Failure, $f(t)$; (2) Reliability, $R(t)$; Failure rate, λ; and (4) Hazard rate, λ'. Since it is usually important to know how well a point estimate has been defined, some consideration will be given to calculation of confidence limits for normal, lognormal and bionomial functions.

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These notes consider specific cases to show how statistical methods can be used in analyzing test data.

1. Testing with normal units

A mechanical part is being used where friction, mechanical loading and temperature are the principal failure causing stresses. Assume that tests to failure have been conducted for these mechanical parts resulting in the data shown in table I.

(a) Calculate the mean-time-between-failures and standard deviation.

(b) What are the hazard rate at 85.3 K hours and failure rate during the next 10.3 K hour interval?

(c) What are the failure and reliability time functions?

The mean-time-between-failures and standard deviation can be calculated for the data given in table I as follows:

$$\text{(a)}\quad \bar{t} = \frac{\sum_{f=1}^{n} t_f}{n}$$

where

$\bar{t}$ mean-time-between-failures

t_f time-to-failure

n number of observations

therefore, using the data from table I

$$\bar{t} = \frac{750\ \text{K}}{10} = 75\ \text{K hours}$$

and

$$\sigma = \left[\frac{\sum_{f=1}^{n} t_f^2 - \frac{\left(\sum_{f=1}^{n} t_f^2\right)^2}{n}}{n - 1} \right]^{1/2}$$

where σ = unbiased standard deviation

$$\left(\sum_{f=1}^{n} t_f\right)^2 = 7.50 \times 10^2 \text{ K hr}$$

$$\left(\sum_{f=1}^{n} t_f\right)^2 = (7.5 \times 10^2)^2 = 5.625 \times 10^5 \text{ (K hr)}^2$$

Therefore,

$$\sigma = \left(\frac{57213 - 56250}{9}\right)^{1/2} = \left(\frac{963}{9}\right)^{1/2} = 10.3 \text{ K hr}$$

(b) The hazard rate, λ' and failure rate λ are calculated as follows:

$$\lambda' = \frac{\text{Normal ordinate at 85.3 K hr}}{\text{Normal area 85.3 K hr to } \infty}$$

Let $\overline{Y}_1$ = Normal ordinate at 85.3 K hr

Z_1 = Standardized normal variable

$$= \frac{t - \bar{t}}{\sigma} = \frac{(85.3 - 75.0) \text{ K hr}}{10.3 \text{ K hr}} = +1.0$$

Table 4 (p. 352 of ref. 5) for $Z = +1.0$ gives $\bar{Y}_1 = 0.242$. The scale constant K_s for this problem is:

$$K_s = \frac{n\theta}{\sigma}$$

where θ = class interval

$$Y_1 = f(t_1) = K_s\bar{Y}_1 = \frac{10 \times 1 \text{ F}}{10.3 \text{ K hr}} \times 0.242 = 2.35 \times 10^{-4} \text{ F/hr}$$

Let $R(t_1)$ = Normal area 85.3 hr to ∞. From table 3 (p. 351 of ref. 5) for $Z_1 = +1.0$

$$Q(t_1) = 0.841 \text{ area from } -\infty \text{ to } Z_1$$

Since $Q(t_1) + R(t_1) = 1.000$

$$R(t_1) = 1.000 - 0.841 = 0.159$$

$$\lambda' = \frac{2.35 \times 10^{-4} \text{ F/hr}}{1.59 \times 10^{-1}} = 1\ 47 \times 10^{-3} \text{ Failures/hr}$$

and

$$\lambda = \frac{1}{h}\left[1 - \frac{R(t_2)}{R(t_1)}\right] \qquad h = 10.3 \text{ K hr}$$

$$R(t_2) = \text{Normal area } 95.6 \text{ K hr to } \infty$$

$$Z_2 = \frac{(95.6 \text{ K} - 75.0) \text{ K hr}}{10.3 \text{ K hr}} = +2.0$$

From table 3, $Q(t_2) = 0.977$ and $R(t_2) = 0.023$

$$\lambda = \frac{1}{10.3 \text{ K hr}}\left(1 - \frac{0.023}{0.159}\right) = \frac{8.56 \times 10^{-1}}{1.03 \times 10^{4}} = 8.31 \times 10^{-5} \text{ Failures/hr}$$

(c) The constants for $f(t)$ and $R(t)$ are calculated as follows:

$$\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{1.03 \times 10^{4} \times 2.52} = 3.87 \times 10^{-5}$$

$$2\sigma^2 = 2 \times (1.03 \times 10^{4})^2 = 2.12 \times 10^{8}$$

Therefore,

$$f(t) = 3.87\times10^{-5}\ e^{-(t - 7.5\times10^4)^2/2.12\times10^8}$$

$$R(t) = 3.87\times10^{-5}\int_t^{\infty} e^{-(t - 7.5\times10^4)^2/2.12\times10^8}\,dt$$

2. Determination of parameters and confidence limits

Twenty-five (25) mechanical parts have been tested to failure. The mean-time-between failures has been calculated to be 75 K hours with $\sigma = 10.3$ K hours (see problem 1). (a) What are the upper and lower confidence limits at a 90 percent confidence level?

The upper and lower confidence limits are given by:

$$U = \bar{t} + K_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$L = \bar{t} - K_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where

$\bar{t}$	mean-time-between failures
$K_{\alpha/2}$	normal coefficient
σ	unbiased standard deviation
n	number of samples
$\frac{\alpha}{2}$	area under one tail

For the same problem:

$$1 - \alpha = 0.90 \qquad \alpha = 0.10 \qquad \frac{\alpha}{2} = 0.05$$

$K_{\alpha/2} = 1.64$ from table 3 of reference 5

$$U = 75\text{ K} + \frac{1.64\times10.3\text{ K}}{\sqrt{25}} = 78.4\text{ K hr}$$

$$L = 75\ K - \frac{1.64 \times 10.3\ K}{\sqrt{25}} = 71.6\ K\ hrs$$

This means that 90 percent of the time the mean-time-between-failures estimate $\bar{t}$ for problem 1 with a larger sample size will be between 71 600 and 78 400 hours. It should be noted that the sample size for problem 1 was only 10 parts. If possible, try to keep $n \geq 25$ for estimating normal parameters with the above equations.

If the sample size, $n < 25$ then use should be made of the student's t distribution (see ref. 6). If problem 2 is reworked for a smaller sample size of 10, it will be interesting to see the effect that sample size has on the size of confidence intervals.

$$U = \bar{t} + t_{\alpha/2} \frac{s}{\sqrt{n}}$$

$$L = \bar{t} - t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where

$t_{\alpha/2}$ student t coefficient

s standard deviation

For this case with $\alpha/2 = 0.05$, $\nu = n - 1 = 10 - 1 = 9$

$t_{\alpha/2} = 2.26$ from table IV of reference 6, page 243

$$s = \left(\frac{57213 - 56250}{10}\right)^{1/2} = 9.82\ K\ hr$$

$$U = 75\ K\ hr + \frac{2.26 \times 9.82\ K\ hr}{\sqrt{10}} = 82.0\ K\ hr$$

$$L = 75\ K\ hr - \frac{2.26 \times 9.82\ K\ hr}{\sqrt{10}} = 68.0\ K\ hr$$

It will be noted that the smaller sample size gives a larger interval of uncertainty for $\bar{t}$.

3. Testing with lognormal units

A cable used as guy supports for sail experiments in wind tunnel testing exhibited the time-to-failure performance data given in table II.

(a) Write the failure and reliability functions.

(b) What is the hazard rate at 5715 hours?

(c) What is the failure rate during the next 3000 hours?

a. The essential steps for solving this problem are given below:

(1) Table 2 gives the median rank for each ordered position.

(2) Plot on lognormal probability graph paper (probability × 2 log cycles) median ranks against failure age as shown in figure 1.

(3) If a straight line can be fit to these plotted points, then the time-to-failure function is lognormal.

(4) The mean-time-between-failures is calculated by $\bar{t}' = \log_e(\bar{t})$ where $\bar{t} = 6970$ hours as shown in figure 1 for a median ranks of 50 percent, hence $\bar{t}' = 8.84$.

(5) The standard deviation is calculated by

$$\sigma_{t'} = \left[\frac{\log_e t_U - \log_e t_L}{3}\right]$$ where $t_U = 49\,500$ hours and $t_L = 1020$ hours

as shown in figure 1 for a median and [1-rank] of 93.3 percent; hence

$$\sigma_{t'} = \left[\frac{15.4 - 6.93}{3}\right] = 2.82.$$ Using these constants the expressions

for f(t) and R(t) can be obtained.

$$f(t) = \frac{1.47\times10^{-1}}{t} e^{-(t'-8.84)^2/1.59\times10}$$

$$R(t) = 1.47\times10^{-1}\int_{\log_e(t)}^{\infty} e^{-(t'-8.84)^2/1.59\times10}\,dt$$

b. The lognormal ordinate required for λ' can be calculated as follows:

$$Z_2 = \left(\frac{t' - \overline{t'}}{\sigma_{t'}}\right) = \left(\frac{8.66 - 8.84}{2.82}\right) = -0.064$$

$\overline{Y}_2$ = 0.398 from table 4 of reference 5

$$Y_2 = \frac{N\overline{Y}_2}{\sigma_{t'}} = \frac{10 \times 0.398}{2.82} = 1.41$$

$$f(t') = \frac{Y_2}{t} = \frac{1.41}{5.715\times10^3} = 2.47\times10^{-4} \text{ Failures/hr}$$

The lognormal area from t' to infinity can be obtained directly from figure 1 using the [1-rank] scale. Enter the time-to-failure ttf ordinate at 5715 hours; project over to the lognormal life function Q(t) and up to the [1-rank] abscissa value of 0.638. Therefore the the hazard rate λ' at 5715 hours is:

$$\lambda' = \frac{2.47\times10^{-4}}{6.38\times10^{-1}} = 3.87\times10^{-4} \text{ Failures/hr}$$

c. The failure rate during the next 3000 hours is calculated knowing the $R(t_1) = 0.638$ at ttf = 5715 hours and obtaining $R(t_2) = 0.437$ from figure 1 at ttf = 8715 hours. Therefore,

$$\lambda = \frac{1}{3\times10^3}\left(1 - \frac{0.437}{0.638}\right) = 1.05\times10^{-4} \text{ Failures/hr}$$

4. Determination of confidence limits

It has been shown that the guy supports of problem 3 exhibited a reliability of 0.628 at a ttf of 5715 hours. Consider now the procedure for determining the confidence band on this lognormal estimate. The data needed for the graphical construction of the 90 percent confidence lines on the lognormal graph of figure 1 is also given in table II. The steps necessary to graphically construct the confidence lines in figure 1 are as follows:

(1) Enter the ranks axis with the first 5 percent rank value hitting Q(t) the lognormal life function shown in figure 1; ordered sample number 3, 5 percent rank 8.7.

(2) Draw a vertical line to intersect Q(t) at point 1 as shown in figure 1.

(3) Draw a horizontal line to cross the corresponding median rank; ordered sample number 3, median rank 25.9.

(4) The intersection point (point 2 in fig. 1) of step 3 and the median rank line is one point on the 95 percent confidence line.

(5) Repeat steps 1 through 4 until the desired time-to-failure is covered, 5715 hours in this case.

(6) The 5 percent confidence line is obtained in a similar manner. Enter the ranks axis with the 95 percent failure rank, 25.9 for ordered sample number 1.

(7) Draw a vertical line which intersects Q(t) at point 3.

(8) Draw a horizontal line to cross the corresponding median rank; ordered sample number 1, median rank 6.7.

(9) The intersection point (point 4 in figure 1) of these two lines is one point on the 5 percent confidence line.

(10) Repeat steps 6 through 9 until the desired time-to-failure is covered.

At 5715 hours the 90 percent confidence interval for $Q(t)$ is from figure 1: 19.7 percent, 69.4 percent. Hence, a 90 percent confidence interval for $R(t)$ at 5715 hours is 0.803 to 0.306. Incidentally, this graphical procedure for finding confidence intervals is completely general and can be used on other types of life test diagrams.

5. Testing with the binomial and Poisson events

The binomial and Poisson distributions are discrete functions of the number of failures N_f which occur rather than time t. A summary of these frequency functions is given in figure 1 of reference 4.

A suspicious lot of explosive bolts is estimated to be 15 percent defective due to improper loading density observed in neutron radiography.

(a) Calculate the probability of one defective unit appearing in a flight quantity of four.

(b) Plot the resulting histogram.

(c) What is the reliability from the first defect?

Not much failure density data is available, however, past experience with pyrotechnic devices has shown that the binomial distribution applies. From the given data:

q per unit number of effectives = 0.85

p per unit number of defectives = 0.15

n sample size = 4

N_f possible number of failures = 0, 1, 2, 3, 4

The frequency functions corresponding to these constants are given below:

$$f(N_f) = \frac{n!}{(n - N_f)!\ N_f!}\ p^{N_f}\ q^{n-N_f}$$

$$f(N_f) = \frac{4!}{(4 - N_f)!\ N_f!}\ p^{N_f}\ q^{4-N_f}$$

$$R(N_f) = \sum_{j=N_f}^{n} \frac{n!}{(n - j)!\ j!}\ p^j\ q^{n-j}$$

$$R(N_f) = \sum_{j=N_f}^{4} \frac{4!}{(4 - j)!\ j!}\ p^j\ q^{n-j}$$

One easy method to obtain the binomial expansion coefficients is to make use of Pascal's triangle. Pascal found that there was symmetry to the coefficient development and explained it as shown in table III. Column 1 gives the sample size n. Column 2 gives the possible number of failures. Column 3 gives the binomial expansion coefficients.
The numbers in the dashed triangle in column 3 are obtained by adding the two numbers above the number to get that number; that is, refer to dashed insertion the triangle $3 + 3 = 6$. In expanded form $f(N_f)$ becomes

$$f(N_f) = q^4 + 4q^3p + 6q^2p^2 + 4qp^3 + p^4$$

The probability of one defective unit appearing in the flight quantity of 4 is given by the second term in the expansion; hence,

$$4q^3p = 4(0.85)^3(0.15) = 0.37$$

The resulting histogram for this distribution is shown in figure 2. The probability that 2, 3, or 4 defects will occur as the reliability from the first defect is the sum of the remaining terms in the binomial expansion. This probability can be calculated using the following equation:

$$R(N_f) = \sum_{j=2}^{4} \frac{4!}{(4 - j)!\ j!}\ p^j\ q^{n-j}$$

However, it is simpler to use the histogram graph and sum the probability defects over N_f from 2 to 4. Hence,

$$R(2) = 0.096 + 0.011 + 0.001 = 0.108$$

These explosive bolts in their present form are not suitable for use on a flight spacecraft as the probability of zero defects is only 0.522 much below the usually desired 0.999 for pyrotechnic spacecraft devices.

The Poisson distribution is used to determine the probabilities associated with a specified number of failures in the continuum of time. Complex electrical components have been shown to follow the Poisson distribution.

Ten space power speed controllers were tested during the Sunflower development program. The time-to-failure test data is given in table IV.

(a) Write the Poisson failure density and reliability functions.

(b) What is the probability of five failures in 10 000 hours?

(c) What is the probability that 6, 7, 8, 9, or 10 failures will occur or the reliability from the 5^{th} failure?

a. Using the data given in table IV, this problem can be solved as follows:

$$\bar{t} = \frac{\sum_{i=1}^{10} t_i}{N_f} = \frac{8.586\times10^4}{10} = 8.59\times10^3 \text{ hr/failures}$$

Hence the Poisson failure density function is:

$$f(N_f) = \frac{(t/8.59\times10^3)^{N_f}}{N_f!} e^{-t/8.59\times10^3}$$

and the reliability function is:

$$R(N_f) = \sum_{j=1}^{10} \frac{(t/8.59\times10^3)^j}{j!} e^{-t/8.59\times10^3}$$

b. The probability of five failures f(5) in 10 000 hours makes use of the ratio $(t/\bar{t})$. Using this ratio, f(5) becomes

$$\frac{t}{\bar{t}} = \frac{1.0\times10^4}{8.56\times10^3} = 1.16$$

$$f(5) = \frac{(1.16)^5 e^{-1.16}}{5!} = \frac{2.09 \times 0.314}{1.2\times 10^2} = 5.47\times10^{-3}$$

One easy method to calculate the term $(1.16)^5$ is as follows:

$$\log (1.16)^5 = 5 \log 1.16 = 5(0.148) = 0.740$$

$$(1.16)^5 = 2.09$$

c. The reliability from the 5^{th} to the 10^{th} failure is the sum of the remaining terms in the Poisson expansion. This probability can be calculated using the following equation

$$R(N_f) = \sum_{j=6}^{10} \frac{0.314\,(1.16)^j}{j!}$$

$$R(6) = 0.0013$$

6. Determination of confidence limits

When an estimate is made using discrete distributions, it is expected that additional estimates of the same parameter will be close to the original estimate. It is desirable to be able to determine upper and lower confidence limits at some stated confidence level for discrete distribution estimates. The analytical procedure for determining these intervals is simplified by using specially prepared tables and graphs. Useful tables for the binomial distribution are given in references 5, 8, 9, and 10.

A prior calculation showed that the probability of one defective pyrotechnic unit appearing in a flight quantity of four was 0.37.

What are the upper and lower confidence limits on this estimate at a 90 percent confidence level?

If the number of defectives is r and the confidence level is γ, this problem has the constraints listed below:

$$n = 4 \qquad r = 1 \qquad \gamma = 90 \text{ percent}$$

Using these constraints, the upper U and lower L confidence limits can be obtained from table I in reference 8.

$$L = 0.026$$

$$U = 0.680$$

This means that with a 90 percent confidence the probability of one defective bolt appearing in a flight quantity of four is in the interval from 0.026 to 0.680.

The reliability from the 5th to the 10th failure for speed controllers was found to be 0.0013 in a previous problem. What are the upper and lower confidence limits on this estimate at a 95 percent confidence level?

The variation in $\bar{t}$ can be found by using Chart I, page 23 from reference 10. Enter Chart I on the 5 percent line at the left hand end of the 5 interval, here $T/\bar{t}_1 = 10.5$; then $\bar{t}_1 = 10\,\bar{t}/T/\bar{t}_1 = 8.57\times10^4/10.5 =$ 8160 hours. Using the left hand end of the 4 interval $T/\bar{t}_2 = 9.25$; then $\bar{t}_2 = 8.57\times10^4/9.25 = 9530$ hours. One easy method to find Q(6) is to use figure 6-1 of reference 5, page 61. The $t/\bar{t}$ ratios of interest are 1.05, 1.16, and 1.22, respectively. For these ratios with $N_f = 5$, the values of Q(6) from figure 6-1 are 0.997, 0.9987, and 0.9992, respectively. Since the sum of the last five terms is desired, R(6) is 0.003, 0.0013, and 0.0008, respectively.

This means that the probability of the 5th to the 10th failure of a speed control occurring is in the interval from 0.0008 to 0.003 at a confidence level of 95 percent.

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TABLE I. - TEST DATA FOR A MECHANICAL PART

Ordered sample number	t_f, K hr	t_f^2, (K hr)^2
1	60	3600
2	65	4225
3	68	4624
4	70	4900
5	75	5625
6	75	5625
7	80	6400
8	83	6889
9	85	7225
n = 10	90	8100
Totals	750	57 213

TABLE II. - CABLE TIME-TO-FAILURE DATA

Ordered sample number	Time-to-failure, hr	Median rank[a]	5% Rank[a]	95% Rank[a]
1	1 100	6.7	0.5	25.9
2	1 890	16.2	3.7	39.4
3	2 920	25.9	8.7	50.7
4	4 100	35.5	15.0	60.7
5	5 715	45.2	22.2	69.7
6	8 720	54.8	30.3	77.8
7	12 000	64.5	39.3	85.0
8	17 500	74.1	49.3	91.3
9	23 900	83.3	60.6	96.3
n = 10	46 020	93.3	74.1	99.5

[a]From tables 2, 5, and 15 of reference 7.

TABLE III. - PASCAL'S TRIANGLE

FOR BINOMIAL COEFFICIENTS

Sample size	Possible failure	Binomial coefficients
1	2	1
2	3	1 2 1
3	4	1 3 3 1
n = 4	5	1 4 6 4 1

TABLE IV. - SPEED CONTROLLER

TIME-TO-FAILURE DATA

Ordered sample number	Time-to-failure, hr
1	3 520.0
2	4 671.2
3	6 729.3
4	7 010.0
5	8 510.2
6	9 250.1
7	10 910.0
8	11 220.5
9	11 815.6
10	12 226.4
Total	85 866.3

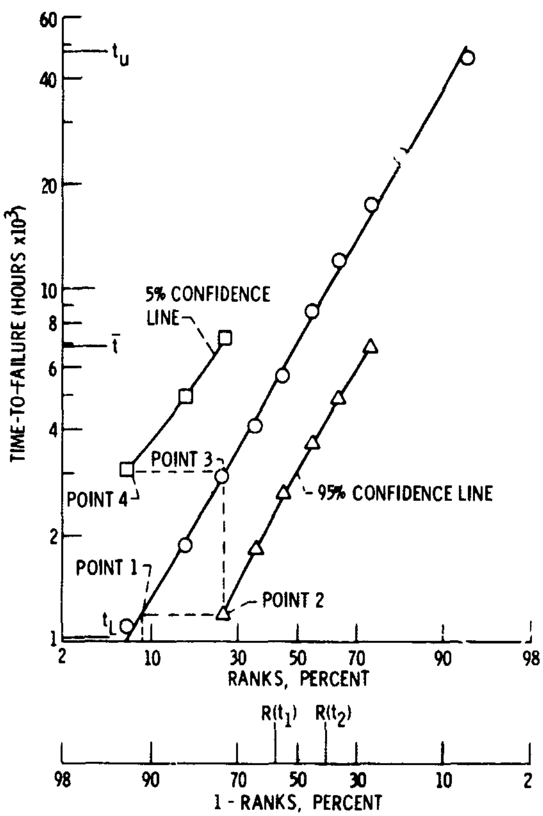

Figure 1. - Lognormal life test diagram.

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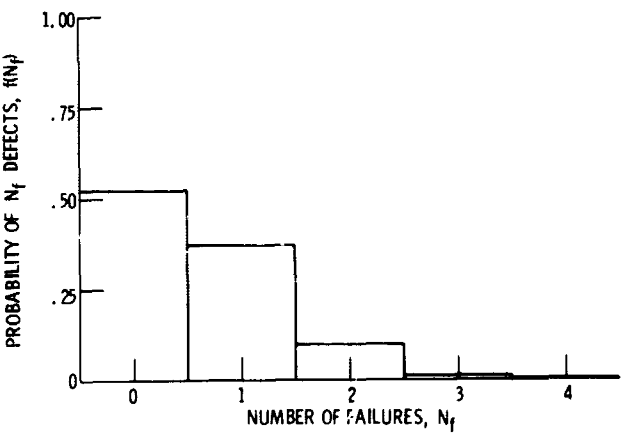

Figure 2. - Explosive bolts histogram.

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