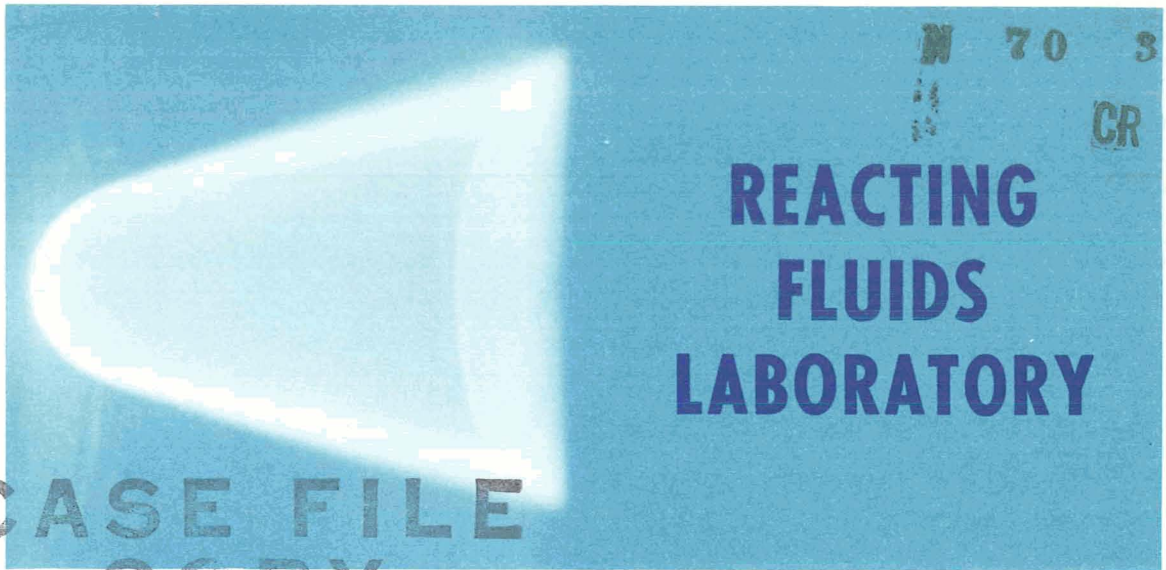


TECHNICAL REPORT

ON

NASA GRANT NGR 19-001-059

ANALYSIS of THE INTERACTION of ABLATING PROTECTION SYSTEMS AND STAGNATION REGION HEATING

Part I: Formulation, Derivation and Development of the Analysis of the Interaction of Ablating Protection Systems and Stagnation Region Heating

by

C. D. Engel, D. D. Esch

R. C. Farmer, R. W. Pike

Department of Chemical Engineering

NASA-RFL TR-70-1

January 1, 1970

LOUISIANA STATE UNIVERSITY

Reacting Fluids Laboratory
Department of Chemical Engineering
Louisiana State University
Baton Rouge, Louisiana

TECHNICAL REPORT

FORMULATION, DERIVATION AND DEVELOPMENT OF THE ANALYSIS OF THE INTERACTION OF ABLATING PROTECTION SYSTEMS AND STAGNATION REGION HEATING

by

C. D. Engel, Research Associate
D. D. Esch, Research Associate
R. C. Farmer, Associate Professor
R. W. Pike, Associate Professor

on

N.A.S.A. Grant NGR 19-001-059

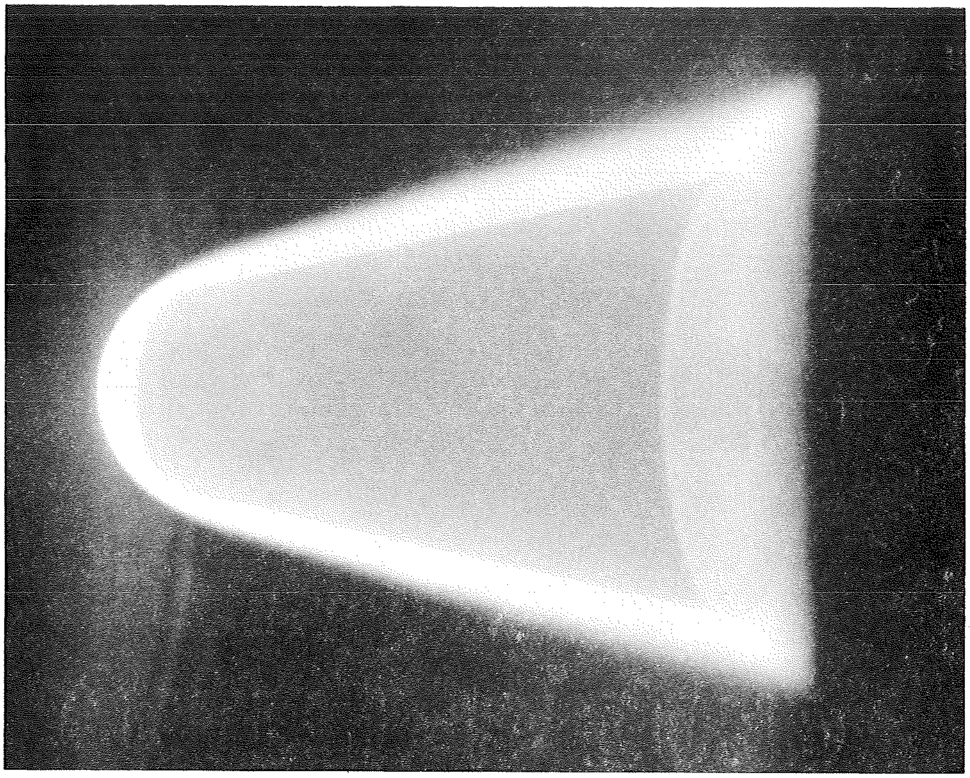
ANALYSIS OF THE INTERACTION OF ABLATING PROTECTION SYSTEMS AND STAGNATION REGION HEATING

NASA - RFL - TR 70-1

January 1, 1970

Photograph of a blunt body in a multiple arc jet wind tunnel showing the ionized shock layer about the body. (Mach no. 7) Courtesy of: T. A. Barr, Jr., U. S. Army Missile Command, Redstone Arsenal, Alabama, 1969.

Photograph of a cross section of a nylon-phenolic resin ablator. Courtesy of: C. W. Stroud, NASA TN D-4817, 1968.



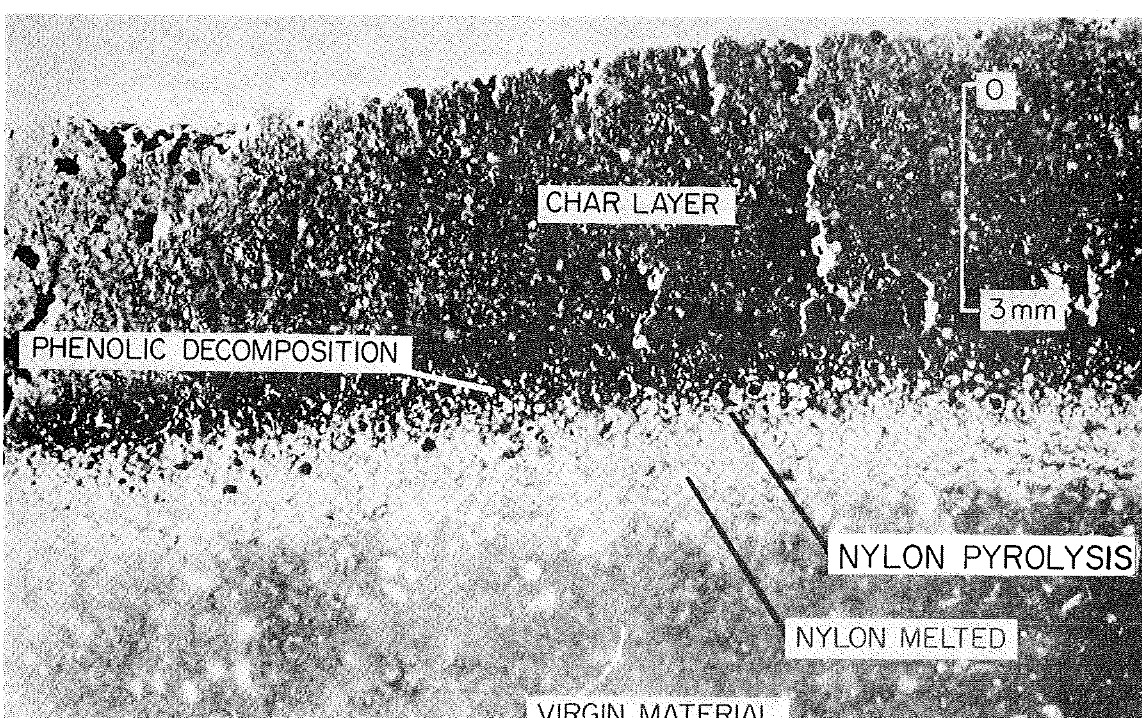

O
CHAR LAYER
3mm
PHENOLIC DECOMPOSITION
NYLON PYROLYSIS
NYLON MELTED
VIRGIN MATERIAL

TABLE OF CONTENTS

	Page
ABSTRACT	
NOMENCLATURE	1
SECTION	
I. CONSERVATION EQUATIONS OF A MULTICOMPONENT, RADIATING, CHEMICALLY REACTING FLUID	I 1
II. CONSERVATION EQUATIONS IN GENERAL ORTHOGONAL COORDINATES	II 1
III. CONSERVATION EQUATIONS IN BODY ORIENTED COORDINATES	III 1
IV. THIN VISCOUS SHOCK LAYER EQUATIONS	IV 1
V. TRANSPORT AND THERMODYNAMIC PROPERTIES	V 1
VI. SUMMARY	VI 1
APPENDIX A. DERIVATION OF SPECIES DIFFUSION FLUXES AS AN EXPLICIT FUNCTION	A 1
APPENDIX B. CORRELATION OF BIFURCATION FACTOR AND BINARY DIFFUSION COEFFICIENTS	B 1
APPENDIX C. SIMPLIFICATION OF THE RELATIONSHIP FOR EVALUATION OF EFFECTIVE MULTICOMPONENT DIFFUSION COEFFICIENTS	C 1
DISTRIBUTION	

ABSTRACT

The conservation equations for a multicomponent, radiating, chemically reacting fluid in local thermodynamic equilibrium are derived. From the conservation equations the thin shock layer equations are developed. These equations appropriately describe the flow in a shock layer produced by a blunt body during a hyperbolic entry, atmospheric encounter. Special attention is given to radiation, chemical reaction and mass transport fluxes. Stagnation line equations are derived from the shock layer equations by taking appropriate limits and are discussed from a mathematical view point as initial conditions for the shock layer equations. Boundary conditions for these equations are derived and discussed in the light of ablator response coupling with the flow-field.

ACKNOWLEDGMENTS

The authors would like to give special acknowledgment to James Breedlove for his meticulous efforts in drafting the equations and figures in this report. Special thanks is also given to Jane Gremillion and other secretaries who typed this report.

NOMENCLATURE*

English

$\bar{B}$	Property flux vector (property x L)/(L^3 x t)+
B_v	Planckian radiation intensity (m/t^2 x no. of particles)
$\mathbb{B}$	Property generation term (property)/(L^3 x t)
b	Generalized property per unit mass term (property/m)
C_i	Mass fraction (mass of i /unit mass of fluid)
	$C_i = \frac{\rho_i}{\rho} = \frac{Y_i M_i}{M} = \frac{n_i M_i}{\rho}, \quad \sum_i C_i = 1$
c	Velocity of light (L/t)
C_p	Specific heat at constant pressure (L^2/t^2 x T)
D_{ij}	Multicomponent diffusion coefficient (L^2/t)
D_i	Effective multicomponent diffusion coefficient (L^2/t)
D_i^T	Thermal diffusion coefficient (m/L x t)
$\mathbb{D}_{ij}$	Binary diffusion coefficient (L^2/t)
E	Stagnation internal energy (mL^2/t^2)
$\bar{\bar{e}}$	Strain tensor (defined in Eq. 2.18) (1/t)
F_i	Diffusion factor for specie i
f	Velocity function, u/u_s
f	Photon distribution function (no. of photons x t)/L^3
G_j	Gibbs free energy (mL^2/t^2 x mole of j)
$\bar{g}_i$	Gravitational force vector acting on a unit mass of specie i (L/t^2)
g	Enthalpy function, H/H_s
H	Total enthalpy, $H = h + \bar{V}^2/2$ (L^2/t^2)

h Static enthalpy, $h = U + P/\rho$ (L^2/t^2), also Planck's constant

h_1, h_2, h_3 Stretching functions in the ξ_1, ξ_2, ξ_3 directions respectively¢

$\bar{J}_i$ Mass flux vector of species i $(m/L^2 \times t)$

J Radiative emission term defined in Eq. 1.23 $(m/t^2 \times L)$

k' Coefficient of thermal conductivity without diffusion effects $(mL/t^3 \times t)$

k Ordinary coefficient of thermal conductivity $(mL/t^3 \times T)$

k_c Boltzmann's Constant (mL^2/t^2T)

L_{lm} Generalized transport coefficient (property/L x t x driving force)

M_i Molecular weight of species i (mass of i /mole of i)

N Number density (particles/L^3)

n_i Molal volume (moles of i /L^3)

n_t Molal density (total moles/L^3)

$\bar{n}$ Unit normal vector

P Static pressure $(m/L \times t^2)$ or (F/L^2)#

P_r Prandtl number, $C_p \mu / k$

$\bar{\bar{P}}_R$ Radiative stress tensor defined by Eq. 1.12 $(m/L \times t^2)$

Q Internal energy per unit mass, including chemical energy (L^2/t^2)

$\bar{q}_R$ Radiative heat flux vector defined by Eq. 1.22 (m/t^3) or $(E/L^2 \times t)$#

$\dot{q}_c$ Convective energy flux to a surface (m/t^3)

$\dot{q}_R$ Radiative energy flux to a surface (m/t^3)

$\bar{q}_D$ Diffusional energy flux vector defined by Eq. 1.13 (m/t^3)

R Body nose radius (L)

R_e Reynolds number, $\rho_{s,0} U_\infty R / \mu_{s,0}$

$\mathbb{R}$ Universal gas constant (mL^3/t^2 x T x no. of moles)

$\tilde{\mathbb{R}}$ $\mathbb{R}/M$, mean molecular weight weighted gas constant (L^2/t^2 x T)

r Cylindrical body radius defined in Fig. 3.1 (L)

S Induction or resistance energy generation term defined by Eq. 1.12 (L^2/t^3)

T Thermodynamic temperature (T)

t Time (t)

U_∞ Freestream velocity (L/t)

$\mathbb{U}_\nu$ Spectral radiant energy density (m/L x t x no. of particles)

u Component of $\bar{V}$ in the ξ_1 direction (parallel to the body surface) (L/t)

$\forall$ Volume (L^3)

$\bar{V}$ Velocity vector, $u\bar{i} + v\bar{j} + w\bar{k}$ (L/t)

v Component of $\bar{V}$ in the ξ_2 direction (normal to the body surface) (L/t)

w Component of $\bar{V}$ in the ξ_3 direction (L/t)

x Body oriented coordinate corresponding to ξ_1 (L)

y Body oriented coordinate corresponding to ξ_2 (L)

Y_i Mole fraction of species i, $Y_i = n_i/n_t$, $\sum_i Y_i = 1$

z Body oriented coordinate corresponding to ξ_3 (L)

Greek

α_ν Volumetric absorption coefficient, effective (L^2 x no. of particles/L^3)

β Generalized property, (mass, momentum, or energy/L^3)

∇ Del operator (defined Tab. 3.1) (1/L)

δ Shock detachment distance (L)

$\tilde{\delta}$ Transformed shock detachment distance

E_n Exponential integral function of order n defined by Eq. 4.51

ϵ Difference between the body and shock angle $\epsilon = \theta - \phi$ (radians)

ϵ_i Characteristic interaction energy (a Lennard-Jones parameter) (mL^2/t^2)

ζ Distance along an arc (L)

η Dorodnitzyn variable

θ Body angle (radians)

I_1 Integral of f^2, (where f is the velocity function)

I_2 Integral of f, (where f is the velocity function)

I_ν Spectral radiation intensity (m/t^2 x no. of particles x no. of steardians)

$\bar{\bar{I}}$ Unit tensor

ι Direction cosine

K Absorption coefficient (L^2 x no. of particles/L^3)

κ Local body curvature (1/L)

$\tilde{\kappa}$ $1 + \kappa y$

Λ Diffusional or radiative flux divergence (see Eq. 4.1) (m/L x t^2)

λ $(\tilde{\mu} - 2/3\,\mu)$ (m/L x t)

M $\sum_i Y_i M_i$ mean molecular weight of the mixture (m of mixture/mole of mixture)

μ Ordinary viscosity (m/L x t)

$\tilde{\mu}$ Bulk viscosity (m/L x t)

ν Frequency (1/t)

ξ Orthogonal coordinate$^{¢}$; nondimensional x- coordinate when not subscripted

ρ Density (m/L^3), $\rho = n_+ M$

ρ_i	Partial density of species i, $\rho_i = n_i M_i$ (m of i $/L^3$)
$\bar{\rho}$	Density ratio across shock ρ_∞ / ρ_s
σ	Radiative absorption cross section (L^2)
σ_i	Collision diameter of specie i, Å (L)
Υ_ν	Optical depth at frequency ν
$\bar{\bar{\tau}}$	Viscous stress tensor defined by Eq. 2.18 $(m/L \times t^2)$
ϕ	Shock angle (radians)
$\bar{X}_m$	Generalized driving force (driving force/L)
$\bar{\Omega}_1$	Unit vector in the direction of photon propogation
Ω	Solid angle (stearadians)
Ω_{ij}^{ls}	Collision integral of colliding species i and j
ω_i	Generation of species i $(m/L^3 \times t)$

Subscripts

a	Atmospheric (sea level) quantities
e	Edge conditions
i	Species i
n	Normal component
t	Tangential component or total quantity
w	Wall quantities
o	Stagnation line quantities
1, 2, 3	Directions corresponding to positive ξ_1, ξ_2, ξ_3 respectively
∞	Freestream conditions
s	Quantities immediately behind the shock

Superscripts

D	Diffusion
g	Gravitational
p	Pressure
A	0 or 1 denoting two-dimensional or axisymetric respectively (an exponent)
T	Thermal
*	Denotes dimensional variables in Section IV
o	Standard state quantity

Script letters

h_i	Partial molal enthalpy $(mL^2/t^2$ x moles of i)
h	Molal enthalpy $(mL^2/t^2/$ x total moles)

---

* Symbols not listed are defined where used.

\+ Abbreviations mean: m, mass; L, length; t, time; T, temperature; F, force; E, energy.

¢ The product $h_i \xi_i$ has unit of L.

\# Note $g_c = (mL/t^2 \times F)$ and $J = (FL/E)$ have been used.

SECTION I

CONSERVATION EQUATIONS OF A MULTICOMPONENT, RADIATING, CHEMICALLY REACTING FLUID

The conservation laws for mass, momentum, and energy will be presented for a continuum, multicomponent fluid whose internal degrees of freedom are in thermodynamic equilibrium. The assumption of thermodynamic equilibrium implies that no matter how small a volume of fluid we are interested in there are enough molecules within the volume to give meaningful average properties and that regardless of the flow velocities of interest a temperature may be ascribed to the fluid. This is roughly equivalent to assuming the first postulate of nonequilibrium thermodynamics, see Fitts Ref. 1.1.

A general property balance can be made on an element of volume $\forall$ moving with the stream velocity $\overline{V}$ similar to that of Brodkey, Ref. 1.2. The property (mass, momentum, or energy) per unit volume is designated by β. The flux of a property through a control surface is denoted by $\overline{B}$ (property x length) / (volume x time), and the generation of a property within the control volume is denoted by $\mathbb{B}$ (property) / (volume x time). The differential form of the general property balance can be written in terms of the above definitions.

$$\frac{\partial\beta}{\partial t} + \nabla\cdot\beta\overline{V} + \nabla\cdot\overline{B} - \mathbb{B} = 0 \qquad (1.1)$$

As stated the general property balance equation was derived for a moving control volume. For a control volume stationary in space there is a convective flow through the control volume which is identical to the motion term associated with the control volume. This means that if the general balance is derived for a moving control volume it may be used for a fixed control volume, with $\overline{B}$ maintaining exactly its same definition. This allows $\overline{B}$ to be interpreted as a diffusive flow. This is obvious for the moving control volume, but the practice of lumping all kinds of effects into this elusive flux term for a fixed control volume effectively redefines $\overline{B}$ as a catch-all. Therefore, the general balance equation is stated in the form for a moving control volume, but it is fully intended to be used to describe a stationary volume in space.

The general property balance Eq. 1.1 can also be written:

$$\underset{(1)}{\frac{\partial\beta}{\partial t}} + \underset{(2)}{\bar{V}\cdot\nabla\beta} + \underset{(3)}{\beta\nabla\cdot\bar{V}} + \underset{(4)}{\nabla\cdot\bar{B}} - \underset{(5)}{\mathbb{B}} = 0 \tag{1.2}$$

The meaning of these terms is:

(1) The accumulation of β within a control volume (C.V.)

(2) The convective flow of β thru the C.V.

(3) The dilation of the flow thru the C.V., i.e. the change of β when the fluid is compressed or expanded

(4) The diffusional flux into and out of the C.V.

(5) The generation of β within the C.V.

Using Eq. 1.2 and specifying β, $\bar{B}$, and $\mathbb{B}$ we now can write the conservation equations. Consider first the conservation of mass by specifying $\beta = \rho$ (mass / volume), $\bar{B} = \mathbb{B} = 0$.
Substitution into Eq. 1.2 yields

Continuity:

$$\frac{\partial\rho}{\partial t} + \bar{V}\cdot\nabla\rho + \rho\nabla\cdot\bar{V} = 0 \tag{1.3}$$

Before proceeding to the other conservation equations let us rewrite the general property balance equation in another form by substituting $\beta = b\rho$ into Eq. 1.2. By using this substitution and noting the continuity equation appears as a product of b, the general property balance relation can be expressed as:

$$\rho\frac{Db}{Dt} + \nabla\cdot\bar{B} - \mathbb{B} = 0 \qquad (1.4)^{+}$$

This equation will be used to evaluate the remainder of the conservation equations.

Consider now species conservation by specifying

$$b = C_i$$
$$\bar{B} = \bar{J}_i$$
$$\mathbb{B} = \dot{\omega}_i$$

where

$$\sum \bar{J}_i = 0$$

+ $\frac{Db}{Dt}$ is the substantial derivative of b which equals $\frac{\partial b}{\partial t} + \bar{V}\cdot\nabla b$.

Substitution of the above relations into Eq. 1.4 yields

Species Continuity:

$$\rho \frac{DC_i}{Dt} + \nabla \cdot \bar{J}_i - \dot{\omega}_i = 0 \tag{1.5}$$

Let us accept the second postulate of thermodynamics of irreversible processes which states that if the fluid is not too far from equilibrium, fluxes and currents are linear homogeneous functions of the driving force. Using this postulate the mass flux vector of specie i, Eq. 1.5, can be written as the sum of contributing vectors.

$$\bar{J}_i = (L_{D1} \bar{X}_1)_i + (L_{D2} \bar{X}_2)_i + \cdots \tag{1.6}$$

where

L_{lm} = Transport Coefficient

$\bar{X}_m$ = Driving Force

and where subscript " $\mathbf{D}$ " indicates diffusional transport coefficients.

The number of necessary terms to consider can only be discussed in reference to a particular application. Four terms are stated below from Bird *et al.*, Ref. 1.3, for consideration.

$$(L_{11}\bar{X}_1)_i = \bar{J}_i^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_j M_i M_j D_{ij} \left[Y_i \sum_{\substack{k=1 \\ k \neq j}} \left(\frac{\partial G_j}{\partial Y_k}\right)_{T, \forall, Y_l,\ l \neq j,k} \nabla Y_k \right] \tag{1.7}$$

$$(L_{12}\bar{X}_2)_i = \bar{J}_i^{(T)} = -D_i^T \nabla \ln T \tag{1.8}$$

$$(L_{13}\bar{X}_3)_i = \bar{J}_i^{(P)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum M_i M_j D_{ij} \left[Y_j M_j \left(\frac{\bar{\forall}_j}{M_j} - \frac{1}{\rho}\right) \nabla P \right] \tag{1.9}$$

$$(L_{14}\bar{X}_4)_i = \bar{J}_i^{(g)} = \frac{-n_t^2}{\rho \mathbb{R} T} \sum M_i M_j D_{ij} \left[Y_j M_j \left(\bar{g}_j - \sum \frac{\rho_k}{\rho} \bar{g}_k\right) \right] \tag{1.10}$$

where

n_t = Concentration in total no. of moles/volume (C in Ref. 1.3)

Y_j = Mole fraction (X_j in Ref. 1.3)

G_j = Gibbs' free energy

D_{ij} = Multicomponent diffusion coefficient

D_i^T = Thermal diffusion coefficient

Eq. 1.7 expresses the mass diffusion vector. Since the driving force is of the same measure as the flux, they are called "conjugate". The conjugate transport coefficients, L_{aa}, are the largest, i.e. mass is diffused primarily by mass concentration gradients. Eq's. 1.8, 1.9, and 1.10 represent the mass flux vector contribution from thermal diffusion, pressure diffusion, and forced diffusion respectively. There are also fluxes due to inertia and viscous terms, but they are very small, see appendix in Fitts Ref. 1.1. Electrical and magnetic effects can also create fluxes.

The definition of flux as a linear function of coefficients and potentials and the realization that fluxes are tensors of various ranks leads one to speculate on what type of cross effects can exist. Curie's theorem states that "fluxes whose tensorial characters differ by an odd integer cannot interact in isotropic systems," Ref. 1.1. This means that the mass flux tensor and the heat flux tensor which are both vectors are not coupled to the reaction rate tensor (a scalar), or the momentum flux tensor (a second order tensor) but may be coupled to each other. Also, it should be observed that momentum flux tensor either as a second order tensor or in contracted form as a scalar may be coupled to the reaction rate tensor.

With the foregoing information in mind consider the conservation of momemtum. For substitution into the general balance equation

$$b = \beta/\rho = \bar{V}$$

$$\bar{B} = \bar{\bar{\tau}} - \bar{\bar{I}}P + \bar{\bar{P}}_R$$

$$\mathbb{B} = \sum_i \rho_i \bar{g}_i$$

Using Eq. 1.4 for momentum conservation yields

<u>Momentum</u>:

$$\rho\frac{D\bar{V}}{Dt} + \nabla\cdot(\bar{\bar{\tau}} - \bar{\bar{I}}P + \bar{\bar{P}}_R) - \sum_i \rho_i \bar{g}_i = 0 \qquad (1.11)$$

Note that in the above equation the radiative pressure tensor, $\bar{\bar{P}}_R$, is included for completeness. This term is negligible for practically all nonnuclear problems.

Let us now apply the general balance equation to conservation of energy by specifying

$$b = Q + \frac{\overline{V}^2}{2} + \sum_i \rho_i \overline{V} \bullet g_i = E \qquad \text{(energy / mass)}$$

$$\overline{B} = \overline{q}_D \qquad \text{(energy / volume)(length / time)}$$

$$\mathbb{B} = \nabla \bullet \overline{q}_R - \nabla \bullet (\overline{\overline{\tau}} - \overline{\overline{I}}P + \overline{\overline{P}}_R) \bullet \overline{V} + \sum_i \overline{g}_i \bullet \overline{J}_i$$
$$- S\rho \qquad \text{(energy / volume - time)}$$

= generation by radiation + pressure tensors + external forces + heat sources internal to the C.V.; i.e. induction heating, resistance heating, etc.

Substitution of the above into Eq. 1.4 yields the total internal energy form of the energy equation

$$\rho \frac{DE}{Dt} + \nabla \bullet \overline{q}_D + \nabla \bullet \overline{q}_R - \nabla \bullet (\overline{\overline{\tau}} - \overline{\overline{I}}P + \overline{\overline{P}}_R) \bullet \overline{V}$$
$$+ \sum_i \overline{g}_i \bullet \overline{J}_i - S\rho = 0 \tag{1.12}$$

where

$\overline{q}_D$ = diffusional heat flux vector

$\overline{q}_R$ = radiative heat flux vector

$\sum_i \overline{g}_i \bullet \overline{J}_i$ = heat generated in the system by a gravitational field

Let us investigate further the diffusional and radiative heat flux vectors. Again imposing restrictions from thermodynamics of irreversible processes, the diffusional heat flux vector may be written as a sum of vectors

$$\overline{q}_D = (L_{T1}\overline{X}_1) + (L_{T2}\overline{X}_2) + \cdots \tag{1.13}$$

where

$$L_{T1}\overline{X}_1 = \sum_i h_i \overline{J}_i - k' \nabla T$$

$L_{T2}\overline{X}_2$ = energy transport due to the Dufour effect

The $L_{T1}\overline{X}_1$ term is the conjugate term for this flux vector. It should be noted however that the right hand side definition is an arbitrary one. The Dufour effect arises due to the consideration of the Soret effect in mass diffusion. Additional cross effects from other coefficients and potentials will not be considered.

Radiative transfer of heat is propogated in an entirely different manner than diffusional heat transfer. Diffusional heat transfer mechanism depends on gradients in the gas, such as temperature, species, pressure or external forces as pointed out by Planck, Ref. 1.4. Radiative transfer of heat is in itself entirely independent of these gradients in the medium through which it passes. In general, radiation is a far more complicated phenomenon than diffusional heat transfer. The reason for this is that the state of the radiation at a given instant and at a given point of the gas can not be represented by a single vector as the diffusional mechanisms can. All radiative energy rays which at a given time pass through the same point in a gas are independent of each other. Therefore, to specify completely the state of the radiation at a point the radiation intensity must be known in all directions which pass through the point under consideration.

Special attention will now be given to the development of the radiative flux and flux divergence terms which are needed in the evaluation of energy conservation. Starting with the basic concepts of radiative transfer in an absorbing and emitting medium, Ref. 1.4 and 1.5, a definition of the spectral radient energy density is developed.

Let $f(\nu,\overline{r},\overline{\Omega}_1,t)d\nu d\Omega$ be the number of photons in the frequency interval ν to $\nu + d\nu$, contained at time t in the volume element $d\forall$ located about the point $\overline{r}$, and having a direction of motion within an element of solid angle $d\Omega$ about the unit vector $\overline{\Omega}_1$. The function f is called the distribution function. For this definition to be meaningful the linear dimensions of the volume element must be larger than the largest wavelength c/ν.

Each photon possesses an energy $h\nu$. Therefore, the <u>spectral radiant energy density</u> may be defined as the radiant energy of frequency ν included in a unit spectral interval and contained in a unit volume at the point $\overline{r}$ and at the time t by:

$$\mathbb{U}_\nu(\overline{r},t) = h\nu\int_{4\pi} f\, d\Omega \tag{1.14}$$

In a like manner, the spectral radiation intensity can be defined. First recall each photon possesses a velocity equal to that of light c. Therefore the quantity

$$h\nu c\ f(\nu, \bar{r}, \bar{\Omega}_1, t)\, d\nu\, d\Omega \tag{1.15}$$

represents the radiant energy in the spectral interval $d\nu$ passing through a unit area in a unit time in the direction within the solid angle $d\Omega$ about $\bar{\Omega}_1$. The area is located at $\bar{r}$ and is normal to $\bar{\Omega}_1$. This statement is not necessarily obvious. In order to clearly indicate how and what area is located at point $\bar{r}$ let us follow the derivation of the spectral radiant energy density given by Planck Ref. 1.4.

Consider an infinitely small element of volume $d\forall$, at the point which has an arbitrary shape Fig. 1.1. In order to allow for all rays to pass through the volume $d\forall$, we can construct about any point $\bar{r}$ of $d\forall$ a sphere of radius σ . The radius σ is selected to be large compared with the linear dimensions of $d\forall$ but still so small that no appreciable absorption or scattoring of radiation occurs in the sphere. Each ray which reaches $d\forall$ must then originate from some point on the surface of the sphere.

Let us determine the amount of energy contained in $d\forall$ which originated from an element of surface area da . The surface area is chosen such that its linear dimensions are small compared to those of $d\forall$. Consider the cone of rays which start at a particular point on da and meet the volume $d\forall$. This cone consists of an infinite number of conical elements with a common vertex at a point on da each cutting out of the volume $d\forall$ a certain element of length s . The solid angle of such a conical element is $\Delta A/\sigma^2$ where ΔA denotes the area of cross section normal to the axis of the cone at a distance σ from the vertex Fig. 1.1.

In order to find the energy radiated through an element of area let us first define $h\nu cf$

$$I_\nu(\bar{r}, \bar{\Omega}_1, t) \equiv h\nu cf \tag{1.16}$$

which is called the spectral radiation intensity. Using Eq. 1.15 and 1.16 the monochromatic energy which has passed through da and is in $d\forall$ is:

$$I_\nu\, d\Omega\, (s/c)\, da = h\nu c f\, d\Omega\, (s/c)\, da \tag{1.17}$$

where $d\Omega = \frac{\sum \Delta A}{\sigma^2}$ and S is the path length in $d\forall$. The energy which enters the conical element in $d\forall$ spreads out into a volume ΔAs. Summing up over all conical elements which originate in da and enter $d\forall$ yields

$$\frac{I_\nu}{c}\frac{da}{\sigma^2}\sum \Delta As \;=\; \frac{I_\nu}{c}\frac{da}{\sigma^2}\,d\forall \;=\; \frac{I_\nu}{c}\,d\Omega\,d\forall \qquad (1.18)$$

This represents the entire monochromatic radiant energy ontained in volume $d\forall$ resulting from radiation through the element of area da. To determine the total monochromatic radiant energy contained in $d\forall$ we must integrate over all elements of area da contained in the surface of the sphere. For the procedure of this integration observe Fig. 1.2. In this case the increment in solid angle $d\Omega = \frac{da}{\sigma^2}$ which corresponds to a cone with a vertex at $\bar{r}$. Integrating the right hand side of Eq.'s 1.18 yields the total energy:

$$\frac{d\forall}{c}\int I_\nu \, d\Omega$$

The monochromatic radiant energy density is obtained by dividing by $d\forall$.

$$\mathbb{U}_\nu \;=\; \frac{1}{c}\int I_\nu \, d\Omega \qquad (1.19)$$

Since the radius σ does not appear in Eq. 1.19 we can think of I_ν as the intensity of radiation at the point $\bar{r}$ itself or the intensity of radiation passing thru a unit area at $\bar{r}$ in the direction $\bar{\Omega}_1$. This clarifies a difficult concept which is avoided in many derivations.

From the definition of I_ν it follows that the radiation heat flux is a vector of magnitude

$$q_R(\bar{r},t) \;=\; \int c\,\mathbb{U}_\nu \, d\nu \;=\; \iint I_\nu \, d\Omega \, d\nu \qquad (1.20)$$

in the direction $\bar{\Omega}_1$ of photon propagation. Let the normal to any surface thru point $\bar{r}$ be called $\bar{n}$. Therefore the magnitude of the heat flux passing thru a unit surface area normal to $\bar{n}$ from photon propagation in the $\bar{\Omega}_1$ direction is:

$$(\bar{n}\cdot\bar{q}_R) \;=\; q_R(\bar{r},\bar{n},t) \;=\; \iint_{4\pi} (\bar{n}\cdot\bar{\Omega}_1)\, I_\nu\,(\bar{r},\bar{\Omega}_1,t)\, d\Omega\, d\nu \qquad (1.21)$$

Finally the radiative flux vector can be written

$$\bar{q}_R(\bar{r},t) = \int_{\nu=0}^{\infty}\int_{\Omega=0}^{4\pi} I_\nu(\bar{r},\bar{\Omega}_1,t)\,\bar{\Omega}_1\,d\Omega\,d\nu \tag{1.22}$$

Therefore $\bar{q}_R$ is defined at any point $\bar{r}_1$ and time t in space.

For the use of the radiative heat flux vector in the energy equation, it is desirable to be able to calculate a component of $\bar{q}_R$ in any coordinate direction of an orthogonal coordinate system and to calculate $\nabla \cdot \bar{q}_R$. These calculations may be accomplished in a more expeditious fashion by first writing the <u>equation</u> of <u>radiative transfer</u>.

The radiative transfer equation states that the rate of radiative energy accumulated in a volume element plus the rate that it flows thru the element equals the rate of generation within the element. The generation of radiative energy is accomplished by emission and absorption. The general property balance can be used by defining

$$\beta = I_\nu$$

$$\bar{B} = 0$$

$$\mathbb{B} = c\left[\mathbb{J}_\nu\left(1 + \frac{c^2}{2h\nu^3} I_\nu\right) - KI_\nu\right]$$

Substituting into Eq. 1.2

$$\frac{\partial I_\nu}{\partial t} + (c\bar{\Omega}_1)\cdot\nabla I_\nu + I_\nu\nabla\cdot(c\bar{\Omega}_1) = c\left[\mathbb{J}_\nu\left(1 + \frac{c^2}{2h\nu^3} I_\nu\right) - KI_\nu\right] \tag{1.23}$$

using the vector identity

$$(c\bar{\Omega}_1)\cdot\nabla I_\nu + I_\nu\nabla\cdot(c\bar{\Omega}_1) = (c\bar{\Omega}_1)\cdot\nabla I_\nu$$

we can write

$$\frac{1}{c}\left[\frac{\partial I_\nu}{\partial t} + c\bar{\Omega}_1\cdot\nabla I_\nu\right] = \mathbb{J}_\nu\left(1 + \frac{c^2}{2h\nu} I_\nu\right) - KI_\nu \tag{1.24}$$

which is identical to the expression given by Zel'dovich and Raizer Ref. 1.4. In order to simplify Eq. 1.24 the following observations are made. The emission term $\mathcal{J}_v$ can be expressed

$$\mathcal{J}_v = \alpha_v B_v \tag{1.25}$$

by using Kirchoff's law and assuming local thermodynamic equilibrium. Note that the effective volumetric absorption coefficient

$$\alpha_v = K\left[1 - \exp\left(-h\nu/k_c T\right)\right] \tag{1.26}$$

is the product of the absorption coefficient and the induced emission term. Therefore the emission term $\mathcal{J}_v$ has both spontaneous and induced emission taken into account. The spontaneous emission term is the Planck function.

$$B_v = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(-h\nu/k_c T\right) - 1} \tag{1.27}$$

Using these definitions Eq. 1.24 can be rewritten as:

$$\frac{1}{c}\frac{\partial I_v}{\partial t} + \bar{\Omega}_1 \cdot \nabla I_v = \alpha_v (B_v - I_v) \tag{1.28}$$

If the radiative transfer Eq. 1.28 is multiplied by $d\Omega$ and integrated over all directions the conservation of radiation equation is obtained

$$\frac{\partial \mathbb{U}_v}{\partial t} + \nabla \cdot \bar{q}_{R,v} = c\alpha_v (\mathbb{U}_{vp} - \mathbb{U}_v) \tag{1.29}$$

where

$$\mathbb{U}_{vp} = \frac{4\pi B_v}{c}$$

Let us assume

$$\frac{\partial \mathbb{U}_v}{\partial t} = \frac{1}{c}\int \frac{\partial I_v}{\partial t}\, d\Omega = 0$$

since c is very large. Then we may solve Eq. 1.26 for the radiative flux divergence.

$$\nabla \cdot \bar{q}_R(\bar{r}_1) = \int_0^\infty \alpha_v \left| 4\pi B_v - \int_0^{4\pi} I_v(\bar{r})\, d\Omega \right| d\nu \tag{1.30}$$

The contribution of the radiative flux divergence term in the energy equation has important mathematical ramifications. It should be noticed that the flux divergence term is evaluated by integrating over all space. The other terms in the energy equation are differentials calculated locally. The radiative flux divergence term therefore makes the energy equation an integro-differential equation.

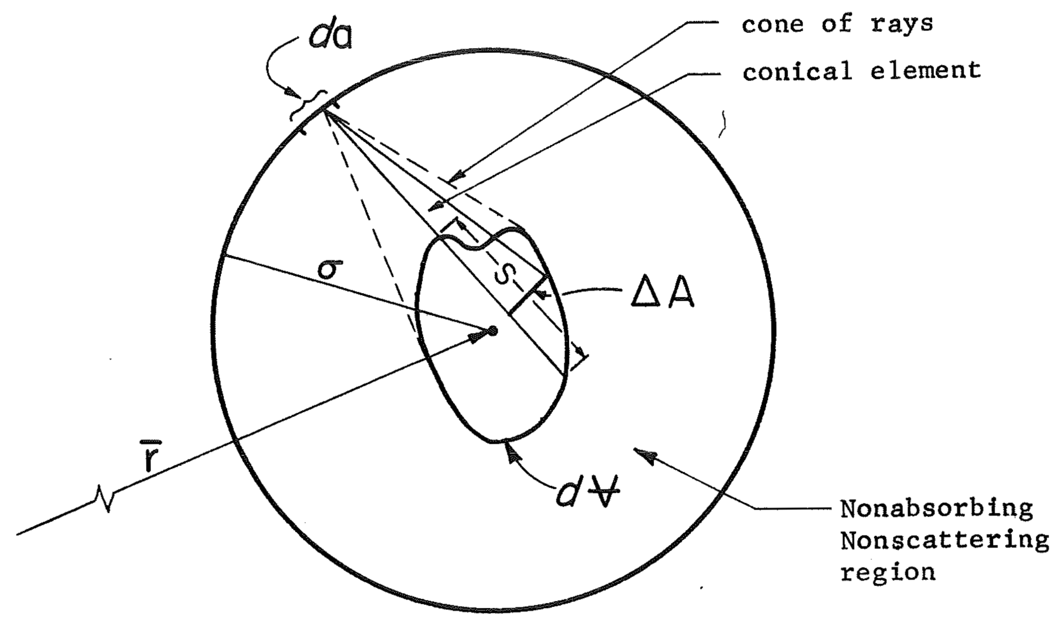


__Fig. 1.1__ Radiation to $d\forall$ from its surroundings.

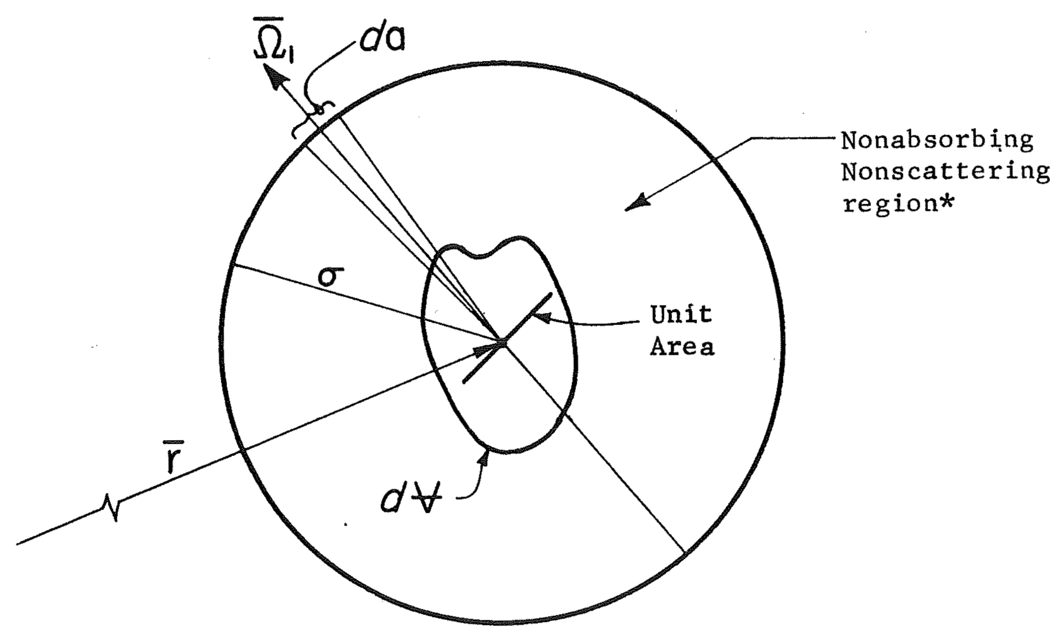


__Fig. 1.2__ Geometric relations for calculation of rediation to $d\forall$.

*Radiation in minus $\overline{\Omega}_1$ direction to the unit area equals the radiation from the the unit area in the minus $\overline{\Omega}_1$ direction.

Section 1

References

1.1 Fitts, Donald D. Nonequilibrium Thermodynamics. New York: McGraw-Hill Book Company, Inc., 1962.

1.2 Brodkey, Robert S. The Phenomena of Fluid Motions. Reading, Massachusetts: Addison-Wesley Publishing Company, 1967.

1.3 Bird, R. Byron, Warren E. Stewart, and Edwin N. Lightfoot. Transport Phenomena. New York: John Wiley & Sons, Inc., 1966.

1.4 Zel'dovich, Ya. B., and Yu. P. Raizer. Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Vol. 1, ed. by Wallace D. Hayes and Ronald F. Probstein, New York: Academic Press, 1966.

1.5 Viskanta, R. "Heat Transfer in Thermal Radiation Absorbing and Scattering Media," Ph.D. thesis, Purdue University, also Argonne National Laboratory Report ANL-6170, May 1960.

1.6 Vincenti, Walter G. and Charles H. Kruger, Jr. Introduction to Physical Gas Dynamics. New York: John Wiley and Sons, Inc., 1965.

SECTION II

CONSERVATION EQUATIONS IN GENERAL ORTHOGONAL COORDINATES

From the previous section we have a vector formulation of the basic conservation equations for a reacting, radiating, conducting fluid. Most flow problems are represented by the conservation equations in orthogonal coordinates. In this section the basic conservation laws will be written in general curvilinear orthogonal coordinates. This permits one to sel-select a useful coordinate system for a particular problem and thus determine the appropriate coordinate stretching functions. Substitution of the stretching functions into the conservation equations in curvilinear orthogonal coordinates will yield the appropriate governing equations for the problem of interest.

Table 2.1 presents a set of vector operations in vector geometry notation. By using the information in this table we are able to write the conservation equations in curvilinear orthogonal coordinates. The statement of these equations have been made in part by Back, Tsien, Brodkey, Ref's: 2.1, 2.2, and 2.3 respectively, and others.

The conservation equations can be stated in time independent vector form as follows.

Global Continuity:

$$\nabla \cdot \rho \overline{V} = 0 \tag{2.1}$$

Species Continuity:

$$\nabla \cdot (\rho_i \overline{V}) + \nabla \cdot \overline{J}_i = \omega_i \tag{2.2}$$

Momentum:

$$\rho(\overline{V} \cdot \nabla)\overline{V} + \nabla \cdot (\overline{\overline{\tau}} - \overline{\overline{I}}P + \overline{\overline{P}}_R) - \sum_i \rho_i \overline{g}_i = 0 \tag{2.3}$$

Energy: (Total internal energy per unit mass)

$$\rho(\overline{V} \cdot \nabla)E + \nabla \cdot (\overline{q}_D + \overline{q}_R) + \sum_i \overline{g}_i \cdot \overline{J}_i - \nabla \cdot (\overline{\overline{\tau}} - \overline{\overline{I}}P + \overline{\overline{P}}_R) \cdot \overline{V} - S\rho = 0 \tag{2.4}$$

Energy: (Total enthalpy per unit mass)

$$\rho(\bar{V}\bullet\nabla)H + \nabla\bullet(\bar{q}_D + \bar{q}_R) + \sum_i \bar{g}_i \bullet \bar{J}_i - \nabla\bullet(\bar{\bar{\tau}} + \bar{\bar{P}}_R)\bullet\bar{V} - S\rho = 0 \tag{2.5}$$

For the purpose of writing the conservation equations in curvilinear orthogonal coordinates, the coordinates are chosen to be ξ_1, ξ_2, and ξ_3 corresponding to ξ_1, ξ_2, and ξ_3 of Tab. 2.1 respectively. The elements of length in the respective coordinate directions are $h_1\xi_1$, $h_2\xi_2$, and $h_3\xi_3$ such that a differential arc length can be expressed as

$$(d\zeta)^2 = h_1^2(d\xi_1)^2 + h_2^2(d\xi_2)^2 + h_3^2(d\xi_3)^2 \tag{2.6}$$

where h_1, h_2, and h_3 are called the "stretching functions" in the respective coordinate directions. In the following equations u, v, and w are the velocity components of $\bar{V}$ in the direction of increasing ξ_1, ξ_2, and ξ_3.

Applying the ∇ operator from Tab. 2.1 the global continuity equation becomes

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 \rho u)}{\partial\xi_1} + \frac{\partial(h_1 h_3 \rho v)}{\partial\xi_2} + \frac{\partial(h_1 h_2 \rho w)}{\partial\xi_3}\right] = 0 \tag{2.7}$$

In a similar manner the species continuity equation can be written

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 \rho_i u)}{\partial\xi_1} + \frac{\partial(h_1 h_3 \rho_i v)}{\partial\xi_2} + \frac{\partial(h_1 h_2 \rho_i w)}{\partial\xi_3}\right] + \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_1 h_3 J_{i,\xi_1})}{\partial\xi_1} + \frac{\partial(h_1 h_3 J_{i,\xi_2})}{\partial\xi_2} + \frac{\partial(h_1 h_2 J_{i,\xi_3})}{\partial\xi_3}\right] - \omega_i = 0 \tag{2.8}$$

In order to evaluate the above equation the components J_{i,ξ_1}, J_{i,ξ_2}, and J_{i,ξ_3} of the mass flux vector $\bar{J}_i$ must be specified. The mass flux vector for a wide range of fluid problems consist of four parts

$$\bar{J}_i = \bar{J}_i^{(D)} + \bar{J}_i^{(P)} + \bar{J}_i^{(g)} + \bar{J}_i^{(T)} \tag{2.9}$$

where $\bar{J}_i^{(D)}$ represents concentration diffusion, $\bar{J}_i^{(P)}$ pressure diffusion, $\bar{J}_i^{(g)}$ forced diffusion, and $\bar{J}_i^{(T)}$ thermal diffusion. In the present derivation we will restrict our attention to thermal and concentration diffusion. These modes of mass transport are the dominant ones for most fluid dynamics problems.

The expressions for there two mass flux vectors are

$$\bar{J}_i^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_{j=1}^{n} M_i M_j D_{ij} \left[Y_j \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\partial G_j}{\partial Y_k} \right)_{\substack{T, \forall, Y_l \\ l \neq j,k}} \nabla Y_k \right] \tag{1.7}$$

$$\bar{J}_i^{(T)} = -D_i^T \nabla \ln T \tag{1.8}$$

The transformed components of the above equations are

$$J_{i,\xi_1}^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_{j=1}^{n} M_i M_j D_{ij} \left[Y_j \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\partial G_j}{\partial Y_k} \right)_{\substack{T, \forall, Y_l \\ l \neq j,k}} \frac{1}{h_1} \frac{\partial Y_k}{\partial \xi_1} \right]$$

$$J_{i,\xi_2}^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_{j=1}^{n} M_i M_j D_{ij} \left[Y_j \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\partial G_j}{\partial Y_k} \right)_{\substack{T, \forall, Y_l \\ l \neq j,k}} \frac{1}{h_2} \frac{\partial Y_k}{\partial \xi_2} \right] \tag{2.10}$$

$$J_{i,\xi_3}^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_{j=1}^{n} M_i M_j D_{ij} \left[Y_j \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\partial G_j}{\partial Y_k} \right)_{\substack{T, \forall, Y_l \\ l \neq j,k}} \frac{1}{h_3} \frac{\partial Y_k}{\partial \xi_3} \right]$$

$$J_{i,\xi_1}^{(T)} = \frac{-D_i^T}{h_1} \frac{\partial (\ln T)}{\partial \xi_1}$$

$$J_{i,\xi_2}^{(T)} = \frac{-D_i^T}{h_2} \frac{\partial (\ln T)}{\partial \xi_2} \tag{2.11}$$

$$J_{i,\xi_3}^{(T)} = \frac{-D_i^T}{h_3} \frac{\partial (\ln T)}{\partial \xi_3}$$

For substitution into the species continuity equation

$$J_{i,\xi_1} = J_{i,\xi_1}^{(D)} + J_{i,\xi_1}^{(T)}$$

$$J_{i,\xi_2} = J_{i,\xi_2}^{(D)} + J_{i,\xi_2}^{(T)} \tag{2.12}$$

$$J_{i,\xi_3} = J_{i,\xi_3}^{(D)} + J_{i,\xi_3}^{(T)}$$

This completes the necessary operations to explicitly write the species continuity equation in general orthogonal coordinates.

Before writing the momentum and energy equations in general orthogonal coordinates the radiation pressure tensor and external force field terms are dropped. The resulting vector form of the two equations are

Momentum:

$$\rho(\bar{V} \cdot \nabla)\bar{V} + \nabla \cdot (\bar{\bar{\tau}} - \bar{\bar{I}}P) = 0 \tag{2.13}$$

Energy:

$$\rho(\bar{V} \cdot \nabla)H + \nabla \cdot (\bar{q}_D + \bar{q}_R) - \nabla \cdot (\bar{\bar{\tau}}) \cdot \bar{V} = 0 \tag{2.14}$$

If the need to account for the additional effects should arise, the appropriate terms could be added to the governing equations in an analogous manner to the terms which will be considered.

Using the definitions in Tab. 2.1, the momentum equation can be written in the three orthogonal directions.

ξ_1 — momentum:

$$\begin{aligned}
&\frac{u}{h_1}\frac{\partial u}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial u}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial u}{\partial \xi_3} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{uw}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} \\
&- \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} - \frac{w^2}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} + \frac{1}{\rho}\frac{1}{h_1}\frac{\partial P}{\partial \xi_1} \\
&- \frac{1}{\rho}\left[\frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 \tau_{11})}{\partial \xi_1} + \frac{\partial(h_1 h_2 \tau_{12})}{\partial \xi_2} + \frac{\partial(h_1 h_3 \tau_{13})}{\partial \xi_3}\right)\right. \\
&\left. + \frac{\tau_{12}}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{\tau_{13}}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{\tau_{22}}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} - \frac{\tau_{33}}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1}\right] = 0
\end{aligned} \tag{2.15}$$

ξ_2 — momentum:

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial v}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial v}{\partial \xi_3} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} + \frac{vw}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}$$

$$- \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} - \frac{w^2}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2} + \frac{1}{\rho}\frac{1}{h_2}\frac{\partial P}{\partial \xi_2} \tag{2.16}$$

$$- \frac{1}{\rho}\left[\frac{1}{h_1 h_2 h_3}\left(\frac{\partial (h_2 h_3 \tau_{12})}{\partial \xi_1} + \frac{\partial (h_1 h_3 \tau_{22})}{\partial \xi_2} + \frac{\partial (h_1 h_2 \tau_{32})}{\partial \xi_3}\right)\right.$$

$$\left. + \frac{\tau_{12}}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} + \frac{\tau_{23}}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3} - \frac{\tau_{11}}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} - \frac{\tau_{33}}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2}\right] = 0$$

ξ_3 — momentum:

$$\frac{u}{h_1}\frac{\partial w}{\partial \xi_2} + \frac{v}{h_2}\frac{\partial w}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial w}{\partial \xi_3} + \frac{wv}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} + \frac{wv}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2}$$

$$- \frac{u^2}{h_1 h_3}\frac{\partial h_1}{\partial \xi_2} - \frac{w^2}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3} + \frac{1}{\rho}\frac{1}{h_3}\frac{\partial P}{\partial \xi_3} \tag{2.17}$$

$$- \frac{1}{\rho}\left[\frac{1}{h_1 h_2 h_3}\left(\frac{\partial (h_2 h_3 \tau_{13})}{\partial \xi_1} + \frac{\partial (h_1 h_3 \tau_{23})}{\partial \xi_2} + \frac{\partial (h_1 h_2 \tau_{33})}{\partial \xi_3}\right)\right.$$

$$\left. + \frac{\tau_{31}}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} + \frac{\tau_{23}}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2} - \frac{\tau_{11}}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{\tau_{22}}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right] = 0$$

In the above equations, the subscripts 1, 2, and 3 in the symmetric stress tensor denote the coordinate directions ξ_1, ξ_2, and ξ_3 respectively. In order to evaluate the three momentum equations the components of the viscous stress tensor must be defined. For a Stokes' fluid the stress tensor is defined by, Ref. 2.3, in terms of the rate of strain tensor $\overline{\overline{e}}$.

$$\overline{\overline{\tau}} = f(\overline{\overline{e}})$$

The simplest form for this equation in three dimensions is

$$\overline{\overline{\tau}} = A\overline{\overline{I}} + B\overline{\overline{e}} + C\overline{\overline{e}}\cdot\overline{\overline{e}} \tag{2.18}$$

For a Newtonian fluid

$$A = -(\tfrac{2}{3}\mu - \tilde{\mu})\nabla\cdot\overline{V}, \qquad B = +\mu, \quad C = 0$$

The stress tensor may now be written as

$$\overline{\overline{\tau}} = \lambda\overline{\overline{I}}\nabla\cdot\overline{V} + \mu\overline{\overline{e}} \tag{2.20}$$

The components of the stress tensor are

$$\left.\begin{aligned} \tau_{11} &= \lambda\nabla\cdot\overline{V} + \mu e_{11} \\ \tau_{22} &= \lambda\nabla\cdot\overline{V} + \mu e_{22} \\ \tau_{33} &= \lambda\nabla\cdot\overline{V} + \mu e_{33} \end{aligned}\right\} \tag{2.21a}$$

$$\left.\begin{aligned} \tau_{12} &= \tau_{21} = \mu e_{12} \\ \tau_{13} &= \tau_{31} = \mu e_{13} \\ \tau_{23} &= \tau_{32} = \mu e_{23} \end{aligned}\right\} \tag{2.21b}$$

Which may be written

$$\tau_{11} = \frac{\lambda}{h_1h_2h_3}\left[\frac{\partial(h_2h_3u)}{\partial\xi_1} + \frac{\partial(h_1h_3v)}{\partial\xi_2} + \frac{\partial(h_1h_2w)}{\partial\xi_3}\right] + 2\mu\left[\frac{1}{h_1}\frac{\partial u}{\partial\xi_1} + \frac{v}{h_1h_2}\frac{\partial h_1}{\partial\xi_2} + \frac{w}{h_3h_1}\frac{\partial h_1}{\partial\xi_3}\right] \tag{2.22}$$

$$\tau_{22} = \frac{\lambda}{h_1h_2h_3}\left[\frac{\partial(h_2h_3u)}{\partial\xi_1} + \frac{\partial(h_1h_3v)}{\partial\xi_2} + \frac{\partial(h_1h_2w)}{\partial\xi_3}\right] + 2\mu\left[\frac{1}{h_2}\frac{\partial v}{\partial\xi_2} + \frac{w}{h_2h_3}\frac{\partial h_2}{\partial\xi_3} + \frac{u}{h_1h_2}\frac{\partial h_2}{\partial\xi_1}\right] \tag{2.23}$$

$$\tau_{33} = \frac{\lambda}{h_1h_2h_3}\left[\frac{\partial(h_2h_3u)}{\partial\xi_1} + \frac{\partial(h_1h_3v)}{\partial\xi_2} + \frac{\partial(h_1h_2w)}{\partial\xi_3}\right] + 2\mu\left[\frac{1}{h_3}\frac{\partial w}{\partial\xi_3} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial\xi_1} + \frac{v}{h_2h_3}\frac{\partial h_3}{\partial\xi_2}\right] \tag{2.24}$$

$$\tau_{12} = \tau_{21} = \mu\left[\frac{h_2}{h_1}\frac{\partial}{\partial \xi_1}\left(\frac{v}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{u}{h_1}\right)\right] \tag{2.25}$$

$$\tau_{13} = \tau_{31} = \mu\left[\frac{h_1}{h_3}\frac{\partial}{\partial \xi_3}\left(\frac{u}{h_1}\right) + \frac{h_3}{h_1}\frac{\partial}{\partial \xi_1}\left(\frac{w}{h_3}\right)\right] \tag{2.26}$$

$$\tau_{23} = \tau_{32} = \mu\left[\frac{h_3}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{w}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial \xi_3}\left(\frac{v}{h_2}\right)\right] \tag{2.27}$$

With the preceding definition of the stress tensor, the momentum equations become three equations expressed in the three components of the velocity vector.

The total enthalpy form of the energy equation Eq. 2.14 may be written in general orthogonal coordinates by noting the form of the three operators expressed in Tab. 2.1.

$$\begin{aligned}
\rho\left[\frac{u}{h_1}\frac{\partial H}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial H}{\partial \xi_2} + \frac{w}{h_3}\frac{\partial H}{\partial \xi_3}\right] &= \frac{-1}{h_1 h_2 h_3}\left[\frac{\partial (h_2 h_3 q_{D,1})}{\partial \xi_1}\right. \\
&+ \left.\frac{\partial (h_1 h_3 q_{D,2})}{\partial \xi_2} + \frac{\partial (h_1 h_2 q_{D,3})}{\partial \xi_3}\right] - \frac{1}{h_1 h_2 h_3}\left[\frac{\partial (h_2 h_3 q_{R,1})}{\partial \xi_1}\right. \\
&+ \left.\frac{\partial (h_1 h_3 q_{R,2})}{\partial \xi_2} + \frac{\partial (h_1 h_2 q_{R,3})}{\partial \xi_3}\right] + \frac{1}{h_1 h_2 h_3}\left[\frac{\partial \{h_2 h_3 (\tau_{11} u + \tau_{21} v + \tau_{31} w)\}}{\partial \xi_1}\right. \\
&+ \left.\frac{\partial \{h_1 h_3 (\tau_{12} u + \tau_{22} v + \tau_{32} w)\}}{\partial \xi_2} + \frac{\partial \{h_1 h_2 (\tau_{12} u + \tau_{23} v + \tau_{33} w)\}}{\partial \xi_3}\right]
\end{aligned} \tag{2.28}$$

The components of the shear stress have been defined in the discussion of the momentum equation. Therefore only the components of the heat flux vectors are left to be defined to provide a complete statement of the energy equation.

The heat flux vector as handled previously will be described as the sum of the diffusional and radiative heat flux vectors. The diffusional heat flux vector can be expressed as a function of the mass flux vector by simple manipulation of the equation given by

$$\bar{q}_D = -k'\nabla T + \sum_i h_i \bar{J}_i - N k_c T \sum_i \frac{D_i^T}{N_i M_i} \nabla \left(\frac{N_i}{N}\right)$$

where k' is not the ordinary thermal conductivity coefficient. The usual form of the diffusional heat flux vector is written in terms of diffusion velocities or mass flux vectors. This form eliminates $\nabla\left(\frac{N_i}{N}\right)$ from the preceding equation and adds a term to k' yielding the ordinary thermal conductivity. This step also introduces the binary diffusion coefficient into the Dufour effect term. Following Hirschfelder et. al., Ref. 2.4, and substituting for the diffusion velocities yields:

$$\bar{q}_D = -k\nabla T + \sum_i h_i \bar{J}_i - \frac{P}{N^2}\sum_i \sum_{j\neq i} \frac{N_i}{M_i}\frac{D_i^T}{\mathbb{D}_{ij}}\left(\frac{\bar{J}_j}{Y_j M_j} - \frac{\bar{J}_i}{Y_i M_i}\right) \tag{2.29}$$ #

where $\mathbb{D}_{ij}$ is the binary diffusion coefficient

$$\mathbb{D}_{ij} = \frac{3(M_i + M_j)P}{16N^2 M_i M_j \Omega_{ij}^{(1,1)}} \tag{2.30}$$

The diffusional heat flux vector contains terms which respectively represent conductive energy flux, diffusional energy flux, and diffusion-thermo (Dufour) energy flux. The Dufour effect is kept in the above equation to be consistent with keeping the Soret effect in the species continuity equation. At this point it is appropriate to point out that the thermal conductivity in the conductive flux term is in general a tensor. For the case of an isentropic fluid, the conductivity reduces to a scalar. This is the form used in the diffusional energy flux vector above.

Having stated the vector form of the diffusional heat flux vector, the components needed in the energy equation can be expressed.

$$q_{D,l} = -\frac{k}{h_l}\frac{\partial T}{\partial \xi_l} + \sum_i h_i J_{i,\xi_l} - \frac{P}{N^2}\sum_i \sum_{j\neq i} \frac{N_i}{M_i}\frac{D_i^T}{\mathbb{D}_{ij}}\left(\frac{J_{j,\xi_l}}{Y_j M_j} - \frac{J_{i,\xi_l}}{Y_i M_i}\right) \tag{2.31}$$

The perfect gas equation of state has been used to replace $k_c T$ in these equations from Ref. 2.4 with P/N.

$$q_{D,2} = -\frac{k}{h_2}\frac{\partial T}{\partial \xi_2} + \sum_i h_i J_{i,\xi_2}$$

$$- \frac{P}{N^2} \sum_i \sum_{j \neq i} \frac{N_i}{M_i} \frac{D_i^T}{\mathbb{D}_{ij}} \left(\frac{J_{j,\xi_2}}{Y_j M_j} - \frac{J_{i,\xi_2}}{Y_i M_i} \right) \tag{2.32}$$

$$q_{D,3} = -\frac{k}{h_3}\frac{\partial T}{\partial \xi_3} + \sum_i h_i J_{i,\xi_3}$$

$$- \frac{P}{N^2} \sum_i \sum_{j \neq i} \frac{N_i}{M_i} \frac{D_i^T}{\mathbb{D}_{ij}} \left(\frac{J_{j,\xi_3}}{Y_j M_j} - \frac{J_{i,\xi_3}}{Y_i M_i} \right) \tag{2.33}$$

where the components of the mass flux vector used in the above expression are defined in the discussion of the species continuity equation.

To calculate the components of the radiative flux vector q_{R,ξ_i} where ξ_i is an orthogonal coordinate, let us integrate Eq. 1.27.

$$\bar{q}_R(\bar{r}_1) = \int_{\bar{r}_0}^{\bar{r}_1} \nabla \cdot \bar{q}_R \, dr$$

$$= \int_{\bar{r}_0}^{\bar{r}_1} \nabla \cdot \bar{q}_R (h_1 \bar{e}_1 d\xi_1 + h_2 \bar{e}_2 d\xi_2 + h_3 \bar{e}_3 d\xi_3) \tag{2.34}$$

Note that $\nabla \cdot \bar{q}_R$ is a scalar independent of coordinate system. The flux components may be written:

$$q_{R,\xi_i} = \int_{\bar{r}_0}^{\bar{r}_1} (\nabla \cdot \bar{q}_R) h_i \, d\xi_i \tag{2.35}$$

or by substituting from Eq. 1.27

$$q_{R,1} = \int_{\xi(\bar{r}_0)}^{\xi(\bar{r}_1)} \int_0^\infty \alpha_\nu \left| 4\pi B_\nu - \int_0^{4\pi} I_\nu(\bar{r}) \, d\Omega \right| d\nu \, h_1 d\xi_1 \tag{2.36}$$

$$q_{R,2} = \int_{\xi(\bar{r}_0)}^{\xi(\bar{r}_1)} \int_0^\infty \alpha_\nu \left| 4\pi B_\nu - \int_0^{4\pi} I_\nu(\bar{r}) \, d\Omega \right| d\nu \, h_2 d\xi_2 \tag{2.37}$$

$$q_{R,3} = \int_{\xi(\bar{r}_0)}^{\xi(\bar{r}_1)} \int_0^\infty \alpha_\nu \left| 4\pi B_\nu - \int_0^{4\pi} I_\nu(\bar{r}) \, d\Omega \right| d\nu \, h_3 d\xi_3 \tag{2.38}$$

In addition to the general conservation equations an equation of state is needed to specify the relationship between pressure and temperature. A reasonable approximation for the thermal behavior of a gaseous mixture is the ideal gas equation of state.

$$P = n_t \mathbb{R} T \tag{2.39}$$

Let us rewrite this equation in several forms which might be of help in furthering our development. These expressions follow those of Scala, Ref. 2.5.

$$P = \rho \breve{\mathbb{R}} T \tag{2.40}$$

where

$$\breve{\mathbb{R}} = \frac{\mathbb{R}}{M}$$

It follows that

$$\breve{\mathbb{R}} = \frac{1}{\rho} \sum \rho_i \breve{\mathbb{R}}_i = \sum C_i \breve{\mathbb{R}}_i = \mathbb{R} \sum_i C_i / M_i \tag{2.41}$$

$$M = \frac{1}{n_t} \sum_i n_i M_i = \sum_i Y_i M_i = \frac{1}{\sum_i C_i / M_i} \tag{2.42}$$

Other forms of the ideal gas equation of state are

$$\left.\begin{aligned} P_i &= \rho_i \breve{\mathbb{R}}_i T \\ P &= n_t M \breve{\mathbb{R}} T \\ P_i &= N_i k_c T \\ P &= N k_c T \qquad \text{where} \quad \sum_i N_i = N \end{aligned}\right\} \tag{2.43}$$

This last expression has been used previously to state Eq.'s 2.29 and 2.30.

TABLE 2.1

Vector Operations in Vector Geometry Notation

Vector Operation	Vector Geometry
1. Vector, $\bar{a}$ =	$\bar{E}_1 A_1 + \bar{E}_2 A_2 + \bar{E}_3 A_3$
2. Scalar or dot product, $\bar{a} \cdot \bar{b}$ =	$\dfrac{A_1 B_1}{h_1^{\,2}} + \dfrac{A_2 B_2}{h_2^{\,2}} + \dfrac{A_3 B_3}{h_3^{\,2}}$
3. Vector or cross product, $\bar{a} \times \bar{b}$ =	$(A_2 B_3 - A_3 B_2)\bar{E}_1 + (A_3 B_1 - A_1 B_3)\bar{E}_2 + (A_1 B_2 - A_2 B_1)\bar{E}_3$
4. Gradient of scalar, ∇U =	$\dfrac{\bar{E}_1}{h_1}\dfrac{\partial U}{\partial \xi_1} + \dfrac{\bar{E}_2}{h_2}\dfrac{\partial U}{\partial \xi_2} + \dfrac{\bar{E}_3}{h_3}\dfrac{\partial U}{\partial \xi_3}$
5. Gradient of vector, $\nabla \bar{a}$ =	W_{ij}; elements of W_{ij} are: $W_{ii} = \dfrac{1}{h_i}\dfrac{\partial A_i}{\partial \xi_i} + \dfrac{1}{h_j}\dfrac{\partial h_i}{\partial \xi_j}\dfrac{A_j}{h_i} + \dfrac{1}{h_k}\dfrac{\partial h_i}{\partial \xi_k}\dfrac{A_k}{h_i}$ $W_{ij} = \dfrac{1}{h_j}\dfrac{\partial A_i}{\partial \xi_j} - \dfrac{1}{h_i}\dfrac{\partial h_j}{\partial \xi_i}\dfrac{A_j}{h_j}$
6. Divergence of $\bar{a}$, $\nabla \cdot \bar{a}$ =	$\dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial(h_2 h_3 A_1)}{\partial \xi_1} + \dfrac{\partial(h_1 h_3 A_2)}{\partial \xi_2} + \dfrac{\partial(h_1 h_2 A_3)}{\partial \xi_3}\right]$
7. Curl of $\bar{a}$, $\nabla \times \bar{a}$ =	$\dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\bar{E}_1 & h_2\bar{E}_2 & h_3\bar{E}_3 \\ \partial/\partial\xi_1 & \partial/\partial\xi_2 & \partial/\partial\xi_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$

(Table 2.1 continued)

8. Laplacian of U,

$$\nabla^2 U = \nabla\cdot(\nabla U) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}\left(\frac{h_2h_3}{h_1}\frac{\partial U}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_3h_1}{h_2}\frac{\partial U}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1h_2}{h_3}\frac{\partial U}{\partial\xi_3}\right)\right]$$

9. Laplacian of $\bar{a}$,

$$\nabla^2\bar{a} = \nabla(\nabla\cdot\bar{a}) - \nabla\times(\nabla\times\bar{a}) = \left\{\frac{1}{h_1}\frac{\partial}{\partial\xi_1}(\nabla\cdot\bar{a}) + \frac{1}{h_2h_3}\left[\frac{\partial}{\partial\xi_3}\left\{\frac{h}{h_3h_1}\left(\frac{\partial h_1A_1}{\partial\xi_3} - \frac{\partial h_3A_3}{\partial\xi_1}\right)\right\} - \frac{\partial}{\partial\xi_2}\left\{\frac{h_3}{h_1h_2}\left(\frac{\partial h_2A_2}{\partial\xi_1} - \frac{\partial h_1A_1}{\partial\xi_2}\right)\right\}\right]\right\}\bar{E}_1$$

$$+\left\{\frac{1}{h_2}\frac{\partial}{\partial\xi_2}(\nabla\cdot\bar{a}) + \frac{1}{h_3h_1}\left[\frac{\partial}{\partial\xi_1}\left\{\frac{h_3}{h_1h_2}\left(\frac{\partial h_2A_2}{\partial\xi_1} - \frac{\partial h_1A_1}{\partial\xi_2}\right)\right\} - \frac{\partial}{\partial\xi_3}\left\{\frac{h_1}{h_2h_3}\left(\frac{\partial h_3A_3}{\partial\xi_2} - \frac{\partial h_2A_2}{\partial\xi_3}\right)\right\}\right]\right\}\bar{E}_2$$

$$+\left\{\frac{1}{h_3}\frac{\partial}{\partial\xi_3}(\nabla\cdot\bar{a}) + \frac{1}{h_1h_2}\left[\frac{\partial}{\partial\xi_2}\left\{\frac{h_1}{h_2h_3}\left(\frac{\partial h_3A_3}{\partial\xi_2} - \frac{\partial h_2A_2}{\partial\xi_3}\right)\right\} - \frac{\partial}{\partial\xi_1}\left\{\frac{h_1}{h_3h_1}\left(\frac{\partial h_1A_1}{\partial\xi_3} - \frac{\partial h_3A_3}{\partial\xi_1}\right)\right\}\right]\right\}\bar{E}_3$$

(Table 2.1 continued)

10. $(\bar{a} \cdot \nabla)U =$

$$\frac{A_1}{h_1}\frac{\partial U}{\partial \xi_1} + \frac{A_2}{h_2}\frac{\partial U}{\partial \xi_2} + \frac{A_3}{h_3}\frac{\partial U}{\partial \xi_3}$$

11. $(\bar{a} \cdot \nabla)\bar{b} =$

$$\frac{1}{h_1}\left\{\left(A_1\frac{\partial B_1}{\partial \xi_1} + A_2\frac{\partial B_2}{\partial \xi_1} + A_3\frac{\partial B_3}{\partial \xi_1}\right) - \frac{A_2}{h_2}\left(\frac{\partial h_2 B_2}{\partial \xi_1} - \frac{\partial h_1 B_1}{\partial \xi_2}\right)\right.$$

$$\left. + \frac{A_3}{h_3}\left(\frac{\partial h_1 B_1}{\partial \xi_3} - \frac{\partial h_3 B_3}{\partial \xi_1}\right)\right\}\bar{E}_1 + \frac{1}{h_2}\left\{\left(A_1\frac{\partial B_1}{\partial \xi_2} + A_2\frac{\partial B_2}{\partial \xi_2} + A_3\frac{\partial B_3}{\partial \xi_2}\right)\right.$$

$$\left. - \frac{A_3}{h_3}\left(\frac{\partial h_3 B_3}{\partial \xi_2} - \frac{\partial h_2 B_2}{\partial \xi_3}\right) + \frac{A_1}{h_1}\left(\frac{\partial h_2 B_2}{\partial \xi_1} - \frac{\partial h_1 B_1}{\partial \xi_2}\right)\right\}\bar{E}_2$$

$$+ \frac{1}{h_3}\left\{\left(A_1\frac{\partial B_1}{\partial \xi_3} + A_2\frac{\partial B_2}{\partial \xi_3} + A_3\frac{\partial B_3}{\partial \xi_3}\right) - \frac{A_1}{h_1}\left(\frac{\partial h_1 B_1}{\partial \xi_3} - \frac{\partial h_3 B_3}{\partial \xi_1}\right)\right.$$

$$\left. + \frac{A_2}{h_2}\left(\frac{\partial h_3 B_3}{\partial \xi_2} - \frac{\partial h_2 B_2}{\partial \xi_3}\right)\right\}\bar{E}_3$$

(Table 2.1 continued)

12. Divergence of a symmetric tensor, $\nabla \cdot \overline{\overline{W}}$
Note: $W_{ij} = W_{ji}$

$$\nabla\cdot\overline{\overline{w}} = (\nabla\cdot\overline{\overline{w}}\ {}_1\overline{E}_1 + (\nabla\cdot\overline{\overline{w}})_2\overline{E}_1 + (\nabla\cdot\overline{\overline{w}})_3\overline{E}_1$$

$$(\nabla\cdot\overline{\overline{w}})_1 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3W_{11}) + \frac{\partial}{\partial\xi_2}(h_1h_3W_{12}) + \frac{\partial}{\partial\xi_3}(h_1h_2W_{13})\right]$$
$$+ \frac{W_{12}}{h_1h_2}\frac{\partial h_1}{\partial\xi_2} + \frac{W_{31}}{h_1h_3}\frac{\partial h_1}{\partial\xi_3} - \frac{W_{22}}{h_1h_2}\frac{\partial h_2}{\partial\xi_1} - \frac{W_{33}}{h_1h_3}\frac{\partial h_3}{\partial\xi_1}$$

$$(\nabla\cdot\overline{\overline{w}})_2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3W_{12}) + \frac{\partial}{\partial\xi_2}(h_1h_3W_{22}) + \frac{\partial}{\partial\xi_3}(h_1h_2W_{23})\right]$$
$$+ \frac{W_{12}}{h_1h_2}\frac{\partial h_2}{\partial\xi_1} + \frac{W_{23}}{h_2h_3}\frac{\partial h_2}{\partial\xi_3} - \frac{W_{11}}{h_1h_2}\frac{\partial h_1}{\partial\xi_2} - \frac{W_{33}}{h_2h_3}\frac{\partial h_3}{\partial\xi_2}$$

$$(\nabla\cdot\overline{\overline{w}})_3 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3W_{31}) + \frac{\partial}{\partial\xi_2}(h_1h_3W_{32}) + \frac{\partial}{\partial\xi_3}(h_1h_2W_{33})\right]$$
$$+ \frac{W_{31}}{h_1h_3}\frac{\partial h_3}{\partial\xi_1} + \frac{W_{23}}{h_2h_3}\frac{\partial h_3}{\partial\xi_2} - \frac{W_{11}}{h_1h_2}\frac{\partial h_1}{\partial\xi_3} - \frac{W_{22}}{h_2h_3}\frac{\partial h_2}{\partial\xi_3}$$

(Table 2.1 continued)

13. $\overline{\overline{w}}:(\nabla\overline{b}) =$

For $\overline{\overline{w}}$ being a symmetric tensor

$$
\begin{aligned}
& W_{11}\left(\frac{1}{h_1}\frac{\partial B_1}{\partial \xi_1} + \frac{B_2}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{B_3}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3}\right) \\
+ \; & W_{22}\left(\frac{1}{h_2}\frac{\partial B_2}{\partial \xi_2} + \frac{B_1}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} + \frac{B_3}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right) \\
+ \; & W_{33}\left(\frac{1}{h_3}\frac{\partial B_3}{\partial \xi_3} + \frac{B_1}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} + \frac{B_2}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2}\right) \\
+ \; & W_{12}\left(\frac{1}{h_2}\frac{\partial B_1}{\partial \xi_2} + \frac{1}{h_1}\frac{\partial B_2}{\partial \xi_1} - \frac{B_2}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} - \frac{B_1}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2}\right) \\
+ \; & W_{13}\left(\frac{1}{h_3}\frac{\partial B_1}{\partial \xi_3} + \frac{1}{h_1}\frac{\partial B_3}{\partial \xi_1} - \frac{B_3}{h_1 h_3}\frac{\partial h_3}{\partial \xi_1} - \frac{B_1}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3}\right) \\
+ \; & W_{23}\left(\frac{1}{h_3}\frac{\partial B_2}{\partial \xi_3} + \frac{1}{h_2}\frac{\partial B_3}{\partial \xi_2} - \frac{B_3}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2} - \frac{B_2}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right)
\end{aligned}
$$

14. $\nabla.\overline{\overline{w}}.\overline{b} =$

For $\overline{\overline{w}}$ being a symmetric tensor

$$
\begin{aligned}
& \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial \xi_1}\left[h_2 h_3 (W_{11}B_1 + W_{21}B_2 + W_{31}B_3)\right] + \frac{\partial}{\partial \xi_2}\left[h_1 h_3 (W_{12}B_1 + W_{22}B_2\right.\right. \\
& \left.\left. + W_{32}B_3)\right] + \frac{\partial}{\partial \xi_3}\left[h_1 h_2 (W_{13}B_1 + W_{23}B_2 + W_{33}B_3)\right]\right\}
\end{aligned}
$$

---

scalar	U			tensor	$\overline{\overline{w}}$ }	components	$\{W_{ij}$
unit vectors	$\overline{E}_1, \overline{E}_2, \overline{E}_3$			orthogonal coordinates	ξ_1, ξ_2, ξ_3		
vectors	$\overline{a}$, $\overline{b}$ }	components	$\{A_1, A_2, A_3$; B_1, B_2, B_3	stretching functions	h_1, h_2, h_3		

Section 2

References

2.1 Back, L. H. "Conservation Equations of a Viscous, Heat-Conducting Fluid in Curvilinear Orthogonal Coordinates," JPL Technical Report 32-1332 (Sept. 1968).

2.2 Emmons, H. W., Ed. Fundamentals of Gas Dynamics, Vol. III, Ch. I, Tsien, H. S., "The Equations of Gas Dynamics," Princeton University Press, Princeton, New Jersey, 1958.

2.3 Brodkey, R. S. The Phenomena of Fluid Motions, Addison-Wesley Co., Reading, Massachusetts (1967).

2.4 Hirschfelder, C. T. Curtiss and R. B. Bird. Molecular Theory of Gases and Liquids, John Wiley & Sons, Inc., New York (1954).

2.5 Scala, S. M. "The Equations of Motion in a Multicomponent Chemically Reacting Gas," Aerophysics Operation Research Memo No. 5, General Electric Co., Missile and Ordinance Systems Dept., Philadelphia, Pennsylvania (1957).

SECTION III

CONSERVATION EQUATIONS IN BODY ORIENTED COORDINATES

In order to describe the flow over blunt bodies moving at hypersonic velocities, it is found convenient to solve the conservation equations in orthogonal body oriented coordinate systems. The type of body under consideration, i.e. three-dimensional, axisymmetric or two dimensional, thus determine the stretching functions, h_1, h_2, h_3, discussed in Sec. II. The class of bodies considered in this development are axisymmetric or two-dimensional and have the following stretching functions, see Tab. 3.1:

$$\left.\begin{array}{ll} \xi_1 = x, & h_1 = 1 + \kappa y \\ \xi_2 = y, & h_2 = 1 \\ \xi_3 = z, & h_3 = r \end{array}\right\} \text{AXISYMMETRIC} \qquad (3.1)$$

$$\left.\begin{array}{ll} \xi_1 = x, & h_1 = 1 + \kappa y \\ \xi_2 = y, & h_2 = 1 \\ \xi_3 = z, & h_3 = 1 \end{array}\right\} \text{TWO-DIMENSIONAL} \qquad (3.2)$$

where κ is the local body curvature and r is defined in Fig. 3.1. Using Fig. 3.1 the following relationship may be found

$$r = r_w + y \sin\theta \qquad (3.3)$$

$$dr = \sin\theta \, dy + \tilde{\kappa} \cos\theta \, dx \qquad (3.4)$$

where

$$\tilde{\kappa} \equiv 1 + \kappa y \qquad (3.5)$$

$$(d\zeta)^2 = \tilde{\kappa}^2 (dx)^2 + (dy)^2 + (r^A dz)^2 \qquad (3.6)$$

Substituting the stretching functions 3.1 and 3.2 and relationships 3.3 and 3.5 into the general conservation equations for a multicomponent continuum gas in general orthogonal coordinates given in Section II yields the following equations.

Continuity:

$$\frac{\partial(\rho u r^A)}{\partial x} + \frac{\partial(\tilde{\kappa} r^A \rho v)}{\partial y} = 0 \qquad (3.7)$$

Species Continuity:

$$\frac{\partial(r^A \rho C_i u)}{\partial x} + \frac{\partial(\tilde{\kappa} r^A \rho C_i v)}{\partial y} = \frac{-\partial(r^A J_{i,x})}{\partial x}$$

$$- \frac{\partial(\tilde{\kappa} r^A J_{i,y})}{\partial y} + \tilde{\kappa} r^A \omega_i \qquad (3.8)$$

where $J_{i,x}$ and $J_{i,y}$ are the mass flux components of species i in the x and y direction respectively. The mass flux vector is the sum of two vectors neglecting force diffusion and pressure diffusion.

$$\bar{J_i} = \bar{J_i}^{(D)} + \bar{J_i}^{(T)} \qquad (3.9)$$

The components are

concentration diffusion:

$$J_{i,x}^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_{j=1}^{n} M_i M_j D_{ij} \left[Y_i \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\partial G_j}{\partial Y_k} \right)_{\substack{T, \forall, Y_l \\ l \neq j,k}} \frac{1}{\tilde{\kappa}} \frac{\partial Y_k}{\partial x} \right] \qquad (3.10)$$

$$J_{i,y}^{(D)} = \frac{n_t^2}{\rho \mathbb{R} T} \sum_{j=1}^{n} M_i M_j D_{ij} \left[Y_i \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{\partial G_j}{\partial Y_k} \right)_{\substack{T, \forall, Y_l \\ l \neq j,k}} \frac{\partial Y_k}{\partial y} \right] \qquad (3.11)$$

thermal diffusion:

$$J_{i,x}^{(T)} = - \frac{D_i^T}{\tilde{\kappa}} \frac{\partial \ln T}{\partial x} \qquad (3.12)$$

$$J_{i,y}^{(T)} = - D_i^T \frac{\partial \ln T}{\partial x} \qquad (3.13)$$

The two momentum equations can be expressed in the following manner.

X - momentum

$$\rho r^A u \frac{\partial u}{\partial x} + \rho \tilde{\kappa} r^A v \frac{\partial v}{\partial y} - \rho \kappa r^A uv$$

$$+ r^A \frac{\partial P}{\partial x} - \frac{\partial(r^A \tau_{xx})}{\partial x} - \frac{\partial(\tilde{\kappa} r^A \tau_{xy})}{\partial y} \qquad (3.14)$$

$$- r^A \kappa \tau_{xy} + \tau_{zz} \frac{\partial r^A}{\partial x} = 0$$

y - momentum

$$\rho r^A u \frac{\partial v}{\partial x} + \rho \tilde{\kappa} r^A v \frac{\partial v}{\partial y} - \rho \kappa r^A u^2$$
$$+ \tilde{\kappa} r^A \frac{\partial P}{\partial y} - \frac{\partial (r^A \tau_{xy})}{\partial x} - \frac{\partial (\tilde{\kappa} r^A \tau_{yy})}{\partial y} \tag{3.15}$$
$$+ \kappa r^A \tau_{xx} + \tilde{\kappa} \tau_{zz} \frac{\partial r^A}{\partial y} = 0$$

where the components of the stress tensor are

$$\tau_{xx} = \frac{\lambda}{\tilde{\kappa} r^A} \left[\frac{\partial (r^A u)}{\partial x} + \frac{\partial (\tilde{\kappa} r^A v)}{\partial y} \right]$$
$$+ \frac{2\mu}{\tilde{\kappa}} \left[\frac{\partial u}{\partial x} + \kappa v \right] \tag{3.16}$$

$$\tau_{yy} = \frac{\lambda}{\tilde{\kappa} r^A} \left[\frac{\partial (r^A u)}{\partial x} + \frac{\partial (\tilde{\kappa} r^A v)}{\partial y} \right]$$
$$+ 2\mu \frac{\partial v}{\partial y} \tag{3.17}$$

$$\tau_{zz} = \frac{\lambda}{\tilde{\kappa} r^A} \left[\frac{\partial (r^A u)}{\partial x} + \frac{\partial (\tilde{\kappa} r^A v)}{\partial y} \right]$$
$$+ 2\mu \left[\frac{u}{\tilde{\kappa} r^A} \frac{\partial r^A}{\partial x} + \frac{v}{r^A} \frac{\partial r^A}{\partial y} \right] \tag{3.18}$$

$$\tau_{xy} = \tau_{yx} = \mu \left[\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{\kappa}{\tilde{\kappa}} u \right] \tag{3.19}$$

The above stress components are also used in the energy equation.

energy:

$$\rho r^A u \frac{\partial H}{\partial x} + \rho \mathcal{K} r^A v \frac{\partial H}{\partial y} = - \frac{\partial (r^A q_{D,x})}{\partial x} - \frac{\partial (\mathcal{K} r^A q_{D,y})}{\partial y}$$

$$- \frac{\partial (r^A q_{R,x})}{\partial x} - \frac{\partial (\mathcal{K} r^A q_{R,y})}{\partial y} + \frac{\partial}{\partial x}\left[r^A u \tau_{xx} + r^A v \tau_{xy} \right] \quad (3.20)$$

$$+ \frac{\partial}{\partial y}\left[\mathcal{K} r^A u \tau_{xy} + \mathcal{K} r^A v \tau_{yy} \right]$$

The components of the diffusional heat flux vector are

$$q_{D,x} = - \frac{k}{\mathcal{K}} \frac{\partial T}{\partial x} + \sum_i h_i J_{i,x}$$
$$- \frac{P}{N^2} \sum_i \sum_{j \neq i} \frac{N_i}{M_i} \frac{D_i^T}{\mathbb{D}_{ij}} \left(\frac{J_{j,x}}{Y_j M_j} - \frac{J_{i,x}}{Y_i M_i} \right) \quad (3.21)$$

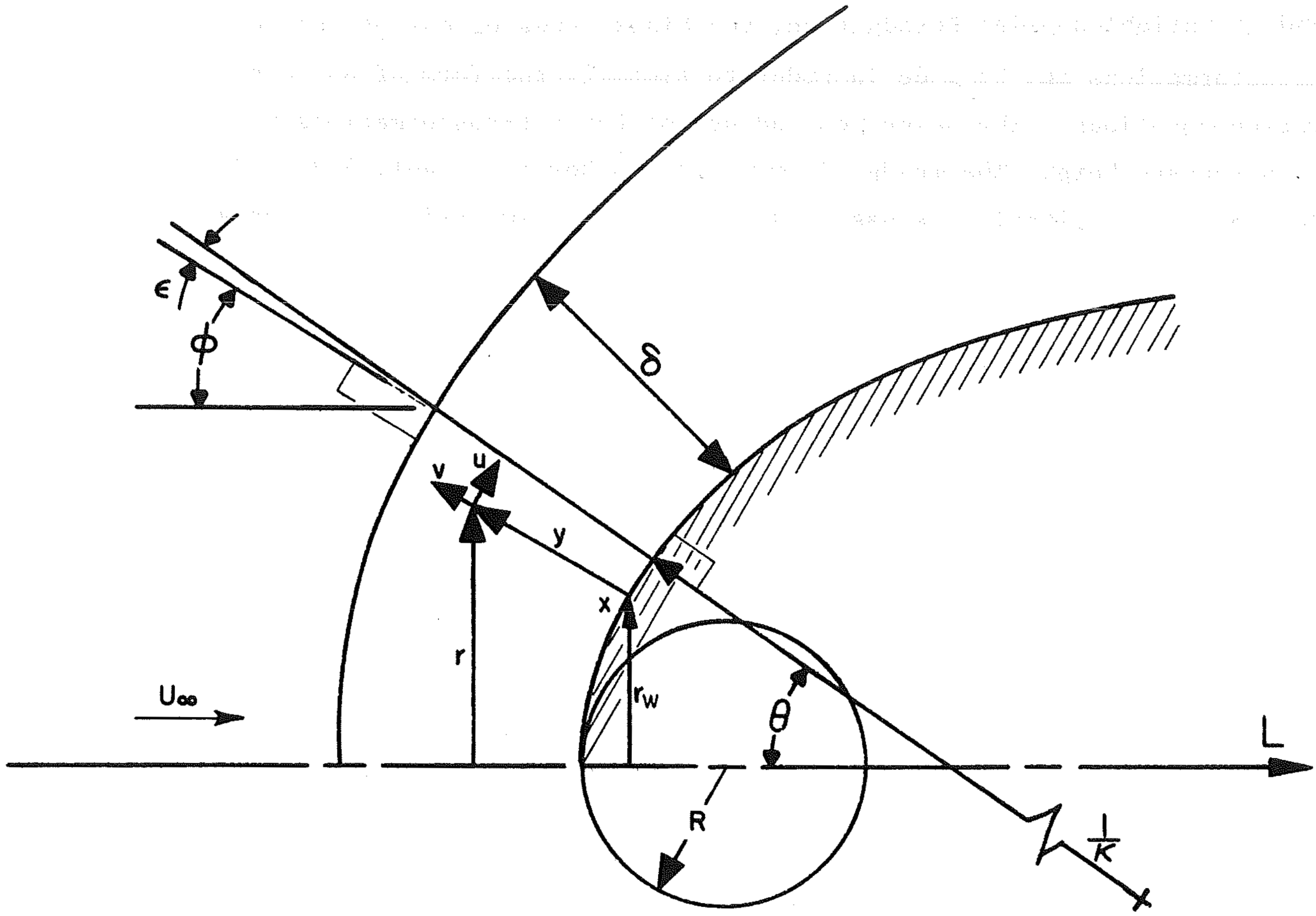
$$q_{D,y} = - k \frac{\partial T}{\partial y} + \sum_i h_i J_{i,y}$$
$$- \frac{P}{N^2} \sum_i \sum_{j \neq i} \frac{N_i}{M_i} \frac{D_i^T}{\mathbb{D}_{ij}} \left(\frac{J_{j,y}}{Y_j M_j} - \frac{J_{i,y}}{Y_i M_i} \right) \quad (3.22)$$

From eq. 2.36 and eq. 2.37 the components of the radiative flux vector are:

$$q_{R,x} = \int_{x(\bar{r}_o)}^{x(\bar{r}_i)} \int_0^\infty \alpha_\nu \left(4\pi B_\nu - \int_0^{4\pi} I_\nu(\bar{r})\, d\Omega \right) d\nu\, \mathcal{K}\, dx \quad (3.23)$$

$$q_{R,y} = \int_{y(\bar{r}_o)}^{y(\bar{r}_i)} \int_0^\infty \alpha_\nu \left(4\pi B_\nu - \int_0^{4\pi} I_\nu(\bar{r})\, d\Omega \right) d\nu\, dy \quad (3.24)$$

The statement of these vector components completes the set of conservation equations expressed in body oriented orthogonal coordinates. By the use of the stretching functions listed in Tab. 3.1, the conservation equations can be written in the coordinate system desired by following the method used for the case under consideration in this section. Subsequent transformation of independent variables using Dorodnitsyn, Von Mises, Lees or one of many other transformations may be made in order to simplify the form of the conservation equations. The selection and use of these transformations will not be discussed here. The reader is referred to Dorrance, Ref. 3.2, and Hansen, Ref. 3.3, for suitable discussion and listing of similarity transformations.



$$\operatorname{Tan}\epsilon = \frac{d\delta}{(1+\kappa\delta)\,dx} \qquad \frac{d\theta}{dx} = \kappa(x)$$

$$\delta = \int_0^x (1+\kappa\delta)\operatorname{Tan}\epsilon\,dx + \delta_0 \qquad \theta = \int_0^x \kappa(x)\,dx$$

Body-Oriented Coordinate System

Fig. 3.1

Table 3.1 Coordinate systems and stretching Functions

1. Orthogonal coordinate system, and 2. orthogonal coordinates ξ_1, ξ_2, ξ_3	Rectangular coordinates			Stretching functions h_1 h_2 h_3		
	x	y	z	h_1	h_2	h_3
Cylindrical r, θ, z	$r \cos \theta$	$r \sin \theta$	z	1	r	1
Spherical r, ϕ, θ	$r \cos \theta \sin \phi$	$r \sin \theta \sin \phi$	$r \cos \phi$	1	r	$r \sin \phi$
Parabolic cylindrical ξ, η, z	$\frac{1}{2}(\xi^2 - \eta^2)$	$\xi\eta$	z	$\sqrt{\xi^2 + \eta^2}$	$\sqrt{\xi^2 + \eta^2}$	1
Local coordinates along surface x, y, z	-	-	-	$1 + \kappa y$	1	1
Local coordinates along surface Symmetric about axis x, y, ϕ	-	-	-	$1 + \kappa y$	1	r

Note: Additional coordinate systems are considered by Back, Ref. 3.1.

Section 3

References

3.1 Back, L. H. "Conservation Equations of a Viscous, Heat-Conducting Fluid in Curvilinear Orthogonal Coordinates," JPL Technical Report 32-1332 (Sept. 1968).

3.2 Dorrance, W. H. Viscous Hypersonic Flow, McGraw Hill Co., Inc., New York (1962).

3.3 Hansen, A. G. Similarity Analyses of Boundary Value Problems in Engineering, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1964).

SECTION IV

THIN VISCOUS SHOCK LAYER EQUATIONS

In the previous section the conservation equations were written in orthogonal body oriented coordinates. It was pointed out that the motivation to use such a coordinate system was for ease in describing flow over a blunt body. Let us expand on this point and the details of the problem to be solved.

The aerothermal environment of a blunt vehicle entering a planetary atmosphere at hypersonic velocities has received a good deal of attention in recent years. This attention has been centered around the prediction of the heat transferred to the surface of a vehicle during its atmospheric encounter. The most severe heating occurs at and near the leading face of a vehicle. For this reason special attention has been given to the calculation of the flow field and its associated energy transfer to the surface near the front of the vehicle.

In order to determine the proper mathematical model to describe the flow field developed by a blunt body moving at hypersonic velocities, one must assess the behavior of the gas that the vehicle will encounter. Fig. 4.1 based on the work of Ref. 4.1 presents the flight regimes which are encountered by a body during atmospheric entry. The regimes can be grouped into two gasdynamic domains - continuum and noncontinuum. Hayes and Probstein, Ref. 4.2, demonstrates the continuum domain can be divided into five regimes: (1) classical boundary layer, (2) vorticity interaction, (3) fully viscous, (4) incipient merged layer, and (5) fully merged layer. The behavior of the gas flowing over a body in the five continuum regimes can be described using the equations developed in the previous sections. Let us consider further the characteristics of fluid flow in the five continuum regimes.

1. Boundary layer regime: The classical boundary layer equations are a valid approximation of the viscous effects for high Reynolds numbers corresponding to lower altitudes. Viscous effects dominate near the wall in a region which is small compared to the shock layer thickness. Vorticity generated by shock curvature is therefore negligible thus not affecting the boundary layer flow.
2. Vorticity interaction becomes important at lower Reynolds numbers where shock generated vorticity becomes significant in respect to viscous effects near the body. Here the outer region of the shock layer, usually considered the invicid layer, becomes coupled through momentum transport

to the higher shear region near the body, usually thought of as the boundary layer. The high shear region near the body is also larger than experienced at higher Reynolds numbers.

3. Viscous layer Regime: Viscous effects from the body interaction are spread throughout the shock layer. This occurs at lower Reynolds numbers and correspondingly higher altitudes than the vorticity interaction regime. Viscous dissipation at the shock is still small in comparison to dissipation at the body. This condition is true so long as the ratio of the mean free path behind the shock over the shock layer thickness is much smaller than the square root of the density ratio across the shock wave, Ref. 4.2. This implies that the Rankine-Hugoniot shock wave equations are valid for determination of shock layer boundary conditions.
4. Incipient merged layer regime: The incipient merged layer begins when dissipative effects at the shock are significant. The shock wave is thin relative to the shock layer thickness but the Rankine-Hugoniot relations must be modified to account for viscous effects at the shock boundary.
5. Fully merged layer regime: At higher altitudes and low Reynolds numbers a distinct shock does not exist. The free stream mean free path over the major body radius is approximately one or less. The flow behaves continuously from the free stream to the body. Above this altitude range continuum concepts are no longer applicable and the flow goes through a transition to free molecular flow.

The foregoing discussion of the five continuum flow regimes follows in part the reasoning of Hayes and Probstein, Ref. 4.2. This reasoning proceeded under the assumption that radiative energy transport and ablative mass injection were negligible. In the present development these two effects are the primary flow field-body interaction mechanisms which are to be assessed when coupled to the viscous mechanisms. Fig. 4.1 shows the flight regimes where radiative heating to a one foot body becomes significant. For the most part, significant ablation rates are also encountered in these regimes when using present day charring ablators such as carbon phenolic or nylon phenolic. Therefore, let us make additional observations about the flow characteristics in these continuum flight regimes where the effects of ablation and radiative energy transfer in the shock layer are important. In proceeding, our attention will be restricted to the first three flight regimes, where the heating rates to a vehicle's surface are the most significant.

Significant radiative energy transfer has several important effects on the shock layer behavior. First, radiative transfer couples the energy equation and thus the thermal boundary layer over the entire shock layer. This is apparent by recalling that the flux divergence term in the energy equation is evaluated by an integration over all space in the shock layer. This effect has been demonstrated by several authors including Ref. 4.3 and 4.4. Further, the thermal boundary layer exists from the shock to the body for all three flight regimes in the radiative coupled domain. Secondly, radiative energy transfer produces nonadiabatic or energy loss effects. Principally, radiant energy is lost through the transparent shock wave. Thirdly, the effect of radiative transfer in the shock wave is coupled through the energy equation to the momentum equation. Although this coupling effect is not altogether negligible, it does not change the conclusions obtained about momentum transfer in the shock layer in the first three flight regimes. Therefore, even though the viscous effects may be approximated through boundary layer concepts with possible modifications of edge conditions in the vorticity layer regime, the energy transport occurs over the entire shock layer. In the viscous layer regime both viscous and energy transport are significant over the entire shock layer.

Appreciable mass injection rates of ablation products results in additional effects on energy and momentum transfer within the shock layer. High mass addition rates tends to enlarge the region of shear dominated flow near the body. Libby, Ref. 4.5 showed experimentally and theoretically that in the boundary layer regime, boundary layer concepts could be applied when mass injection or suction rates were quite large. This study did not include the effects of radiation, but since energy transport does not change the character of momentum transport these conclusions are also valid insofar as momentum transfer is concerned for radiative coupled shock layers. Mass injection has other effects such as reduction of shear at the wall, Ref. 4.4, and reduction of heat transfer at the wall, Ref. 4.4, 4.5 and many others. These effects although of great importance do not change the basic characteristics of momentum or energy transfer in the shock layer.

We may conclude that for flight conditions in the radiative coupled domain where ablation rates are also significant, the character of the momentum transfer is essentially the same as without these effects. However, the characteristics of energy transfer are significantly different in that the entire shock layer must be considered in all three flight regimes.

With the foregoing statements as background the problem which we wish to solve can be stated. The basic conservation equations stated in the previous section are appropriate to describe the flow of a continuum reacting and radiating gas mixture over a blunted surface when thermodynamic equilibrium exists. For the present work, we will determine the reduced set of equations which describe the flow in a shock layer over a blunt body when the outer boundary of the shock layer is a shock wave described by the Rankine-Hugoniot equations. Thus the equations governing the flow in the shock layer will be applicable to the three higher Reynolds number regimes both in and out of the radiation coupled domain. The prime concern and motivation for obtaining this set of equations is to describe the heat transfer mechanisms which produce surface heating such that surface heating conditions can be predicted by numerical calculation.

In order to determine the appropriate set of equations which realistically approximate the flow situation just described, an order of magnitude assessment of the terms in the basic conservation equations is needed. This is properly carried out by first nondimensionalizing the conservation equations. The following nondimensional variables are introduced which are appropriate to the problem under consideration.

$$
\begin{aligned}
&\xi = \frac{x^*}{R^*} \qquad y = \frac{y^*}{R^*} \qquad u = \frac{u^*}{U_\infty^*} \qquad v = \frac{v^*}{U_\infty^*} \\
&\rho = \frac{\rho^*}{\rho_{s,o}^*} \qquad \mu = \frac{\mu^*}{\mu_{s,o}^*} \qquad \lambda = \frac{\lambda^*}{\mu_{s,o}^*} \qquad \delta = \frac{\delta^*}{R^*} \\
&r = \frac{r^*}{R^*} \qquad \kappa = \kappa^* R^* \qquad P = \frac{P^*}{\rho_\infty^*(U_\infty^*)^2} \qquad H = \frac{H^*}{H_s^*} \\
&h = \frac{h_s^*}{H_s^*} \qquad h = \frac{h^*}{h_s^*} \quad \text{where} \quad H_s^* = \frac{1}{2} U_\infty^{*2} \\
&\tilde{\kappa} = 1 + \kappa y \qquad \omega_i = \frac{R^* \omega_i^*}{\rho_\infty^* U_\infty^*} \qquad J_i = \frac{J_i^*}{\rho_\infty^* U_\infty^*} \qquad \Lambda_{R,x} = \frac{\Lambda_{R,x}^*}{\rho_\infty^*(U_\infty^*)^3} \\
&\Lambda_{R,y} = \frac{\Lambda_{R,y}^*}{\rho_\infty^*(U_\infty^*)^3} \qquad \Lambda_{D,x} = \frac{\Lambda_{D,x}^*}{\rho_\infty^*(U_\infty^*)^3} \qquad \Lambda_{D,y} = \frac{\Lambda_{D,y}^*}{\rho_\infty^*(U_\infty^*)^3}
\end{aligned}
\qquad (4.1)
$$

where $\Lambda^*_{R,x} = \frac{\partial}{\partial x^*}(r^{*A} q^*_{R,x})$ $\Lambda^*_{R,y} = \frac{\partial}{\partial y^*}(r^{*A}\tilde{\kappa} q^*_{R,y})$

$\Lambda^*_{D,x} = \frac{\partial}{\partial x^*}(r^{*A} q^*_{D,x})$ $\Lambda^*_{D,y} = \frac{\partial}{\partial y^*}(r^{*A}\tilde{\kappa} q^*_{R,y})$

It should be noted that the equations in the previous sections are in dimensional form. In this section a superscript * will denote dimensional variables unless it is explicitly stated otherwise.

The dimensional global continuity equation is:

$$\frac{\partial}{\partial x^*}(r^{*A}\rho^* u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A}\rho^* v^*) = 0 \tag{4.2}$$

Using the dimensionless variables stated in Eq. 4.1 the above equation may be written as

$$\rho^*_\infty U^*_\infty \frac{R^*}{R^*}\frac{\partial}{\partial \xi}(r^A \rho u) + \rho^*_\infty U^*_\infty \frac{R^*}{R^*}\frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho v) = 0$$

Dividing by $\rho^*_\infty U^*_\infty$ yields the dimensionless form

$$\frac{\partial}{\partial \xi}(r^A \rho u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho v) = 0 \tag{4.3}$$

From Eq. 3.8, the dimensional species continuity equation is:

$$\frac{\partial}{\partial x^*}(r^{*A}\rho^* C_i u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A}\rho^* C_i v^*) = \frac{-\partial}{\partial x^*}(r^{*A} J^*_{i,x}) - \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} J^*_{i,y}) + \tilde{\kappa} r^{*A}\omega^*_i \tag{4.4}$$

Introducing dimensionless variables gives

$$\rho_\infty^* U_\infty^* \frac{R^*}{R^*}\left[\frac{\partial}{\partial \xi}(r^A \rho C_i u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho C_i v)\right]$$
$$= -\rho_\infty^* U_\infty^* \frac{R^*}{R^*}\left[\frac{\partial}{\partial \xi}(r^A J_{i,x}) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A J_{i,y})\right] +$$
$$\rho_\infty^* U_\infty^* \frac{R^*}{R^*} \tilde{\kappa} r^A \omega_i$$

which yields

$$\frac{\partial}{\partial \xi}(r^A \rho C_i u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho C_i v) = -\frac{\partial}{\partial \xi}(r^A J_{i,x}) - \frac{\partial}{\partial y}(\tilde{\kappa} r^A J_{i,y}) + \tilde{\kappa} r^A \omega_i \tag{4.5}$$

Substituting Eqs. 3.16, 3.18 and 3.19 into Eq. 3.15 yields the dimensional x-momentum equation

$$\begin{aligned}
&\rho^* r^{*A} u^* \frac{\partial u^*}{\partial x^*} + \rho^* \tilde{\kappa} r^{*A} v^* \frac{\partial u^*}{\partial y^*} + \rho^* \kappa^* r^{*A} u^* v^* \\
&+ r^{*A} \frac{\partial P^*}{\partial x^*} - \frac{\partial}{\partial x^*}\left(r^{*A}\left[\frac{\lambda^*}{\tilde{\kappa} r^{*A}} \left\{ \frac{\partial}{\partial x}(r^{*A} u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*)\right\}\right.\right. \\
&\left.\left. + \frac{2\mu^*}{\tilde{\kappa}}\left\{ \frac{\partial u^*}{\partial x^*} + \kappa^* v^*\right\}\right]\right) - \frac{\partial}{\partial y^*}\left(\tilde{\kappa} r^{*A} \mu^* \left[\frac{1}{\tilde{\kappa}} \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right.\right. \\
&\left.\left. - \frac{\kappa^*}{\tilde{\kappa}} u^*\right]\right) - \kappa r^{*A} \mu^* \left[\frac{1}{\tilde{\kappa}} \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} - \frac{\kappa^*}{\tilde{\kappa}} u^*\right] \\
&+ \frac{\lambda^*}{\tilde{\kappa} r^{*A}}\left[\frac{\partial}{\partial x^*}(r^{*A} u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*)\right]\left[\frac{\partial r^{*A}}{\partial x^*}\right] \\
&+ \frac{2\mu^* u^*}{\tilde{\kappa} r^{*A}}\left[\frac{\partial r^{*A}}{\partial x^*}\right]^2 + 2\mu^* \frac{v^*}{r^{*A}} \frac{\partial r^{*A}}{\partial x^*} \frac{\partial r^{*A}}{\partial y^*} = 0
\end{aligned} \tag{4.6}$$

Proceeding as before the dimensionless variables are introduced.

$$
\begin{aligned}
&\rho_\infty^* R^* \frac{U_\infty^{*2}}{R^*}\left[\rho r^A u \frac{\partial u}{\partial \xi}\right] + \rho_\infty^* R^* \frac{U_\infty^{*2}}{R^*}\left[\rho \tilde{\kappa} r^A v \frac{\partial u}{\partial y}\right] \\
&+ \rho_\infty^* R^* U_\infty^{2*} \frac{1}{R^*}\left[\rho \kappa r^A u v\right] + \frac{R^*}{R^*} \rho_\infty^* U_\infty^{*2}\left[r^A \frac{\partial P}{\partial \xi}\right] \\
&- R^* \mu_{s,o}^* \frac{R^* U_\infty^*}{R^* R^* R^*}\left[\frac{\partial}{\partial \xi}\left(\frac{\lambda}{\tilde{\kappa}}\left\{\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right\}\right)\right] \\
&- R^* \frac{\mu_{s,o}^* U_\infty^*}{R^* R^*}\left[\frac{\partial}{\partial \xi}\left(\frac{2 r^A \mu}{\tilde{\kappa}}\left\{\frac{\partial u}{\partial \xi} + \kappa v\right\}\right)\right] \\
&- \frac{\mu_{s,o}^* R^* U_\infty^*}{R^* R^*}\left[\frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \mu\left\{\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y}\right\}\right)\right] \\
&+ \frac{R^* \mu_{s,o}^* U_\infty^*}{R^* R^*}\left[\frac{\partial}{\partial y}(\kappa r^A \mu u)\right] \\
&- \frac{1}{R^*} R^* \mu_{s,o}^* \frac{U_\infty^*}{R^*}\left[\kappa r^A \mu\left(\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa}{\tilde{\kappa}} u\right)\right] \\
&+ \mu_{s,o}^* \frac{R^* U_\infty^* R^*}{R^* R^* R^*}\left[\frac{\lambda}{\tilde{\kappa} r^A}\left(\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)\left(\frac{\partial r^A}{\partial \xi}\right)\right] \\
&+ \mu_{s,o}^* \frac{U_\infty^* R^* R^*}{R^* R^* R^*}\left[\frac{2 \mu u}{\tilde{\kappa} r^A}\left(\frac{\partial r^A}{\partial \xi}\right)^2 + 2\mu \frac{v}{r^A} \frac{\partial r^A}{\partial y} \frac{\partial r^A}{\partial \xi}\right] = 0
\end{aligned}
$$

By letting

$$
R_e = \frac{\rho_\infty^* U_\infty^* R^*}{\mu_{s,o}^*} \tag{4.7}
$$

and cancelling dimensionless terms yields the dimensionless ξ-momentum equation.

$$
\begin{aligned}
&\rho r^A u \frac{\partial u}{\partial \xi} + \rho \tilde{\kappa} r^A v \frac{\partial u}{\partial y} + \rho \kappa r^A uv + r^A \frac{\partial P}{\partial \xi} \\
&- \frac{1}{R_e} \Big\{ \frac{\partial}{\partial \xi}\Big(\frac{\lambda}{\tilde{\kappa}} \frac{\partial}{\partial \xi}(r^A u) + \frac{\lambda}{\tilde{\kappa}} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\Big) \\
&+ \frac{\partial}{\partial \xi}\Big(\frac{2 r^A \mu}{\tilde{\kappa}} \Big[\frac{\partial u}{\partial \xi} + \kappa v \Big]\Big) \\
&+ \frac{\partial}{\partial y}\Big(r^A \mu \frac{\partial v}{\partial \xi} + \tilde{\kappa} r^A \mu \frac{\partial u}{\partial y} - \kappa r^A \mu u \Big) \\
&+ \kappa r^A \mu \Big(\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa}{\tilde{\kappa}} u \Big) \\
&- \frac{\lambda}{\tilde{\kappa} r^A} \Big(\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\Big) \Big(\frac{\partial r^A}{\partial \xi} \Big) \\
&- 2 \frac{\mu u}{\tilde{\kappa} r^A} \Big(\frac{\partial r^A}{\partial \xi} \Big) - 2 \mu \frac{v}{r^A} \frac{\partial r^A}{\partial \xi} \frac{\partial r^A}{\partial y} \Big\} = 0
\end{aligned}
\tag{4.8}
$$

The y-momentum equation can be stated in dimensional form from Eqs. 3.15 → 3.19.

$$
\begin{aligned}
&\rho^* r^{*A} u^* \frac{\partial v^*}{\partial x^*} + \rho^* r^{*A} \tilde{\kappa} v^* \frac{\partial v^*}{\partial y^*} - \rho^* \kappa^* r^{*A} u^{*2} \\
&+ \tilde{\kappa} r^{*A} \frac{\partial P^*}{\partial y^*} - \frac{\partial}{\partial x^*}\Big(r^{*A} \mu^* \Big[\frac{1}{\tilde{\kappa}} \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} - \frac{\kappa^*}{\tilde{\kappa}} u^* \Big] \Big) \\
&- \frac{\partial}{\partial y^*}\Big(\lambda^* \frac{\partial}{\partial x^*}(r^{*A} u^*) + \lambda^* \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*) \Big) \\
&- \frac{\partial}{\partial y^*}\Big(2 \tilde{\kappa} r^{*A} \mu^* \frac{\partial v^*}{\partial y^*} \Big) + \lambda^* \frac{\kappa^*}{\tilde{\kappa}} \Big[\frac{\partial}{\partial x^*}(r^{*A} u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*) \Big] \\
&+ 2 \mu^* \frac{\kappa^* r^{*A}}{\tilde{\kappa}} \Big[\frac{\partial u^*}{\partial x^*} + \kappa^* v^* \Big] \\
&+ \frac{\lambda^*}{r^{*A}} \Big[\frac{\partial}{\partial x^*}(r^{*A} u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*) \Big] \Big[\frac{\partial r^{*A}}{\partial y^*} \Big] \\
&+ 2 \mu^* \frac{u^*}{r^{*A}} \Big[\frac{\partial r^{*A}}{\partial y^*} \Big] \Big[\frac{\partial r^{*A}}{\partial x^*} \Big] + 2 \mu^* \frac{\tilde{\kappa} v^*}{r^{*A}} \Big[\frac{\partial r^{*A}}{\partial y^*} \Big]^2 = 0
\end{aligned}
\tag{4.9}
$$

Introduction of nondimensional variables into the y-momentum equations follows the same procedure and pattern as in the x-momentum equation. The resulting nondimensional y-momentum equation is:

$$
\begin{aligned}
&\rho r^A u \frac{\partial v}{\partial \xi} + \rho r^A \tilde{\kappa} v \frac{\partial v}{\partial y} - \rho \kappa r^A u^2 + \tilde{\kappa} r^A \frac{\partial P}{\partial y} \\
&- \frac{1}{R_e}\left\{ \frac{\partial}{\partial \xi}\left(r^A \mu \left[\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa}{\tilde{\kappa}} u \right] \right) \right. \\
&+ \frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial \xi}(r^A u) + \lambda \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right) + \frac{\partial}{\partial y}\left(2 \tilde{\kappa} r^A \mu \frac{\partial v}{\partial y} \right) \\
&- \frac{\lambda \kappa}{\tilde{\kappa}} \left[\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right] - 2\mu \frac{\kappa r^A}{\tilde{\kappa}} \left[\frac{\partial u}{\partial \xi} + \kappa v \right] \\
&- \frac{\lambda}{r^A} \left[\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right] \left[\frac{\partial r^A}{\partial y} \right] - 2\mu \frac{u}{r^A} \left[\frac{\partial r^A}{\partial y} \right]^2 \\
&\left. - 2\mu \frac{\tilde{\kappa} v}{r^A} \frac{\partial r^A}{\partial \xi} \frac{\partial r^A}{\partial y} \right\} = 0
\end{aligned}
\tag{4.10}
$$

Using Eqs. 3.20 with 3.16, 3.17 and 3.19 the energy equation can be written in dimensional form:

$$
\begin{aligned}
&\rho^* r^{*A} u^* \frac{\partial H^*}{\partial x^*} + \rho^* \tilde{\kappa} r^{*A} v^* \frac{\partial H^*}{\partial y^*} = -\Lambda^*_{D,x} - \Lambda^*_{D,y} - \Lambda^*_{R,x} \\
&- \Lambda^*_{R,y} + \frac{\partial}{\partial x^*}\left(\frac{\lambda^* u^*}{\tilde{\kappa}} \left[\frac{\partial}{\partial x^*}(r^{*A} u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*) \right] \right. \\
&+ 2\mu^* r^{*A} \frac{u^*}{\tilde{\kappa}} \left[\frac{\partial u^*}{\partial x^*} + \kappa^* v^* \right] + r^{*A} v^* \mu^* \left[\frac{1}{\tilde{\kappa}} \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right. \\
&\left. \left. - \frac{\kappa^* u^*}{\tilde{\kappa}} \right] \right) + \frac{\partial}{\partial y^*}\left(\tilde{\kappa} r^{*A} \mu^* u^* \left[\frac{1}{\tilde{\kappa}} \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} - \frac{\kappa^*}{\tilde{\kappa}} u^* \right] \right. \\
&\left. + \lambda^* v^* \left[\frac{\partial}{\partial x^*}(r^{*A} u^*) + \frac{\partial}{\partial y^*}(\tilde{\kappa} r^{*A} v^*) \right] + 2\tilde{\kappa} r^{*A} \mu^* v^* \frac{\partial v^*}{\partial y^*} \right)
\end{aligned}
\tag{4.11}
$$

where the diffusional and radiative flux divergence terms, Λ^*, are defined in Eq. 4.1. Substitution of the nondimensional ratios from Eq. 4.1 yields

$$\rho_\infty^* \frac{R^*}{R^*} U_\infty^* \frac{U_\infty^{*2}}{2} \left[r^A \rho u \frac{\partial H}{\partial \xi} + \tilde{\kappa} r^A \rho v \frac{\partial H}{\partial y} \right] = -\rho_\infty^* (U_\infty^*)^3 \left[\Lambda_{D,x} + \Lambda_{D,y} + \Lambda_{R,x} + \Lambda_{R,y} \right] + \mu_{s,o}^* \frac{U_\infty^{*2}}{R^*} \frac{R^*}{R^*} \left\{ \frac{\partial}{\partial \xi} \left(\frac{\lambda u}{\tilde{\kappa}} \left[\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right] + 2\mu r^A \frac{u}{\tilde{\kappa}} \left[\frac{\partial u}{\partial \xi} + \kappa v \right] + r^A \mu v \left[\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa u}{\tilde{\kappa}} \right] \right) + \frac{\partial}{\partial y} \left(\tilde{\kappa} r^A \mu u \left[\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa u}{\tilde{\kappa}} \right] + \lambda v \left[\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right] + 2\tilde{\kappa} r^A \mu v \frac{\partial v}{\partial y} \right) \right.$$

Introducing the Reynolds number the nondimensional energy equation can be written:

$$r^A \rho u \frac{\partial H}{\partial \xi} + \tilde{\kappa} r^A \rho v \frac{\partial H}{\partial y} = -2 \left[\Lambda_{D,x} + \Lambda_{D,y} + \Lambda_{R,x} + \Lambda_{R,y} \right] + \frac{2}{R_e} \left\{ \frac{\partial}{\partial \xi} \left(\frac{\lambda u}{\tilde{\kappa}} \left[\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right] + 2 r^A \mu \frac{u}{\tilde{\kappa}} \left[\frac{\partial u}{\partial \xi} + \kappa v \right] + r^A \mu v \left[\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa u}{\tilde{\kappa}} \right] \right) + \frac{\partial}{\partial y} \left(\tilde{\kappa} r^A \mu u \left[\frac{1}{\tilde{\kappa}} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial y} - \frac{\kappa u}{\tilde{\kappa}} \right] + \lambda v \left[\frac{\partial}{\partial \xi}(r^A u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A v) \right] + 2 \tilde{\kappa} r^A \mu v \frac{\partial v}{\partial y} \right) \right. \tag{4.12}$$

Having stated the nondimensional conservation equations we are confronted with the problem of estimating the relative magnitude of the terms in each equation.

According to the results of Hayes and Probstein Ref. 4.2 the gas behind a bow shock of a hypervelocity vehicle is a continuum for freestream Reynolds numbers greater than 100 based on principle body radius. Further, the standoff distance nondimensionalized by body radius for flight Reynolds numbers greater than 100 has been shown to be approximately equal to the density ratio across the bow shock. It can be quite simply shown that the density ratio for hypersonic Mach numbers is of the order of one tenth and less for dissociating gases. These stated relationships can be expressed as follows:

$$R_e > 100, \qquad \frac{\delta^*}{R^*} \simeq \bar{\rho} \leq .10 \tag{4.13}$$

Since we are concerned with a thin layer with respect to the body radius, Prandtls concepts for the relative order of magnitude of terms in the conservation equations can be employed. From Schlichting Ref. 4.6, the relationships for the relative order of nondimensionalized terms may be written.

$$\begin{array}{llll}
u \sim 1 & v \sim \bar{\rho} & \kappa \sim 1 & \tilde{\kappa} \sim 1 \\
H \sim 1 & \rho \sim 1 & \mu \sim 1 & C_i \sim 1 \\
\dfrac{\partial u}{\partial \xi} \sim 1 & \dfrac{\partial^2 u}{\partial \xi^2} \sim 1 & \dfrac{\partial v}{\partial \xi} \sim \bar{\rho} & \dfrac{\partial^2 v}{\partial \xi^2} \sim \bar{\rho} \\
\dfrac{\partial u}{\partial y} \sim \dfrac{1}{\bar{\rho}} & \dfrac{\partial^2 u}{\partial y^2} \sim \dfrac{1}{\bar{\rho}^2} & \dfrac{\partial v}{\partial y} \sim 1 & \dfrac{\partial^2 v}{\partial y^2} \sim \dfrac{1}{\bar{\rho}} \\
\dfrac{\partial C_i}{\partial \xi} \sim 1 & \dfrac{\partial^2 C_i}{\partial \xi^2} \sim 1 & \dfrac{\partial C_i}{\partial y} \sim \dfrac{1}{\bar{\rho}} & \dfrac{\partial^2 C_i}{\partial y^2} \sim \dfrac{1}{\bar{\rho}^2} \\
J_{i,x} \sim \dfrac{1}{R_e}\dfrac{\partial C_i}{\partial \xi} & J_{iy} \sim \dfrac{1}{R_e}\dfrac{\partial C_i}{\partial y} & \dfrac{\partial r^A}{\partial \xi} \sim 1 & \dfrac{\partial r^A}{\partial y} \sim \bar{\rho}
\end{array} \tag{4.14}$$

Using the above estimates the relative order of magnitude of the terms in the four conservation equations have been determined.

Global continuity

$$\overset{O[1]}{\frac{\partial}{\partial \xi}(r^A \rho u)} + \overset{O[1]}{\frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho v)} = 0 \tag{4.15}$$

Species continuity

$$\overset{O[1]}{\frac{\partial}{\partial \xi}(r^A \rho C_i u)} + \overset{O[1]}{\frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho C_i v)} = \overset{O\left[\frac{1}{R_e}\right]}{-\frac{\partial}{\partial \xi}(r^A J_{ix})} \tag{4.16}$$

$$\overset{O\left[\frac{1}{\bar{\rho}^2 R_e}\right]}{-\frac{\partial}{\partial y}(\tilde{\kappa} r^A J_{iy})} + \tilde{\kappa} r^A \omega_i$$

ξ - Momentum

$$\overset{O[1]}{\rho r^A u \frac{\partial u}{\partial \xi}} + \overset{O[1]}{\rho \tilde{\kappa} r^A v \frac{\partial u}{\partial y}} + \overset{O[\bar{\rho}]}{\rho \kappa r^A u v} + r^A \frac{\partial P}{\partial \xi}$$

$$- \frac{1}{R_e} \left\{ \overset{O[1]}{\frac{\partial}{\partial \xi}\left(\frac{\lambda}{\tilde{\kappa}} \frac{\partial}{\partial \xi}(r^A u)\right)} + \overset{O[1]}{\frac{\partial}{\partial \xi}\left(\frac{\lambda}{\tilde{\kappa}} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)} \right.$$

$$+ \overset{O[1]}{\frac{\partial}{\partial \xi}\left(\frac{2 r^A \mu}{\tilde{\kappa}} \frac{\partial u}{\partial \xi}\right)} + \overset{O[\bar{\rho}]}{\frac{\partial}{\partial \xi}\left(\frac{2 r^A \mu \kappa v}{\tilde{\kappa}}\right)} + \overset{O[\bar{\rho}]}{\frac{\kappa}{\tilde{\kappa}} r^A \mu \frac{\partial v}{\partial \xi}} \tag{4.17}$$

$$+ \overset{O[\bar{\rho}]}{\frac{\partial}{\partial y}\left(r^A \mu \frac{\partial v}{\partial \xi}\right)} + \overset{O\left[\frac{1}{\bar{\rho}^2}\right]}{\frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \mu \frac{\partial u}{\partial y}\right)} - \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}(\kappa r^A \mu u)}$$

$$+ \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\kappa r^A \mu \frac{\partial u}{\partial y}} - \overset{O[1]}{\frac{\kappa^2}{\tilde{\kappa}} r^A \mu u} - \overset{O[1]}{\frac{\lambda}{\tilde{\kappa} r^A} \frac{\partial}{\partial \xi}(r^A u)\left(\frac{\partial r^A}{\partial \xi}\right)}$$

$$\left. - \overset{O[1]}{\frac{\lambda}{\tilde{\kappa} r^A} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\left(\frac{\partial r^A}{\partial \xi}\right)} - \overset{O[1]}{2 \frac{\mu u}{\tilde{\kappa} r^A}\left(\frac{\partial r^A}{\partial \xi}\right)^2} - \overset{O[\bar{\rho}^2]}{\frac{2 \mu v}{r^A} \frac{\partial r^A}{\partial \xi} \frac{\partial r^A}{\partial y}} \right\} = 0$$

$\underline{y}$ - Momentum

$$
\begin{aligned}
&\overset{O[\bar{\rho}]}{\rho r^A u \frac{\partial v}{\partial \xi}} + \overset{O[\bar{\rho}]}{\rho r^A \tilde{\kappa} v \frac{\partial v}{\partial y}} - \overset{O[1]}{\rho \kappa r^A u^2} + \tilde{\kappa} r^A \frac{\partial P}{\partial y} \\
&- \frac{1}{R_e} \left\{ \overset{O[\bar{\rho}]}{\frac{\partial}{\partial \xi}\left(\frac{r^A \mu}{\tilde{\kappa}} \frac{\partial v}{\partial \xi}\right)} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial \xi}\left(r^A \mu \frac{\partial u}{\partial y}\right)} - \overset{O[1]}{\frac{\partial}{\partial \xi}\left(\frac{r^A \mu \kappa u}{\tilde{\kappa}}\right)} \right. \\
&+ \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial \xi}(r^A u)\right)} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)} - \overset{O[1]}{\frac{\lambda \kappa}{\tilde{\kappa}} \frac{\partial}{\partial \xi}(r^A u)} \\
&- \overset{O[1]}{\frac{\lambda \kappa}{\tilde{\kappa}} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)} - \overset{O[1]}{2\mu \frac{\kappa r^A}{\tilde{\kappa}} \frac{\partial u}{\partial \xi}} - \overset{O[\bar{\rho}]}{2\mu \frac{\kappa^2}{\tilde{\kappa}} r^A v} \qquad (4.18) \\
&- \overset{O[\bar{\rho}]}{\frac{\lambda}{r^A} \frac{\partial}{\partial \xi}(r^A u)\left(\frac{\partial r^A}{\partial y}\right)} - \overset{O[\bar{\rho}]}{\frac{\lambda}{r^A} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\left(\frac{\partial r^A}{\partial y}\right)} - \overset{O[\bar{\rho}^2]}{2\mu \frac{u}{r^A}\left(\frac{\partial r^A}{\partial y}\right)^2} \\
&\left. - \overset{O[\bar{\rho}^2]}{2\mu \frac{\tilde{\kappa} v}{r^A} \frac{\partial r^A}{\partial \xi} \frac{\partial r^A}{\partial y}} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(2\tilde{\kappa} r^A \mu \frac{\partial v}{\partial y}\right)} \right\} = 0
\end{aligned}
$$

Energy

$$
\begin{aligned}
&\overset{O[1]}{r^A \rho u \frac{\partial H}{\partial \xi}} + \overset{O[1]}{\tilde{\kappa} r^A \rho v \frac{\partial H}{\partial y}} = -2\Big[\overset{O\left[\frac{1}{R_e}\right]}{\Lambda_{D,x}} + \overset{\left[\frac{1}{\bar{\rho}^2 R_e}\right]}{\Lambda_{D,y}} \\
&+ \overset{O\left[\frac{1}{R_e}\right]}{\Lambda_{R,x}} + \overset{O\left[\frac{1}{\bar{\rho}^2 R_e}\right]}{\Lambda_{R,y}} \Big] + \frac{2}{R_e}\Big\{ \overset{O[1]}{\frac{\partial}{\partial \xi}\left(\frac{\lambda u}{\tilde{\kappa}} \frac{\partial}{\partial \xi}(r^A u)\right)} \\
&+ \overset{O[1]}{\frac{\partial}{\partial \xi}\left(\frac{\lambda u}{\tilde{\kappa}} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)} + \overset{O[1]}{\frac{\partial}{\partial \xi}\left(2 r^A \mu \frac{u}{\tilde{\kappa}} \frac{\partial u}{\partial \xi}\right)} \\
&+ \overset{O[\bar{\rho}]}{\frac{\partial}{\partial \xi}\left(2 r^A \frac{\kappa}{\tilde{\kappa}} \mu u v\right)} + \overset{O[\bar{\rho}^2]}{\frac{\partial}{\partial \xi}\left(r^A \mu \frac{v}{\tilde{\kappa}} \frac{\partial v}{\partial \xi}\right)} \\
&+ \overset{O[1]}{\frac{\partial}{\partial \xi}\left(r^A \mu v \frac{\partial u}{\partial y}\right)} - \overset{O[\bar{\rho}]}{\frac{\partial}{\partial \xi}\left(r^A \frac{\kappa}{\tilde{\kappa}} \mu u v\right)} + \overset{O[\bar{\rho}]}{\frac{\partial}{\partial y}\left(r^A \mu u \frac{\partial v}{\partial \xi}\right)} \\
&+ \overset{O\left[\frac{1}{\bar{\rho}^2}\right]}{\frac{\partial}{\partial y}\left(\tilde{\kappa} r \mu u \frac{\partial u}{\partial y}\right)} - \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\kappa r^A \mu u^2\right)} + \overset{O[1]}{\frac{\partial}{\partial y}\left(\lambda v \frac{\partial}{\partial \xi}(r^A u)\right)} \\
&+ \overset{O[1]}{\frac{\partial}{\partial y}\left(\lambda v \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)} + \overset{O[1]}{\frac{\partial}{\partial y}\left(2 \tilde{\kappa} r^A \mu v \frac{\partial v}{\partial y}\right)} \Big\}
\end{aligned}
\tag{4.19}
$$

Using the lower limit on Reynolds number we observe

$$\frac{1}{R_e} \approx \bar{\rho}^2 \approx \frac{1}{100}$$

At this lower limit on Reynolds number, let us drop all terms of order $\bar{\rho}^2$ and higher in all equations except the y-momentum equation. In the y-momentum equation terms of order $\bar{\rho}^2$ are retained for a specific reason. Along the stagnation line, $\xi = 0$, the u component of velocity is zero. Thus the y-momentum equation is of one order lower at $\xi = 0$. It is appropriate in this case to consider terms of two orders of magnitude in this equation namely $\bar{\rho}$ and $\bar{\rho}^2$. The resulting conservation equations are:

Global continuity

$$\frac{\partial}{\partial \xi}(r^A \rho u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho v) = 0 \tag{4.15}$$

Species continuity

$$\frac{\partial}{\partial \xi}(r^A \rho C_i u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho C_i v) = -\frac{\partial}{\partial y}(\tilde{\kappa} r^A J_{iy}) + \tilde{\kappa} r^A \omega_i \tag{4.20}$$

ξ -momentum

$$\underbrace{\rho r^A u \frac{\partial u}{\partial \xi}}_{O[1]} + \underbrace{\rho \tilde{\kappa} r^A v \frac{\partial u}{\partial y}}_{O[1]} + \underbrace{\rho \kappa r^A uv}_{O[\bar{\rho}]} + r^A \frac{\partial P}{\partial \xi}$$

$$- \frac{1}{R_e}\left\{ \underbrace{\frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \mu \frac{\partial u}{\partial y}\right)}_{O\left[\frac{1}{\bar{\rho}^2}\right]} - \underbrace{\frac{\partial}{\partial y}(\kappa r^A \mu u)}_{O\left[\frac{1}{\bar{\rho}}\right]} + \underbrace{\kappa r^A \mu \frac{\partial u}{\partial y}}_{O\left[\frac{1}{\bar{\rho}}\right]} \right\} = 0 \tag{4.21}$$

$\underline{y}$ -momentum ($O[\bar{\rho}]$ and larger terms)

$$
\begin{aligned}
&\overset{O[\bar{\rho}]}{\rho r^A u \frac{\partial v}{\partial \xi}} + \overset{O[\bar{\rho}]}{\rho r^A \tilde{\kappa} v \frac{\partial v}{\partial y}} - \overset{O[1]}{\rho \tilde{\kappa} r^A u^2} + \tilde{\kappa} r^A \frac{\partial P}{\partial y} \\
&- \frac{1}{R_e}\Big\{ \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial \xi}\left(r^A \mu \frac{\partial u}{\partial y}\right)} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial \xi}(r^A u)\right)} \\
&+ \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(2 \tilde{\kappa} r^A \mu \frac{\partial v}{\partial y}\right)}\Big\} = 0
\end{aligned}
\qquad (4.22)
$$

$\underline{y}$ -momentum ($O[\bar{\rho}^2]$ and larger terms)

$$
\begin{aligned}
&\overset{O[\bar{\rho}]}{\cancelto{0}{\rho r^A u \frac{\partial v}{\partial \xi}}} + \overset{O[\bar{\rho}]}{\rho r^A \tilde{\kappa} v \frac{\partial v}{\partial y}} - \overset{O[1]}{\cancelto{0}{\rho \kappa r^A u^2}} + \tilde{\kappa} r^A \frac{\partial P}{\partial y} \\
&- \frac{1}{R_e}\Big\{ \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial \xi}\left(r^A \mu \frac{\partial u}{\partial y}\right)} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial \xi}(r^A u)\right)} \\
&+ \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\lambda \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)\right)} + \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(2 \tilde{\kappa} r^A \mu \frac{\partial v}{\partial y}\right)} - \overset{O[1]}{\frac{\partial}{\partial \xi}(r^A \mu \kappa u)} \\
&- \overset{O[1]}{\frac{\lambda \kappa}{\tilde{\kappa}} \frac{\partial}{\partial \xi}(r^A u)} - \overset{O[1]}{\frac{\lambda \kappa}{\tilde{\kappa}} \frac{\partial}{\partial y}(\tilde{\kappa} r^A v)} - \overset{O[1]}{2\mu \frac{\kappa r^A}{\tilde{\kappa}} \frac{\partial u}{\partial \xi}}\Big\} = 0
\end{aligned}
\qquad (4.23)
$$

At $\xi = 0$ the above equation has terms which are of order $\bar{\rho}$ and $\bar{\rho}^2$. Two terms which can be directly eliminated from this equation when $u = 0$ at $\xi = 0$ are indicated by arrows. It is interesting to note that the convective and viscous terms are of the same order along the stagnation line.

Energy

$$\overset{O[1]}{r^A \rho u \frac{\partial H}{\partial \xi}} + \overset{O[1]}{\tilde{\kappa} r^A \rho v \frac{\partial H}{\partial y}} = -2\left(\overset{O[1]}{\Lambda_{D,y}} + \overset{O[1]}{\Lambda_{R,y}}\right)$$

$$+ \frac{2}{R_e}\left\{ \overset{O\left[\frac{1}{\bar{\rho}^2}\right]}{\frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \mu u \frac{\partial u}{\partial y}\right)} - \overset{O\left[\frac{1}{\bar{\rho}}\right]}{\frac{\partial}{\partial y}\left(\kappa r^A \mu u^2\right)} \right\} \tag{4.24}$$

The simplified set of conservation equations, Eqs. 4.15, 4.20 - 4.24 form a set of partial differential equations (neglecting the radiative terms) which are valid for Reynolds numbers greater than 100. It is obvious that the terms which have been dropped due to order of magnitude reasoning become less significant as the Reynolds number is increased. These "thin shock layer" equations are the same as second order boundary layer equations with curvature terms.

To this point little has been said about the bulk viscosity which appears in the λ term in the momentum and energy equations. This has been done for the sake of generality. However, to interpret the pressure in our equations as the local thermodynamic pressure Stokes' postulate

$$2\mu^* + 3\lambda^* = 0 \tag{4.25}$$

must be accepted. The bulk viscosity $\tilde{\mu}$ is a direct indication of the departure of the mean pressure from the thermodynamic pressure expressed by the equation of state Ref. 4.7. Further, Laitone Ref. 4.7 points out that by accepting Stokes postulate for compressible flows we are at best restricted to monatomic gases. This appears to be a rather stringent assumption until one examines the type of behavior a polyatomic gas must exhibit to significantly deviate from monatomic behavior. To a first approximation the bulk viscosity $\tilde{\mu}$ characterizes the dependence of pressure on the rate of change of density

Ref. 4.8. Gases which exhibit showly excited internal degrees of freedom (i.e. rotational or vibrational) in flows which have rapid changes in the state of the fluid, the pressure cannot follow the changes in density and differs from its value for thermodynamic equilibrium. Thus, accepting Stokes' postulate for bulk viscosity is consistent with our basic assumption of local thermodynamic equilibrium used throughout this development. Henceforth, we will use

$$\lambda^* = -\frac{2}{3}\mu^* \qquad (4.26)$$

in our equations. In thin shock layer equations Stokes' relation is needed only for the y-momentum equation. The order analysis has eliminated all terms containing λ in both the x-momentum and energy equation.

In addition to the simplifications from the order of magnitude analysis, further simplification of the radiative flux divergence term in the energy equation is necessary in order to solve the set of integro-partial differential equations in a practical manner. Without additional simplification the conservation equations are elliptic. Two assumptions are made here in order to evaluate the radiative flux divergence term $\Lambda_{R,y}$.

- The shock layer geometry is approximated locally by an infinite plane slab.
- The shock layer is assumed to be locally one-dimensional in that radiative transport characteristics vary only across the infinite plane slab.

It has been shown that this one-dimensional plane slab model can be used in obtaining quantitatively valid results Ref. 4.9. Further, this model identically satisfies the order of magnitude analysis which dropped $\Lambda_{R,y}$. The mathematical development of this model follows that presented by Spradley and Engel Ref. 4.10 with the exception that boundary conditions are left general following the work of R. and M. Goulard Ref. 4.11.

We note that dimensional equations will be used throughout the rest of this section without the superscript * notation unless the superscript is needed for clarity.

Let us consider the radiative transfer Eq. 1.27

$$\frac{1}{c}\frac{\partial I_v}{\partial t} + \bar{\Omega}_1 \cdot \nabla I_v = \alpha_v (B_v - I_v)$$

Following the assumptions of Section I let

$$\frac{1}{c}\frac{\partial I_v}{\partial t} = 0$$

Therefore our transfer equation can be written

$$\bar{\Omega}_1 \nabla I_v = \alpha_v (I_v - B_v) \tag{4.27}$$

By imposing the one-dimensional approximation, the radiative transfer equation for the y-direction may be written

$$((\bar{j}\iota) \cdot \left(\bar{j}\frac{dI_v}{dy}\right) = \iota \frac{dI_v}{dy} = \alpha_v (I_v - B_v) \tag{4.28}$$

For the one-dimensional problem the absorption and emission characteristics vary only in one direction, y. This fact is sufficient information to solve Eq. 4.28 for the specific intensity by integration in y. We will see later that although the specific intensity is evaluated one-dimensionally the radiative flux and flux divergence must be evaluated over all space. Consequently the flux divergence is integrated over an infinite plane slab which has the same intensity variation across the slab at any station down the slab.

In order to clarify the solution of Eq. 4.28, Fig. 4.2 is presented. From Fig. 4.2 we observe

$$dy = \cos\Psi \, ds = \iota \, ds \tag{4.29}$$

where $\iota \equiv \cos\Psi$

By defining the optical depth as

$$\Upsilon_v = \int \alpha_v \, dy \tag{4.30}$$

and using Eq. 4.29 the radiative transfer equation can be rewritten

$$\iota \frac{dI_v}{d\Upsilon_v} = I_v - B_v \tag{4.31}$$

The radiative transfer Eq. 4.31 can be solved formally by using the variable coefficient method:

$$I_v = C(\tau_v)\exp(\tau_v/\iota)$$

Substitution of the above relation into Eq. 4.31 and solving for the constant $C(\tau_v)$ yields

$$C(\tau_v) = C(\tau_{v1}, \tau_{v2}) - \int B_v \exp(-\tau_v/\iota)\, d\frac{\tau_v}{\iota} \tag{4.32}$$

Thus the general expression for the specific intensity is

$$I_v = C(\tau_{v,1}, \tau_{v,2})\exp(\tau_v/\iota) - \exp(\tau_v/\iota)\int_{\tau_{v,1}}^{\tau_{v,2}} B_v \exp(-\tau_v/\iota)\frac{d\tau}{\iota} \tag{4.33}$$

Splitting the integration into two parts and evaluating boundary conditions yields

$$I_v = I_v^+ + I_v^-$$

where

$$\begin{aligned} I_v^+ = & -\int_{\tau_{v,w}}^{\tau_v} B_v \exp(-(\hat{\tau}_v - \tau_v)/\iota)\frac{d\hat{\tau}_v}{\iota} \\ & + I_v^+(\tau_{v,w})\exp(-(\tau_{v,w} - \tau_v)/\iota) \end{aligned} \tag{4.34}$$

$$\begin{aligned} I_v^- = & -\int_{\tau_v}^{\tau_{v,s}} B_v \exp(-(\hat{\tau}_v - \tau_v)/\iota)\frac{d\hat{\tau}_v}{\iota} \\ & + I_v^-(\tau_{v,s})\exp(-(\tau_{v,s} - \tau_v)/\iota) \end{aligned} \tag{4.35}$$

The above equations describe the radiation field in terms of temperature through Planck's function B_v for a nonscattering gas. The quantities $I_v^-(\tau_{v,s})$ and $I_v^+(\tau_{v,w})$ are boundary conditions and the exponentials represent attenuation over optical path length.

Using Eqs. 4.34 and 4.35 for the specific intensity, the radiative flux and flux divergence may be evaluated. Recalling from Section I the radiative flux term can be expressed as

$$\bar{q}_R(\bar{r}) = \int_0^\infty \int_0^{4\pi} I_\nu \bar{\Omega}_1 \, d\Omega \, d\nu \tag{4.36}$$

For the geometry under consideration the unit vector $\bar{\Omega}_1$ can be repaced by the direction cosine ι . From Fig. 4.3 we note that

$$d\Omega = \sin\Psi \, d\Psi \, d\Theta$$

and

$$\iota = \cos\Psi$$

Therefore

$$d\Omega = -d\iota \, d\Theta \tag{4.37}$$

Substitution of Eq. 4.37 into 4.36 yields

$$q_{R,y} = -\int_0^\infty \int_{-1}^{1} \int_0^{2\pi} I_\nu \, d\Theta \, \iota \, d\iota \, d\nu \tag{4.38}$$

Simplifying for the one dimensional case by integration in $d\Theta$ yields

$$q_{R,y} = -2\pi \int_0^\infty \int_{-1}^{1} I_\nu \, \iota \, d\iota \, d\nu \tag{4.39}$$

It is convenient to split the integration in Eq. 4.39.

$$q^{+}_{R,v} = 2\pi \int_{0}^{1} I^{+}_{v}\, \iota\, d\iota$$

$$q^{-}_{R,v} = 2\pi \int_{0}^{-1} I_{v}\, \iota\, d\iota \tag{4.40}$$

Thus the manocromatic heat flux is the

$$q_{R,v} = q^{+}_{R,v} - q^{-}_{R,v}$$

Substituting Eq. 4.34 and 4.35 into 4.40 yields

$$q^{+}_{R,v} = -2\pi \int_{\Upsilon_{v,w}}^{\Upsilon_{v}} B_{v} E_{2}(\hat{\Upsilon}_{v} - \Upsilon_{v})\, d\hat{\Upsilon}_{v} + 2q^{+}_{R,v}(\Upsilon_{v,w})\, E_{3}(\Upsilon_{v,w} - \Upsilon_{v}) \tag{4.41}$$

$$q^{-}_{R,v} = -2\pi \int_{\Upsilon_{v}}^{\Upsilon_{v,s}} B_{v} E_{2}(\Upsilon_{v} - \hat{\Upsilon}_{v})\, d\hat{\Upsilon}_{v} + 2q^{-}_{R,v}(\Upsilon_{v,s})\, E_{3}(\Upsilon_{v,s} - \Upsilon_{v}) \tag{4.42}$$

where the direction cosine, ι , dependence is expressed in terms of the exponential integral function of order n.

$$E_{n} = \int_{0}^{1} \iota^{n-2} \exp(-t/\iota)\, dt \tag{4.43}$$

Let us examine the radiative flux equation given in Section III

$$q_{R,y} = \int_{y(\bar{r}_{o})}^{y(\bar{r}_{1})} \int_{0}^{\infty} \alpha_{v} \left(4\pi B_{v} - \int_{0}^{4\pi} I_{v}(\bar{r})\, d\Omega \right) d\nu\, dy \tag{4.44}$$

Differentiating with respect to y we obtain

$$\frac{\partial q_{R,y}}{\partial y} = \int_{0}^{\infty} \alpha_{v} \left(4\pi B_{v} - \int_{0}^{4\pi} I_{v}\, d\Omega \right) d\nu \tag{4.45}$$

which is the radiative flux divergence in the y direction. In our energy equation, Eq. 4.24, we have the term

$$\Lambda_{R,y} = \frac{\partial}{\partial y}(\tilde{\kappa}\, r^A q_{R,y})$$

Due to the one-dimensional planar slab approximation this term will be represented by

$$\frac{\partial}{\partial y}(\tilde{\kappa}\, r^A q_{R,y}) \cong \tilde{\kappa}\, r^A \frac{\partial q_{R,y}}{\partial y} + \cancelto{\approx 0}{q_{R,y}\frac{\partial \tilde{\kappa}\, r^A}{\partial y}} \tag{4.46}$$

As a result of this approximation, an evaluation of Eq. 4.45 is sufficient to describe the radiative transfer influence in the energy equation.

In order to evaluate Eq. 4.45, the intensity at a fixed point y and in a direction defined by Θ and ι is integrated over all solid angles. Substituting for the solid angle, the integration for a one-dimensional plane slab can be readily carried out.

$$\frac{\partial q_{R,y}}{\partial y} = \int_0^\infty \alpha_\nu \left(2\pi \int_{-1}^{1} I_\nu \, d\iota - 4\pi B_\nu\right) d\nu \tag{4.47}$$

where the inner integral is

$$\begin{aligned}
\int_1^{-1} I_\nu \, d\iota = & -\int_1^0 \left[\int_{\Upsilon_{\nu,w}}^{\Upsilon_\nu} B_\nu \exp\left(\frac{-(\hat{\Upsilon}_\nu - \Upsilon_\nu)}{\iota}\right) \frac{d\hat{\Upsilon}}{\iota} \right] d\iota \\
& + \int_1^0 I^+(\Upsilon_{\nu,w}) \exp\left(\frac{-(\Upsilon_{\nu,w} - \Upsilon_\nu)}{\iota}\right) d\iota \\
& - \int_0^1 \left[\int_{\Upsilon_\nu}^{\Upsilon_{\nu,w}} B_\nu \exp\left(\frac{-(\hat{\Upsilon}_\nu - \Upsilon_\nu)}{\iota}\right) \frac{d\hat{\Upsilon}_\nu}{\iota} \right] d\iota \\
& + \int_0^{-1} I_\nu^-(\Upsilon_{\nu,s}) \exp\left(\frac{-(\Upsilon_{\nu,s} - \Upsilon_\nu)}{\iota}\right) d\iota
\end{aligned} \tag{4.48}$$

Eq. 4.48 can be simplified by interchanging the order of integration as substituting the exponential integral function.

$$\int_{1}^{-1} I_v \, d\iota = \int_{\Upsilon_{v,w}}^{\Upsilon_v} B_v E_1(\hat{\Upsilon}_v - \Upsilon_v) \, d\hat{\Upsilon}_v - I_v^+(\Upsilon_{v,w}) E_2(\Upsilon_{v,w} - \Upsilon_v) + \int_{\Upsilon_v}^{\Upsilon_{v,w}} B_v E_1(\hat{\Upsilon}_v - \Upsilon_v) \, d\hat{\Upsilon} - I_v^-(\Upsilon_{v,s}) E_2(\Upsilon_v - \Upsilon_{v,s}) \tag{4.49}$$

Substituting Eq. 4.49 into 4.47 provides an expression for the radiative flux divergence in a one-dimensional slab.

$$\frac{-\partial q_R}{\partial y} = \int_0^\infty 2\pi a_v \left[\int_{\Upsilon_{v,w}}^{\Upsilon_v} B_v E_1(\Upsilon_v - \hat{\Upsilon}_v) \, d\hat{\Upsilon}_v + I_v^+(\Upsilon_{v,w}) E_2(\Upsilon_v - \Upsilon_{v,w}) + \int_{\Upsilon_v}^{\Upsilon_{v,w}} B_v E_1(\hat{\Upsilon}_v - \Upsilon_v) \, d\hat{\Upsilon} + I_v^-(\Upsilon_{v,s}) E_2(\Upsilon_{v,s} - \Upsilon_v) - 2B_v \right] d\nu \tag{4.50}$$

where the exponential integral function E_n has the following characteristics.

$$E_n(t) = E_n(-t) \quad \text{for } n = 1, 3, 5, 7, \cdots$$
$$E_n(t) = -E_n(-t) \quad \text{for } n = 2, 4, 6, 8, \cdots \tag{4.51}$$

Eq. 4.50 is valid for arbitrary boundary conditions with the exception that only one boundary reflection of a photon packet is allowed. In practice, for a shock layer solution, the subscript "w" is interpreted as conditions at the wall or body and "s" as conditions at the shock. Under this interpretation $I^{-}(\tau_{\nu,s})=0$ barring precursor radiation and the optical depth at the wall $\tau_{\nu,w}=0$. Further, for the case of a perfect absorbing wall $I^{+}(0)=0$. These simplifications are the usual ones made in describing radiation transport in a radiating shock layer. Making these simplifications reduces Eq. 4.50 to Eq. B.31 of Ref. 4.10.

The one-dimensional planar slab approximations which result in Eq. 4.50 have important ramifications to our shock layer problem. Radiation calculations can be made using Eq. 4.50 at each ξ location independent of other ξ locations. This makes the thin shock layer equations a set of parabolic integro-differential equations which can be solved using marching schemes which are used for classical boundary layer equations.

An observation concerning the planar slab approximation is in order at this point. This approximation eliminates all curvature effects from the radiation calculation. A more appropriate approximation for most axisymmetrically blunted vehicles would be a concentric sphere approximation for the boundaries of the shock layer as proposed by Viskanta Ref. 4.12. For a two-dimensional body the corresponding approximation is quite obviously concentric cylinder boundaries. However, as pointed out by Viskanta Ref. 4.12 comparatively little attention has been given to radiative transfer in curvilinear systems. The paper by Viskanta analyzed the steady state radiative transfer between two concentric, gray, opaque spheres separated by a gray absorbing and emitting medium which generated heat uniformily. He concluded, for constant absorption coefficients, that curvature effects were evident for concentric sphere radii ratios as high as .99. This corresponds approximately to a shock standoff distance of $\delta/R \approx .01$. Nominal hypersonic standoff distances are $.04 \leq \delta/R \leq .10$. From Viskanta's work we are led to expect that curvature effects may be significant for both radiative flux and flux divergence in a typical shock layer. The actual magnitude of these effects are difficult to assess because of the constant absorption coefficient assumption and differences in boundary conditions for the problems under consideration. Thus an accurate assessment of curvature effects on shock layer radiative transport, to the authors' knowledge, is absent today. It is felt that using a concentric sphere model is analogus to including both first and second order effects whereas the infinite parallel plate model includes only first order effects. However, for the present we will use the infinite parallel plate model in our development.

As a result of the order of magnitude analysis, the bulk viscosity assumption, and the radiative transfer model the thin shock layer equations may be written in the following dimensional form:

Global continuity

$$\frac{\partial}{\partial x}(r^A \rho u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho v) = 0 \tag{4.52}$$

Species continuity

$$\frac{\partial}{\partial x}(r^A \rho C_i u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho C_i v) = -\frac{\partial}{\partial y}(\tilde{\kappa} r^A J_{i,y}) + \kappa r^A \omega_i \tag{4.53}$$

X - Momentum

$$\rho r^A u \frac{\partial u}{\partial x} + \rho \tilde{\kappa} r^A v \frac{\partial u}{\partial y} + \rho \kappa r^A u v = -r^A \frac{\partial P}{\partial x} + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \mu \frac{\partial u}{\partial y}) - \kappa u \frac{\partial r^A \mu}{\partial y} \tag{4.54}$$

y - Momentum ($O[\bar{\rho}]$ and larger terms)

$$\begin{aligned} \rho r^A u \frac{\partial v}{\partial x} + \rho \tilde{\kappa} r^A v \frac{\partial v}{\partial y} - \rho \kappa r^A u^2 &= -\tilde{\kappa} r^A \frac{\partial P}{\partial y} \\ &+ \frac{\partial}{\partial x}(r^A \mu \frac{\partial u}{\partial y}) - \frac{2}{3}\frac{\partial}{\partial y}(\mu \frac{\partial r^A u}{\partial x}) + \frac{4}{3}\frac{\partial}{\partial y}(\kappa r^A \mu \frac{\partial v}{\partial y}) \\ &- \frac{2}{3}\frac{\partial}{\partial y}(\kappa r^A \mu v + \tilde{\kappa} \mu v \frac{\partial r^A}{\partial y}) \end{aligned} \tag{4.55}$$

$\underline{y}$ — Momentum $(O[\bar{\rho}^2]$ and larger terms)

$$\rho r^A u \frac{\partial v}{\partial x} + \rho r^A \tilde{\kappa} v \frac{\partial v}{\partial y} - \rho \kappa r^A u^2 = -\tilde{\kappa} r^A \frac{\partial P}{\partial y}$$
$$+ \frac{\partial}{\partial x}(r^A \mu \frac{\partial u}{\partial y}) - \frac{2}{3}\frac{\partial}{\partial y}(\mu \frac{\partial r^A u}{\partial x}) + \frac{4}{3}\frac{\partial}{\partial y}(\tilde{\kappa} r^A \mu \frac{\partial v}{\partial y})$$
$$- \frac{2}{3}\frac{\partial}{\partial y}(\kappa r^A \mu v) - \frac{\partial}{\partial x}(r^A \kappa \mu u) + \frac{2}{3}\mu \frac{\kappa}{\tilde{\kappa}}\frac{\partial}{\partial x}(r^A u) \tag{4.56}$$
$$+ \frac{2}{3}\mu \frac{\kappa}{\tilde{\kappa}}\frac{\partial}{\partial y}(\tilde{\kappa} r^A v) - 2\mu \frac{\kappa}{\tilde{\kappa}} r^A \frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial}{\partial y}(\tilde{\kappa}\mu v \frac{\partial r^A}{\partial y})$$

Energy

$$r^A \rho u \frac{\partial H}{\partial x} + \tilde{\kappa} r^A \rho v \frac{\partial H}{\partial y} = \frac{-\partial}{\partial y}\Big[\tilde{\kappa} r^A \Big\{-k \frac{\partial T}{\partial y} + \sum_i h_i J_{i,y}$$
$$- \frac{P}{N^2}\sum_i \sum_{j\neq i} \frac{N_i}{M_i}\frac{D_i^T}{\mathbb{D}_{ij}}(\frac{J_{j,y}}{Y_j M_j} - \frac{J_{i,y}}{Y_i M_i})\Big\}\Big] - \tilde{\kappa} r^A \frac{\partial q_{R,y}}{\partial y} \tag{4.57}$$
$$\frac{\partial}{\partial y}(\tilde{\kappa} r^A \mu u \frac{\partial u}{\partial y}) - \frac{\partial}{\partial y}(\kappa r^A \mu u^2)$$

Let us now examine the simplifications which are needed to obtain the classical boundary layer equations from the thin shock layer equations stated above. First, let us drop all terms of order $\bar{\rho}$ or smaller. The resulting equations are:

Global continuity

$$\frac{\partial}{\partial x}(\rho r^A u) + \frac{\partial}{\partial y}(\rho \tilde{\kappa} r^A v) = 0 \tag{4.52}$$

Species continuity

$$\frac{\partial}{\partial x}(r^A \rho C_i u) + \frac{\partial}{\partial y}(\tilde{\kappa} r^A \rho C_i v) = -\frac{\partial}{\partial y}(\tilde{\kappa} r^A J_{i,y}) + \tilde{\kappa} r^A \omega_i \tag{4.53}$$

X - Momentum (first order)

$$\rho r^A u \frac{\partial u}{\partial x} + \rho \tilde{\kappa} r^A v \frac{\partial u}{\partial y} = -r^A \frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[\tilde{\kappa} r^A \mu \frac{\partial u}{\partial y}\right] \quad (4.58)$$

y - Momentum (first order)

$$\rho \kappa u^2 = \tilde{\kappa} \frac{\partial P}{\partial y} \quad (4.59)$$

Energy (first order)

$$r^A \rho u \frac{\partial H}{\partial x} + \tilde{\kappa} r^A \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\tilde{\kappa} r^A k \frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \left\{\sum_i h_i J_{i,y} - \frac{P}{N^2}\sum_i \sum_{j\neq i} \frac{N_i}{M_i}\frac{D_i^T}{\mathbb{D}_{i,j}}\left(\frac{J_{j,y}}{Y_j M_j} - \frac{J_{i,y}}{Y_i M_i}\right)\right\}\right) - \tilde{\kappa} r^A \frac{\partial q_{R,y}}{\partial y} + \frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \mu u \frac{\partial u}{\partial y}\right) \quad (4.60)$$

Additional simplifications can be made by neglecting the boundary layer thickness in comparison to the local body radius. This implies

$$\kappa \to 0, \quad \tilde{\kappa} \to 1, \quad \text{and} \quad r^A \to r_w^A \quad (4.61)$$

Using these limits, Eqs. 4.52, 4.53, 4.58→ 4.60 can be written:

Global continuity (B.L.)

$$\frac{\partial}{\partial x}(r_w^A \rho u) + r_w^A \frac{\partial}{\partial y}(\rho v) = 0 \quad (4.62)$$

Species continuity (B.L.)

$$\frac{1}{r_w^A}\frac{\partial}{\partial x}(r_w^A \rho C_i u) + \frac{\partial}{\partial y}(\rho C_i v) = -\frac{\partial}{\partial y}(J_{i,y}) + \omega_i \quad (4.63)$$

X - Momentum (B.L.)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad (4.64)$$

y - Momentum (B.L.)

$$0 = \frac{\partial P}{\partial y} \quad (4.65)$$

Energy (B.L.)

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}(\bar{\kappa} k \frac{\partial T}{\partial y}) - \frac{\partial}{\partial y}\Big\{\sum_i h_i J_{i,y} - \frac{P}{N^2}\sum_i \sum_{i \neq j} \frac{N_i}{M_i}\frac{D_i^T}{\mathbb{D}_{ij}}\Big(\frac{J_{j,y}}{Y_j M_j} - \frac{J_{i,y}}{Y_i M_i}\Big)\Big\} - \frac{\partial q_{R,y}}{\partial y} + \frac{\partial}{\partial y}(\mu u \frac{\partial u}{\partial y}) \qquad (4.66)$$

Equations 4.62 through 4.66 are essentially the same as the boundary layer (B.L.) equations which are given by Fay and Riddell Ref. 4.13, Dorance Ref. 4.14 and others. The boundary layer equations for a flat plate are obtained by simply noting that r_w^A is not a function of X. We can conclude from the foregoing simplifications of the thin shock layer equations that the classical Prandtl type boundary equations contain only first order terms which exhibit no normal component curvature effects.

Having stated the thin shock layer and boundary layer equations, the appropriate boundary conditions for the two sets of equations can now be discussed. Figure 4.4 presents a sketch of the various regions and boundaries of particular interest in the thin shock layer problem. We note that in addition to the shock layer region the char layer and decomposition zone are important in our problem. These regions are important because the momentum, energy and mass transfer in the char and decomposition regions are intimately coupled to the transfer in the shock layer. Theoretically we could consider all the processes which take place between the shock wave and the virgin plastic of the body and attempt to solve the governing equations for this boundary valued problem. However, it is more practical to divide the solution of this general problem into a shock layer and material response problem and iterate on the boundary conditions at the material surface. Therefore, it is important to realize what information is available from the material response solution which could be used for boundary conditions of the thin shock layer equations. This is accomplished by using surface balances. With this perspective of the general problem in mind, the nature of the thin shock layer equations and boundary conditions will be discussed.

As noted previously, the thin shock equations are a set of parabolic integro-differential equations with initial values given along $X = 0$, the stagnation line. Because the shock wave location is not known before hand, the blunt body problem is mathematically referred to as a free boundary problem. Given initial conditions along the stagnation line and boundary conditions along the body, the thin shock layer equations can theoretically be solved with a simultaneous development of the shock geometry and corresponding shock boundary conditions. The shock geometry (see Fig. 3.1) can be obtained by carrying out the following integration.

$$\delta^* = \int_0^{x^*} (1 + \kappa^* \delta^*) \tan \epsilon \, dx^* \tag{4.67}$$

In practice another technique has been used to determine the shock geometry Ref. 4.3, 4.10 and others. The shock geometry is assumed and specified in terms of $d\epsilon/dx$. Iterations are made around the body until the input and output shock geometry coincide.

If the shock geometry is known the Rankine-Hugoniot equations can be used to obtain some of the shock boundary conditions. The development of these equations in curvilinear coordinates follows directly from Ref. 4.10. The dimensional Rankine-Hugoniot equations written in rectangular coordinates are:

Continuity

$$\rho_\infty^* V_{\infty,n}^* = \rho_s^* V_{s,n}^* \tag{4.68}$$

Momentum

(normal) $$\rho_\infty^* V_{\infty,n}^{*2} + P_\infty^* = \rho_s^* V_{s,n}^{*2} + P_s^* \tag{4.69}$$

(tangential) $$V_{\infty,\tau}^* = V_{s,\tau}^* \tag{4.70}$$

Energy

$$\frac{1}{2} V_{\infty,n}^{*2} + h_\infty^* = \frac{1}{2} V_{s,n}^{*2} + h_s^* \tag{4.71}$$

Using Fig. 4.5 the above equations can be written in body oriented coordinates. From geometry we have

$$V_s^* = V_{s,\tau}^* \sin\epsilon - V_{s,n}^* \cos\epsilon \tag{4.72}$$

$$u_s^* = V_{s,\tau}^* \cos\epsilon - V_{s,n}^* \sin\epsilon \tag{4.73}$$

where

$$V_{\infty,n}^* = U_\infty^* \cos\phi$$

$$V_{s,n}^* = \bar{\rho}\, U_\infty^* \cos\phi$$

$$V_{s,\tau}^* = V_{\infty,\tau}^* = U_\infty^* \sin\phi$$

Substituting for $V_{\infty,n}^*$, $V_{s,n}^*$ and $V_{\infty,\tau}^*$ Eqs. 4.72 and 4.73 yield

$$V_s^* = U_\infty^* \sin\phi \sin\epsilon - \bar{\rho}\, U_\infty^* \cos\phi \cos\epsilon \tag{4.74}$$

$$u_s^* = U_\infty^* \sin\phi \cos\epsilon - \bar{\rho}\, U_\infty^* \cos\phi \sin\epsilon \tag{4.75}$$

The pressure behind the shock can be obtained by using the normal momentum equation and substituting for $V_{\infty,n}^*$ and $V_{s,n}^*$.

$$\rho_\infty^* (U_\infty^* \cos\phi)^2 + P_\infty^* = \rho_s^* (\bar{\rho}\, U_\infty^* \cos\phi)^2 + P_s^* \tag{4.76}$$

By substituting for $V_{\infty,n}^*$ and $V_{s,n}^*$ the energy equation can be written

$$h_s^* = \frac{1}{2} U_\infty^* (1 - \bar{\rho}^2) \cos^2\phi + h_\infty^* \tag{4.77}$$

It can be shown that Eq. 4.77 is a simplified form of

$$h_s^* = \frac{U_\infty^{*2}}{2} - \frac{1}{2}(u_s^{*2} + V_s^{*2}) + h_\infty^* \tag{4.78}$$

Nondimensionalizing Eqs. 4.74 through 4.78 and dropping P_∞ and h_∞ which are order $(\bar{\rho}^2)$ yields the following shock boundary conditions.

$$v_s = \sin\phi \sin\epsilon - \bar{\rho}\cos\phi\cos\epsilon \tag{4.79}$$

$$u_s = \sin\phi\cos\epsilon + \bar{\rho}\cos\phi\sin\epsilon \tag{4.80}$$

$$P_s = (1-\bar{\rho})\cos^2\phi \tag{4.81}$$

$$h_s = (1-\bar{\rho}^2)\cos^2\phi \tag{4.82}$$

or

$$h_s = 1 - (u_s^2 + v_s^2) \tag{4.83}$$

It is important to realize that the Rankine-Hugoniot relations are valid only if strong precursor radiation effects do not become important. The shock conditions can be more adequately described for the strong precursor radiation problem with modified Rankine-Hugoniot relations presented by Zeldovich and Raezer Ref. 4.15. This restriction in effect provides an upper Mach number limit on the boundary conditions of the present analysis. However, significant precursor radiation effects are not experienced in air below flight velocities of approximately 60,000 to 65,000 ft./sec. as demonstrated by Lasher and Wilson Ref. 4.16. Therefore, the Rankine-Hugoniot relations provide satisfactory boundary conditions for the outer edge of the thin shock layer equations for many problems of current interest in atmospheric entry. Let us now write the shock boundary conditions at $y = \delta$.

$$\begin{aligned} u &= u_s \\ v &= v_s \\ P &= P_s \\ h &= h_s \text{ or } g_s = 1 \\ C_i &= C_{is}(P_s, h_s) \quad \text{(Assuming chemical equilibrium)} \\ I_\nu^-(\Upsilon_{\nu,s}) &= 0 \end{aligned} \tag{4.84}$$

The Rankine-Hugoniot equations provide expressions for u_s, v_s, P_s, and h_s. The equation of state and freestream mass fraction provides the additional information needed to determine the post shock mass fractions assuming chemical equilibrium. The specific intensity coming through the shock towards the body is specified as zero. We note that in total four boundary conditions are needed for the energy equation because of its integro-differential nature. Thus two boundary conditions, enthalpy and specific intensity, have been specified at the shock.

The corresponding body surface boundary conditions can be written for $y = 0$:

$$
\begin{aligned}
u &= 0 \\
v &= v_w \\
P &= P_w \\
h &= h_w \quad \text{or} \quad g = g_w \\
C_i &= C_{i,w} \\
&\begin{cases} I_\nu^+(\tau_{\nu,w}) = B_\nu \\ \tau_{\nu,w} = 0 \end{cases}
\end{aligned} \tag{4.85}
$$

The boundary conditions specified in 4.84 and 4.85 are sufficient to solve the thin shock layer equations. However, substitution of equivalent boundary conditions for some surface conditions is found to be practical. For example the normal velocity at the wall is usually replaced by $(\rho v)_w$. Of greater practical importance is the wall boundary condition on pressure. This pressure is not known <u>a priori</u>. An equivalent boundary condition is then needed. There are at least two suitable boundary conditions which might be used in lieu of pressure. These are the normal pressure gradient at the shock or the normal pressure gradient at the body. The normal pressure gradient at the shock could be specified by evaluating the inviscid y – momentum equation at the shock using the Rankine-Hugoniot equations. The normal pressure gradient at the body could be set equal zero from boundary layer theory. Each of these conditions would involve some degree of approximation. To evaluate the pressure gradient at the shock an approximate form of the continuity equation is needed. Correspondingly the zero normal pressure gradient assumption at the wall neglects the wall velocity head at the body which would push the true stagnation pressure point off the body. However, each of the approximations appears to be consistent with the order of magnitude analysis. An additional complicating factor arises when one observes what boundary condition is needed in the material response analysis. The pressure at the outer wall is usually specified as a boundary condition Ref. 4.17. Thus by specifying a slope in the flow field analysis the pressure at the surface will be calculated whereas in the material response analysis it is specified. Ideally one would like to know and specify the pressure boundary condition for both problems. This would eliminate iterating on this variable between the two solutions.

In addition to the boundary conditions discussed above, additional boundary conditions have been used when integral techniques are used to solve the governing equations. The number of additional boundary conditions used is dictated in this case by the order of polynomial selected to represent the velocity or enthalpy profiles. Some typical boundary conditions which have been used for this purpose Ref. 4.10, 4.18 are:

<u>momentum</u>:

<u>variable obtained</u>	<u>by</u>
$\int_0^\delta u\,dy$	Total mass balance
$\left(\dfrac{\partial u}{\partial y}\right)_s$	X – momentum evaluated behind the shock or vorticity evaluated behind the shock
$\left(\dfrac{\partial^2 u}{\partial y^2}\right)_s = 0$	Assuming no viscous dissipation behind the shock
$\left(\dfrac{\partial^2 u}{\partial y^2}\right)_w$	X – momentum evaluated at the body surface

energy:

variable obtained	by
$\left(\frac{\partial g}{\partial y}\right)_s$	Energy Eq. evaluated behind the shock
$\left(\frac{\partial^2 g}{\partial y^2}\right)_s$	Differentiation of the energy Eq. evaluated behind the shock assuming a concentric shock
$\left(\frac{\partial^4 g}{\partial y^4}\right)_s = 0$	Third differerential of the energy Eq. evaluated behind the shock assuming a concentric shock and the radiative flux divergence is proportional to the total enthalpy to a constant power.
$\left(\frac{\partial^2 g}{\partial y^2}\right)_w$	The energy equation evaluated at the wall neglecting curvature effects

Typical boundary conditions for the boundary layer equations can now be discussed in terms of the ones used for the shock layer equations. Outer boundary conditions along a line between the shock and the body known as the B.L. edge are used rather than the Rankine-Hugoniot equations. These edge conditions are usually obtained using some inviscid layer analysis which is bounded by a shock and a streamline. The method of characteristics is used for the supersonic portion of the flow and typically a Belostserkovskii technique is used for the near stagnation subsonic flow. These methods provide the following B.L. edge conditions

$$
\begin{aligned}
u &= u_e \\
v &= v_e = 0 \\
P &= P_e \\
h &= h_e \text{ or } g = g_e = 1 \\
C_i &= C_{ie}(P_e, h_e) \quad (\text{assuming chemical equilibrium}) \\
& I_\nu^-(\Upsilon_{\nu,e}) \quad (\text{usually not used})
\end{aligned}
\tag{4.86}
$$

The B.L. wall boundary conditions can be written:

$$
\begin{aligned}
&u = u_w = 0 \\
&\rho v = (\rho v)_w \\
&P = P_w = P_e \\
&h = h_w \text{ or } g = g_w \\
&C_i = C_{iw} \\
&I_w^+(\Upsilon_{v_i w}) = B_v \quad \text{(usually not used)}
\end{aligned}
\tag{4.87}
$$

If the spectral intensity is eliminated from the previous two sets of boundary conditions they are equivalent to those presented in Chapter 1 of Ref 4.14. One can observe that the problem of iterating on pressure between a boundary layer solution and material response solution is eliminated. However, this problem is left unresolved in that the correct edge pressure can be obtained accurately only through an iteration procedure between the inviscid flow analysis and the boundary layer analysis. It is also significant to point out that, although usually not attempted, it is computationally rather difficult to handle B.L. and inviscid flows which are coupled by radiative transfer. In addition to the geometrical integration problems the boundary condition on specific intensity or radiative flux is not a single value but a frequency dependent function which must be matched at the B.L. edge.

To this point we have not discussed how initial values for the T.S.L. equations may be determined. This problem is of near equal importance to the entire shock layer problem and will be discussed in the remainder of this section. To obtain initial values for the shock layer solution, a reduced set of the T.S.L. equations must be solved at $X = 0$ along y , the stagnation line. The solution of this set of equations is of major importance because (1) the highest heating rates and pressures on a body are experienced at the stagnation point (2) any distributional shock layer solution because of its parabolic nature is only as valid as its initial values and (3) the T.S.L. equations along characteristics $X =$ constant reduce to ordinary differential equations like at the stagnation line. Thus by developing a stagnation line solution an important problem is solved and a great deal of the work is completed which is applicable to the total shock layer problem. This is primarily why the stagnation line problem has received a great deal of attention in the past decade.

The solution to the stagnation line (S.L.) problem by direct methods has been approached in two ways. The work of Ho and Probstein Ref. 4.19 typifies the stagnation region solutions which use expansions of the dependent variables in X to obtain the stagnation and near stagnation line equations. The work of Hoshizaki and Wilson Ref. 4.3 typifies the stagnation line solutions which determine the stagnation line equations by formally taking the limit of the terms in the T.S.L. equations at $X = 0$ using symmetry conditions. The latter method is used in this development.

Let us first examine the global continuity equation in expanded dimensional form.

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\tilde{\kappa}\rho v) + \frac{\rho u}{r}\frac{\partial r}{\partial x} + \tilde{\kappa}\frac{\rho v}{r}\frac{\partial r}{\partial y} = 0 \tag{4.88}$$

As $X \to 0$ the following limit is approached

$$\lim_{X\to 0}\frac{1}{r}\frac{\partial r}{\partial x} = \lim_{X\to 0}\frac{\partial^2 r}{\partial x^2}\Big/\frac{\partial r}{\partial x} = -\left[\frac{\kappa \sin\theta}{\cos\theta}\right]_{x=0} = 0 \tag{4.89}$$

assuming a spherically shaped body at $X = 0$. Also, note that

$$\frac{1}{r}\frac{\partial r}{\partial y} = \left[\frac{\sin\theta}{(1/\kappa + y)\sin\theta}\right] = \frac{\kappa}{1+\kappa y} = \frac{\kappa}{\tilde{\kappa}} \tag{4.90}$$

Using these conditions the global continuity equation can be rewritten.

<u>Global continuity</u> (S.L.)

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\tilde{\kappa}\rho v) + \kappa\rho v = 0 \tag{4.91}$$

The species continuity equation can be rewritten by subtracting the global continuity Eq. 4.52 from the left hand side of Eq. 4.53.

$$r^A\rho u\frac{\partial C_i}{\partial x} + r^A\tilde{\kappa}\rho v\frac{\partial C_i}{\partial y} = -\frac{\partial}{\partial y}(\tilde{\kappa}r^A J_{i,y}) + \tilde{\kappa}r^A\omega_i \tag{4.92}$$

Noting that at $X = 0$, $U = 0$ and using Eq. 4.90 in Eq. 4.92 yields

Species continuity (S.L.)

$$\tilde{\kappa}\rho v \frac{\partial C_i}{\partial y} = -\frac{\partial}{\partial y}(\tilde{\kappa} J_{i,y}) - \kappa J_{i,y} + \tilde{\kappa}\omega_i \tag{4.93}$$

or by noting $\dfrac{\partial \tilde{\kappa}}{\partial y} = \kappa$

$$\rho v \frac{\partial C_i}{\partial y} = -\frac{\partial}{\partial y}(J_{i,y}) - \frac{2\kappa}{\tilde{\kappa}} J_{i,y} + \omega_i \tag{4.94}$$

Now consider the X – momentum Eq. 4.54

$$r^A \rho u \frac{\partial u}{\partial x} + \tilde{\kappa} r^A \rho v \frac{\partial u}{\partial y} + \kappa r^A \rho u v = -r^A \frac{\partial P}{\partial x}$$
$$+ \frac{\partial}{\partial y}\left(\tilde{\kappa} r^A \mu \frac{\partial u}{\partial y}\right) - \kappa u \frac{\partial}{\partial y}(r^A \mu)$$

By evaluating the above equation at $X = 0$, relatively little information is obtained. Along the stagnation line $U = 0$ for all y ; therefore

$$\left(\frac{\partial u}{\partial y}\right)_{X=0} = 0 \tag{4.95}$$

Using this information in Eq. 4.54 yields

$$\left(\frac{\partial P}{\partial x}\right)_{X=0} = 0 \tag{4.96}$$

which agrees identically with the Rankine-Hugoniot equations for a symmetrical shock (i.e. $\phi = 0$ at $X = 0$). The reduction of Eq. 4.54 to 4.96 along the stagnation line yields the expected physical interpretation that no momentum is transfered in the X- direction at the stagnation line. Rather than using this reduced form of the momentum equation the rate of change of momentum in the X- direction is usually used. Therefore let us differentiate the X - momentum equation with respect to X and determine its limiting form along the stagnation line.

$$
\begin{aligned}
& r^A \rho u \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(r^A \rho u)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\tilde{\kappa} r^A \rho v)\frac{\partial u}{\partial y} \\
& + \tilde{\kappa} r^A \rho v \frac{\partial^2 u}{\partial x \partial y} + \kappa r^A \rho v \frac{\partial u}{\partial x} + u \frac{\partial}{\partial x}(\kappa r^A \rho v) = \\
& - r^A \frac{\partial^2 P}{\partial x} - \frac{\partial P}{\partial x}\frac{\partial r^A}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(\tilde{\kappa} r^A \mu)\frac{\partial u}{\partial y} + \tilde{\kappa} r^A \mu \frac{\partial u}{\partial x \partial y}\right) \\
& - \frac{\partial}{\partial x}(\kappa u)\frac{\partial}{\partial y}(r^A \mu) - \kappa u \frac{\partial}{\partial x \partial y}(r^A \mu)
\end{aligned}
\tag{4.97}
$$

After some manipulation and substitution for limit quantities Eq. 4.97 reduces to

$$
\begin{aligned}
& \frac{\partial}{\partial y}\left(\mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right) + \left[2\kappa\mu - \tilde{\kappa}\rho v\right]\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
& - \kappa\left[\rho v + \left[\frac{\kappa}{\tilde{\kappa}} + 1\right]\frac{\partial \mu}{\partial y}\right]\left(\frac{\partial u}{\partial x}\right) - \rho\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial^2 P}{\partial x^2} = 0
\end{aligned}
\tag{4.98}
$$

For substitution into Eq. 4.98 the (S.L.) global continuity Eq. 4.91 may be rewritten.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= -\left[\frac{1}{\rho}\frac{\partial}{\partial y}(\tilde{\kappa}\rho v) + \kappa v\right] \\
&= -\left[\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\kappa v\right]
\end{aligned}
\tag{4.99}
$$

Combining Eqs. 4.98 and 4.99 yields

X – Momentum (S.L.)

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\tilde{\kappa}\rho v) + \kappa v\right)\right) + \left[2\kappa u - \tilde{\kappa}\rho v\right]\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\tilde{\kappa}\rho v)\right.$$

$$\left. + \kappa v\right) - \kappa\left[\rho v + \left(\frac{\kappa}{\tilde{\kappa}} + 1\right)\frac{\partial \mu}{\partial y}\right]\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\tilde{\kappa}\rho v) + \kappa v\right)$$

$$+ \rho\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\tilde{\kappa}\rho v) + \kappa v\right)^2 + \left(\frac{\partial^2 P}{\partial x^2}\right)_{x=0} = 0 \qquad (4.100)$$

This is a third order inhomogenous ordinary differential equation where the rate of change of the pressure gradient in the X – directions is an unspecified function of y.

The y - momentum equation can be evaluated directly by substitution of the stagnation line limit quantities. The S.L. normal momentum equation to order $\bar{\rho}^2$ is

y – Momentum (S.L.)

$$\rho\tilde{\kappa} v \frac{\partial v}{\partial y} = -\tilde{\kappa}\frac{\partial P}{\partial y} - \mu\frac{\partial}{\partial y}\left[\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\kappa v\right]$$

$$+ \frac{2}{3}\frac{\partial}{\partial y}\left[\frac{\mu}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\mu\kappa v\right] + \frac{2}{3}\frac{\kappa}{\tilde{\kappa}}\mu\left[\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\kappa v\right]$$

$$+ \frac{4}{3}\kappa\mu\frac{\partial v}{\partial y} + \frac{4}{3}\tilde{\kappa}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) - \frac{4}{3}\kappa v\frac{\partial \mu}{\partial y} - \frac{4}{3}\frac{\kappa^2}{\tilde{\kappa}}\mu v$$

$$+ \left\{\left[\frac{4}{3}\mu\frac{\kappa}{\tilde{\kappa}} + \mu\kappa\right]\left[\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\kappa v\right] + \frac{4}{3}\mu\frac{\kappa^2}{\tilde{\kappa}}v + \frac{2}{3}\mu\kappa\frac{\partial v}{\partial y}\right\} \qquad (4.101)$$

where the terms in the brackets $\{\ \}$ are the terms of order $\bar{\rho}^2$. By dropping these terms only terms of order $\bar{\rho}$ remain. Since some of the terms of order

$\bar{\rho}$ have been expanded in Eq. 4.101 a few of the terms will combine.

<u>y- Momentum (S.L.)</u>

$$\rho\tilde{\kappa}v\frac{\partial v}{\partial y} = -\tilde{\kappa}\frac{\partial P}{\partial y} - \mu\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\kappa v\right)$$

$$+ \frac{2}{3}\frac{\partial}{\partial y}\left(\frac{\mu}{\rho}\frac{\partial}{\partial y}(\rho v) + 2\mu\kappa v\right) + \mu\kappa\left(\frac{2}{\tilde{\kappa}} + 1\right)\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right.$$

$$\left. + 2\kappa v\right) + 2\kappa\mu\frac{\partial v}{\partial y} + \frac{4}{3}\tilde{\kappa}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) - \frac{4}{3}\kappa v\frac{\partial \mu}{\partial y} \tag{4.102}$$

It is obvious that either with or without the second order terms the y-momentum equation is second order, inhomogenous, ordinary differential equation with variable coefficients. Given a solution to the energy equation (i.e. an enthalpy or temperature profile) in principle the x- and y- momentum equations could be solved for the normal velocity and the normal pressure gradient given the rate of change of the pressure gradient in the x- direction as a function of y.

The energy Eq. 4.57 can be reduced to the S.L. energy equation by inspection.

$$\kappa\rho v\frac{\partial H}{\partial y} = -\left(1 + \frac{\kappa}{\tilde{\kappa}}\right)\frac{\partial}{\partial y}\left[-\tilde{\kappa}k\frac{\partial T}{\partial y} + \tilde{\kappa}\sum_i h_i J_{i,y}\right.$$

$$\left. - \frac{\tilde{\kappa}P}{N^2}\sum\sum\frac{N_i}{M_i}\frac{D_i}{\mathbb{D}_{ij}}\left(\frac{J_{j,y}}{Y_j M_j} - \frac{J_{i,y}}{Y_i M_i}\right)\right] - \tilde{\kappa}\frac{\partial q_{R,y}}{\partial y} \tag{4.103}$$

This is a second order, ordinary integrodifferential equation. It is interesting to note that the S.L. energy equation has no viscous dissipation terms in it.

The S.L. conservation equation, obtained from the T.S.L. equations, are a set of four ordinary differential equations in five unknowns (i.e. ρ, v, H, P and C_i). In addition to the conservation equations, the caloric equation of state is available to provide another independent equation. The global continuity equation was used to eliminate the tangential velocity gradient in the momentum equations and therefore is not needed in a solution of the S.L.equations. It can be used <u>post priori</u> to provide initial conditions for the T.S.L equations. For a S.L.solution the rate of change of the pressure gradient in the tangential direction must be specified as a function of the normal direction. Comment on how this might be specified is reserved until we have considered the reduction of the B.L. equations to S.L. equations.

The first order boundary layer equations can be evaluated at the stagnation line by keeping only first order terms and dropping normal curvature effects in the thin shock layer S.L. equations. The resulting equations are:

<u>Global continuity</u> (B.L., S.L.)

$$\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v) \tag{4.104}$$

<u>Species continuity</u> (B.L., S.L.)

$$\rho v \frac{\partial C_i}{\partial y} = -\frac{\partial}{\partial y}(J_{i,y}) + \omega_i \tag{4.105}$$

<u>X - Momentum</u> (B.L., S.L.)

$$\begin{aligned}\frac{\partial}{\partial y}\left[\mu \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right)\right] - \rho v \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right) \\ + \rho\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right)^2 + \left(\frac{\partial^2 P}{\partial x^2}\right)_{x=0} = 0\end{aligned} \tag{4.106}$$

<u>y - Momentum</u> (B.L., S.L.)

$$\frac{\partial P}{\partial y} = 0 \tag{4.107}$$

Energy (B.L., S.L.)

$$\rho v \frac{\partial H}{\partial y} = -\frac{\partial}{\partial y}\Big[-k\frac{\partial T}{\partial y} + \sum_i h_i J_{i,y} - \frac{P}{N^2}\sum\sum \frac{N_i}{M_i}\frac{D_i^T}{\mathbb{D}_{ij}}\left(\frac{J_{j,y}}{Y_j M_j} - \frac{J_{i,y}}{Y_i M_i}\right)\Big] - \frac{\partial q_{R,y}}{\partial y} \tag{4.108}$$

Since $\frac{\partial P}{\partial y} = 0$ FOR ALL y AT $x = 0$, $\frac{d^2 P}{d x^2}$ is a constant and may be evaluated at any y station. If the B.L. equations are evaluated over the whole shock layer as done at the S.L.by Dirling, Rigdon and Thomas Ref. 4.20 we may use the Rankine-Hugoniot relations to determine this constant.

From Eq. 4.81 the dimensional pressure behind the shock can be expressed as

$$P_s = (1-\bar{\rho})\cos^2\phi \; \rho_\infty U_\infty^2 \tag{4.109}$$

differentiating we get

$$\frac{\partial^2 P_s}{\partial x^2} = -2(1-\bar{\rho})\left(\frac{\partial \phi}{\partial x}\right)^2\left[\cos^2\phi - \sin^2\phi\right]\rho_\infty U_\infty^2 \tag{4.100}$$

at $x = 0$, $\phi = 0$ by symmetry. Therefore

$$\left(\frac{\partial^2 P_s}{\partial x^2}\right)_{x=0} = -2(1-\bar{\rho})\left(\frac{\partial \phi}{\partial x}\right)^2_{x=0} \rho_\infty U_\infty^2 \tag{4.111}$$

In order to get the B.L. momentum equation into a more common form let us express the rate of change of the pressure gradient in terms of the velocity gradient behind the shock. From Eq. 4.75, the dimensional tangential velocity behind the shock is

$$u_s = \left[\sin\phi \,\cos\epsilon + \bar{\rho}\cos\phi\,\sin\epsilon\right] U_\infty$$

from which we can obtain

$$\left(\frac{d u_s}{\partial x}\right)_{x=0} = \left[\frac{\partial \phi}{\partial x} + \bar{\rho}\frac{\partial \epsilon}{\partial x}\right] U_\infty{}_{x=0} \qquad (4.112)$$

The rate of change of the pressure gradient, Eq. 4.111, can be rewritten in terms of the velocity gradient.

$$\left(\frac{\partial^2 P_s}{\partial x^2}\right)_{x=0} = -2\rho_\infty(1-\bar{\rho})\left[\left(\frac{\partial u_s}{\partial x}\right)_{x=0} - U_\infty\bar{\rho}\left(\frac{\partial \epsilon}{\partial x}\right)_{x=0}\right]^2 \qquad (4.113)$$

If the shock is assumed to be concentric to the body at $X=0$ then

$$\left(\frac{\partial \epsilon}{\partial x}\right)_{x=0} = 0 \Rightarrow \left(\frac{\partial \phi}{\partial x}\right)_{x=0} = 1 \qquad (4.114)$$

This gives a Newtonian velocity gradient used in many B.L. analysis. Instead of applying this condition behind the shock most analyses apply this condition at the edge of the B.L. which is at some intermediate station between the shock and body. Using the concentric assumption Eq. 4.106 may be written

<u>$X-$ Momentum</u> (B.L., S.L.)

$$\frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right)\right] - \rho v\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right)$$
$$+ \rho\left(\frac{1}{\rho}\frac{\partial}{\partial y}(\rho v)\right)^2 - 2\rho_\infty(1-\bar{\rho})\left(\frac{\partial u_s}{\partial x}\right)_{x=0} = 0 \qquad (4.115)$$

It has been demonstrated that the thin shock layer and boundary layer equations can be reduced to ordinary differential equations along the S.L. without resorting to similarity transformations. By doing so one important difference in the resulting two sets has become apparent. The stagnation line B.L. equations are completely specified by boundary conditions at the surface and outer edge. However, an unknown function of y remains in the T.S.L. equations which cannot be determined, without approximation, by outer and inner boundary conditions. The undetermined function as stated previously is

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_{x=0} = F(y) \tag{4.116}$$

This function like the rate of change of the shock angle is, by physical interpretation, determined by the flow downstream. The downstream flow is to be calculated by specifying these S.L. conditions such that initial conditions may be determined. The problem is complicated further by the fact that there is no apparent theoretically based means of iterating on this function such that it could be assumed and corrected until some satisfactory convergence is obtained. The derivation of the S.L. boundary layer equations demonstrates that to a first approximation the function $F(y)$ is a constant which can be evaluated at the shock by specifying the shock geometry. For usual boundary layer problems the edge tangential velocity gradient is specified rather than the rate of change of the pressure gradient at the B.L. edge. The velocity gradient has been correlated as a function of flight conditions and body shape for many cases to be used in blunt body B.L. solutions in order to specify this unknown downstream influence a priori.

In shock layer solutions the shock wave has been considered concentric by Refs. 4.3, 4.4, 4.9, 4.10, 4.16, 4.18, 4.19, 4.20 and many others. Furthermore, most of these analyses set the function, Eq. 4.116, equal to a constant. The full extent of influence of these assumptions has not been determined for radiation and ablation coupled flows although some radiative coupled results are presented in Ref. 4.21. This is the point where engineering judgement and or experimental results must be used in order to make the mathematical model useful.

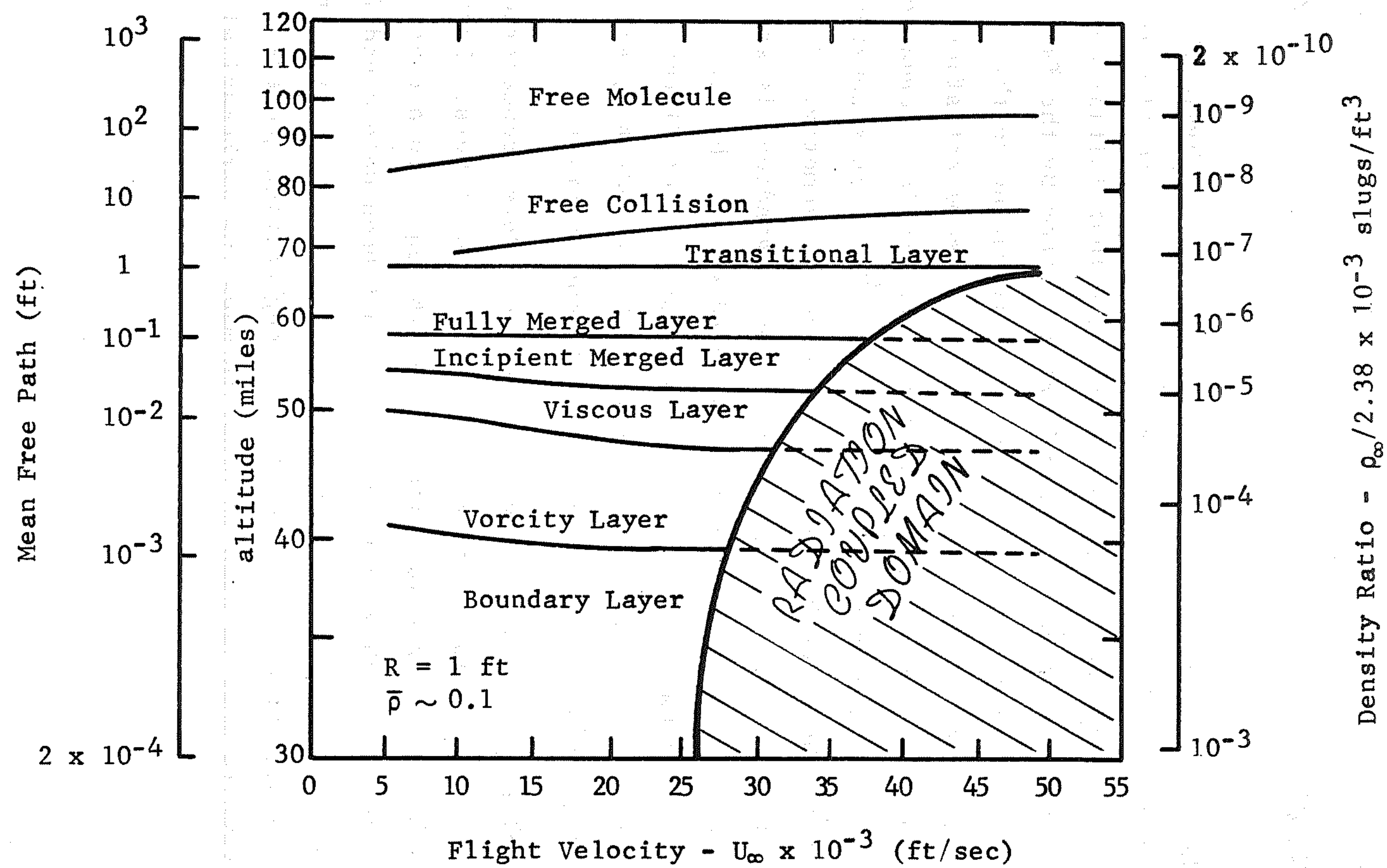


Figure 4.1 Flight Regimes (Based on Ref. 4.1)

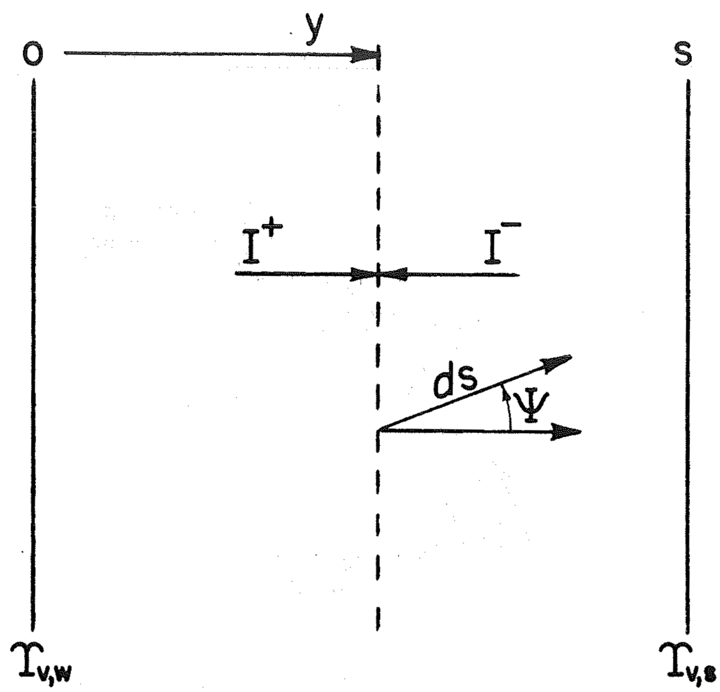


Fig. 4.2 Radiating Slab Nomenclature

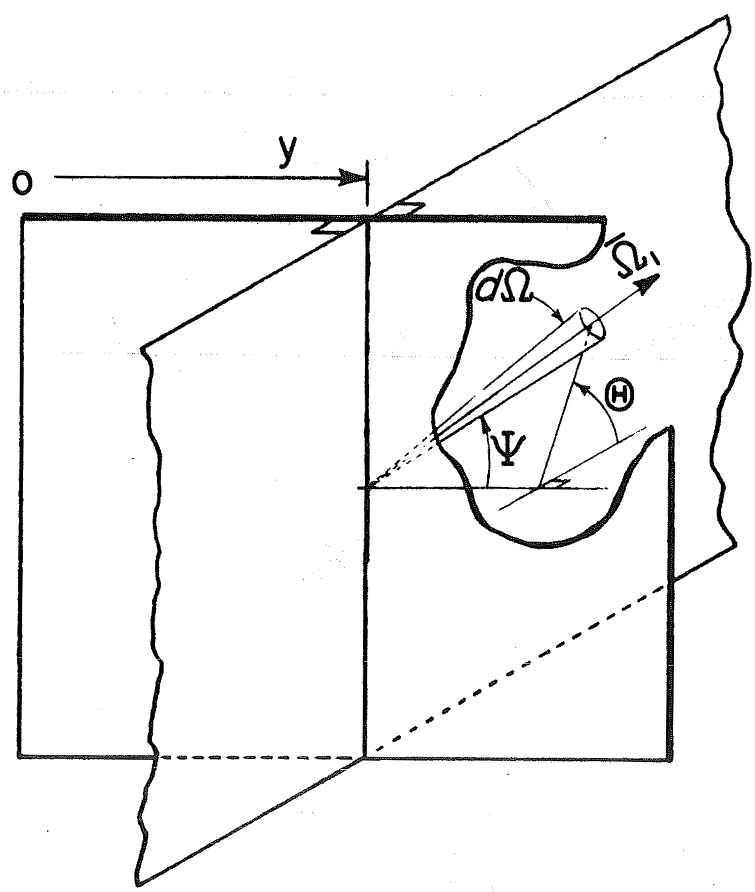
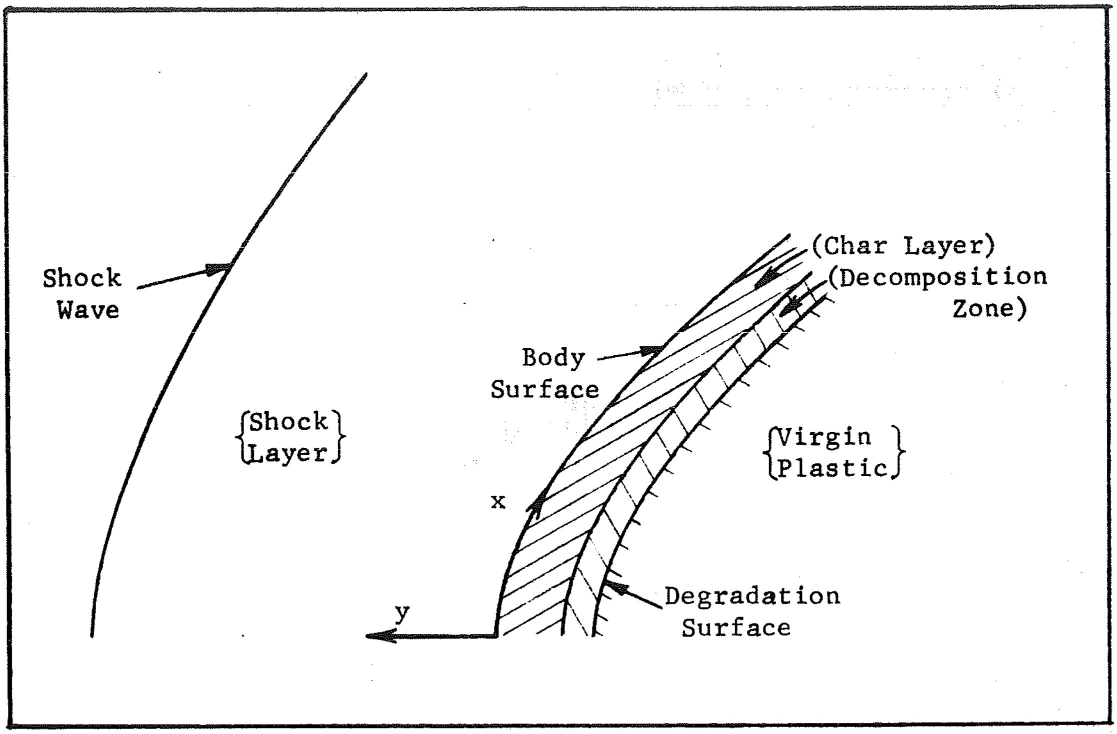
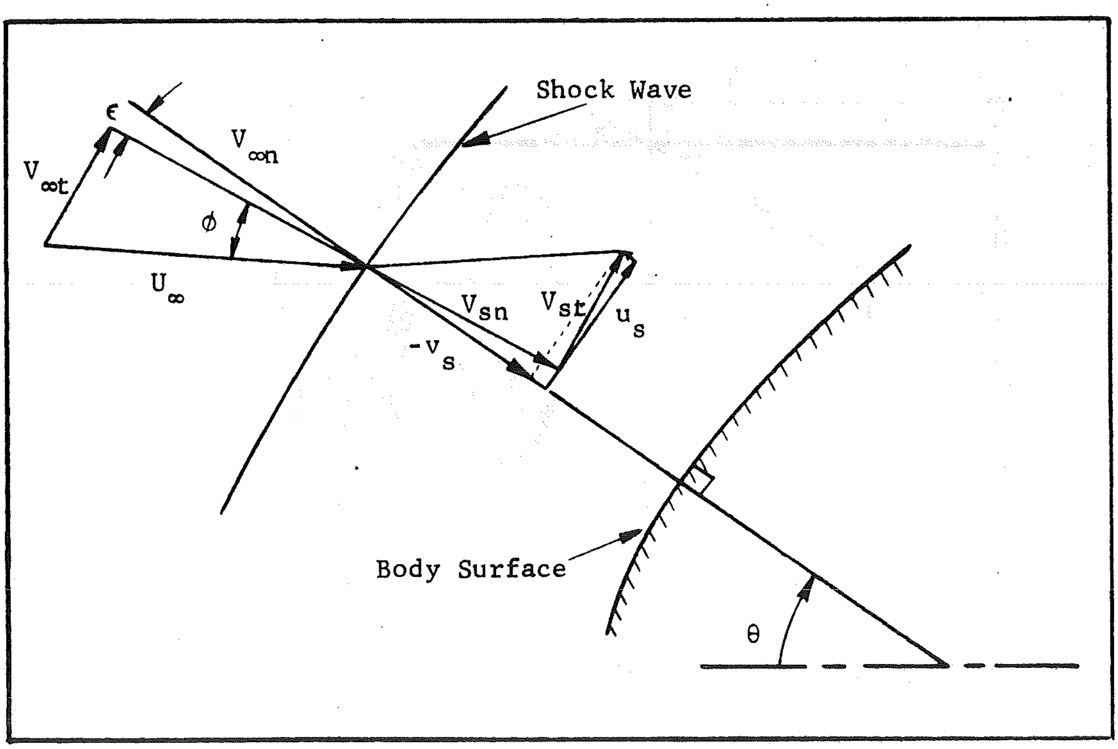


Fig. 4.3 Radiating Slab Geometry



Schematic Representation of the Interaction of Shock Heated Air and a Charring Ablator

Fig. 4.4



Resolution of velocity components in a body oriented coordinate system

Fig. 4.5

Section 4

References

4.1 Probstein, R. F. "Shock Wave and Flow Field Development in Hypersonic Recentry," Am. Rocket Soc. Jour., 31, 184-94 (Feb. 1961).

4.2 Hayes, W. D. and R. F. Probstein. Hypersonic Flow Theory 2nd Ed, Ch. 5, Academic Press, New York (1966).

4.3 Hoshizaki, H. and K. H. Wilson. "Convective and Radiation Heat Transfer During Superorbital Entry," AIAA Jour., 5(1), 25 (Jan. 1967)

4.4 Spradley, L. W. and C. D. Engel, "The Effects of Shock Layer Radiation and Viscous Coupling on the Total Heating Rate to a Reacting Body," NASA CR-61224 (April, 1968).

4.5 Libby, P. A. "The Homogeneous Boundary Layer at an Axisymmetric Stag-Point with Large Rates of Injection," Jour. of Aerospace Sci., p.48 (Jan. 1962).

4.6 Schlichting, H. Boundary Layer Theory 4th Ed., McGraw Hill Col, Inc. New York (1960).

4.7 Laitone, E. V. "On Equations of Motion for a Compressible Viscous Gas", Jour. of Aeronautical Sciences, p. 846, (Sept. 1956).

4.8 Zeldovich, Ya. B. and Yu. P. Raizer. Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, Ch. 2, Academic Press, N. Y. (1966).

4.9 Hoshizaki, H. and L. E. Lasher. "Convective and Radiative Heat Transfer to an Ablating Body," AIAA Thermophysics Specialist Conference, Paper No. 67-327 (April 1967).

4.10 Spradley, L. W. and C. D. Engel. "Formulation of a Method for Predicting Coupled Convective and Radiative Heat Transfer About a Blunt Body," NASA CR-61200 (April 1968).

4.11 Goulard, R. and M. "One-Dimensional Energy Transfer in Radiant Media," J. Heat Mass Transfer, Vol. 1, p. 81 (1960).

4.12 Viskanta, R. "Radiative Transfer Between Two Concentric Gray Spheres Separated by Radiating Gas," in Advanced Heat Transfer, Ed. B. T. Chao, (1969).

4.13 Fay, J. A. and F. R. Riddell, "Theory of Stagnation Point Heat Transfer in Dissociated Air," J. Aero./Space Sci., 25(2), 73-85 (1958)

4.14 Dorrance, W. H. Viscous Hypersonic Flow, Ch. 2, McGraw Hill Co., Inc., New York (1962).

4.15 Zeldovich, Ya. B. and Yu. P. Raizer. Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, Ch. 7, Academic Press, N. Y. (1962).

4.16 Lasher, L. E. and K. H. Wilson. "Effect of Shock Percursor Heating on Radiative Flux to Blunt Bodies," NASA CR-1265 (Dec 1968).

4.17 April, G. A. Evaluation of the Energy Transfer in the Char Zone During Ablation, Ph.D. Dissertation, Louisiana State University, Baton Rouge, Louisiana (1969).

4.18 Hoshizaki, H. and K. H. Wilson. "The Viscous, Radiating Shock About a Blunt Body," AIAA Jour., 3, 1614 (Sept. 1965).

4.19 Ho, H. T. and R. F. Probstein. "The Compressible Viscous Shock Layer in Rarefied Hypersonic Flow" in Rarefied Gas Dynamics, Ed. L. Talbot, Academic Press, N. S. (1961).

4.20 Rigdon, W. S., R. B. Dirling, and M. Thomas. "Radiation and Convective Heating During Atmospheric Entry," NASA CR-1170 (Sept. 1968).

4.21 Spradley, L. S. and C. D. Engel. "A Study of the Accuracy of the Radiation-Coupled Shock Layer Program Including an Analysis of Error Sources," LMSC/HREC D 148807, Lockheed Missiles and Space Company, Huntsville, Ala. (April 1969).

SECTION V

TRANSPORT AND THERMODYNAMIC PROPERTIES

Transport and Thermodynamic Properties

The reliability of the flow field calculations in the current study is highly dependent on the values used for the various transport and thermodynamic properties. With this in mind, it is desirable to attain the ultimate in accuracy; however, some complications do exist. There is very little data in the temperature range of interest in this work, and the data that does exist is subject to some scrutiny due to experimental difficulties at these higher temperatures. Therefore, it becomes necessary to rely heavily on rigorous kinetic theory for the estimation of these properties.

Generally, investigators in this area have resorted to the classical Chapman-Enskog kinetic theory relations for estimation of the required transport properties. The modification of these relationships to account for polyatomic reacting mixtures results in very cumbersome equations. In some cases, as will be shown, there are simplifications which can be applied without substantial loss in accuracy. At this point, it becomes desirable to optimize between accuracy and computation time. A wide variety of methods for estimation of these properties has been developed in just this manner. In this work, an attempt has been made to combine some advantages of each of these methods into one for optimum results to accurately compute high temperature transport properties with reasonable computational convenience. The properties to be discussed in this section are mixture viscosity coefficients, thermal conductivity of polyatomic reacting mixtures, multicomponent diffusion, heat capacity and enthalpy.

Viscosity Coefficients

From first order kinetic theory the coefficient of viscosity of a pure monatomic gas can be given by the following relationship from Hirschfelder, et al., Ref. 5.9 (p. 528):

$$\mu_i \times 10^7 = 266.93 \frac{\sqrt{M_i T}}{\sigma_i^2 \Omega_{ii}^{(2,2)}} \qquad \text{g/cm - sec} \tag{5.1}$$

where

T = temperature, ^oK

M_i = molecular weight of i

σ_i = collision diameter of species i, Å

$\Omega_{ii}^{(2,2)}$ = Lennard-Jones collision integral for viscosity, a function of reduced temperature, $k_c T/\epsilon_i$

ϵ_i = characteristic interaction energy (a Lennard-Jones parameter)

The basic concepts of these collision properties have been discussed extensively by various authors, Ref. 5.9, 5.12.

The coefficient of viscosity for binary mixtures can be calculated by a similar equation, Ref. 5.9 (p. 529).

$$\mu_{ij} \times 10^7 = 266.93 \frac{\sqrt{\frac{2M_iM_jT}{M_i+M_j}}}{\sigma_{ij}^2\,\Omega_{ij}^{(2,2)}}, \quad \text{g/cm sec} \qquad (5.2)$$

where

$$\sigma_{ij}^2 = \left[\tfrac{1}{2}(\sigma_i + \sigma_j)\right]^2$$

$$\epsilon_{ij} = \sqrt{\epsilon_i\,\epsilon_j}$$

Although the above formula was derived for monatomic gases, it has also been found to be "remarkably accurate for polyatomic gases", Ref. 5.3 (p. 23).

The calculation of viscosity for multi-component gas mixtures according to rigorous kinetic theory results in the evaluation of the following equation, Ref. 5.9 (p. 531):

$$\mu_{mix} = - \frac{\begin{vmatrix} H_{11} & H_{12} & \cdot & \cdot & \cdot & H_{1\nu} & X_1 \\ H_{12} & & & & & \cdot & \cdot \\ \cdot & & & & & \cdot & \cdot \\ \cdot & & & & & \cdot & \cdot \\ \cdot & & & & & \cdot & \cdot \\ H & H & \cdot & \cdot & \cdot & H_{\nu\nu} & X_2 \\ X_1 & X_2 & \cdot & \cdot & \cdot & X_\nu & 0 \end{vmatrix}}{\begin{vmatrix} H_{11} & H_{12} & \cdot & \cdot & \cdot & \cdot & H_{1\nu} \\ H_{12} & H_{22} & \cdot & \cdot & \cdot & \cdot & H_{2\nu} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ H_{1\nu} & H_{2\nu} & \cdot & \cdot & \cdot & \cdot & H_{\nu\nu} \end{vmatrix}} \qquad (5.3)$$

where

$$H_{ii} = \frac{Y_i}{\mu_i} + \sum_{\substack{k=1 \\ k \neq i}}^{\nu} \frac{2Y_i Y_k}{\mu_{ik}} \frac{M_i M_k}{(M_i + M_k)^2} \left[\frac{5}{3A_{ik}} + \frac{M_k}{M_i} \right] \qquad (5.4)$$

and

$$A_{ik} = \Omega_{ik}^{(2,2)} / \Omega_{ik}^{(1,1)} \qquad (5.5)$$

where the quantity $\Omega_{ik}^{(1,1)}$ is the Lennard-Jones collision integral for diffusion.

$$H_{ij} = - \frac{2 Y_i Y_j M_i M_j}{\mu_{ij}(M_i + M_j)^2} \left[\frac{5}{3A_{ij}} - 1 \right] \qquad i \neq j \qquad (5.6)$$

As will be subsequently discussed, A_{ij} can be taken equal to $5/3$ and the result is that the non-diagonal elements vanish. Expansion of the remaining determinants gives:

$$\mu_{mix} = \sum_{i=1}^{\nu} \frac{Y_i^2}{\frac{Y_i^2}{\mu_i} + 2.0 \sum_{\substack{k=1 \\ k \neq i}}^{\nu} \frac{Y_i Y_k \mathbb{R} T}{\rho M_i \mathbb{D}_{ij}}} \tag{5.7}$$

It has been shown by Buddenberg and Wilke, Ref. 5.5, that Eq. 5.7 provides a very good approximation to the rigorous analysis if the 2.0 is replaced by 1.385. The resulting equation is then

$$\mu_{mix} = \sum_{i=1}^{\nu} \frac{Y_i^2}{\frac{Y_i^2}{\mu_i} + 1.385 \sum_{\substack{k=1 \\ k \neq i}}^{\nu} \frac{Y_i Y_k \mathbb{R} T}{\rho M_i \mathbb{D}_{ij}}} \tag{5.8}$$

The use of Eq. 5.8 now permits a simple estimate of mixture viscosities without resorting to the evaluation of determinants.

<u>Thermal Conductivities of Polyatomic Reacting Mixtures</u>

The following equation is suggested by Hirschfelder, Ref. 5.9 (p. 534), for calculations if species thermal conductivities of inert, monatomic gases

$$k_{mono_i} = \frac{1.981 \times 10^{-4} \sqrt{T/M_i}}{\sigma_i^2 \, \Omega^{(2,2)}} \tag{5.9}$$

where k_{mono} is the thermal conductivity in g-cal/cm sec ^{o}K, T is the absolute temperature in ^{o}K, M_i the molecular weight of i, σ_i the low-velocity collision diameter in Å, and $\Omega^{(2,2)}$ is the reduced collision integral.

Polyatomic thermal conductivities can be obtained by multiplying Eq. 5.9 by an appropriate Eucken-type correction, Ref. 5.3, which accounts for the transfer of energy between internal degrees of freedom and translational motion. Applying a Eucken-type correction, the thermal conductivity of a polyatomic gas can be written as,

$$k = k_{mono} + k_{int} \tag{5.10}$$

where k_{int} is the thermal conductivity contribution due to the internal degrees of polyatomic molecules.

Similarly the thermal conductivities of mixtures of polyatomic molecules can be determined from

$$k_{mix} = k_{mono\text{-}mix} + k_{int\text{-}mix} \tag{5.11}$$

Using the same averaging procedure as in Eq. 5.8,

$$k_{mono\text{-}mix} = \sum_{i=1}^{\nu} \frac{Y_i k_{i\,mono}}{Y_i + 1.385\mu_i \frac{\mathbb{R}T}{M_i} \sum_{k=1}^{\nu} \left(\frac{Y_i}{\mathbb{D}_{ik}} - \frac{Y_i}{\mathbb{D}_{ii}} \right)} \tag{5.12}$$

where $k_{i\,mono}$ is calculated with Eq. 5.9.

In Ref. 5.10 (p. 366), Hirschfelder shows that

$$k_{int\text{-}mix} = \sum_{i=1}^{\nu} \frac{k_{i_{int}} \frac{Y_i}{\mathbb{D}_{ii}}}{\sum_{k}^{\nu} \frac{Y_k}{\mathbb{D}_{ik}}} \tag{5.13}$$

$$\text{where } k_{i_{int}} = \frac{6}{5}\left(\frac{C_{p_i}}{\mathbb{R}} - \frac{5}{2} \right) A_{i,i} \frac{\mathbb{R}}{M_i} \mu_i \tag{5.14}$$

The quantity, $A_{i,i}$, is defined by Eq. 5.5. Gomez, Ref. 5.8, reports that $A_{i,i}$ is approximately constant for a wide range of temperatures (500-7000°K). Typical values are $A_{i,i}$ = 1.10 in Ref. 5.11 and $A_{i,i}$ = 1.13 in Ref. 5.1.

Equations 5.4 through 5.7 can be combined to give

$$k_{mix} = \frac{\rho}{T}\left[\frac{15}{4}\sum_{i=1}^{\nu}\frac{\frac{5}{6}\frac{Y_i M_i \mathbb{D}_{ii}}{A_{ii}}}{Y_i + (1.385)\frac{5}{6}\frac{\mathbb{D}_{ii}}{A_{ii}}\sum_{k=1}^{\nu}\left(\frac{Y_k}{\mathbb{D}_{ik}} - \frac{Y_i}{\mathbb{D}_{ii}}\right)} \right.$$

$$\left. + \frac{1}{\mathbb{R}}\sum_{i=1}^{\nu}\frac{Y_i(\rho_i - \frac{5}{2}\mathbb{R})}{\sum_{k=1}^{\nu}\frac{Y_i}{\mathbb{D}_{ik}}}\right] \tag{5.15}$$

In reacting systems, thermal conductivities may be considerably higher than in "frozen" or "non-reacting" systems. This behavior has been discussed in detail in a work by Brokaw, Ref. 5.4. Molecules that diffuse because of concentration gradients transfer heat in the form of chemical enthalpy. These gradients exist because the gas composition varies with temperature. A gas may absorb heat by dissociating as the temperature is raised; heat is then transferred as the molecules diffuse to a low temperature region and recombine releasing the heat absorbed at the high temperature.

Rigorous equations have been derived that predict the thermal conductivity of reacting gas mixtures, Ref. 5.6. These equations become very involved for anything but the simplest systems, which makes their use impractical. However, a simplification does exist. It has been shown in Ref. 5.6 that the ratio of the equilibrium thermal conductivity to equilibrium heat capacity is "nearly equal" to the ratio of frozen thermal conductivity and heat capacity. Using this property, the thermal conductivity of a reacting mixture of polyatomic species could be estimated from

$$k_r = k_f \frac{Cp_{r_{mix}}}{Cp_{f_{mix}}} \tag{5.16}$$

Diffusional Coefficients

From the first-order kinetic theory, binary diffusion coefficients are expressible with the following relationship from Bird, et al., Ref. 5.3 (p. 539):

$$\mathbb{D}_{ij} = 26.28 \times 10^{-4} \frac{T^{3/2}\left(\frac{M_i+M_j}{2M_iM_j}\right)^{1/2}}{P\,\sigma_{ij}^2\,\Omega_{ij}^{(1,1)}}, \quad \text{cm}^2/\text{sec} \tag{5.17}$$

where $\Omega_{ij}^{(1,1)}$ is the Lennard-Jones collision integral for diffusion. By combining this equation with Eq. 5.5, the following relationship is obtained:

$$\mathbb{D}_{ij} = 26.28 \times 10^{-4} \frac{T^{3/2}\left(\frac{M_i+M_j}{2M_iM_j}\right)^{1/2} A_{ij}}{P\,\sigma_{ij}^2\,\Omega_{ij}^{(2,2)}} \tag{5.18}$$

The mass diffusion fluxes are given implicitly in Hirschfelder, et al., Ref. 5.6 (p. 718), by the Stefan-Maxwell relations.

$$\frac{\partial Y_i}{\partial y} = \sum_j \frac{Y_i Y_j}{P\,\mathbb{D}_{ij}}\left[\frac{J_{j,y}}{C_j} - \frac{J_{i,y}}{C_i}\right] \tag{5.19}$$

where

Y_i = mole fraction of species i

$J_{i,y}$ = mass flux by molecular diffusion

$\mathbb{D}_{ij}$ = binary diffusion coefficient of species i and j

C_i = mass fraction of species i

Use of these relations with the boundary layer conservation equations is awkward even in the absence of thermal diffusion effects as a result of the implicit behavior of $J_{i,y}$ on mole fractions and their gradients. Furthermore, the Stefan-Maxwell relations cannot be arranged into an explicit relationship for $J_{i,y}$ through introduction of $\mathbb{D}_{ij}$ because the contributions of species i and j to the binary diffusion coefficient are inseparable.

In Ref. 5.2, Bird showed that a bifurcation (separation) of the effects of species i and j permits explicit solution of the Stefan-Maxwell relations for J_i in terms of gradients and properties of species i and of the system as a whole. The empirical approximation used for this simplification was

$$\mathbb{D}_{ij} = \frac{\bar{D}}{F_i F_j} \tag{5.20}$$

where

$\bar{D}$ = the coefficient of self diffusion of a selected reference species

$F_i F_j$ = diffusion factors for species i and j

This relationship was also used by Kendall _et al._, Ref. 5.1, who found that Bird's approximation provided many computational conveniences (in particular, speed, storage, and input requirements). Its adaptation to an explicit solution of the Stefan-Maxwell relations is a strong point in favor of using the empirical relationship in Eq. 5.20 for determining _diffusion coefficients_ in the current study.

Using Eq. 5.20, the explicit solution of the Stefan-Maxwell relations can be developed as shown in Appendix A. The result is expressed as

$$J_{i,y} = -\frac{\rho\bar{D}}{\psi_1}\left[\frac{\psi_2}{M}\frac{\partial z_i}{\partial y} + \frac{(z_i - C_i)}{M}\frac{\partial \psi_2}{\partial y} + C_i\left(\frac{1}{F_i^2}\frac{dF_i}{dT} - \psi_4\right)\frac{dT}{dy}\right] \tag{5.21}$$

where

$$z_i = M_i Y_i / F_i \psi_2 = M C_i / F_i \psi_2 \tag{5.22}$$

$$\psi_1 = \sum_j Y_j F_j = M \sum_j (F_j C_j / M_j) \tag{5.23}$$

$$\psi_2 = \sum_j M_j Y_j / F_j = M \sum_j (C_j / F_j) \tag{5.24}$$

$$\psi_4 = \sum_j (C_j / F_j^2)(dF_j / dT) \tag{5.25}$$

It was found by Kendall, Ref. 5.1, and confirmed by Gomez, Ref. 5.8, that it often is consistent with the level of the approximation to consider F_i independent of temperature. Eq. 5.21 then becomes

$$J_{i,y} = -\frac{\rho \bar{D} \psi_2}{\psi_1 M}\left[\frac{\partial z_i}{\partial y} + \frac{(z_i - C_i)}{\psi_2}\frac{\partial \psi_2}{\partial y}\right] \tag{5.26}$$

This relationship can be modified for later computational convenience to a form which is analogous to Fick's Law. The resulting expression is

$$J_{i,y} = -D_i \frac{dC_i}{dy} \tag{5.27}$$

where

$$D_i = \frac{\rho \bar{D} \psi_2}{\psi_1 M}\left[\frac{\partial z_i}{\partial C_i} + \frac{(z_i - C_i)}{\psi_2}\frac{\partial \psi_2}{\partial C_i}\right] \tag{5.28}$$

Using Eqs. 5.22 and 5.24 with the previous equation, it is shown in Appendix C that

$$D_i = \frac{\rho \bar{D}}{\psi_1}\left[\frac{(1 - C_i)}{F_i} + \frac{C_i}{M_i}\left(\psi_2 - \frac{M}{F_i}\right)\right] \tag{5.29}$$

Calculation of $\bar{D}$ and F_i:

The reference diffusion coefficient is simply the coefficient of self-diffusion for the reference species.

$$\bar{D} = 26.38 \times 10^{-4} \frac{T^{3/2} M_{ref} A_{ref}}{P \sigma_{ref}^2 \Omega_{ref}^{(2,2)}} \tag{5.30}$$

where the variables are defined as previously done for Eq. 5.17. Here the subscript "ref" designates a reference species. The F_i are then determined by a least-squares correlation of $\mathbb{D}_{ij}$, calculated by means of Eq. 5.17, for all diffusing pairs in the chemical system of interest. Such a method of correlation is developed in Appendix B.

Thermodynamic Properties:

For the current study an expression of the thermodynamic data in polynomial form is desired. For example:

$$\frac{C_p^\circ}{\mathbb{R}} = a_1 + a_2 T + a_3 T^2 + \bullet \bullet \bullet \bullet \tag{5.31}$$

$$\frac{h^\circ}{\mathbb{R}T} = a_1 + \frac{a_2}{2} T + \frac{a_3}{3} T^2 + \bullet \bullet \bullet \bullet \tag{5.32}$$

C_p° = pure component heat capacity at constant pressure for standard state

h° = sum of sensible enthalpy and chemical energy of formation of a pure substance at standard state.

The constants for the preceding equations, calculated from polynomial curve-fits of experimental data, can be obtained from a number of sources, Ref. 5.11 and 5.13. For higher temperatures, i.e. 8000-100,000°K, these constants can be obtained from curve-fits based on statistical thermodynamic considerations. The relationships required for these calculations can be found in many sources; Refs: 5.14, 5.15, 5.16, 5.17, and 5.18. For gas mixtures, the following relationships are generally used, Ref. 5.9.

$$C_{p_{mix}} = \sum_j Y_j C_{p_j} + T \sum_j \left(\frac{h_j}{\mathbb{R}T}\right)\left(\frac{\partial Y_j}{\partial T}\right)_p \tag{5.33}$$

$$h_{mix} = \sum_j Y_j h_j \tag{5.34}$$

Section 5

References

5.1 Bartlett, E. P., Kendall, R. M., and Rindal, R. A., "An Analysis of the Coupled Chemically Reacting Boundary Layer and Charring Ablator," Part IV, NASA CR-1063.

5.2 Bird, R. B., "Diffusion in Multicomponent Gas Mixtures," NASA TT-F-10, 925.

5.3 Bird, R. B., Stewart, W. E., and Lightfoot, E. N., Transport Phenomena, John Wiley & Sons, Inc., N. Y., 1960.

5.4 Browkaw, Richard S., " 'Thermal Conductivity' and Chemical Kinetics," Journal of Chem. Phys., Vol. 35, No. 5. November 1961, p. 1569.

5.5 Buddenberg, J. W. and C. R. Wilke, Ind. Eng. Chem., 41, 1345 (1949).

5.6 Butler, J. N., and Brokaw, R. S., "Thermal Conductivity of Gas Mixtures in Chemical Equilibrium," Journal of Chem. Phys., Vol. 26, No. 6, June 1957.

5.7 D'Amur, I. and Mason, E. A., "Properties of Gases at Very High Temperatures," The Physics of Fluids, Vol. 1, No. 5, Sept.-Oct., 1958, pp. 370-383.

5.8 Gomez, A. V. and Mills, A. F., TRW Proj. Tech. Rept. TASK E-458, 30 June 1969.

5.9 Hirschfelder, J. O., Curtiss, C. F.,and Bird, R. B., Molecular Theory of Gases and Liquids, 2nd Printing, Corrected, with notes added, John Wiley and Sons, Inc., N. Y. (1964).

5.10 Hirschfelder, J. O., "Heat Conductivity in Polyatomic, Electronically Excited or Chemically Reacting Mixtures," 6th Symposium (International) on Combustion, Reinhold Publishing Corp., N. Y. (1957).

5.11 McBride, B. J., et al., NASA SP-3001.

5.12 Svehla, R. A., "Estimated Biscosities and Thermal Conductivities of Gases at High Temperatures," NASA TR R-132 (1962).

5.13 Svehla. R. A., NASA SP-3011.

5.14 Crawford, F. H., Heat, Thermodynamics, and Statistical Physics, (New York; Harcourt. Brace, and World, Inc., 1963).

5.15 Engel, C. D. and Spradley, L. W. "Approximate Thermodynamic Properties of Hydrogen Helium Mixtures From 1000° to 100,000°K", LMSC/HREC D148816, Lockheed Missiles & Space Co., May 1969.

5.16 Jevons W., Report on Band Spectra of Diatomic Molecules, Physical Society, London, 1932.

5.17 Hansen, C. F., Approximations for the Thermodynamic and Transport Properties of High Temperature Air, NASA TR R-50.

5.18 Herzberg, G., Molecular Spectra and Molecular Structure, 2nd ed., Van Nostrand, New York, Vol. I, 1950.

SECTION VI

SUMMARY

In the preceding sections the conservation equations were developed with particular emphasis on maintaining generality in the mass transport, chemical reaction, and radiation terms. The flight regimes for hyperbolic entry and the corresponding fluid dynamic behavior were discussed to provide a bases for an order of magnitude analysis. This provided a sound foundation for the derivation of the shock layer equation which govern the phenomenon of current interest.

The shock layer equations, which describe the flow between the body and the detached shock wave, were stated both with and without second order effects. Furthermore, the classical boundary layer equations were shown to be a simplified set of shock layer equations. The shock layer and boundary layer equation which are valid for flow around the body were reduced to stagnation line equations by taking appropriate limits. The stagnation line equations were discussed in detail because of their importance on their own right and because they provide the needed initial condition for around the body solutions. In essence, the problem has been defined and the pertenent equations derived.

In addition to the flow-field equations, boundary conditions at the shock wave and on the body's surface were described. The boundary conditions on the body's surface were left quite general. To specificately state these conditions requires a mass, momentum and energy balance at the surface. In this manner the flow field calculations and the ablator response calculations are coupled.

Emphasis was placed on the derivation of the radiative flux divergence term which appears in the energy equation. The radiative transfer was modeled using a plane slab approximation. Additional work needs to be done to assess the role of curvature in determining the radiative flux and flux divergence. Moreover, the existing solution must be arranged in a more suitable form for calculations involving both line and continuum transport. In order to explain and demonstrate an appropriate radiative transport calculation scheme the line and continuum transport mechanisms must be explored theoretically. Furthermore, available data of line strengths and continuum absorption cross sections must be compiled and put into usable form for computer program usage.

A theoretical base has been developed for the calculation of mass transport, momentum transport and thermodynamic properties. Prime emphasis was placed on mass and momentum transport theory because of the expected improvements in calculation procedures for these mechanisms. Additional work must be done to examine and demonstrate the appropriate procedures.

Based on this work and the current state of the art, a logical sequence of further developments can be stated as follows: First, a stagnation line solution which includes the effects of diffusion, finite rate chemistry and line-continuum coupled radiation could be developed. This will provide both a worthwhile solution and a means of checking out diffusion, chemistry and radiation calculation schemes. Secondly, an around the body solution which includes these same mechanisms could be developed. The stagnation line solution should be used as initial conditions for the around the body calculations. In both the stagnation and around the body solution surface boundary conditions could be specified by mass, momentum, and energy balances at the surface for direct compatability with material response solutions.

APPENDIX A

DERIVATION OF SPECIES DIFFUSION FLUXES

AS AN EXPLICIT FUNCTION

This appendix contains the development of an explicit solution to the Stefan-Maxwell equations for the diffusive mass fluxes in terms of gradients and properties of the respective species and of the system as a whole. The development of these equations has been previously shown in much less detail by Kendall et al., in NASA CR-1063. The Stefan-Maxwell equations are written as follows:

$$\frac{\partial Y_i}{\partial y} = \sum_j \frac{Y_i Y_j}{\rho \mathcal{D}_{ij}} \left[\frac{J_{j,y}}{C_j} - \frac{J_{i,y}}{C_i} \right] \tag{A-1}$$

The species mass fractions are related to the mole fractions by the following relationship:

$$Y_i M_i = C_i M \tag{A-2}$$

Substituting Eq. A-2 and the bifurcation relationship, Eq. 5.20, into Eq. A-1 gives

$$\frac{\partial Y_i}{\partial y} = \sum_j \frac{C_i C_j M^2 F_i F_j}{\rho M_i M_j \bar{D}} \left[\frac{J_{j,y}}{C_j} - \frac{J_{i,y}}{C_i} \right]$$

which simplifies to (dropping the subscript y on $J_{i,y}$)

$$\frac{\partial Y_i}{\partial y} = \frac{M^2}{\rho \bar{D}} \left(\frac{C_i F_i}{M_i} \sum_j \frac{J_j F_j}{M_j} - \frac{F_i J_i}{M_i} \sum_j \frac{C_j F_j}{M_j} \right) \tag{A-3}$$

multiplying each side of Eq. A-3 by M_i/F_i and summing over all i gives

$$\sum_j \frac{M_j}{F_j} \frac{\partial Y_j}{\partial y} = \sum_i \frac{M^2 C_i}{\rho \bar{D}} \sum_j \frac{J_j F_j}{M_j} - \sum_i \frac{M^2}{\rho \bar{D}} J_i \sum_j \frac{C_j F_j}{M_j} \tag{A-4}$$

By noting that the sum of the mass fractions is unity and that the sum of the diffusive fluxes is zero, the following relationship is obtained from Eq. A-4:

$$\sum_j \frac{J_j F_j}{M_j} = \frac{\rho \bar{D}}{M^2} \sum_j \frac{M_j}{F_j} \frac{\partial Y_j}{\partial y} \tag{A-5}$$

Substitution of Eq. A-5 into Eq. A-3 gives

$$\frac{\partial Y_i}{\partial y} = \frac{C_i F_i}{M_i} \sum_j \frac{M_j}{F_j} \frac{\partial Y_j}{\partial y} - \frac{M^2}{\rho \bar{D}} \frac{F_i J_i}{M_i} \sum_j \frac{C_j F_j}{M_j} \tag{A-6}$$

Define the following new quantities:

$$Z_i \equiv \frac{M_i Y_i}{F_i \Psi_2} \tag{A-7}$$

$$\Psi_1 \equiv \sum_j Y_j F_j \tag{A-8}$$

$$\Psi_2 \equiv \sum_j \frac{M_j Y_j}{F_j} \tag{A-9}$$

$$\Psi_4 \equiv \sum_j \left(\frac{C_j}{F_j^2}\right)\left(\frac{dF_j}{dT}\right) \tag{A-10}$$

Multiplying Eq. A-7 by Ψ_2 and differentiating with respect to y gives

$$\frac{M_i}{F_i} \frac{\partial Y_i}{\partial y} - \frac{M_i Y_i}{F_i^2} \frac{\partial F_i}{\partial y} = \Psi_2 \frac{\partial Z_i}{\partial y} + Z_i \frac{\partial \Psi_2}{\partial y} \tag{A-11}$$

Rearranging Eq. A-11,

$$\frac{\partial Y_i}{\partial y} = \psi_2 \frac{F_i}{M_i}\frac{\partial Z_i}{\partial y} + Z_i \frac{F_i}{M_i}\frac{\partial \psi_2}{\partial y} + \frac{\psi_2}{F_i}\frac{\partial F_i}{\partial y} \tag{A-12}$$

Combining Eq. A-12 with Eq. A-6 gives

$$\frac{\psi_2 F_i}{M_i}\frac{\partial Z_i}{\partial y} + \frac{Z_i F_i}{M_i}\frac{\partial \psi_2}{\partial y} + \frac{Y_i}{F_i}\frac{\partial F_i}{\partial y} = \frac{C_i F_i}{M_i}\sum_j \frac{M_j}{F_j}\left[\frac{\psi_2 F_j}{M_j}\frac{\partial Z_j}{\partial y}\right.$$

$$\left. + \frac{Z_j F_j}{M_j}\frac{\partial \psi_2}{\partial y} + \frac{Y_j}{F_j}\frac{\partial F_j}{\partial y}\right] - \frac{M}{\rho \bar{D}}\frac{F_i J_i}{M_i}\sum_j \frac{C_j F_j}{M_j} \tag{A-13}$$

Rearranging,

$$\frac{M_i^2 J_i}{\rho \bar{D}}\sum_j \frac{C_j F_j}{M_j} = -\psi_2 \frac{\partial Z_i}{\partial y} - Z_i \frac{\partial \psi_2}{\partial y} - \frac{M_i Y_i}{F_i^2}\frac{\partial F_i}{\partial y} + C_i \sum_j \psi_2 \frac{\partial Z_j}{\partial y}$$

$$+ C_i \sum_j Z_j \frac{\partial \psi_2}{\partial y} + C_i \sum \frac{M_i Y_i}{F_j^2}\frac{\partial F_j}{\partial y} \tag{A-14}$$

and

$$J_i = -\frac{\rho \bar{D}}{M \psi_1}\left[\psi_2 \frac{\partial Z_i}{\partial y} + (Z_i - C_i)\frac{\partial \psi_2}{\partial y} + \frac{M_i Y_i}{F_i^2}\frac{\partial F_i}{\partial y} - C_i \sum_j \psi_2 \frac{\partial Z_j}{\partial y}\right.$$

$$\left. + C_i \sum_j \frac{M_j Y_j}{F_j^2}\frac{\partial F_j}{\partial y}\right] \tag{A-15}$$

The following terms in Eq. A-15 can be rewritten as indicated:

$$\frac{M_i Y_i}{F_i^2}\frac{\partial F_i}{\partial y} = \frac{C_i M}{F_i^2}\frac{\partial F_i}{\partial y} = \frac{C_i M}{F_i^2}\frac{\partial F_i}{\partial T}\frac{\partial T}{\partial y} \qquad \text{(A-16)}$$

$$C_i \sum_j \psi_2 \frac{\partial Z_j}{\partial y} = C_i \psi_2 \frac{\partial}{\partial y}\sum_j Z_j = 0 \qquad \text{(A-17)}$$

$$C_i \sum_j \frac{M_j Y_j}{F_j^2}\frac{\partial F_i}{\partial y} = C_i M \psi_4 \frac{\partial T}{\partial y} \qquad \text{(A-18)}$$

Substitution of Eqs. A-16, A-17 and A-18, into Eq. A-15 gives

$$J_i = -\frac{\rho \bar{D}}{\psi_1}\left[\frac{\psi_2}{M}\frac{\partial Z_i}{\partial y} + \frac{(Z_i - C_i)}{M}\frac{\partial \psi_2}{\partial y} + C_i\left(\frac{1}{F_i^2}\frac{dF_i}{dT} - \psi_4\right)\frac{\partial T}{\partial y}\right] \qquad \text{(A-19)}$$

where J_i is the mass flux due to molecular diffusion.

APPENDIX B

CORRELATION OF BIFURCATION FACTOR AND BINARY DIFFUSION COEFFICIENTS

The least squares analysis which is developed in this appendix is used to determine the best empirical constants for the bifurcation relationship discussed in Section V. The bifurcation approximation is written as (Eq. 5.20)

$$\mathbb{D}_{ij} = \frac{\bar{D}}{F_i F_j} \tag{B-1}$$

or in terms of logarithms,

$$\log \mathbb{D}_{ij} = \log \bar{D} - \log F_i F_j \tag{B-2}$$

In Ref. 5.1, Kendall uses F_i in the form

$$F_i = \left(\frac{M_i}{M_{ref}}\right)^B \tag{B-3}$$

Substituting Eq. B-3 into Eq. B-2 gives

$$\log \mathbb{D}_{ij} = \log \bar{D} + B \log \left(\frac{M_i M_j}{M_{ref}^2}\right) \tag{B-4}$$

which is of the form

$$y_k = A + B x_k \tag{B-5}$$

Least squares analysis of this linear relationship results in

$$B = \sum_{k=1}^{N} (x_k - \bar{x})(y_k - \bar{y}) \Big/ \sum_{k=1}^{N} (x_k - \bar{x})^2 \tag{B-6}$$

where N = all possible interactions of species present,

$$\chi_k = \log\left(\frac{M_{ref}^2}{M_i M_j}\right) \quad \text{, k representing a particular combination of species i and j,} \tag{B-7}$$

$$y_k = \log D_{ij} \tag{B-8}$$

$$\bar{\chi} = \sum \chi_k / N \tag{B-9}$$

$$\bar{y} = \sum y_k / N \tag{B-10}$$

Having calculated B for a particular chemical system and a selected reference species (dependent upon a secondary best-fit analysis), the bifurcation factors are available from Eq. B-3.

APPENDIX C

SIMPLIFICATION OF THE RELATIONSHIP FOR EVALUATION OF EFFECTIVE MULTICOMPONENT DIFFUSION COEFFICIENTS

The diffusive mass flux of species i can be calculated by means of Eq. 5.26.

$$J_{i,y} = -\frac{\rho \bar{D} \psi_2}{\psi_1 M}\left[\frac{\partial z_i}{\partial y} + \frac{(z_i - c_i)}{\psi_2}\frac{\partial \psi_2}{\partial y}\right] \tag{C-1}$$

As will be shown it is more convenient to express this flux in terms of an effective multicomponent diffusion coefficient and a concentration gradient.

$$J_{i,y} = -D_i \frac{dC_i}{dy} \tag{C-2}$$

where

$$D_i = \frac{\rho \bar{D} \psi_2}{\psi_1 M}\left[\frac{\partial z_i}{\partial C_i} + \frac{(z_i - c_i)}{\psi_2}\frac{\partial \psi_2}{\partial C_i}\right] \tag{C-3}$$

Recalling from Eqs. 5.22 and 5.24 that

$$z_i = \frac{M_i Y_i}{F_i \psi_2} = \frac{M C_i}{F_i \psi_2} \tag{C-4}$$

and

$$\psi_2 = \sum_j \frac{M_j Y_j}{F_j} = M \sum_j \frac{C_j}{F_j} \tag{C-5}$$

Since $F_i \neq F_i(C_i)$, then

$$\frac{\partial z}{\partial C_i} = \frac{F_i \psi_2 M + F_i \psi_2 C_i \left(\frac{\partial M}{\partial C_i}\right) - M C_i F_i \left(\frac{\partial \psi_2}{\partial C_i}\right)}{F_i^2 \psi_2^2} \tag{C-6}$$

where

$$\frac{\partial M}{\partial C_i} = \frac{\partial}{\partial C_i}\left(\sum_j \frac{C_j}{M_j}\right)^{-1} = -\frac{1}{M_i}\left(\sum \frac{C_j}{M_j}\right)^{-2} = -\frac{M^2}{M_i} \tag{C-7}$$

and

$$\frac{\partial \psi_2}{\partial C_i} = \frac{\partial}{\partial C_i}\left(M \sum_j \frac{C_j}{F_j}\right) = M \frac{\partial}{\partial C_i} \sum_j \frac{C_j}{F_j} + \frac{\partial M}{\partial C_i} \sum \frac{C_j}{F_j}$$

$$\frac{\partial \psi_2}{\partial C_i} = \frac{M}{F_i} - \frac{M^2}{M_i} \sum_j \frac{C_j}{F_j} = \frac{M}{F_i} - \frac{M \psi_2}{M_i} \tag{C-8}$$

Therefore the substitution Eqs. C-7 and C-8 into Eq. C-6 results in the following relationship:

$$\frac{\partial z_i}{\partial C_i} = \frac{F_i \psi_2 M - F_i \psi_2 C_i \frac{M^2}{M_i} - M C_i F_i \left(\frac{M}{F_i} - \frac{M \psi_2}{M_i}\right)}{F_i^2 \psi_2^2}$$

$$\frac{\partial z_i}{\partial C_i} = \frac{F_i \psi_2 M - M^2 C_i}{F_i^2 \psi_2^2} \tag{C-9}$$

Substitution of Eqs. C-4, C-8 and C-9, into Eq. C-3 and combining terms gives

$$D_i = \frac{\rho \bar{D}}{\psi_1}\left[\frac{(1 - C_i)}{F_i} + \frac{C_i}{M_i}\left(\psi_2 - \frac{M}{F_i}\right)\right] \tag{C-10}$$

DISTRIBUTION

5 Headquarters Contracts Division
Code DHC-4
National Aeornautics and Space
Administration
Washington, D. C. 20546

2 Structures Research Division
Langley Research Center, NASA
Hampton, Virginia

3 Mr. Robert T. Swann, Grant Monitor
Entry Structures Branch, M.S. 206
Langley Research Center, NASA
Hampton, Virginia

1 Mr. James Moss
Entry Structures Branch, M.S. 206
Langley Research Center, NASA
Hampton, Virginia

1 Mr. Fred J. DeMeritte, Chief
Aerothermodynamics Group Code RV-1
National Aeronautics and Space
Administration
Washington, D. C. 20546

1 Mr. John E. Green
Aerothermodynamics Group Code RV-1
National Aeronautics and Space
Administration
Washington, D. C. 20546

1 Mr. Dennis Gardner
TRW Systems
Houston Operations
Space Park Drive
Houston, Texas 77058

1 Mr. Allen Forney
R-Aero-ATA
George C. Marhsall Space Flight
Center
Huntsville, Alabama 35805

1 Mr. Lawrence Spradley
Lockheed Missile and Space Division
Huntsville Research and Engineering
Center
Huntsville, Alabama 35805

1 Dr. Raymond Edleman
General Applied Science Laboratory Inc.
Merrick and Stewart Avenue
Westbury, Long Island, New York

1 Mr. K. H. Wilson
Lockheed Missile and Space
Company
Palo Alto, California

1 Dr. M. Thomas
McDonnell Douglas Astronautics
Company-Western Division
Santa Monica, California

1 Dr. Robert Goulard
School of Aeronautical and Engineering
Sciences
Purdue University
Lafayette, Indiana 47907

1 Dr. R. W. Richardson, Dean
College of Engineering
Louisiana State University

1 Dr. Elvin J. Dautin, Director
Division of Engineering Research
Louisiana State University

1 Dr. Jesse Coates, Head
Department of Chemical Engineering
Louisiana State University

Reacting Fluids Laboratory

4 Authors
1 G. Perez
1 E. G. del Valle
1 M. Ozelsel