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70-412L-1
May 19, 1970

Mr. Thomas B Murtagh
Code FM8
NASA/Manned Spacecraft Center
Advanced Mission Design Branch
Houston, Texas 77058

Dear Mr Murtagh

Pursuant to the technical report provisions and the schedule of paragraph 5 1 of contract NAS-9-10386 the first quarterly progress report is hereby submitted This describes the period of work from the beginning of the contract

Nineteen copies of this report are enclosed Additional copies have been distributed as indicated below

Very truly yours,

Donald C Fraser

Dr. Donald C Fraser
Program Manager

DCF jw

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First Quarterly Progress Report, Contract NAS-9-10386

INTRODUCTION

This quarterly report to the technical monitor of the referenced contract is organized as follows

Chapter I The ability to make large plane changes during escape maneuvers can greatly increase the operational flexibility of interplanetary or lunar missions The fuel penalties for large plane changes can be kept small by going to long duration, multi-burn escape maneuvers This chapter contains an analytic solution for the minimum fuel trajectory for circular initial orbits and fixed maneuver durations The optimum solution is shown to require either three or four burns This section will constitute the major part of a paper to be presented at the AIAA-AAS Astrodynamics Conference in Santa Barbara in August 1970 The remainder of the paper will appear in the next quarterly progress report

Chapter II A complete set of variational equations in universal form involving classical orbital elements is derived In particular, a universal form of the variational equations for the time of nodal passage is derived which is free of singularities and in universal form It is further shown how all mixed secular terms can be avoided (in the spirit of Born and Christensen) in the use of variational equations via the true anomaly difference rather than the eccentric anomaly difference

Chapter III The conservative nature of the proposed filter algorithm requires that the spacecraft equations of motion be written in such a way that the perturbing gravitational specific force along the trajectory does not appear This chapter discusses the general elimination process which replaces as a driving term one of the components of the perturbing specific force with a term depending on the perturbing specific potential energy A method is presented which allows computations to be done in the original coordinate system and precludes the necessity for the energy-adjoined state vector used in previous work

Chapter IV This chapter presents preliminary numerical results of the comparative performance in the satellite problem of the proposed optimal filter, a standard Kalman filter, and a particularly simple suboptimal filter

The following people have contributed to the effort summarized in this First Quarterly Progess Report

R. H Battin

S R Croopnick

J C Deckert

T N Edelbaum

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I Optimal Nonplanar Escape from Circular Orbits

ABSTRACT

An approximate analytical solution is obtained for minimum impulse transfer between a given circular orbit and a given hyperbolic velocity vector at infinity The transfer time between the first and last impulses is assumed fixed and much larger than the period of the circular orbit A minimum allowable periapsis radius is prescribed The solution is developed as the initial terms in an asymptotic expansion in inverse powers of the transfer time Below a critical inclination the optimal solution requires three impulses, while above this inclination four impulses are required

INTRODUCTION

The problem of nonplanar escape from circular orbits arises when the plane of a circular parking orbit does not contain the desired hyperbolic departure asymptote An execllent summary of previous work on this subject is contained in Ref 1 The following brief summary is based on this Reference

Single impulse transfer is discussed in Refs 2 and 3 The optimal one and two impulse transfers are derived in Ref 3 A suboptimal three impulse transfer is described in Ref 4 The basic idea of the absolute optimal transfer which requires infinite time is discussed in Ref 5

The fuel requirements of the one and two impulse transfer become prohibitively large unless the inclination of the hyperbolic asymptote to the circular orbit plane is quite small These fuel requirements can be reduced to reasonable levels for any inclination if sufficient time is allowed for the transfer The present paper is intended to provide an approximate analytic solution to this problem for large transfer times

The transfer time will be defined to be the time between the first and last impulses The solution is developed as an asymptotic expansion in powers of this transfer time to the minus one third power The zero order term in the asymptotic expansion for the required impulse is that for the absolute minimum solution requiring infinite time This term is the sum of two finite impulses and one or more infinitesimal impulses at infinity (Ref 6)

The first impulse is the difference between circular and escape velocity in the initial circular orbit

$$\lim_{t \to \infty} \Delta V_1 = \sqrt{\frac{2\mu}{R_o}} - \sqrt{\frac{\mu}{R_o}} \tag{1}$$

The last impulse is the difference between the hyperbolic periapsis velocity and escape velocity at the minimum allowable periapsis radius

$$\lim_{t \to \infty} \Delta V_f = \sqrt{\frac{2\mu}{R_p} + V_\infty^2} - \sqrt{\frac{2\mu}{R_p}} \tag{2}$$

The minimum allowable periapsis radius is assumed to be less than or equal to the radius of the circular orbit

The optimal transfer will require either three or four impulses depending upon the eccentricity (e) of the final escape hyperbola and the minimum angle between the hyperbolic asymptote and the plane of the circular orbit (θ) The final escape hyperbola will always have its periapsis at the minimum allowable radius Its eccentricity is given by equation (3)

$$e = -1 + \frac{R_p V_\infty^2}{\mu} \tag{3}$$

The limiting value of true anomaly for this hyperbola is given by equation (4)

$$\lim_{R \to \infty} f = \cos^{-1}\left(\frac{1}{e}\right) \equiv f_\infty \tag{4}$$

$$\frac{\pi}{2} \leq f_\infty \leq \pi$$

If the direction of the hyperbolic asymptote is assumed to pierce the celestial sphere at the north pole, the locus of allowable periapsides will correspond to a small circle of latitude in the southern hemisphere The optimal transfer will be a three impulse transfer if the plane of the circular orbit cuts this small circle ($\theta \leq \pi - f_\infty$) and will be a four impulse transfer if it does not cut the small circle ($\frac{\pi}{2} \geq \theta > \pi - f_\infty$)

The zero order term in the asymptotic expansion of the required total impulse is given by the sum of eqs (1) and (2) for both the three and four impulse transfers The order of the next term depends upon whether the transfer uses three or four impulses For the four impulse case the next term is of the order of transfer time to the minus one third power For the three impulse case the next term is of the order of transfer time to the minus two thirds power It is necessary to use much longer times for the four impulse case than for the three impulse case to obtain close to the absolute minimum total impulse

Three Impulse Transfer

The projection of the circular orbit plane on the celestial sphere will cut the periapsides locus of the escape hyperbola in two points The position of the first and third impulses on the celestial sphere is at the point where the velocity in the circular orbit is directed away from the periapsides locus The position of the second impulse on the celestial sphere is diametrically opposite the position of the other two impulses

The transfer is of the coaxial type which has been extensively studied for ellipse to ellipse transfer in the time open case (Ref 1) The primer vector solution for the current problem will be the same as for the time open case except for higher order correction terms The order of these correction terms will be estimated later in this section

The original orbit plane will have to be rotated through an angle i into a plane specified by the location of the third impulse and the hyperbolic asymptote This angle i is given by spherical trigonometry as eq (5)

$$\sin i = \frac{\sin \theta}{\sin f_{\infty}} \tag{5}$$

$$0 \leq i \leq \frac{\pi}{2}$$

The first impulse is used to inject the vehicle onto a highly eccentric ellipse with the same periapsis radius as the circular orbit This ellipse will be specified by its apoapsis radius R_a and its small inclination to the circular orbit plane i_1 The first impulse has no radial component and only a small out of plane component Its magnitude is given by eq (6)

$$\Delta V_1 = \sqrt{\frac{\mu}{R_o}} \sqrt{1 + \frac{2 R_a}{R_a + R_o} - 2 \sqrt{\frac{2 R_a}{R_a + R_o}} \cos i_1} \tag{6}$$

$$= \sqrt{\frac{2\mu}{R_o}} \left(1 - \frac{1}{\sqrt{2}} - \frac{1}{2} \frac{R_o}{R_a}\right) + 0(R_a^{-2})$$

The optimal inclination angle i_1, is given by eq (7)

$$i_1 = \frac{(\sqrt{2} - 1)\sqrt{R_p}}{\sqrt{R_o + R_p - 2\sqrt{R_o R_p} \cos i}} \frac{R_o}{R_a} \sin i + 0(R_a^{-2}) \tag{7}$$

The second impulse is applied at apoapsis of this first transfer ellipse and is used for two purposes, to produce most of the plane change and to lower the periapsis to R_p Its magnitude is given by eq (8).

$$\Delta V_2 = \sqrt{\frac{2\mu}{R_a} \left[\frac{R_o}{R_o + R_a} + \frac{R_p}{R_p + R_a} - 2\sqrt{\frac{R_o R_p}{(R_o + R_a)(R_p + R_a)}} \cos i_2\right]} \tag{8}$$

$$= \frac{\sqrt{2\mu (R_o + R_p - 2\sqrt{R_o R_p} \cos i)}}{R_a} + 0(R_a^{-2})$$

The inclination change of this maneuver is given by eq (9)

$$i_2 = i - i_1 - i_3 \tag{9}$$

The third impulse is given at periapsis of the second transfer ellipse and injects the vehicle onto the final escape hyperbola at its periapsis The magnitude of this impulse is given by eq (10)

$$\Delta V_3 = \sqrt{\frac{2\mu}{R_p} + V_\infty^2 + \frac{2\mu}{R_p+R_a}\frac{R_a}{R_p} - 2\sqrt{\left(\frac{2\mu}{R} + V_\infty^2\right)\left(\frac{2\mu}{R_p+R_a}\frac{R_a}{R_p}\right)}\cos \imath_3}$$

$$= \sqrt{\frac{2\mu}{R_p} + V_\infty^2} - \sqrt{\frac{2\mu}{R_p}}\left(1 - \frac{1}{2}\frac{R_p}{R_a}\right) + 0\left(R_a^{-2}\right) \tag{10}$$

The small inclination change of this third impulse is given by eq (11)

$$\imath_3 = \frac{\sqrt{R_o}}{\sqrt{R_o + R_p - 2\sqrt{R_o R_p}\cos \imath}} \frac{\sqrt{1 + \frac{R_p V_\infty^2}{2\mu}} - 1}{\sqrt{1 + \frac{R_p V_\infty^2}{2\mu}}} \frac{R_p}{R_a} \sin \imath + 0\left(R_a^{-2}\right) \tag{11}$$

The total time for the maneuver is given by eq (12)

$$t_{13} = \frac{\pi}{\sqrt{8\mu}}\left[(R_o + R_a)^{3/2} + (R_p + R_a)^{3/2}\right] \tag{12}$$

$$= \pi\sqrt{\frac{R_a^3}{2\mu}}\left[1 + \frac{3}{4}\frac{R_o}{R_a} + \frac{3}{4}\frac{R_p}{R_a} + 0\left(R_a^{-2}\right)\right]$$

The total ΔV for the maneuver is given by eqs (13) and (14)

$$\sum \Delta V = \Delta V_{t=\infty} + \frac{\sqrt{2\mu}}{R_a}\left[\frac{\sqrt{R_p}}{2} - \frac{\sqrt{R_o}}{2} + \sqrt{R_o + R_p - 2\sqrt{R_o R_p}\cos \imath}\right] + 0\left(R_a^{-2}\right) \tag{13}$$

$$\sum \Delta V = \Delta V_{t=\infty} + (2\mu)^{1/6}(\pi)^{2/3}\left[\frac{\sqrt{R_p}}{2} - \frac{\sqrt{R_o}}{2} + \sqrt{R_o + R_p - 2\sqrt{R_o R_p}\cos \imath}\right] t_{13}^{-2/3} + 0\left(t^{-4/3}\right) \tag{14}$$

The effect of the time constraint on the solution for the primer vector can be estimated from the value of the Hamiltonian H and the Lagrange multiplier for the mean anomaly λ_M (Ref 7)

$$H = \sqrt{\frac{a^3}{\mu}} \; \lambda_M = - \frac{\partial \sum \Delta V}{\partial t_{13}} = 0 \, (t_{13}^{-5/3}) = 0 \, (R_a^{-5/2}) \tag{15}$$

For the time open problem both H and λ_M will be zero For the time fixed problem they will introduce radial components λ_R into the primer vector The order of these radial components can be estimated from eq (15)

$$\lambda_{R_1} = \sqrt{\frac{a}{\mu}} \, \lambda_M \, \frac{(1-e)}{e} = 0 \, (R_a^{-11/2}) \tag{16}$$

$$\lambda_{R_2} = -4\sqrt{\frac{a}{\mu}} \, \lambda_M \, \frac{(1+e)^2}{e} = 0 \, (R_a^{-7/2}) \tag{17}$$

$$\lambda_{R_3} = \sqrt{\frac{a}{\mu}} \, \lambda_M \, \frac{(1-e)^2}{e} = 0 \, (R_a^{-11/2}) \tag{18}$$

All of these radial components of the primer vector are of high enough order to be neglected

Four Impulse Transfer

In the four-impulse case it is no longer possible to use a coaxial transfer where the highly eccentric ellipses are rotated around their common major axis In the four impulse case the major axis itself must be changed in orientation The rotation of the major axis requires a larger ΔV of order $R_a^{-1/2}$ or $t^{-1/3}$ To a first approximation, the minimum ΔV four impulse transfer will be that transfer that minimizes the rotation of the major axis

The minimum possible rotation of the major axis is through an angle $2\,\phi$ defined by eq (19)

$$2\,\phi = \theta + f_\infty - \pi \tag{19}$$

This angle is the minimum angle between the plane of the circular orbit and the periapsides locus It falls within the limits given by equation (20),

$$0 < 2\,\phi \leq f_{\infty} - \frac{\pi}{2} \leq \frac{\pi}{2} \tag{20}$$

In the following analysis, both ϕ and f_{∞} will be assumed to be of order unity The cases where either or both of these angles are small requires separate investigation

The first of the four impulses is a tangential impulse which is used to leave the initial circular orbit

$$\Delta V_1 = \sqrt{\frac{2\mu}{R_o}} \left(1 - \frac{1}{\sqrt{2}} - \frac{1}{2}\,\frac{R_o}{R_a}\right) + 0\,(R_a^{-2}) \tag{21}$$

This impulse is applied at a point almost opposite the point of closest approach of the trace of the circular orbit and the periapsides locus on the celestial sphere It is applied beyond the diametrically opposite point by an angle f_1 given by equation (22),

$$f_1 = \sqrt{\frac{R_o}{R_a}}\;\frac{1 + 2\cos\phi}{\sqrt{1 + \sin^2\phi}} + 0\,(R_a^{-1}) \tag{22}$$

This angle positions the second impulse at the optimal location on the celestial sphere

The second impulse is applied at a radius given by eq (23)

$$R_2 = R_a \left[\frac{1 + \sin^2\phi}{2} + 0\,(R_a^{-1})\right] \tag{23}$$

The magnitude of the second impulse is given by equation (24)

$$\Delta V_2 = \sqrt{\frac{2\mu}{R_a}} \frac{\sin\phi}{\sqrt{1+\sin^2\phi}} - \frac{\sqrt{2\mu R_p}}{R_a}\left(\frac{\cos\phi}{1+\sin^2\phi}\right)^2 \left[1 + \frac{3e - 1 - \sqrt{2(e+1)}}{4e(1+\sin^2\phi)}\right] \tag{24}$$

$$+ 0(R_a^{-3/2})$$

The eccentricity e is that of the final hyperbola, eq (3) This second impulse has no radial component and is applied at almost right angles to the plane of the circular orbit and the first transfer ellipse The angle with the initial plane is given by eq (25) Note that this impulse removes half of the circumferential velocity of the first transfer ellipse

$$\beta_2 = \frac{\pi}{2} + \sqrt{\frac{R_o}{R_a}} \frac{1}{\sin\phi\sqrt{1+\sin^2\phi}} + 0(R_a^{-1}) \tag{25}$$

The second impulse transfers the vehicle onto an ellipse which is almost at right angles to the plane of the original orbit This ellipse is entered at its semiminor axis The vehicle coasts past apoapsis and the third impulse is applied when the other end of the minor axis is reached The radius will then be the same as given by eq (23) The magnitude of the third impulse is given by equation (26)

$$\Delta V_3 = \Delta V_2 + \frac{2\sqrt{2\mu R_p}}{R_a(1+\sin^2\phi)} \tag{26}$$

The third impulse has no radial component and is directed almost opposite to the circumferential velocity in the second transfer ellipse

$$\beta_3 = \pi + \sqrt{\frac{R}{R_a}} \frac{1}{\sin\phi\sqrt{1+\sin^2\phi}} + 0(R_a^{-1}) \tag{27}$$

This impulse changes the orbit plane by almost 180 degrees It orients the third transfer ellipse so that it is coplanar with the hyperbolic asymptote

The final impulse is given just before periapsis of the third transfer ellipse The true anomaly at which this impulse is applied is given by eq 28

$$f_4 = -2\sqrt{\frac{R_p}{R_a}}\ \frac{\cos\phi}{\sqrt{1+\sin^2\phi}} \tag{28}$$

This impulse is coplanar but has a small radial component directed towards the planet The angle with the local horizontal is given by eq (29)

$$\alpha_4 = \frac{f_4}{4} \tag{29}$$

The magnitude of the fourth impulse is given by eq (30)

$$\Delta V_4 = \sqrt{\frac{2\mu}{R_p} + V_\infty^{\,2}} - \sqrt{\frac{2\mu}{R_p}}$$

$$+ \frac{\sqrt{2\mu R_p}}{2R_a}\left[1 + \frac{\cos^2\phi}{(1+\sin^2\phi)^3}\quad \underline{3e - 1 - \sqrt{2(e+1)}}\right] + 0\,(R_a^{\,-2}) \tag{30}$$

Following the fourth impulse the vehicle is injected onto the correct escape hyperbola slightly before its periapsis

The time spent on the three transfer ellipses is given by eqs (31), (32) and (33)

$$t_{12} \doteq \sqrt{\frac{R_a^{\,3}}{8\mu}}\left[\pi - \cos^{-1}(\sin^2\phi) - \sqrt{1-\sin^4\phi} + 0\,(R_a^{\,-1})\right] \tag{31}$$

$$t_{23} = \sqrt{\frac{R_a^{\,3}}{8\mu}}\left[(1+\sin^2\phi)^{3/2}\,(\pi + 2\cos\phi) + 0\,(R_a^{\,-1})\right] \tag{32}$$

$$t_{34} = t_{12} \tag{33}$$

The total ΔV for the transfer is given by eq (34)

$$\Sigma \Delta V = \Delta V_{t=\infty} + \sqrt{\frac{2\mu}{R_a}} \frac{2 \sin \phi}{\sqrt{1 + \sin^2 \phi}} - \frac{\sqrt{2 \mu R_o}}{2 R_a}$$

$$+ \frac{\sqrt{2 \mu R_p}}{R_a} \left\{ \frac{1}{2} + \frac{2}{1 + \sin^2 \phi} - \frac{\cos^2 \phi}{(1 + \sin^2 \phi)^2} \left[2 + \frac{7}{16} \frac{3e - 1 - \sqrt{2(e+1)}}{e(1 + \sin^2 \phi)} \right] \right\}$$

$$+ 0 (R_a^{-3/2}) \tag{34}$$

Four Impulse Transfer-Derivation of the Lower Order Terms

The terms of order $R_a^{-1/2}$ and $t_{14}^{-1/3}$ in the four impulse case correspond to transfer between two equal unit eccentricity ellipses These are degenerate ellipses having a finite major axis but zero angular momentum and no definable orbit plane The transfer problem is to change the orientation of the major axis so that the fourth impulse may be placed at an optimal position in space The departures of these ellipses from unit eccentricity contribute higher order terms which will be considered in a later section

The transfer is assumed to start with an infinitesimal impulse at periapsis which produces the optimal apoapsis radius for the given transfer time A second impulse is used to transfer from the first unit eccentricity ellipse to an intermediate ellipse A third impulse is then used to transfer onto the second unit eccentricity ellipse An infinitesimal impulse at periapsis is assumed to end the fixed time maneuver The maneuver is assumed to be symmetrical

An approximate analytic solution will be obtained and compared with an exact numerical solution of this problem The assumption used to obtain the approximate solution is that the radial component of the primer vector is of higher order and may be neglected For this problem the order of the Hamiltonian may be found from eq 35

$$H = \sqrt{\frac{a^3}{\mu}}\, \lambda_M = -\frac{\partial\, \Sigma\, \Delta V}{\partial t_{14}} = O\left(t_{14}^{-4/3}\right) = O\left(R_a^{-2}\right) \tag{36}$$

On the unit eccentricity ellipse the radial component of the primer vector is given by eq 37

$$\lambda_R = \sqrt{\frac{a}{\mu}}\, \lambda_M \left[\frac{3\ E \sin E}{1 - \cos E} - 5 - \cos E\right]$$

$$= O\left(R_a^{-3}\right) \tag{37}$$

This term is of higher order and may be neglected to the order of the rest of the analysis

The circumferential component of the primer vector on the unit eccentricity ellipse is independent of the magnitude of the Hamiltonian As a result the fixed-time problem is reduced to a minimum ΔV time-open transfer with circumferential impulses

The latter problem can be solved by the following procedure On the intermediate transfer ellipse, the radial and circumferential velocities are given by eqs (38) and (39)

$$\dot{R} = \sqrt{\frac{\mu}{p}}\, e \sin f \tag{38}$$

$$R\dot{\theta} = \sqrt{\frac{\mu}{p}}\,(1 + e \cos f) \tag{39}$$

The semi-latus rectum and eccentricity are given by eqs (40) and (41)

$$P = \frac{R^4 \dot{\theta}^2}{\mu} \tag{40}$$

$$e = \frac{R^2 R\dot{\theta}}{\mu \sin v} = \frac{R^3 \theta^2 - \mu}{\mu \cos v} \tag{41}$$

Combining these four equations yields eq (42) for the circumferential velocity as a function of radius, radial velocity and true anomaly

$$R\dot{\theta} = \frac{\dot{R} \cot f}{2} + \sqrt{\frac{R^2 \cot^2 f}{4} + \frac{\mu}{R}} \tag{42}$$

On the unit eccentricity ellipses, the radius and radial velocity are connected by eq (43)

$$\frac{\mu}{R} = \frac{R^2}{2} + \frac{\mu}{R_a} \tag{43}$$

As a circumferential impulse will not change radial velocity, eq (43) may be substituted into eq (42) to yield eq (44)

$$R\dot{\theta} = \frac{\dot{R}\cot f}{2} + \sqrt{\frac{\dot{R}^2\cot^2 f}{4} + \frac{\dot{R}^2}{2} + \frac{\mu}{R_a}} \tag{44}$$

This equation may be differentiated with respect to R to find the stationary minimum value of circumferential velocity The optimum value of $\dot{R}$ that yields this stationary minimum is given by eq (45)

$$\dot{R} = \sqrt{\frac{2\mu}{R_a}} \frac{\cos f}{1 + \sin^2 f} \tag{45}$$

The radius corresponding to this radial velocity is given by eq (46)

$$R = R_a \frac{1 + \sin^2 f}{2} \tag{46}$$

This optimum radius is always between the semiminor axis and the apopapsis and is consistent with the necessary conditions on the primer vector for unit eccentricity ellipses The true anomaly f will always fall in the second quadrant so that the radial velocity will be positive at entry onto the transfer ellipse The half-angle of rotation of the major axis is given by eq (47) and always lies in the first quadrant

$$\phi = \pi - f \tag{47}$$

By taking careful note of quadrants and signs eq (48) for the circumferential velocity may be obtained from eq (44)

$$R\dot{\theta} = \sqrt{\frac{2}{R_a}} \frac{\sin\phi}{\sqrt{1 + \sin^2\phi}} = \Delta V_2 = \Delta V_3 \tag{48}$$

The eccentricity and semi-major axis of the transfer ellipse is given by eqs (49) and (50)

$$e = \bar{\cos}\ \phi \tag{49}$$

$$a = R_a \frac{1 + \sin^2 \phi}{2} = R \tag{50}$$

Since the semi-major axis is equal to the radius, the transfer ellipse is entered at its semi-minor axis The time required for these transfers is given by the order $R_a^{3/2}$ term of eqs (31), (32) and (33)

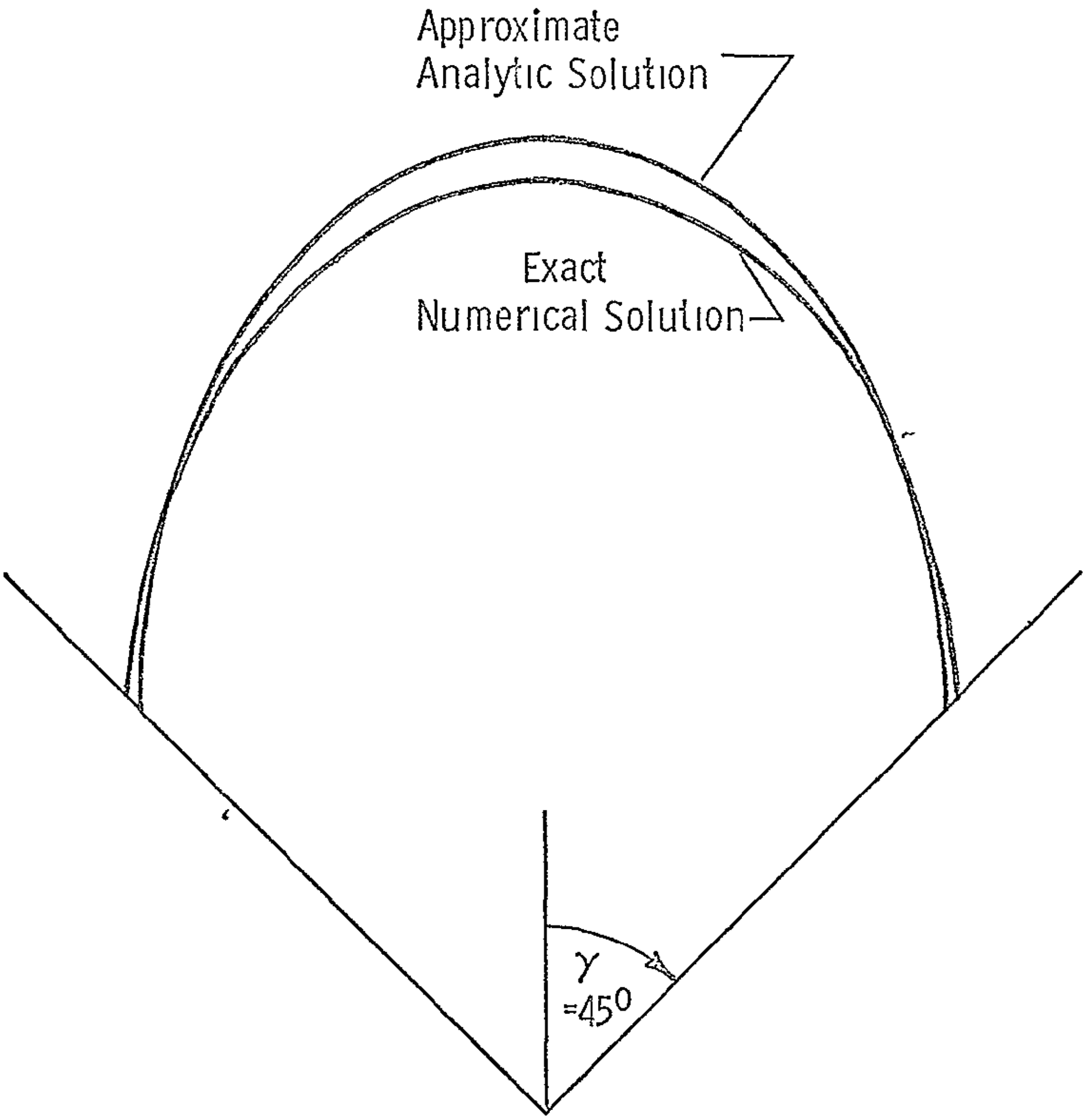
The transfer ellipse of this approximate solution is compared with the transfer ellipse of an exact numerical solution in fig 1 The numerical solution was found by minimizing the cost function $\Delta V \left(\frac{t}{\mu}\right)^{1/3}$ by an accelerated gradient method The figure is drawn for the maximum rotation angle that can occur with circular initial orbits For smaller rotation angles the differences between the exact and approximate solutions will be smaller

Fig 2 illustrates the difference in the cost function between the exact and approximate solution For values of ϕ up to 45° the difference in cost is negligible For larger values of ϕ, which become of interest for elliptic initial orbits, the difference in cost is still quite small, reaching a maximum of 2 3% at ϕ = 90° As this difference occurs in a first order correction term, it would usually represent a negligible portion of the zero order ΔV

For a given apoapsis radius of the unit eccentricity ellipses the approximate solution has a lower ΔV and a longer transfer time Fig 1 is drawn for the same apoapsis radius rather than the same transfer time The ΔV requirements of the two solutions is illustrated in fig 3 The ΔV of the approximate analytic solution is within 0 4% of the absolute minimum ΔV two impulse transfer between unit eccentricity ellipses The latter transfer has not been treated in detail because it requires longer transfer times than the approximate analytic solution

The time for the two transfer modes is illustrated in fig 4 The approximate solution takes somewhat longer for large rotation angles because the radial velocity after the impulse is always positive The numerical solution has negative radial velocities after the impulse for values of ϕ in the vicinity of 90°

Fig 5 and 6 show the radii of the impulses and the angle of the impulses with the horizontal for the two solutions For the approximate solution, the impulse is always circumferential



TRANSFER BETWEEN UNIT ECCENTRICTY ELLIPSES $\gamma = 45^0$

Figure 1

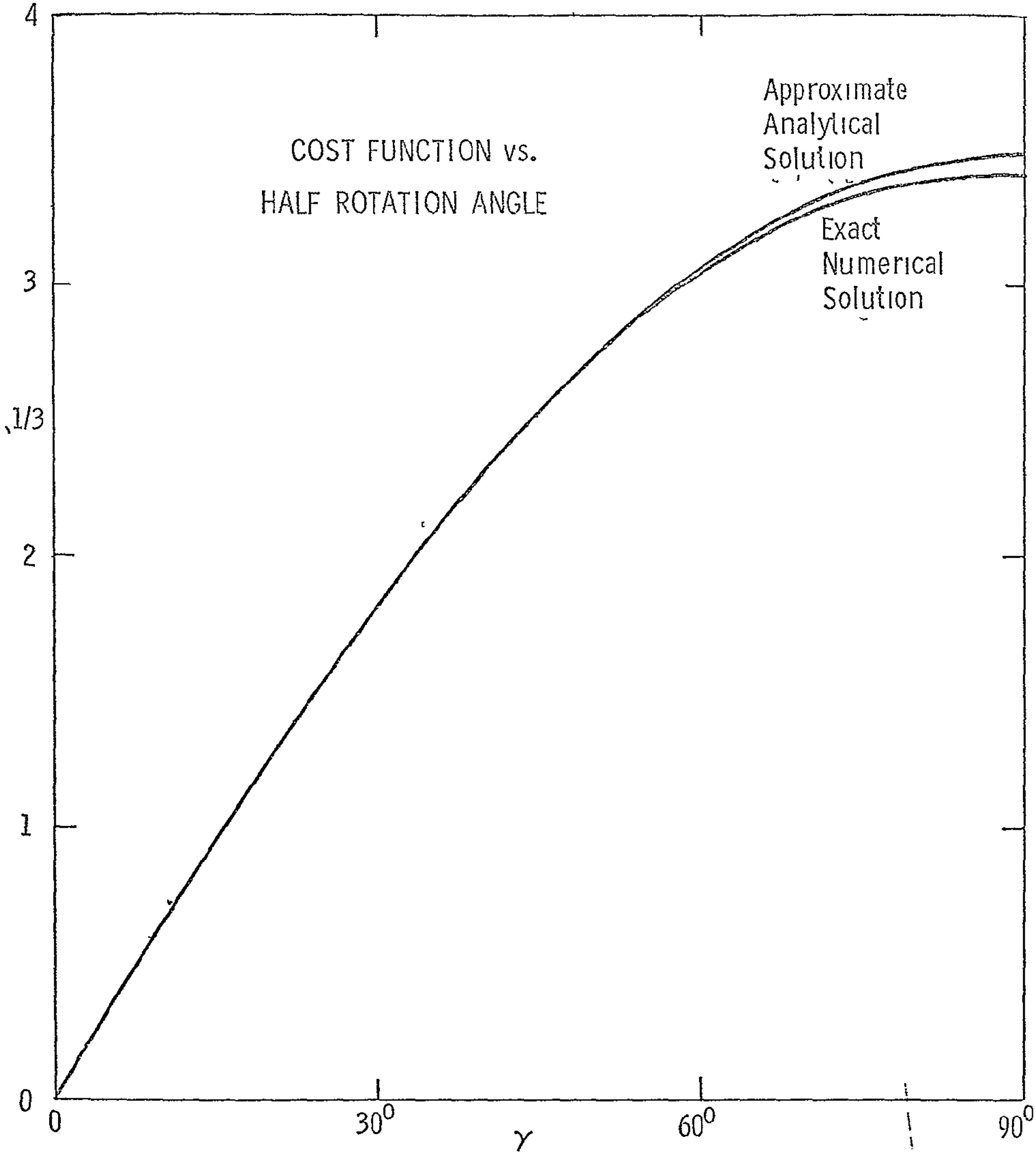
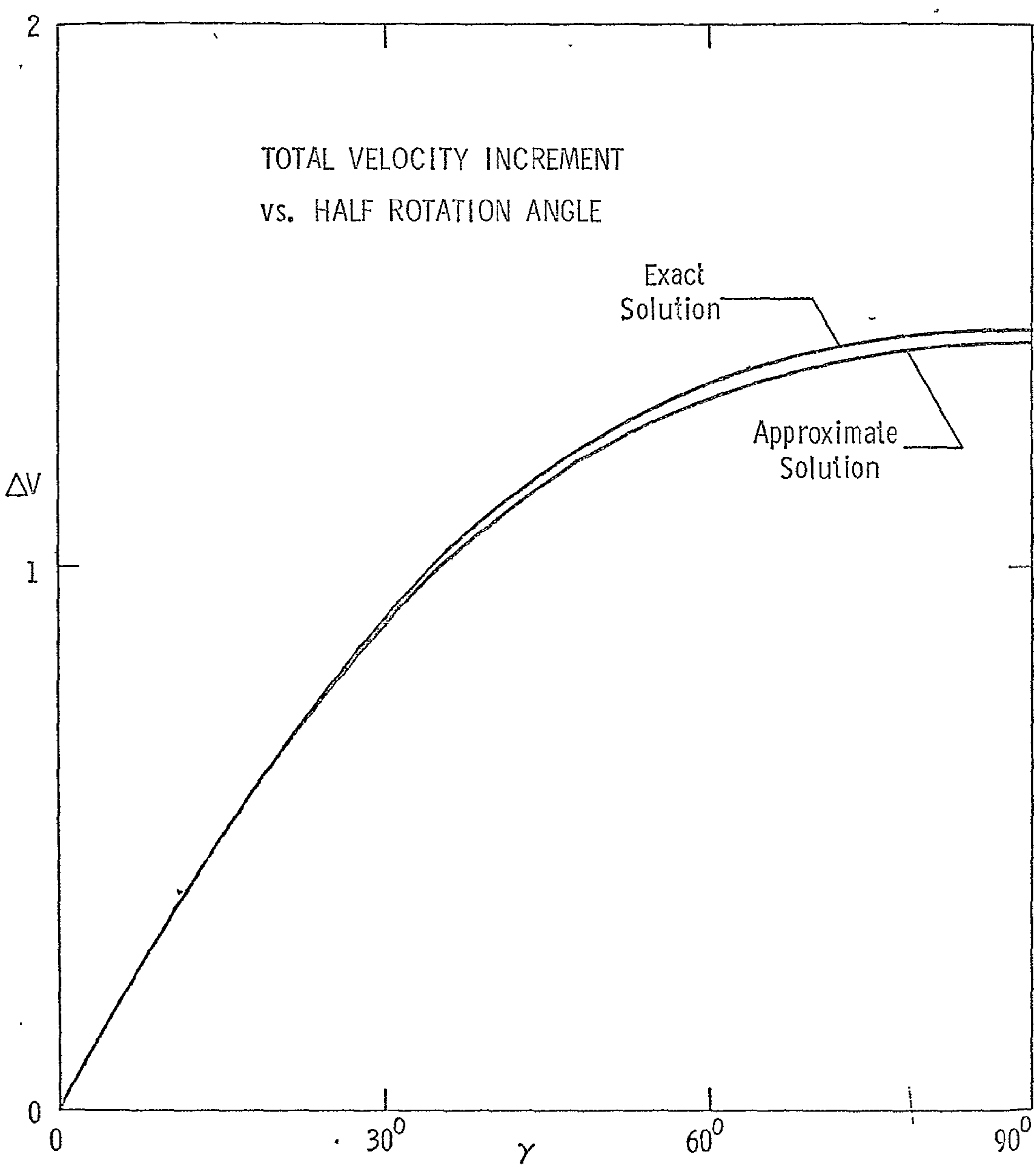
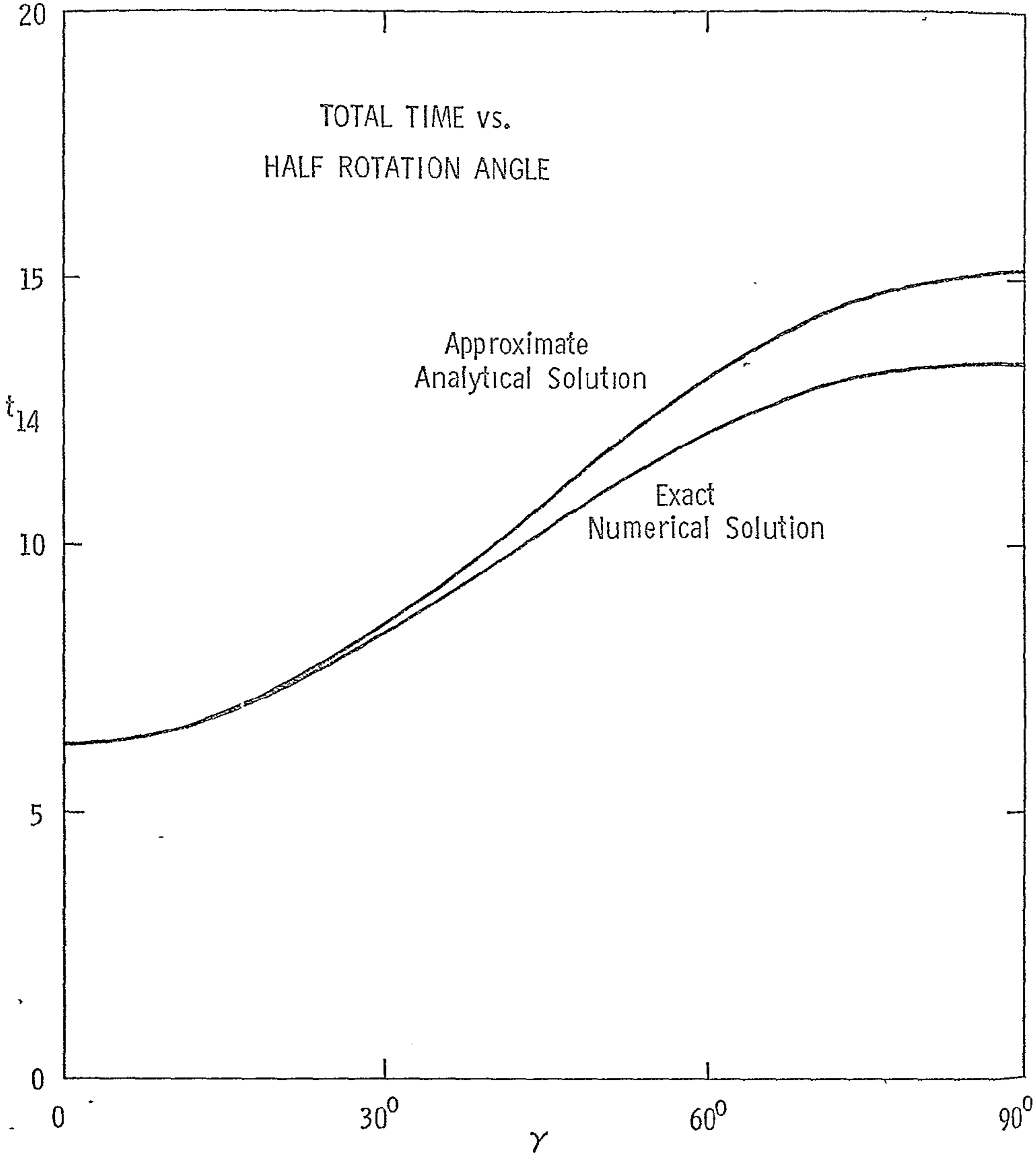

COST FUNCTION vs.
HALF ROTATION ANGLE
Approximate
Analytical
Solution
Exact
Numerical
Solution
4
3
1/3
2
1
0
0
30⁰
60⁰
90⁰
γ

Figure 2



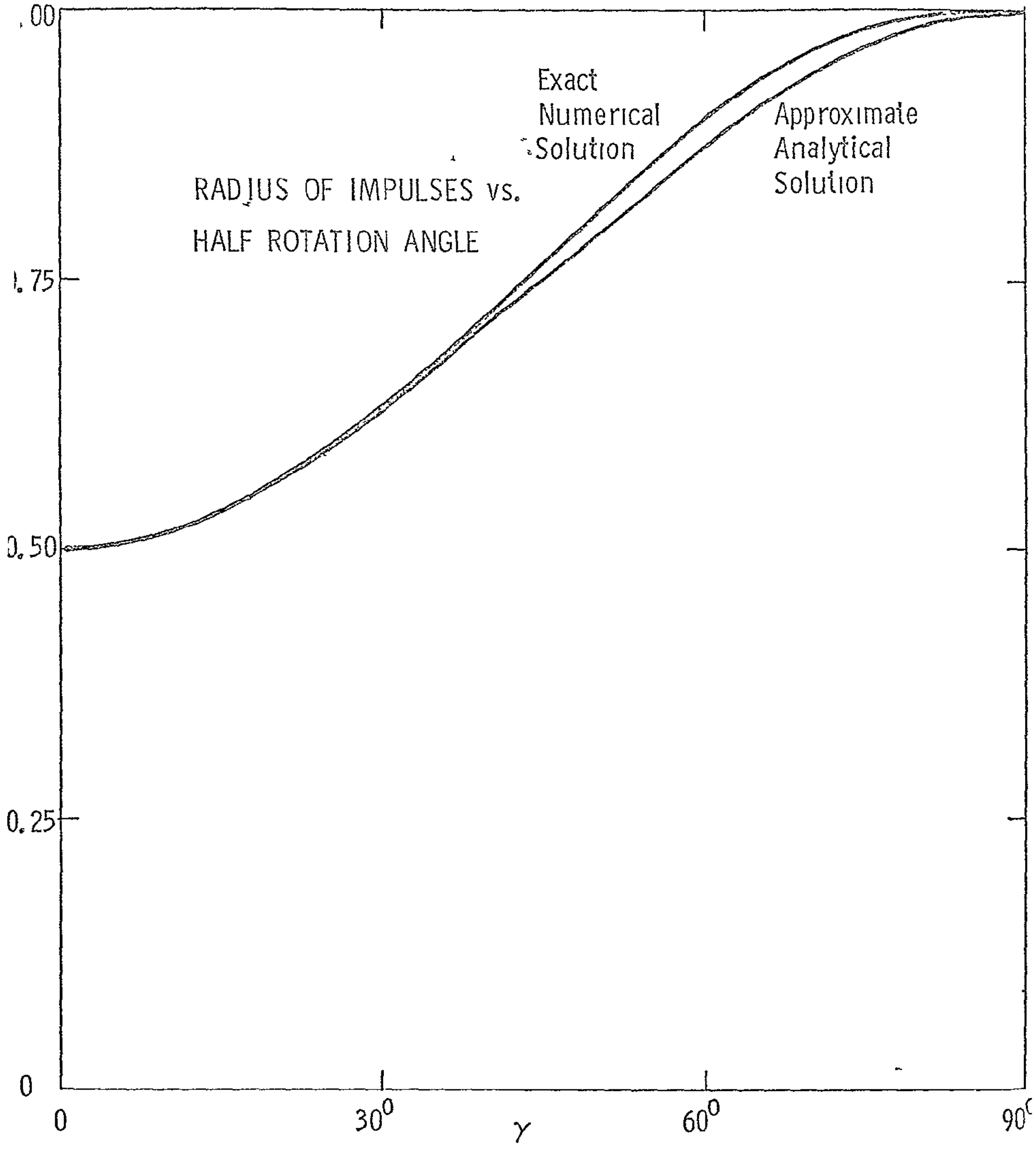
TOTAL VELOCITY INCREMENT
vs. HALF ROTATION ANGLE
Exact Solution
Approximate Solution
ΔV
2
1
0
0
30°
60°
90°
γ

Figure 3



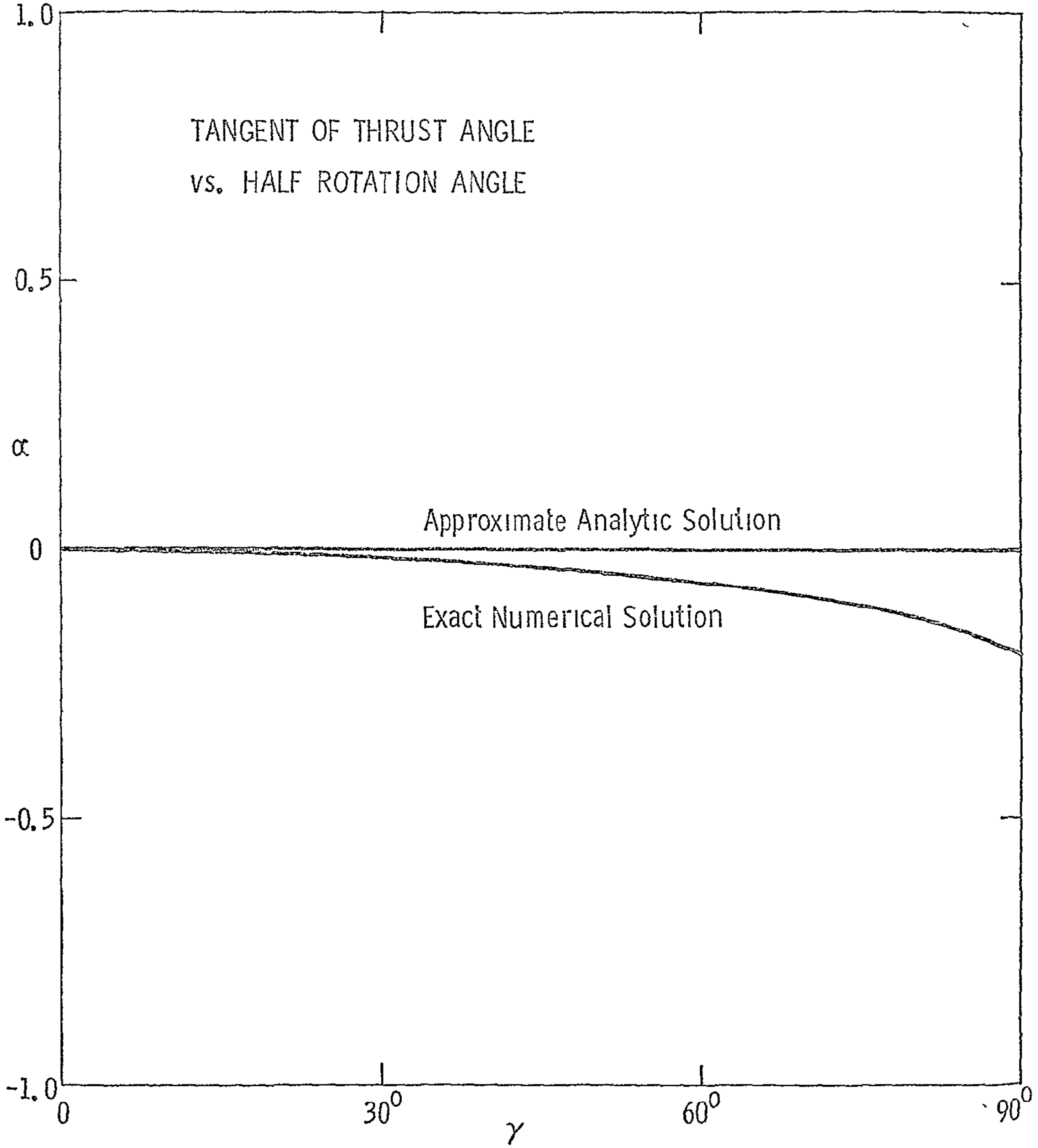
TOTAL TIME vs.
HALF ROTATION ANGLE
Approximate
Analytical Solution
Exact
Numerical Solution
t_{14}
20
15
10
5
0
0
30^0
60^0
90^0
γ

Figure 4



Exact
Numerical
Solution
Approximate
Analytical
Solution
RADIUS OF IMPULSES vs.
HALF ROTATION ANGLE
.00
.75
0.50
0.25
0
0
30⁰
γ
60⁰
90

Figure 5



TANGENT OF THRUST ANGLE
vs. HALF ROTATION ANGLE
1.0
0.5
α
0
-0.5
-1.0
Approximate Analytic Solution
Exact Numerical Solution
0
30°
60°
90°
γ

Figure 6

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II Variation of Paramcters

The position and velocity vectors $\underline{r}$ and $\underline{v}$ can be expressed in terms of their values $\underline{r}_0$ and $\underline{v}_0$ at some epoch time τ as follows

$$\begin{pmatrix} \underline{r}^T \\ \underline{v}^T \end{pmatrix} = \Psi \begin{pmatrix} \underline{r}_0^T \\ \underline{v}_0^T \end{pmatrix} = \begin{pmatrix} F & G \\ F_t & G_t \end{pmatrix} \begin{pmatrix} \underline{r}_0^T \\ \underline{v}_0^T \end{pmatrix} \tag{1}$$

where the scalar quantities F, G, F_t, G_t are defined by

$$F = 1 - \frac{U_2}{r_0} = \frac{1}{r_0} (r\, U_0 - \sigma\, U_1) \tag{2}$$

$$\sqrt{\mu}\, G = r_0 U_1 + \sigma_0 U_2 = r\, U_1 - \sigma\, U_2 = \sqrt{\mu}(t-\tau) - U_3 \tag{3}$$

$$F_t = - \frac{\sqrt{\mu}}{r r_0} U_1 \tag{4}$$

$$G_t = 1 - \frac{U_2}{r} = \frac{1}{r}(r_0\, U_0 + \sigma_0\, U_1) \tag{5}$$

with the notation

$$\sigma = \frac{1}{\sqrt{\mu}} \underline{r} \cdot \underline{v} \tag{6}$$

introduced for convenience. The following relations also obtain

$$r = r_0 U_0 + \sigma_0 U_1 + U_2 \qquad r_0 = r U_0 - \sigma U_1 + U_2 \tag{7}$$

$$\sigma = \sigma_0 U_0 + (1 - \alpha r_0) U_1 \qquad \sigma_0 = \sigma U_0 - (1 - \alpha r) U_1 \tag{8}$$

$$\sqrt{\mu}\,(t-\tau) = r_0 U_1 + \sigma_0 U_2 + U_3 = r U_1 - \sigma U_2 + U_3 \tag{9}$$

The variable x is the generalized anomaly difference and α, the reciprocal of the semimajor axis, is given by

$$\alpha = \frac{2}{r_0} - \frac{v_0^2}{\mu} = \frac{2}{r} - \frac{v^2}{\mu} \tag{10}$$

<u>Variations in the special transcendental functions</u>

The special transcendental functions U_0, U_1, . . . are defined by

$$U_n(x,\alpha) = x^n \left[\frac{1}{n!} - \frac{\alpha x^2}{(n+2)!} + \frac{(\alpha x^2)^2}{(n+4)!} - \cdots \right] \tag{11}$$

and can be shown to be related by the identities

$$U_n + \alpha U_{n+2} = \frac{x^n}{n!} \tag{12}$$

$$U_0^2 + \alpha U_1^2 = 1 \qquad U_1^2 - U_0 U_2 = U_2$$

$$U_0 U_3 - U_1 U_2 = U_3 - x U_2 \qquad U_1 U_3 - U_2^2 = 2 U_4 - x U_3 \tag{13}$$

We may also show that

$$\frac{\partial U_0}{\partial x} = -\alpha U_1 \qquad\qquad \frac{\partial U_0}{\partial \alpha} = -\frac{1}{2} x U_1$$

$$\frac{\partial U_n}{\partial x} = U_{n-1} \qquad\qquad \frac{\partial U_n}{\partial \alpha} = \frac{1}{2}(n U_{n+2} - x U_{n+1}) \tag{14}$$

The procedure for deriving the variational equations for U_0, U_1, . . . is simply to calculate the derivatives using Eq. (14), treating r as constant and replacing $d\underline{v}/dt$ by $\underline{a}_d$ and dx/dt by $d\xi/dt$, where $\underline{a}_d$ is the disturbing acceleration vector and ξ represents the change in x arising solely from changes in the orbital elements. The following relations result

$$\begin{aligned}
\frac{dU_0}{dt} &= -\alpha U_1 \left(\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt}\right) - \frac{1}{2} U_1^2 \frac{d\alpha}{dt} \\
\frac{dU_1}{dt} &= U_0 \left(\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt}\right) - \frac{1}{2} U_1 U_2 \frac{d\alpha}{dt} \\
\frac{dU_2}{dt} &= U_1 \left(\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt}\right) - \frac{1}{2} U_2^2 \frac{d\alpha}{dt} \\
\frac{dU_3}{dt} &= U_2 \frac{d\xi}{dt} + \frac{1}{2}(3 U_5 - x U_4) \frac{d\alpha}{dt}
\end{aligned} \tag{15}$$

where

$$\frac{1}{2}\frac{d\alpha}{dt} = -\frac{1}{\mu} \underline{v} \cdot \underline{a}_d \tag{16}$$

Variations in r_0, σ_0 and τ

Variational equations for r_0, σ_0 and τ may be calculated from Eqs. (7) - (9) by formal differentiation treating t as a constant and using Eqs (15). We find

$$\frac{d\,r_0}{dt} = -\ \sigma_0\left(\frac{d\xi}{dt} + \frac{1}{2}U_3\frac{d\alpha}{dt}\right) - \frac{1}{2}(r + r_0)\,U_2\frac{d\alpha}{dt} - U_1\frac{d\sigma}{dt} \tag{17}$$

$$\frac{d\,\sigma_0}{dt} = -\ (1 - \alpha r_0)\left(\frac{d\xi}{dt} + \frac{1}{2}U_3\frac{d\alpha}{dt}\right) + \frac{1}{2}(r + r_0)\,U_1\frac{d\alpha}{dt} + U_0\frac{d\sigma}{dt} \tag{18}$$

$$\sqrt{\mu}\ \frac{d\,\tau}{dt} = -\ r_0\left(\frac{d\xi}{dt} + \frac{1}{2}U_3\frac{d\alpha}{dt}\right) - \frac{1}{2}\sqrt{\mu}\ c\,\frac{d\alpha}{dt} + U_2\frac{d\sigma}{dt} \tag{19}$$

where, for convenience, we have defined

$$\sqrt{\mu}\ c = 3\,U_5 - x\,U_4 - U_2\sqrt{\mu}\,(t-\tau) \tag{20}$$

The perturbation derivative of σ is found using Eq. (6). Thus

$$\frac{d\sigma}{dt} = \frac{1}{\sqrt{\mu}}\ \underline{r}\cdot\underline{a}_d \tag{21}$$

Variation in the true anomaly difference

The functions U_n are also related to the true anomaly difference θ in the following manner

$$U_0 = 1 - \frac{\alpha\, r r_0}{p}\,(1 - \cos\theta)$$

$$U_1 = \frac{r}{\sqrt{p}}\,\sin\theta - \frac{r\,\sigma_0}{p}\,(1 - \cos\theta)$$

$$U_2 = \frac{r r_0}{p}\,(1 - \cos\theta) \qquad (22)$$

$$U_3 = \sqrt{\mu}\,(t-\tau) - \frac{r r_0}{\sqrt{p}}\,\sin\theta$$

We will now relate the variation ξ in the generalized anomaly difference x and the variation δ in the true anomaly difference θ. Formal differentiation of U_1 and U_2, replacing $d\theta/dt$ by $d\delta/dt$, yields

$$\frac{d\,U_1}{dt} = \left(\frac{r}{\sqrt{p}}\cos\theta - \frac{r\,\sigma_0}{p}\sin\theta\right)\frac{d\delta}{dt} - \frac{1}{p}\left(U_1 - \frac{r}{2\sqrt{p}}\sin\theta\right)\frac{dp}{dt} - \frac{U_2}{r_0}\,\frac{d\sigma_0}{dt}$$

$$\frac{d\,U_2}{dt} = \frac{r r_0}{p}\,\sin\theta\,\frac{d\delta}{dt} + \frac{U_2}{r_0}\,\frac{d r_0}{dt} - \frac{U_2}{p}\,\frac{dp}{dt}$$

Now we substitute from Eqs. (17) and (18) and then compare with Eqs (15) to obtain the following relationships between ξ and δ

$$F\left(\frac{d\xi}{dt} + \frac{1}{2}\,U_3\,\frac{d\alpha}{dt}\right) = \left(\frac{r}{\sqrt{p}}\cos\theta - \frac{r\,\sigma_0}{p}\sin\theta\right)\frac{d\delta}{dt}$$

$$- \frac{1}{p}\left(U_1 - \frac{r}{2\sqrt{p}}\sin\theta\right)\frac{dp}{dt}$$

$$- \frac{U_2}{r_0}\left(\frac{1}{2}\,r\,U_1\,\frac{d\alpha}{dt} + U_0\,\frac{d\sigma}{dt}\right)$$

$$\frac{\sqrt{\mu}\,G}{r_0}\left(\frac{d\xi}{dt} + \frac{1}{2}\,U_3\,\frac{d\sigma}{dt}\right) = \frac{r r_0}{p}\sin\theta\,\frac{d\delta}{dt} - \frac{U_2}{p}\,\frac{dp}{dt}$$

$$- \frac{U_2}{r_0}\left(\frac{1}{2}\,r\,U_2\,\frac{d\alpha}{dt} + U_1\,\frac{d\sigma}{dt}\right)$$

where F and G are defined in Eqs. (2) and (3).

If we multiply the first of these equations by G_t and the second by F_t, subtract and use the identity

$$F G_t - F_t G = 1 \tag{23}$$

we obtain

$$\sqrt{\mu}\left(\frac{d\xi}{dt} + \frac{1}{2}\,U_3\,\frac{d\alpha}{dt}\right) = \left[\sqrt{\mu}\;G_t\left(\frac{r}{\sqrt{p}}\cos\theta - \frac{r\sigma_0}{p}\sin\theta\right) - F_t\,\frac{r r_0^2}{p}\sin\theta\right]\frac{d\delta}{dt}\,.$$

$$+ \left[F_t\,\frac{r_0 U_2}{p} - \frac{\sqrt{\mu}\,G_t}{p}\left(U_1 - \frac{r}{2\sqrt{p}}\sin\theta\right)\right]\frac{dp}{dt}$$

$$- \sqrt{\mu}\,\frac{r}{2 r_0}\,U_1 U_2\,\frac{d\alpha}{dt} - \frac{\sqrt{\mu}}{r r_0}\,U_2 (r_0 + \sigma U_1)\,\frac{d\sigma}{dt}$$

Now

$$p = 2 r_0 - \alpha r_0^2 - \sigma_0^2$$

$$\cos\theta = 1 - \frac{p}{r r_0}\,U_2$$

$$\sin\theta = \frac{\sqrt{p}}{r r_0}\,(r_0 U_1 + \sigma_0 U_2)$$

so that, after simplification, we find

$$\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt} = \frac{r_0}{\sqrt{p}} \frac{d\delta}{dt} - \frac{1}{2r}\left[\frac{1}{r_0} U_1 U_2 + \frac{1}{p}(r_0 U_1 + \sigma_0 U_2)\right]\frac{dp}{dt}$$

$$- \frac{r}{2r_0} U_1 U_2 \frac{d\alpha}{dt} - \frac{U_2}{rr_0}(r_0 + \sigma U_1)\frac{d\sigma}{dt}$$

Finally, substituting from Eqs. (16), (21) and

$$\frac{1}{2}\frac{dp}{dt} = \frac{1}{\mu}(r^2 \underline{v} - \sqrt{\mu}\, \sigma \underline{r}) \cdot \underline{a}_d \tag{24}$$

we obtain the relation

$$\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt} = \frac{\sqrt{\mu}\, r_0}{h}\frac{d\delta}{dt} - \frac{\sqrt{\mu}}{r}\left(\frac{G}{h^2}\underline{h} \times \underline{r} + \frac{U_2}{\mu}\underline{r}\right) \cdot \underline{a}_d \tag{25}$$

Variation in the orbital elements

Five of the six orbital elements are contained in the vector expressions for angular momentum and eccentricity.

$$\underline{h} = \underline{r} \times \underline{v} \tag{26}$$

$$\mu \underline{e} = \underline{v} \times \underline{h} - \frac{\mu}{r}\underline{r} \tag{27}$$

Applying the formal differentiation rule, the following variational equations are obtained

$$\frac{d\underline{h}}{dt} = \underline{r} \times \underline{a}_d \tag{28}$$

$$\mu \frac{d\underline{e}}{dt} = \underline{a}_d \times (\underline{r} \times \underline{v}) + (\underline{a}_d \times \underline{r}) \times \underline{v} \quad . \tag{29}$$

In order to compute $\underline{r}$ and $\underline{v}$, we may use the instantaneous ascending node as a reference point. This is to be preferred over the choice of pericenter since the latter point does not exist for circular orbits.

From the vectors $\underline{h}$ and $\underline{e}$ the following calculations will produce $\underline{r}_0$ and $\underline{v}_0$

$$\underline{i}_n = \text{Unit}\,(\underline{i}_z \times \underline{h}) \tag{30}$$

$$\underline{r}_0 = \frac{h^2}{\mu + \mu\underline{e}\cdot\underline{i}_n}\,\underline{i}_n \tag{31}$$

$$\underline{v}_0 = \frac{1}{h^2}\{\mu\underline{h} \times \underline{i}_n + \mu\underline{e} \times [(\underline{h} \times \underline{i}_n) \times \underline{i}_n]\} \tag{32}$$

Then the vectors $\underline{r}$ and $\underline{v}$ are calculated from Eq. (1) by first solving the universal form of Kepler's equation for the generalized anomaly difference x

$$\sqrt{\mu}\,(t-\tau) = r_0 U_1(x,\alpha) + \sigma_0\, U_2(x,\alpha) + U_3(x,\alpha) \tag{33}$$

where

$$r_0 = \frac{h^2}{\mu + \mu\underline{e}\cdot\underline{i}_n}$$

$$\sigma_0 = \frac{1}{\sqrt{\mu}}\,(\underline{r}_0 \cdot \underline{v}_0) = \frac{r_0}{h^2\sqrt{\mu}}\,\mu\underline{e}\cdot\underline{i}_n \times \underline{h}$$

$$\alpha = \frac{\mu}{h^2}\,(1 - e^2)$$

The sixth orbital element is τ and a variational equation for it must be obtained. Indeed, Eqs (19) and (25) are appropriate for this purpose. Since $\theta = \omega + f$, it follows that

$$\frac{d\delta}{dt} = - \cos i \frac{d\Omega}{dt} = - \frac{\cot i}{h} \underline{i}_n \times \underline{r} \cdot \underline{a}_d \tag{34}$$

Hence,

$$\frac{d\tau}{dt} = \left[\frac{r_0^2}{h^2} \left(\cot i \; \underline{i}_n + \frac{G}{rr_0} \underline{h} \right) \times \underline{r} + \left(1 + \frac{r_0}{r} \right) \frac{U_2}{\mu} \underline{r} + \frac{c}{\mu} \underline{v} \right] \cdot \underline{a}_d \tag{35}$$

or, alternately,

$$\frac{d\tau}{dt} = \left\{ \frac{r_0^2}{h^2} \left[\frac{(\underline{i}_z \cdot \underline{h})(\underline{i}_z \times \underline{h})}{(\underline{i}_z \times \underline{h}) \cdot (\underline{i}_z \times \underline{h})} + \frac{G}{rr_0} \underline{h} \right] \times \underline{r} \right.$$

$$\left. + \left(1 + \frac{r_0}{r} \right) \frac{U_2}{\mu} \underline{r} + \frac{c}{\mu} \underline{v} \right\} \cdot \underline{a}_d \tag{36}$$

Although somewhat in violation of the spirit of the variational method, we may avoid the problem of the sixth orbital element by solving the following differential equation directly for θ

$$\frac{d\theta}{dt} = \frac{h}{r^2} - \frac{1}{h \tan i} \underline{i}_n \times \underline{r} \cdot \underline{a}_d \tag{37}$$

The vectors $\underline{r}$ and $\underline{v}$ are then readily calculated from

$$r = \frac{h^2}{\mu + \mu \underline{e} \cdot \underline{i}_n \cos \theta + \mu \underline{e} \cdot \underline{i}_m \sin \theta}$$

$$\underline{r} = r \cos \theta \; \underline{i}_n + r \sin \theta \; \underline{i}_m \tag{38}$$

$$\underline{v} = \frac{1}{h^2} \underline{h} \times \mu \underline{e} - \frac{1}{h} (\sin \theta \; \underline{i}_n - \cos \theta \; \underline{i}_m)$$

where

$$\underline{i}_m = \frac{1}{h} \underline{h} \times \underline{i}_n$$

Variation in the initial conditions

Equation (1) may be solved for $\underline{r}_0$ and $\underline{v}_0$ simply by replacing $t - \tau$ by $-(t - \tau)$ and x by $-x$ and interchanging the roles of $\underline{r}_0$, $\underline{v}_0$ and $\underline{r}$, $\underline{v}$. We have therefore,

$$\begin{pmatrix} \underline{r}_0^T \\ \\ \underline{v}_0^T \end{pmatrix} = \Psi^{-1} \begin{pmatrix} \underline{r}^T \\ \\ \underline{v}^T \end{pmatrix} = \begin{pmatrix} G_t & -G \\ \\ -F_t & F \end{pmatrix} \begin{pmatrix} \underline{r}^T \\ \\ \underline{v}^T \end{pmatrix} \tag{39}$$

where the determinant of the matrix Ψ is unity.

We may now calculate the perturbation derivatives of $\underline{r}_0$ and $\underline{v}_0$ by formal differentiation of Eq (39) and using relations previously derived We have

$$\frac{d}{dt} \begin{pmatrix} \underline{r}_0^T \\ \\ \underline{v}_0^T \end{pmatrix} = \frac{d\Psi^{-1}}{dt} \begin{pmatrix} \underline{r}^T \\ \\ \underline{v}^T \end{pmatrix} + \Psi^{-1} \begin{pmatrix} \underline{0} \\ \\ \underline{a}_d \end{pmatrix}$$

where

$$\frac{d\Psi^{-1}}{dt} = \begin{pmatrix} \frac{1}{\mu} U_2 \underline{v}^T \underline{a}_d & -\frac{r_0}{\sqrt{\mu}} \left(\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt} \right) \\ \frac{\sqrt{\mu}}{r_0^2} \left(\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt} \right) & \frac{r}{\mu} (\underline{v}_0^T - \underline{v}^T) \underline{a}_d \end{pmatrix} \Psi^{-1}$$

$$+ \begin{pmatrix} -\frac{1}{\mu} U_2 \underline{v}^T \underline{a}_d & \frac{1}{\mu} U_2 \underline{r}^T \underline{a}_d \\ \\ 0 & -\frac{r}{\mu} (\underline{v}_0^T - \underline{v}^T) \underline{a}_d \end{pmatrix}$$

If we require that the time τ associated with the epoch remain fixed, then we have

$$\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt} = \frac{1}{r_0}\left(U_2 \frac{d\sigma}{dt} - \frac{1}{2} \sqrt{\mu}\, c \frac{d\alpha}{dt}\right)$$

from Eq. (19) In this case, we obtain

$$\frac{d\underline{r}_0}{dt} = \frac{1}{\mu}\left\{\left[U_2(\underline{r}_0 - \underline{r}) - c\,\underline{v}_0\right]\underline{v}^T - U_2(\underline{v}_0 - \underline{v})\,\underline{r}^T\right\}\underline{a}_d - G\,\underline{a}_d \tag{40}$$

$$\frac{d\underline{v}_0}{dt} = \frac{1}{r_0^3}(U_2\, \underline{r}_0\underline{r}^T + c\, \underline{r}_0\underline{v}^T)\underline{a}_d + \frac{r}{\mu}(\underline{v}_0 - \underline{v})(\underline{v}_0 - \underline{v})^T\underline{a}_d + F\,\underline{a}_d \tag{41}$$

With appropriate initial conditions specified for $\underline{r}_0$ and $\underline{v}_0$, Eqs. (40) and (41) may be integrated by any appropriate numerical method. For each time step, the corresponding value of x is determined by solution of Kepler's equation (33). The instantaneous position and velocity vectors are then determined from Eq. (1).

On the other hand, τ may be permitted to vary in such a way that

$$\frac{d\xi}{dt} + \frac{1}{2} U_3 \frac{d\alpha}{dt} = 0$$

Then the variational equations take the form

$$\frac{d\underline{r}_0}{dt} = \frac{1}{\mu} U_2 \left[(\underline{r}_0 - \underline{r})\,\underline{v}^T + \underline{v}\,\underline{r}^T\right]\underline{a}_d - G\,\underline{a}_d \tag{42}$$

$$\frac{d\underline{v}_0}{dt} = \frac{r}{\mu}(\underline{v}_0 - \underline{v})(\underline{v}_0 - \underline{v})^T\,\underline{a}_d + F\,\underline{a}_d \tag{43}$$

$$\frac{d\tau}{dt} = \frac{1}{\mu}(c\,\underline{v} + U_2\,\underline{r})\cdot\underline{a}_d \tag{44}$$

We may, of course, avoid solving Kepler's equation, as well as the integration of Eq (44), by instead solving the following differential equation for x

$$\frac{dx}{dt} = \frac{\sqrt{\mu}}{r} + \frac{U_3}{\mu} \underline{v} \cdot \underline{a}_d \tag{45}$$

As pointed out by Born and Christensen, this eliminates the mixed secular term c from the equations.

Finally, we note still the presence of a mixed secular term U_3 in Eq (45). This may be eliminated by solving the following differential equation for θ instead of x

$$\frac{d\theta}{dt} = \frac{h}{r^2} + \frac{1}{h^2}[\sin\theta\, \underline{h} \times \underline{r} + h(1 - \cos\theta)\underline{r}] \cdot \underline{a}_d \tag{46}$$

which is obtained from Eq. (25). The vectors $\underline{r}$ and $\underline{v}$ are calculated from Eq. (1) where

$$\begin{aligned} F &= 1 - \frac{r}{p}(1 - \cos\theta) & G &= \frac{rr_0}{h}\sin\theta \\ F_t &= \frac{\sqrt{\mu}\,\sigma_0}{r_0 p}(1 - \cos\theta) - \frac{\mu}{r_0 h}\sin\theta & G_t &= 1 - \frac{r_0}{p}(1 - \cos\theta) \end{aligned} \tag{47}$$

and

$$r = \frac{p}{1 + \left(\frac{p}{r_0} - 1\right)\cos\theta - \frac{h\sigma_0}{\sqrt{\mu}\, r_0}\sin\theta} \tag{48}$$

III Perturbing Force Component Elimination from the Equations of Motion

This chapter discusses the elimination process which replaces one of the components of perturbing gravitational specific force (i e force per unit mass) driving the spacecraft equations of motion with a driving term depending on the perturbing gravitational specific potential energy (i e potential energy per unit mass) This substitution of one driving term for another in the equations of motion would have no effect on the solutions of the equations if the perturbing gravitational field were exactly known However, if only the orders of magnitude of the perturbing quantities are known (as has been assumed in the earlier work on the approximation of the estimation error covariance matrix), it is advantageous to employ potential energy information rather than force information when possible The reason for this is that two body orbital energy variations are proportional to the integral of the velocity times the perturbing force component along the trajectory and, based on a worst case analysis, the two body energy uncertainty could increase monotonically with time Because it is assumed that only the magnitude and not the direction of the perturbing force is available, the statistical calculations are in effect a worst case analysis and a two body energy uncertainty accumulation does occur Since energy uncertainties are equivalent to orbit period uncertainties, they cause secular position and velocity uncertainties which also grow monotonically with time If the perturbing specific force component which drives the spacecraft two body energy is replaced by the perturbing gravitational specific potential energy as a driving term in the equations of motion, the two body energy uncertainty is in effect limited to the perturbing specific potential energy uncertainty and remains small for all time

In the first section of this chapter, the elimination process is applied to the nonlinear spacecraft equations of motion so that the exact nonlinear equations of motion are obtained in a form in which they are driven by two components of the perturbing specific force and the perturbing specific potential energy This is done to illustrate the elimination process in a straightforward problem The results obtained in this section are not practically useful for two reasons First, the equations of motion must be written in a rather inconvenient coordinate system, called elimination coordinates Second, the nonlinear equations of motion driven by the unknown perturbing specific forces are never used explicitly in navigation calculations This is because the nominal or estimated trajectory is computed between navigation measurements by assuming that the perturbing forces are zero, and the statistical filtering calculations employ only the linearized equations of motion What is needed is a set of linearized equations of motion expressed in a convenient coordinate system with one perturbing specific force replaced by the perturbing specific potential energy

The general elimination problem is discussed in the second section, and a method is developed for carrying out the elimination process on the linearized equations of motion in any coordinate system

The third section presents an example in which the relevant equations are derived for the case in which the velocity magnitude is replaced as a state variable

A Elimination of a Force Component from the Nonlinear Equations of Motion

The spacecraft equations of motion about a planet are

$$\dot{\underline{r}} = \underline{v}$$

$$\dot{\underline{v}} = - \frac{\mu \underline{r}}{|\underline{r}|^3} + \underline{f}$$

where $\underline{r}$ and $\underline{v}$ are the position and velocity of the vehicle, $\underline{f}$ is the perturbing specific force, and μ is the gravitational constant of the planet Since $\underline{f}$ is a gravitational force, it may be expressed in terms of a perturbing gravitational specific potential energy $u(\underline{r}, t)$,

$$\underline{f} = - \underline{\nabla}\ u$$

The specific potential energy depends explicitly on time because of the rotation of the planet, and thus the total specific energy of the vehicle

$$\frac{1}{2} |\underline{v}|^2 - \frac{\mu}{|\underline{r}|} + u(\underline{r},\ t)$$

is not constant However, another energy-like integral of the vehicle motion may be obtained by writing the Hamiltonian function for the vehicle equations of motion expressed in coordinates which rotate with the planet When written in non-rotating coordinates this Hamiltonian takes the form

$$h = \frac{1}{2} |\underline{v}|^2 - \frac{\mu}{|\underline{r}|} + u(\underline{r},\ t) - \underline{\omega}\ (\underline{r} \times \underline{v}) \tag{1}$$

where $\underline{\omega}$ is the rotation rate of the planet It was verified in Reference 1 that it is indeed true that

$$\frac{dh}{dt} = 0$$

Employing this integral of the motion, h, one of the components of $\underline{f}$ can be eliminated from the equations of motion while introducing the perturbing specific potential energy as a new driving variable If h, the vehicle position $\underline{r}$, and two components of the vehicle velocity, say v_1 and v_2, are known it should be possible to solve for the third velocity component v_3 Thus

$$v_3 = w(h,\ \underline{r},\ v_1,\ v_2,\ u)$$

Note that it is necessary to know u to solve for v_3

Defining the vehicle state vector to be

$$\underline{x} = [\underline{r},\ h,\ v_1,\ v_2]^T$$

the vehicle equations of motion take the form

$$r_1 = v_1$$

$$r_2 = v_2$$

$$\dot{r}_3 = w(h, \underline{r}, v_1, v_2, u)$$

$$\dot{h} = 0$$

$$\dot{v}_1 = -\frac{\mu r_1}{|\underline{r}|^3} + f_1$$

$$\dot{v}_2 = -\frac{\mu r_2}{|\underline{r}|^3} + f_2$$

Since the differential equation for v_3 no longer appears in the set of equations of motion, f_3 no longer appears as a driving variable However, a new variable u has appeared to replace f_3

A slightly more complex approach must be taken since it is desired to eliminate the component of force which lies along the vehicle trajectory at each given instant rather than eliminating a force component in a fixed direction in space (The along-track force component is to be eliminated since it is the component which drives the vehicle two body energy) To accomplish this, let θ and ϕ be two angular coordinates which determine the direction of the velocity vector,

$$\underline{v} = |\underline{v}| \, \underline{e}_v(\theta, \phi)$$

where the unit vector $\underline{e}_v$ is given by

$$\underline{e}_v(\theta, \phi) = [\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi]^T$$

In addition, two orthogonal unit vectors $\underline{e}_\theta$ and $\underline{e}_\phi$ perpendicular to $\underline{e}_v$ will be defined at this point by

$$\underline{e}_\theta = [-\sin\theta,\ \cos\theta,\ 0]^T = \frac{1}{\cos\phi}\ \frac{\partial\ \underline{e}_v}{\partial\ \theta}\,.$$

$$\underline{e}_\phi = [-\cos\theta\ \sin\phi,\ -\sin\theta\ \sin\phi,\ \cos\phi] = \frac{\partial\ \underline{e}_v}{\partial\ \phi}$$

Now, given h, $\underline{r}$, θ and ϕ, it should be possible to solve for $|\underline{v}|$,

$$|\underline{v}| = \nu(h,\ \underline{r},\ \theta,\ \phi,\ u)$$

Carrying out the calculations yields

$$\nu = \underline{\omega}\ [\underline{r} \times \underline{e}_v(\theta,\ \phi)] + \sqrt{\{\underline{\omega}\ [\underline{r} \times \underline{e}_v(\theta,\ \phi)]\}^2 + 2\,(h + \frac{\mu}{|\underline{r}|} - u)} \quad (2)$$

Since the along-track component of perturbing specific force drives the magnitude of the velocity, in analogy with the preceeding example the magnitude of the velocity should be eliminated from the state vector to eliminate the along-track specific force component Thus, let the new state vector be

$$y = [\underline{r},\ \theta,\ \phi,\ h]^T$$

This set of state variables is called the elimination coordinate system

To find the vehicle equations of motion in this coordinate system, differentiate the expression

$$\underline{v} = \nu\, \underline{e}_v(\theta,\ \phi)$$

yielding

$$\underline{v} = \dot{\nu}\,\underline{e}_v + \nu\,\theta \frac{\partial \underline{e}_v}{\partial \theta} + \nu\,\phi \frac{\partial \underline{e}_v}{\partial \phi}$$

$$\underline{v} = \nu\,\underline{e}_v + \nu\,\theta \cos\phi\,\underline{e}_\theta + \nu\phi\,\underline{e}_\phi$$

Since $\underline{e}_v$, $\underline{e}_\theta$ and $\underline{e}_\phi$ form an orthogonal set of unit vectors, it follows that

$$\dot{\theta} = \frac{\underline{e}_\theta\ \dot{\underline{v}}}{\nu \cos\phi}, \quad \phi = \frac{\underline{e}_\phi\ \dot{\underline{v}}}{\nu}$$

and the equations of motion in the new coordinate system become

$$\dot{\underline{r}} = \nu(\underline{r}, \theta, \phi, h, u)\,\underline{e}_v(\theta, \phi)$$

$$\dot{\theta} = \left[\frac{-\mu\,\underline{e}_\theta(\theta, \phi)\ \underline{r}}{|\underline{r}|^3} + f_\theta\right] / \cos\phi\ \nu(\underline{r}, \theta, \phi, h, u)$$

$$\dot{\phi} = \left[\frac{-\mu\,\underline{e}_\phi(\theta, \phi)\ \underline{r}}{|r|^3} + f_\phi\right] / \nu(\underline{r}, \theta, \phi, h, u)$$

$$h = 0$$

where $f_\theta = \underline{f}\ \underline{e}_\theta$, $f_\phi = \underline{f}\ \underline{e}_\phi$

Thus the component of perturbing specific force, $f_v = \underline{f}\ \underline{e}_v$, along the velocity vector has been eliminated as a driving variable, but the equations of motion now depend on the perturbing potential energy u

B Force Component Elimination for Linearized Equations of Motion

1) Nonlinear Change of Coordinates

Consider a conservative dynamical system whose equations of motion are

$$\underline{x} = \underline{g}(\underline{x}) + G(\underline{x})\,\underline{f}$$

where $\underline{x}$ is the state vector consisting of the system position and velocity in some coordinate system and the vector $\underline{f}$ represents the components of a conservative specific force derived from a conservative specific potential energy u which acts on the system (Note that if the functions $\underline{g}$ and G depend explicitly on time, the vector $\underline{x}$ can be augmented by adding a state variable x_{n+1} satisfying the differential equation

$$x_{n+1} = 1$$

With the application of the proper initial condition,

$$x_{n+1} = t$$

and the time dependence of $\underline{g}$ and G can be replaced by dependence on x_{n+1} The purpose of this transformation is to show that explicit time dependence of $\underline{g}$ and G has no effect on the form of the resulting equations, hence the transformation need not be employed in actual navigation computations)

It is assumed that the system has an integral

$$H(\underline{x}, u) = \tilde{H}(x) + u$$

that is, at any point on any trajectory of the system

$$\frac{dH}{dt} = 0 \tag{3}$$

It will also be assumed that

$$\frac{\partial \tilde{H}(\underline{x})}{\partial \underline{x}}\,\underline{g}(\underline{x}) = 0 \tag{4}$$

for every $\underline{x}$ This means that $H(\underline{x}, u)$ is also an integral of the system when $\underline{f}$ and u are identically zero Further, consider a nonlinear change of state variables which depends on the value of the specific potential energy u,

$$\underline{x} = \underline{m}(\underline{y}, u)$$

where $\underline{y}$ represents the set of transformed state variables

In the remainder of this section it will be shown, without making use of the explicit form of the specific force $\underline{f}$, that the transformed state vector $\underline{y}$ satisfies a differential equation

$$\dot{\underline{y}} = \underline{w}(\underline{y}, u, \underline{f})$$

in which the specific potential energy u appears as a driving variable as well as the conservative specific force $\underline{f}$ By the proper choice of the transformation $\underline{m}$, it may be possible to make the function $\underline{w}$ in the differential equation above independent of one of the components of the specific force $\underline{f}$, thus substituting the specific potential energy u for one of the specific force components as a driving variable In order to derive the differential equation for $\underline{y}$ it will be necessary to assume that the matrix $\frac{\partial \underline{m}}{\partial \underline{y}}$ is nonsingular along the trajectories of interest This should not be considered a severe restriction for most problems

To derive the differential equation for $\underline{y}$, note that

$$\dot{\underline{x}} = M\dot{\underline{y}} - \underline{e}\,\dot{u}$$

where

$$M(\underline{y}, u) = \frac{\partial \underline{m}(\underline{y}, u)}{\partial \underline{y}}$$

$$\underline{e}(\underline{y}, u) = -\frac{\partial \underline{m}(\underline{y}, u)}{\partial u}$$

Now, since H is an integral of the forced system, evaluation of Eq (3) gives

$$0 = \frac{dH}{dt} = \frac{\partial \widetilde{\Pi}(\underline{x})}{\partial \underline{x}} \left\{ \underline{g}(\underline{x}) + G(\underline{x})\underline{f} \right\} + \frac{du}{dt}$$

and using the fact that $\tilde{H}(\underline{x})$ is an integral of the unforced system, i e Eq. (4), it follows that

$$\frac{du}{dt} = -\frac{\partial \tilde{H}(\underline{x})}{\partial \underline{x}} G(\underline{x})\underline{f} \tag{5}$$

or

$$\frac{du}{dt} = \underline{r}^T(\underline{y}, u)\,\underline{f}$$

where

$$\underline{r}^T(\underline{y}, u) = -\frac{\partial \tilde{H}(\underline{x})}{\partial \underline{x}} G(\underline{x})\Bigg|_{\underline{x} = \underline{m}(\underline{y}, u)} \tag{6}$$

Thus $\underline{y}$ satisfies the differential equation

$$\dot{\underline{y}} = \underline{w}(\underline{y}, u, \underline{f})$$

where

$$\underline{w}(\underline{y}, u, \underline{f}) = M^{-1}(\underline{y}, u)\Big\{\underline{g}(\underline{m}(\underline{y}, u)) + G(\underline{m}(y, u))\underline{f} + \underline{e}(\underline{y}, u)\,\underline{r}^T(\underline{y}, u)\underline{f}\Big\} \tag{7}$$

2) Transformation of Coordinates for Linearized Equations of Motion

Let $\underline{x}_1(t)$ be a trajectory of the system computed with the specific force $\underline{f}$ set equal to zero, i e $\underline{x}_1(t)$ satisfies

$$\frac{d\underline{x}_1(t)}{dt} = \underline{g}(\underline{x}_1)$$

Then to first order, deviations from this trajectory due to initial condition variations and the conservative specific force $\underline{f}$ satisfy the linear differential equation

$$\delta \dot{\underline{x}} (t) = F (t) \delta \underline{x} (t) + G (t) \underline{f} (t) \tag{8}$$

with the initial condition

$$\delta \underline{x} (t_o) = \underline{x} (t_o) - \underline{x}_1 (t_o)$$

and

$$F (t) = \left. \frac{\partial \underline{g} (\underline{x})}{\partial (\underline{x})} \right|_{\underline{x} = \underline{x}_1 (t)}$$

$$G (t) = G (\underline{x}_1 (t))$$

Solving the equation

$$\underline{x}_1 (t) = \underline{m} (\underline{y}_1 (t), 0)$$

for $\underline{y}_1 (t)$ yields a system trajectory in $\underline{y}$ coordinates, and to first order deviations from this trajectory due to initial condition variations and the conservative specific force $\underline{f}$ satisfy the linear differential equation

$$\delta \dot{\underline{y}} (t) = W (t) \delta \underline{y} (t) + \underline{a} (t) u (t) + \Omega (t) \underline{f} (t)$$

with the initial condition determined from the equation

$$\delta \underline{x} (t_o) = M (t_o) \delta \underline{y} (t_o) - \underline{e} (t_o) u (t_o)$$

where

$$W(t) = \left. \frac{\partial \underline{w} (\underline{y}, u, \underline{f})}{\partial \underline{y}} \right|_{\underline{y} = \underline{y}_1 (t), \ u = 0, \ \underline{f} = \underline{0}}$$

$$\underline{a}(t) = \left.\frac{\partial \underline{w}(\underline{y}, u, \underline{f})}{\partial u}\right|_{\underline{y} = \underline{y}_1(t),\ u = 0,\ \underline{f} = \underline{0}}$$

$$\Omega(t) = \left.\frac{\partial \underline{w}(\underline{y}, u, \underline{f})}{\partial \underline{f}}\right|_{\underline{y} = \underline{y}_1(t),\ u = 0,\ \underline{f} = \underline{0}}$$

$$M(t) = M(\underline{y}, u)\Big|_{\underline{y} = \underline{y}_1(t),\ u = 0} = \underline{x}_1(t)$$

Since the $\underline{y}$ coordinate system was employed to eliminate the influence of one of the components of the specific force $\underline{f}$ as a control variable, this effect should be seen in the variational equation for $\underline{y}$, specifically in the matrix $\Omega(t)$

Now from Eq (7) it follows that

$$\Omega(t) = M^{-1}(t)\left\{G(\underline{m}(\underline{y}, u)) + \underline{e}(\underline{y}, u)\,\underline{r}^T(\underline{y}, u)\right\}\Bigg|_{\underline{y} = \underline{y}_1(t),\ u = 0} \tag{9}$$

and since

$$\underline{m}(\underline{y}, u)\Big|_{\underline{y} = \underline{y}_1(t),\ u = 0} = \underline{x}_1(t)$$

it follows that

$$G(\underline{m}(\underline{y}, u))\Big|_{\underline{y} = \underline{y}_1(t),\ u = 0} = G(t)$$

and from Eq (6)

$$\underline{r}^T(\underline{y}, u)\Big|_{\underline{y} = \underline{y}_1(t),\ u = 0} = -\frac{\partial \tilde{H}(\underline{x})}{\partial \underline{x}}\, G(\underline{x})\Bigg|_{\underline{x} = \underline{x}_1(t)} \tag{10}$$

Thus

$$\underline{\Omega}(t) = M^{-1}(t)\left\{ I - \underline{e}(t)\,\frac{\partial \widetilde{H}(\underline{x})}{\partial \underline{x}} \right\} G(\underline{x})\Bigg|_{\underline{x} = \underline{x}_1(t)}$$

where

$$\underline{e}(t) = \underline{e}(y, u)\Bigg|_{\underline{y} = \underline{y}_1(t),\ u = 0}$$

The matrix in brackets above has the form of a projection matrix and by proper manipulation of the vector $\underline{e}$, which depends upon the coordinate transformation $\underline{m}(\underline{y}, u)$, it is possible to reduce the rank of $\Omega(t)$ to one less than the rank of G (t) provided that

$$\frac{\partial \widetilde{H}(\underline{x}_1)}{\partial \underline{x}_1}\, G(t) \neq 0$$

For example, if the effect of a particular specific force vector $\underline{f}_1$ is to be eliminated, the transformation $\underline{m}(\underline{y}, u)$ must be chosen so that

$$\underline{e}(t) = \frac{1}{\dfrac{\partial \widetilde{H}(\underline{x}_1)}{\partial \underline{x}_1}\, G(t)\,\underline{f}_1(t)}\; G(t)\,\underline{f}_1(t)$$

The examples in the preceeding section show that it is indeed possible to do this for the along-track component of specific force

Because the $\underline{y}$ coordinate system was chosen in a special way to eliminate the effect of a particular specific force component, it may not be a particularly convenient coordinate system to use for numerical integration of the variational equations Consider a linear transformation

$$\underline{z}(t) = K(t)\;\delta\,\underline{y}(t)$$

employed to transform the variational equation to a more convenient coordinate system for computation. The transformed variation $\underline{z}$ (t) satisfies the linear differential equation

$$\dot{\underline{z}}(t) = K(t)\ \delta\dot{\underline{y}}(t) + \dot{K}(t)\ \delta\underline{y}(t)$$
$$= K(t)\left[W(t)\ \delta\underline{y}(t) + \underline{a}(t)\,u(t) + \Omega(t)\,\underline{f}(t)\right] + \dot{K}(t)\ \delta\underline{y}(t)$$

or

$$\dot{\underline{z}}(t) = \left[K(t)\,W(t)\,K^{-1}(t) + \dot{K}(t)\,K^{-1}(t)\right]\underline{z}(t) + K(t)\,\underline{a}(t)\,u(t) + K(t)\,\Omega(t)\,\underline{f}(t)$$

If the effect of a specific force component $\underline{f}_1$ has been eliminated in $\underline{y}$ coordinates, it will still be eliminated in $\underline{z}$ coordinates since $\Omega\,\underline{f}_1 = 0$ implies that $K\,\Omega\,\underline{f}_1 = 0$

A particularly convenient choice for the transformation matrix K(t) is M(t) since this transforms $\delta\underline{y}$ back to the original $\underline{x}$ state space. In this case the transformed $\underline{y}$ variation will be denoted $\delta\underline{x}^*$, where

$$\delta\underline{x}^*(t) = M(t)\ \delta\underline{y}(t)$$

Using the following identities which will be proved later,

$$M(t)\ W(t)\ M^{-1}(t) + \dot{M}(t)\ M^{-1}(t) = F(t) \tag{11}$$

$$M(t)\ \underline{a}(t) = \underline{\dot{e}}(t) - F(t)\ \underline{e}(t) \tag{12}$$

$$M(t)\ \Omega(t) = \left[I - \underline{e}(t)\frac{\partial\,\tilde{H}(\underline{x})}{\partial\,\underline{x}}\right] G(\underline{x})\Bigg|_{\underline{x} = \underline{x}_1(t)} \tag{13}$$

it follows that $\delta\underline{x}^*$ satisfies the differential equation

$$\delta \dot{\underline{x}}^*(t) = F(t)\ \delta \underline{x}^*(t) + [\ \dot{\underline{e}}(t) - F(t)\ \underline{e}(t)]\ u(t) + [I - \underline{e}(t) \frac{\partial \widetilde{H}(\underline{x}_1)}{\partial \underline{x}_1}]\ G\ (\underline{x}_1) \underline{f}(t) \tag{14}$$

Transforming the initial condition for the $\delta \underline{y}$ equation gives

$$\delta \underline{x}^*(t_o) = \delta \underline{x}(t_o) + \underline{e}(t_o)\ u(t_o')$$

Since there is nothing special in a physical sense about the initial time t_o, one is led to hypothesize that

$$\delta \underline{x}(t) = \delta \underline{x}^*(t) - \underline{e}(t)\ u(t) \tag{15}$$

to first order for all times t To show that this is indeed correct, it will be shown that $\delta \underline{x}(t)$ as defined in Eq (15) satisfies the correct differential equation, Eq (8), to first order Differentiating Eq (15) gives

$$\delta \dot{\underline{x}}(t) = \delta \dot{\underline{x}}^*(t) - \dot{\underline{e}}(t)\ u(t) - \underline{e}(t)\ \dot{u}(t)$$

and substituting Eq (14) yields

$$\delta \dot{\underline{x}}(t) = F(t)\ \delta \underline{x}(t) + G(t)\ \underline{f}(t) - \underline{e}(t)\ [\dot{u}(t) + \frac{\partial \widetilde{H}(\underline{x}_1)}{\partial \underline{x}_1}\ G\ (\underline{x}_1)\ \underline{f}(t)]$$

Thus $\delta \dot{\underline{x}}$ satisfies the correct differential equation except for the term

$$- \left\{ \dot{u}(t) + \left[\frac{\partial \widetilde{H}(\underline{x})}{\partial \underline{x}}\ G(\underline{x}) \right] \Bigg|_{\underline{x} = \underline{x}_1(t)} \underline{f}(t) \right\} \underline{e}(t)$$

However, by Eq (5)

$$\dot{u}(t) = -\frac{\partial \tilde{H}(\underline{x})}{\partial \underline{x}} G(\underline{x}) \underline{f}(t)$$

Thus the error term becomes

$$\underline{e}(t) \left[\frac{\partial \tilde{H}(\underline{x})}{\partial \underline{x}} G(\underline{x}) - \frac{\partial \tilde{H}(\underline{x}_1)}{\partial \underline{x}_1} G(\underline{x}_1) \right] \underline{f}(t)$$

which is the product of an $\underline{x}$ deviation and $\underline{f}(t)$ and is therefore second order Thus $\delta \underline{x}$ as defined in Eq (15) satisfies the correct differential equation to first order and has the correct initial condition and therefore Eq (15) is correct to first order

Derivation of Eqs (11) - (13) follows (It should be remembered that the partial derivatives are evaluated along the reference trajectory, i e u = 0, $\underline{f}$ = 0) Eq (13) follows directly from Eqs (9) and (10)

To derive Eq (11), note that

$$\underline{w}(\underline{y}, 0, \underline{0}) = M^{-1}(\underline{y}, 0)\, \underline{g}(\underline{m}(\underline{y}, 0))$$

Then, using simplified notation,

$$\frac{\partial \underline{w}}{\partial y_i} = -M^{-1} \frac{\partial M}{\partial y_i} M^{-1} \underline{g} + M^{-1} F \frac{\partial \underline{m}}{\partial y_i}$$

Now

$$\frac{\partial M}{\partial y_i} M^{-1} \underline{g} = \frac{\partial M}{\partial y_i} \underline{w} = \frac{\partial M}{\partial y_i} \frac{d\underline{y}}{dt}$$

$$= \sum_j \frac{\partial^2 \underline{m}(\underline{y}, 0)}{\partial y_i \, \partial y_j} \frac{dy_j}{dt}$$

$$= \frac{d}{dt} \left(\frac{\partial \underline{m}}{\partial y_i} \right)$$

Thus

$$W = \frac{\partial \underline{w}}{\partial \underline{y}} = - M^{-1} M + M^{-1} F M$$

and Eq (11) follows

The proof of Eq (12) is given below By definition

$$\underline{a}(t) = \left. \frac{\partial \underline{w}(\underline{y}, u, \underline{f})}{\partial u} \right|_{\underline{y} = \underline{y}_1(t),\ u = 0,\ \underline{f} = \underline{0}}$$

and when $\underline{f} = \underline{0}$,

$$\underline{w}(\underline{y}, u, \underline{0}) = M^{-1}(\underline{y}, u)\, \underline{g}(\underline{m}(\underline{y}, u))$$

Then

$$\frac{\partial \underline{w}}{\partial u} = -M^{-1} \frac{\partial M}{\partial u} M^{-1} \underline{g} - M^{-1} F \underline{e}$$

or

$$M \underline{a} = - \frac{\partial M}{\partial u} \underline{w} - F \underline{e}$$

Now

$$\left\{ \frac{\partial M}{\partial u} \underline{w} \right\}_i = \sum_j \frac{\partial M_{ij}}{\partial u} w_j$$

$$= \sum_j \frac{\partial^2 m_i}{\partial y_j\, \partial u} w_j$$

$$= - \sum_j \frac{\partial e_i}{\partial y_j} \frac{dy_j}{dt}$$

Since u is identically zero along the reference trajectory, we have

$$\frac{d e_i}{dt} = \sum_j \frac{\partial e_i}{\partial y_j} \frac{d y_j}{dt}$$

and thus

$$\left\{ \frac{\partial M}{\partial u} \underline{w} \right\}_i = - \frac{d e_i}{dt}$$

and finally

$$M \underline{a} = \frac{d \underline{e}}{dt} - F \underline{e}$$

C Velocity Magnitude Elimination Example

In this section we calculate the relevant quantities associated with the velocity magnitude elimination example presented in Section A in order to convey a feeling for the procedure involved in utilizing the elimination technique.

1) Necessary partial derivatives

$$h = \frac{1}{2} \nu^2 - \frac{\mu}{r} - \nu \underline{\omega} \quad (\underline{r} \times \underline{e}_v) + u \tag{16}$$

a) Calculations of $\frac{\partial \nu}{\partial u}$

Because h is an integral of the system, total differentiation of Eq (16) with respect to u yields

$$0 = \nu \frac{\partial \nu}{\partial u} - \underline{\omega} \quad (\underline{r} \times \underline{e}_v) \frac{\partial \nu}{\partial u} + 1$$

Thus

$$\frac{\partial \nu}{\partial u} = \frac{-1}{\nu - \underline{\omega} \quad (\underline{r} \times \underline{e}_v)}$$

or

$$\frac{\partial \nu}{\partial u} = \frac{-|\underline{v}|}{[\underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r})]}$$

b) Calculation of $\frac{\partial \nu}{\partial \underline{r}}$

Differentiation of Eq (16) with respect to $\underline{r}$ gives

$$0 = \nu \frac{\partial \nu}{\partial \underline{r}} + \frac{\mu}{|\underline{r}|^3} \underline{r}^T - \underline{\omega} \quad (\underline{r} \times \underline{e}_v) \frac{\partial \nu}{\partial \underline{r}} + (\underline{\omega} \times \underline{v})^T$$

Thus

$$\frac{\partial \nu}{\partial \underline{r}} = \frac{-[\frac{\mu}{|\underline{r}|^3} \underline{r}^T + (\underline{\omega} \times \underline{v})^T] \quad |\underline{v}|}{[\underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r})]}$$

2) Calculation of the elimination vector, $\underline{e}$

$$\underline{m} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \underline{e}_v \ (\theta, \ \phi) \ \nu \ (\underline{r}, \ \theta, \ \phi, \ h, \ u) \end{bmatrix}$$

$$\underline{y} = [r_1, \ r_2, \ r_3, \ \theta, \ \phi, \ h]^T$$

$$\underline{e} = -\frac{\partial \underline{m}}{\partial \underline{u}} = \begin{bmatrix} \underline{0} \\ -\underline{e}_v \end{bmatrix} \frac{\partial \nu}{\partial u}$$

Thus

$$\underline{e} = \frac{1}{[\underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r})]} \begin{bmatrix} \underline{0} \\ \underline{v} \end{bmatrix}$$

3) Calculation of M

$$M = \frac{\partial \underline{m}}{\partial \underline{y}}$$

Note that

$$\frac{\partial \underline{e}_v}{\partial \theta} = \cos\phi \, \underline{e}_\theta, \quad \frac{\partial \underline{e}_v}{\partial \phi} = \underline{e}_\phi$$

$$M = \left[\begin{array}{c|c|c|c} I_{3\times 3} & \multicolumn{3}{c}{O_{3\times 3}} \\ \hline \underline{e}_v \dfrac{\partial \nu}{\partial \underline{r}} & \nu \cos\phi \, \underline{e}_\theta & \nu \, \underline{e}_\phi & \dfrac{\partial \nu}{\partial h} \underline{e}_u \end{array}\right]$$

$$\det(M) = \nu^2 \cos\phi \frac{\partial \nu}{\partial h} \det[\underline{e}_\theta, \underline{e}_\phi, \underline{e}_v]$$

$$\det[\underline{e}_\theta, \underline{e}_\phi, \underline{e}_v] = (\underline{e}_\theta \times \underline{e}_\phi) \cdot \underline{e}_v = 1$$

Thus

$$\det(M) = \nu^2 \frac{\partial \nu}{\partial h} \cos\phi = \frac{|\underline{v}|^3 \cos\phi}{[\underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r})]}$$

Note that the condition that M be singular corresponds to $\phi = \pi/2$ which implies gimbal lock of the θ, ϕ coordinate system

4) Calculation of $\underline{a}$

$$\underline{a} = \frac{\partial \underline{w}}{\partial u} \qquad \underline{g} = -\frac{\mu \underline{r}}{|\underline{r}|^3}$$

$$\underline{w} = \begin{bmatrix} \nu \underline{e}_v \\ \hdashline \dfrac{\underline{e}_\theta \cdot \underline{g}}{\nu \cos\phi} \\ \hdashline \dfrac{\underline{e}_\phi \quad \underline{g}}{\nu} \\ \hdashline 0 \end{bmatrix}$$

$$\underline{a} = \frac{\partial \nu}{\partial u} \begin{bmatrix} \underline{e}_v \\ \hdashline \dfrac{-\underline{e}_\theta \quad \underline{g}}{\nu^2 \cos\phi} \\ \hdashline \dfrac{-\underline{e}_\phi \quad \underline{g}}{\nu^2} \\ \hdashline 0 \end{bmatrix}$$

5) Calculation of $M \, \underline{a}$

Define $\underline{b} = M \, \underline{a}$

$$\underline{b} = \frac{\partial \nu}{\partial \underline{u}} \begin{bmatrix} \underline{e}_v \\ \\ \underline{c} \end{bmatrix}$$

where

$$\underline{c} = \underline{e}_v \left(\frac{\partial \nu}{\partial \underline{r}} \, \underline{e}_v \right) - \frac{1}{\nu} \left[\underline{e}_\theta \left(\underline{e}_\theta \cdot \underline{g} \right) + \underline{e}_\phi \left(\underline{e}_\phi \quad \underline{g} \right) \right]$$

$$= \underline{e}_v \left(\frac{\partial \nu}{\partial \underline{r}} \, \underline{e}_v \right) - \frac{1}{\nu} \left[I - \underline{e}_v \, \underline{e}_v^{\,T} \right] \underline{g}$$

$$= \frac{(\underline{g} \quad \underline{v}) \, \underline{e}_v}{\left[\underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r}) \right]} - \frac{1}{\nu} \left(I - \underline{e}_v \, \underline{e}_v^{\,T} \right) \underline{g}$$

$$= \frac{1}{|\underline{v}|} \left\{ - \underline{g} + \underline{v} \left[\frac{\underline{v} \quad (2 \, \underline{v} - \underline{\omega} \times \underline{r})}{|\underline{v}|^2 \, \underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r})} \right] (\underline{v} \quad \underline{g}) \right\}$$

Substituting the equation for $\frac{\partial \nu}{\partial \underline{u}}$ gives

$$\underline{b} = \frac{1}{\alpha} \left[\begin{array}{c} - \underline{v} \\ \hline \underline{g} - \left(\dfrac{\alpha + |\underline{v}|^2}{\alpha} \right) \dfrac{(\underline{g} \quad \underline{v}) \, \underline{v}}{|\underline{v}|^2} \end{array} \right]$$

where $\alpha = \underline{v} \quad (\underline{v} - \underline{\omega} \times \underline{r})$

IV Simulation Results Using Position Measurements

In Reference 3, the necessary conditions for the optimal linear incorporation of position measurements into the proposed filter algorithm were derived. In this chapter, preliminary simulation results are presented in which the performance of the new algorithm is compared with a standard Kalman filter and a suboptimal scheme which will be discussed

A Review of Necessary Conditions

Recall that the estimation error is made up of two components, only one of which is correlated with the driving noise $\underline{d}$ which drives the actual state $\underline{x}$,

$$\dot{\underline{x}} = F \underline{x} + \underline{d}$$

The covariance of the correlated component of the estimation error is defined to be P_d, while the covariance matrix of the uncorrelated component is P_n. The covariance of the total estimation error, P, is thus the sum of P_d and P_n

The differential equation for P_n used by the filter is exact and is given by

$$\dot{P}_n = F P_n + P_n F^T \qquad , \; P_n(0) \text{ given}$$

while the differential equation for P_d must be conservatively approximated in the filter by

$$\dot{P}_d = F P_d + P_d F^T + \lambda P_d + Q/\lambda \qquad , \; P_d(0) \text{ given}$$

where $Q = \overline{\underline{d}(t)\, \underline{d}^T(t)}$ and λ is a positive scalar variable to be chosen as the control variable in a matrix optimal control problem

Now consider a scalar position-type measurement m, given by

$$m = \underline{b}^T \underline{x} + n$$

where n is white measurement noise and the only nonzero elements of $\underline{b}$

correspond to position components of $\underline{x}$ The measurement is incorporated into the state estimate, $\hat{\underline{x}}$, using the linear equation

$$\hat{\underline{x}}' = \hat{\underline{x}} + \underline{w}(m - \underline{b}^T\hat{\underline{x}})$$

where a caret over a quantity indicates the estimate of that quantity, a prime denotes the value of a variable just after measurement incorporation, and $\underline{w}$ is the measurement weighting vector to be chosen later

We now formulate the following optimal control problem

Choose the ℓ weighting vectors $\underline{w}_k$ and $\lambda(t)$ such that the cost function

$$J = \text{trace}\left[L\, P(t)\right] \tag{1}$$

is minimized, where $L = L^T \geq 0$, $P(t)$ is the filter estimate of the estimation error covariance matrix at T, ℓ is the number of measurements between 0 and T, and T is some given final time of interest The necessary conditions for the optimality of $\underline{w}_k$ and $\lambda(t)$ are the following

$$\underline{w}_k = (\mu_k P_{dk} + P_{nk})\, \underline{b}_k / \left[\underline{b}_k^T (\mu_k P_{dk} + P_{nk}) \underline{b}_k + q_k\right] \tag{2}$$

$$\mu_k = \exp\left[\int_{t_k}^{T} \lambda(s)\, ds\right] \tag{3}$$

$$\lambda = \sqrt{\text{trace}\left(CQ\right) / \text{trace}\left(C P_d\right)} \tag{4}$$

$$\dot{C} = -CF - F^T C \quad , \quad C(T) = L \tag{5}$$

$$C_k = (I - \underline{b}_k \underline{w}_k^T)\, C_k' \,(I - \underline{w}_k \underline{b}_k^T) \tag{6}$$

$$\dot{P}_d = F P_d + P_d F^T + \lambda P_d + Q/\lambda \tag{7}$$

$$\dot{P}_n = F P_n + P_n F^T \tag{8}$$

$$P'_n = (I - \underline{w}_k \underline{b}_k^T) P_n (I - \underline{b}_k \underline{w}_k^T) + \underline{w}_k \underline{w}_k^T q_k \tag{9}$$

$$P'_d = (I - \underline{w}_k \underline{b}_k^T) P_d (I - \underline{b}_k \underline{w}_k^T) \tag{10}$$

where $q_k = \overline{n_k^2}$, $P_{nk} = P_n(t_k)$, etc , $P_d(0)$, $P_n(0)$, q_k, L, T, Q(t) and $\underline{b}_k$ are given.

B Numerical Optimization Procedure

Assume that we have stored a guess of the optimal time history $\lambda(t)$ The cost, given by Eq (1), will be less using Eqs (2), (3), and (7)-(10) to compute the weighting vectors $\underline{w}_k$ than using any other values of $\underline{w}_k$ with the stored λ history Using these computed values of $\underline{w}_k$ and $P_d(t)$, the cost will be further reduced by using Eqs (4)-(6) to compute a new $\lambda(t)$ time history.

C. A Simple Suboptimal Technique

In Reference 3 the global optimum solution to the problem of minimizing Eq. (1) without measurement incorporation was presented Those results are reproduced here for completeness.

Given the matrix differential equations

$$\dot{P}_n = F P_n + P_n F^T \quad , \quad P_n(0) \text{ given} \tag{11}$$

$$\dot{P}_d = F P_d + P_d F^T + \lambda P_d + Q/\lambda \quad , \quad P_d(0) \text{ given} \tag{12}$$

the control constraint $\lambda > 0$, and the cost function

$$J = \text{trace} \left\{ L[P_n(T) + P_d(T)] \right\} \tag{13}$$

The globally optimum $\lambda(t)$ is given by

$$\lambda = \sqrt{\text{trace} [K Q] / \text{trace} [K P_d]} \tag{14}$$

where

$$\dot{K} = -KF - F^T K \qquad K(T) = L \qquad (15)$$

The proposed suboptimal scheme is as follows Between the (k - 1)st and k^{th} measurements, use Eqs (12), (14), and (15) to compute $\lambda(t)$ without iteration, with the boundary condition on K given at t_k, i e $K(t_k) = L$. Use for $\underline{w}_k$ the standard Kalman weighting vector to minimize $P(t_k)$, i e.

$$\underline{w}_k^* = (P_{nk} + P_{dk})\, \underline{b}_k / [\underline{b}_k^T (P_{nk} + P_{dk}) \underline{b}_k + q_k]$$

This scheme is suboptimal for the cost function given by Eq (1) when there are one or more measurements between 0 and the given final time T

D. Error Analysis

This section contains an analysis which leads to formulas for the direct computation of the true estimation error covariance matrix for a particular nominal trajectory Recall that the differential equation for the uncorrelated estimation error covariance matrix is given exactly by

$$\dot{P}_n = F P_n + P_n F^T \qquad ; \; P_n(0) \text{ given}$$

Thus with the correct initial condition, P_n is known exactly as a function of time. Since the true covariance matrix is the sum of P_n and P_d, the only remaining task is to determine the actual $P_d(t)$, which will be designated here as $P_{da}(t)$

Recall that the differential equation for the correlated estimation error is given by

$$\dot{\underline{e}}_d = F \underline{e}_d - \underline{d} \qquad (16)$$

and $\underline{e}_d$ is updated at the measurement times by

$$\underline{e}_d' = (I - \underline{w}\underline{b}^T)\, \underline{e}_d \qquad (17)$$

where $\underline{w}$ is the measurement weighting vector used by the particular filter of interest.

Now, for a simulation using any particular nominal trajectory, the true trajectory, $\underline{x}(t)$, will be computed using the actual nonlinear equations of motion For this actual trajectory, the disturbance vector $\underline{d}(t)$ arising from gravitational sources is a deterministic function of time, and thus $\underline{e}_d(t)$ is a deterministic function of time This means that $P_{da}(t)$ is deterministic and given by

$$P_{da}(t) = \underline{e}_d(t)\, \underline{e}_d^T(t) \tag{18}$$

In order to compute $P_{da}(t)$ without the direct integration of Eq (16), we introduce the vector $\underline{\tilde{x}}$ We define the initial condition of $\underline{\tilde{x}}$ as

$$\underline{\tilde{x}}(0) = \underline{x}(0) + \underline{e}_d(0) \tag{19}$$

We define the differential equation for $\underline{\tilde{x}}$ to be

$$\dot{\underline{\tilde{x}}} = F\underline{\tilde{x}} \tag{20}$$

and the update equation at the measurement times to be

$$\underline{\tilde{x}}' = (I - \underline{w}\,\underline{b}^T)\underline{\tilde{x}} + \underline{w}\,\underline{b}^T\underline{x} \tag{21}$$

where $\underline{w}$ is the measurement weighting vector used in updating the state estimate of the filter. (Note that if the uncorrelated estimation error is zero at the initial time, $\underline{\tilde{x}}(t)$ corresponds to the state estimate of the filter if perfect measurements are incorporated)

Subtracting the true state differential equation from Eq. (20) and the true state from both sides of Eq (20) gives

$$\dot{\underline{\tilde{x}}} - \dot{\underline{x}} = F(\underline{\tilde{x}} - \underline{x}) - \underline{d} \tag{22}$$

$$(\underline{\tilde{x}} - \underline{x})' = (I - \underline{w}\,\underline{b}^T)(\underline{\tilde{x}} - \underline{x}) \tag{23}$$

Comparing Eqs (22) and (23) with (16) and (17) and noting the initial condition given by Eq (19), we see that

$$(\underset{\sim}{\tilde{x}} - \underset{\sim}{x}) = \underset{\sim}{e}_d$$

and thus for all time

$$P_{da} = (\underset{\sim}{\tilde{x}} - \underset{\sim}{x})(\underset{\sim}{\tilde{x}} - \underset{\sim}{x})^T$$

E. Numerical Results

In this section we present the numerical results of a single computer run which compares the performance of a standard Kalman filter with the optimal filter derived numerically as outlined in Section B and the suboptimal scheme discussed in Section C

The environment used is a tenth order harmonic earth gravitational model with the J_2 term set to zero The equations used by the filters to propagate the state estimates are obtained from the correct inverse square component of the above environment Thus the driving noise $\underline{d}$ is a deterministic vector which arises from all but the inverse square component of the environment, and the reference trajectory, which is computed from the true equations of motion with $\underline{d}$ identically zero, is a conic The particular conic used on this run is a 500, 000 foot circular orbit, 1° from polar

The state vector used by the optimal and suboptimal filters is the adjoined vector in rotating coordinates discussed in Reference 1 consisting of deviations from the nominal position, two components of the deviation in velocity and the deviation in specific energy The state vector used by the Kalman filter consists of deviations from the nominal position and velocity in inertial coordinates

Measurements are taken every 200 seconds and the same measurement is incorporated into each of the three filters (Transformation equations relating the measurement geometry vector for the optimal filter state vector to the Kalman filter state vector were developed in Reference 2) Since the period of the nominal trajectory is roughly 5, 200 seconds, this corresponds to measurements at 14° increments in true anomaly Two measurement types are used planet center to star and planet diameter measurements The sequence of measurements nine planet center to star measurements, one planet diameter measurement, nine planet center to star measurements, one planet diameter measurement, etc. Initially, the following realistic gaussian errors were assumed to corrupt the measurements

Quantity	Standard deviation
Earth phenomena uncertainty	.5 km (light side)
Earth phenomena uncertainty	1 5 km (dark side)
Pointing error, visible sensor	10 arc-seconds
Pointing error, I R sensor	1 arc-minute

The possible stars to be sighted on were limited to the 37 stars used in the Apollo program Of these 37, the star sighted on at a particular measurement time was visible and maximized the ratio of the square of the in-plane component of the geometry vector to the variance of the noise on the measurement.

Because of the relatively high ratio of phenomena uncertainty to altitude of the nominal orbit, a one sigma measurement noise was roughly equivalent to a position error of 15,000 feet Thus the actual measurements were of poor quality and the resulting estimation errors were not significantly different from the no-measurement results presented in Reference 3. In order to provide better measurements while maintaining compatibility with existing software, the variance of the measurement noise, computed using standard techniques and the error sources above, is artificially set to one percent of the computed value

The simulation begins with zero position and velocity errors In order to allow the Kalman filter to incorporate measurements, it is given a non-zero initial covariance matrix. This matrix is diagonal with elements corresponding to mean square estimation errors of 10^6 ft^2 in each position component and 100 ft^2/sec^2 in each velocity component The choice of these numbers is arbitrary and no effort was made to optimize them However, the RMS position error as reflected by this covariance matrix is the same order of magnitude as the actual RSS position deviation from the nominal at the end of the run

The value of Q used in Eq (12) is computed using mean square values of the specific perturbing force and specific potential energy averaged along the entire actual trajectory All cross correlations between the components of the perturbing specific force and the perturbing specific potential energy are set to zero. For both the optimal and suboptimal filters, the initial values of the elements of P_n are zero, while the only non-zero element of P_d is the mean square uncertainty in specific energy which is set equal to the actual mean square uncertainty in specific potential energy It is important to note that these initial conditions reflect the true initial statistics and are not chosen arbitrarily

For both the optimal and suboptimal filters, the matrix L is diagonal with ones corresponding to position components and zeros elsewhere Thus the cost function given by Eq (1) is the estimated RMS position estimation error at the final time T.

The numerical results are summarized in Figures 1 - 4 Figures 1 - 3 show the actual and estimated RMS position estimation errors for the Kalman, suboptimal and optimal filters The actual estimation error is the square root of the sum of the position components of P_n, which is known exactly in the filter and P_{da}, which was shown in Section D to be

$$(\underline{\tilde{x}} - \underline{x})(\underline{\tilde{x}} - \underline{x})^T$$

The estimated RMS position estimation error is the square root of the sum of the diagonal position components of the filter covariance matrix Figure 4 presents the time histories of λ for the optimal and suboptimal filters

F. Discussion and Conclusions

Because of the nature of $\underline{d}$, i e the disturbances arising from a tenth order harmonic earth model with a zero J_2 term, it would seem that the statistical behavior of the filters should not be greatly affected by the orientation of the nominal trajectory with respect to the model

Thus, although these results are preliminary and may be regarded as a single Monte Carlo simulation with respect to the process noise $\underline{d}$, it would still seem appropriate to make a few observations at this point (It should be remembered, however, that the behavior of the Kalman filter for a fixed measurement schedule is completely determined by the initial filter covariance matrix. Since this matrix was chosen somewhat arbitrarily, there is possible room for improvement in the Kalman filter performance)

There are three criteria on which the various filters may be judged.

1) Conservatism -- Is the RMS estimation error, as reflected by the filter covariance matrix, less than or equal to the actual RMS estimation error?

2) How close is the RMS estimation error, as reflected by the filter covariance matrix, to the actual RMS estimation error?

3) Accuracy -- How do the actual RMS estimation errors of the three filters compare?

The Kalman filter remains conservative for less than one orbit The divergence between the estimated and actual position estimation errors becomes pronounced at the end of 1 1/2 orbits The actual RMS estimation error is large for the first 1/2 orbit, due to the large initial filter covariance matrix, but then settles down to a level close to that of the suboptimal filter

The suboptimal filter is always conservative and tracks the true RMS estimation error quite closely The actual RMS estimation error seems to be less than or very near the actual RMS estimation error of both the Kalman and optimum filters for the entire trajectory The estimated RMS estimation error is less than the estimated error for the optimum filter up to the final 1/2 orbit, and does not differ from it substantially for the remainder of the trajectory After an initial transient, the suboptimal λ oscillates closely about an average value

The optimal filter is always conservative and tracks the true RMS estimation error closely, with excellent accuracy after the first orbit On the average, the actual RMS estimation error begins high and drops off slowly throughout the trajectory Little can be said about the resulting optimal λ history

It would appear that the suboptimal filter offers performance superior to the standard Kalman filter while having the advantages of a closed form $\lambda(t)$ solution between measurements It also appears to have a λ history which may be closely approximated by a constant Needless to say, these possibilities must be explored in further simulations

The relatively poor behavior of the optimal filter in regard to the actual RMS estimation error over the entire trajectory can be attributed to the fact that the cost function given by Eq (1) penalizes the filter covariance matrix only at the final time Better overall performance could be attained if the cost function were defined to be the integral of some linear combination of the elements of the filter covariance matrix over the entire trajectory The necessary conditions for this cost function are easily derived and will be discussed in a future report Unfortunately, these necessary conditions do not yield an analytic solution for even the no-measurement case Thus, when measurements are incorporated there is not an explicit formula for the optimum $\lambda(t)$ as a function of the state and costate, and the numerical optimization scheme outlined in Section B is not applicable This would seem to necessitate the use of a gradient-type search algorithm to find the optimal $\lambda(t)$ for an integral cost.

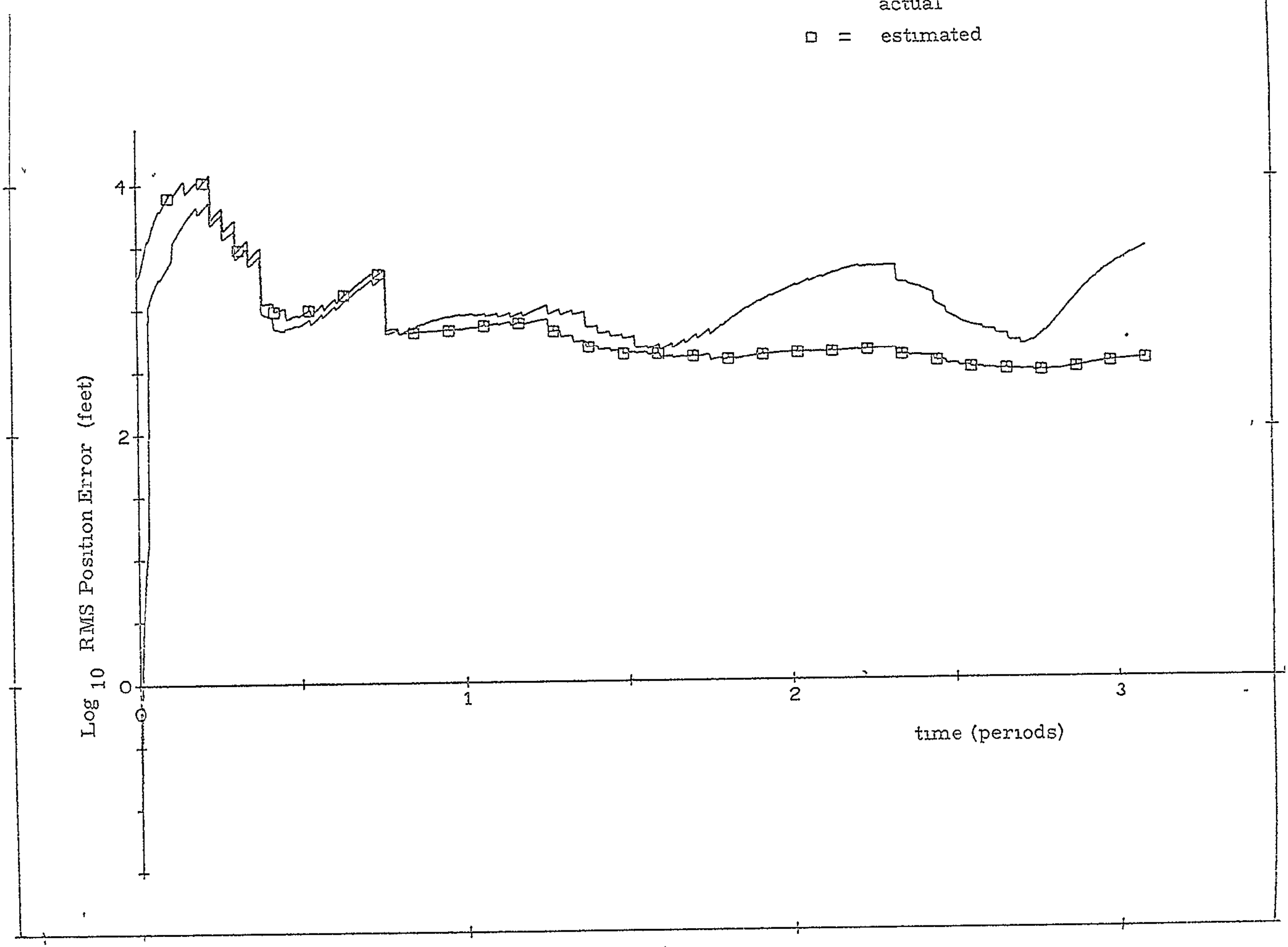

Figure 1 (Kalman)

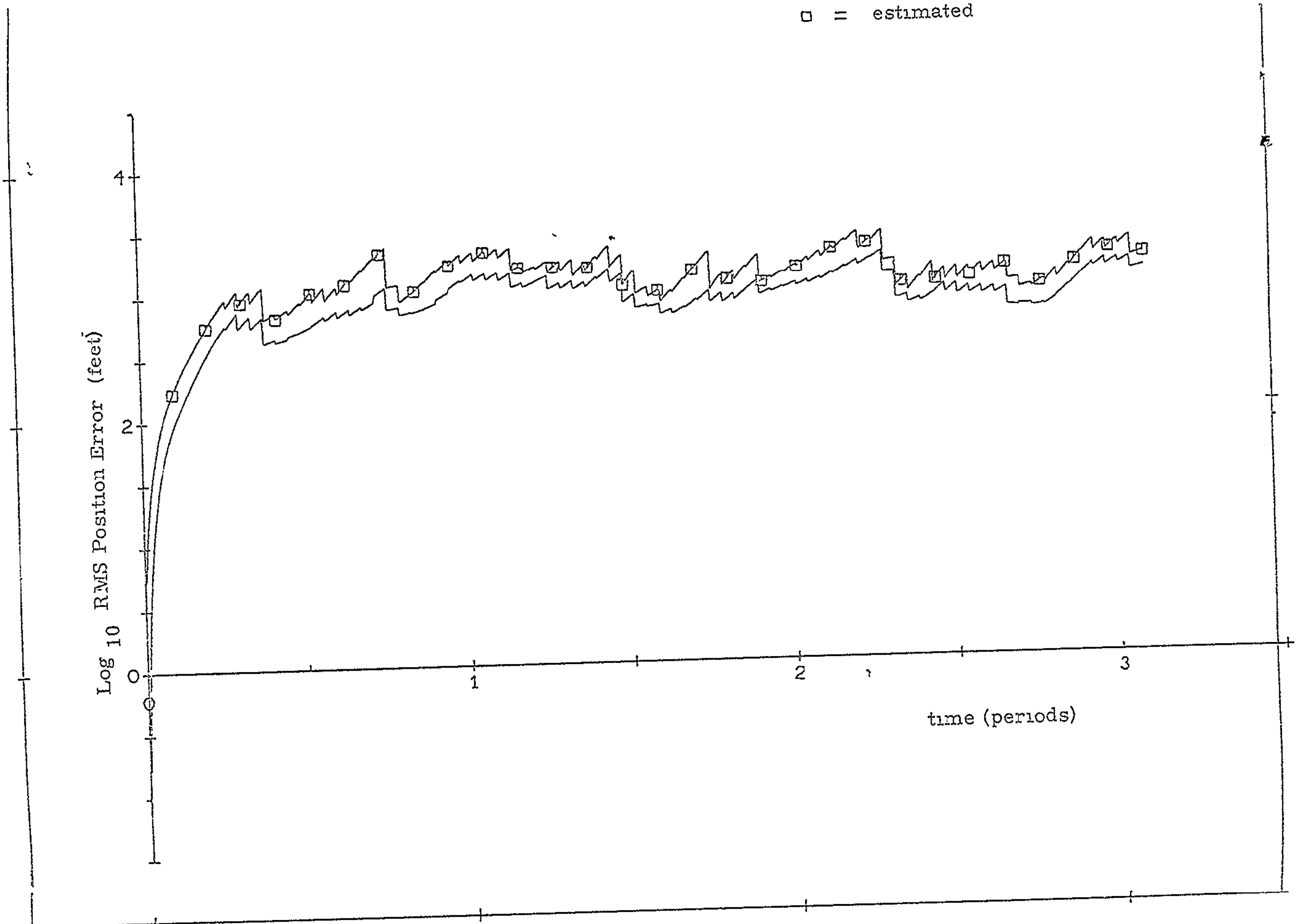

Figure 2 (Suboptimal)

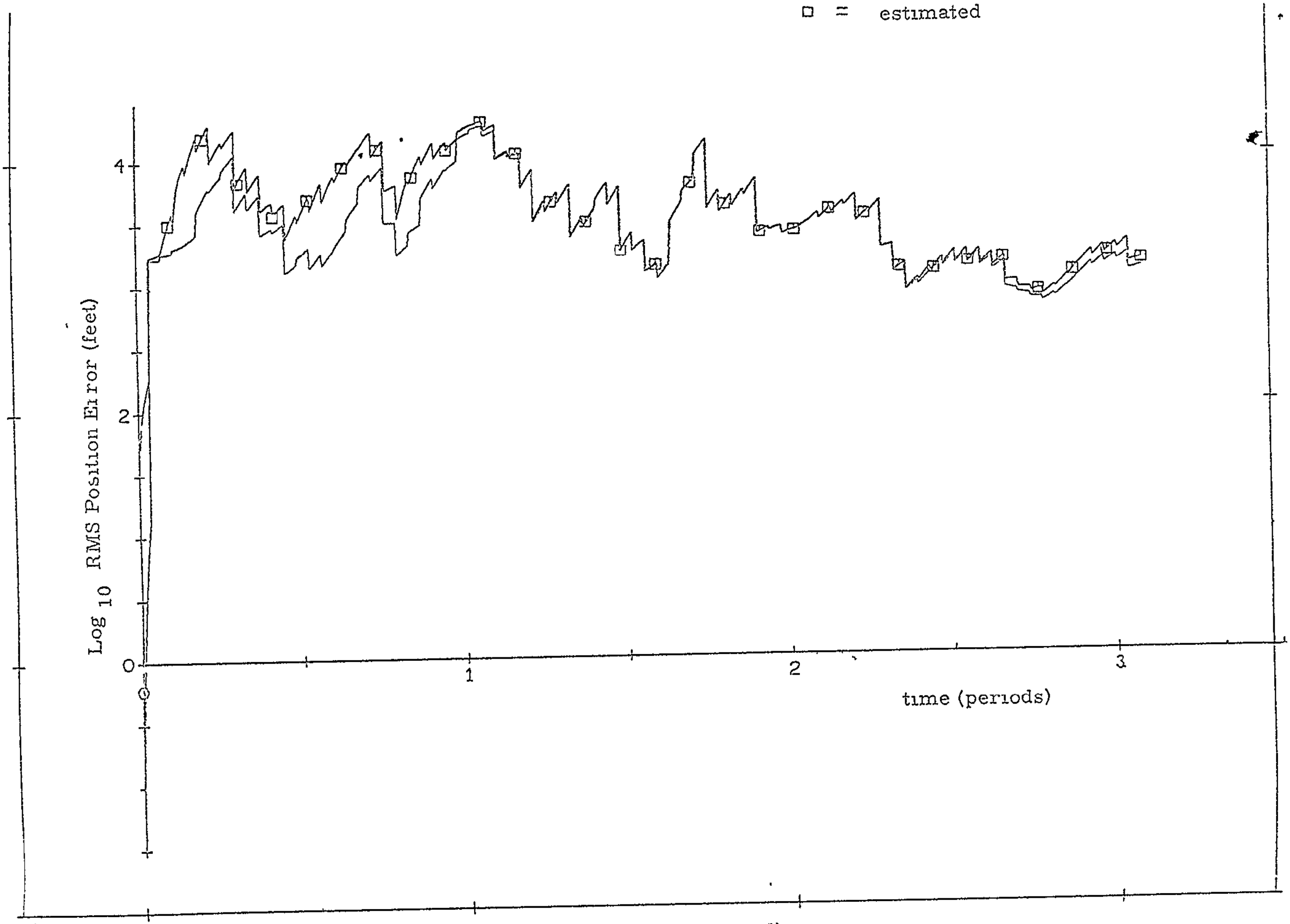

Figure 3 (Optimal)

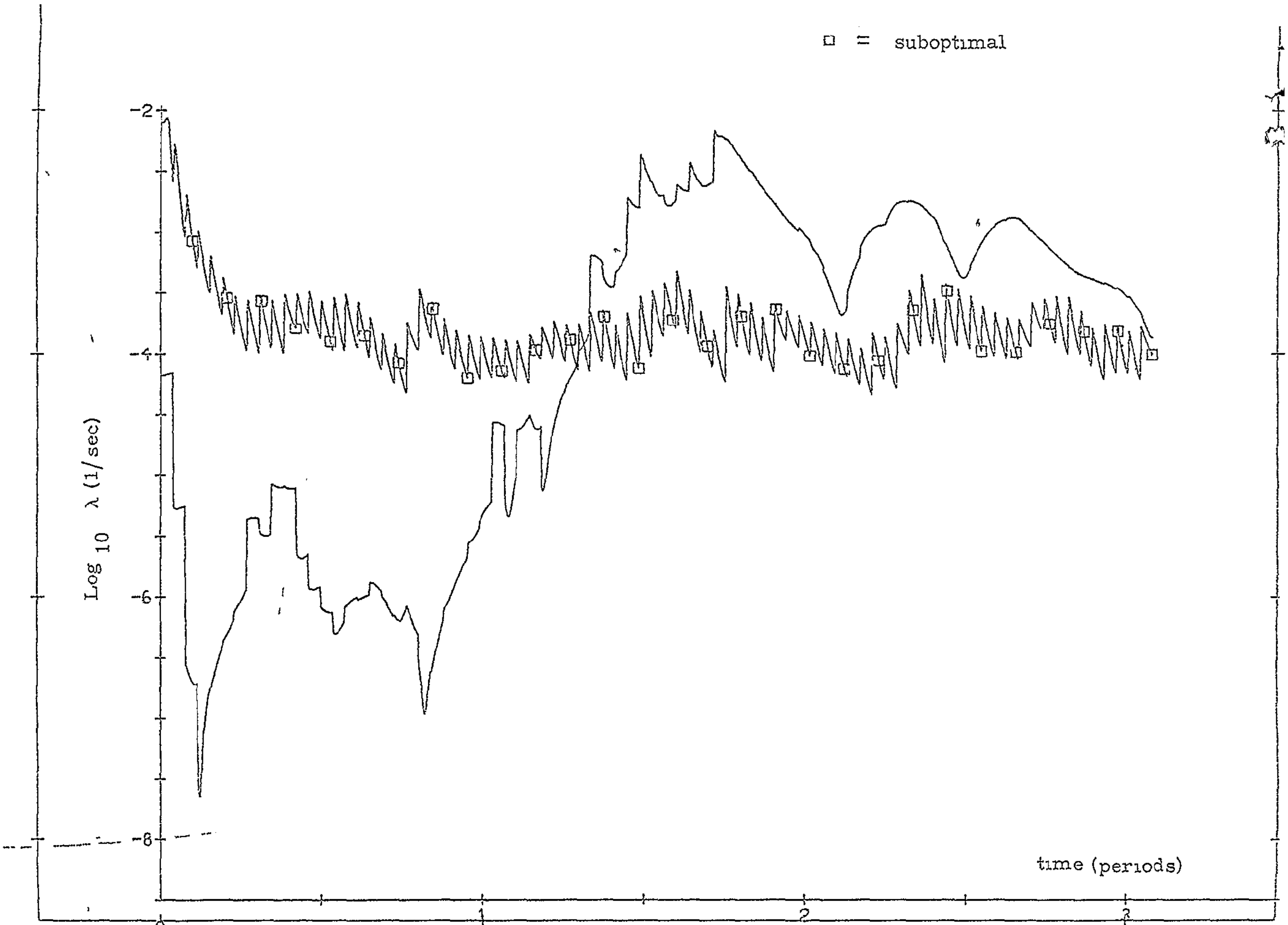

Figure 4