ON THE RIGHT WEAK SOLUTION OF THE CAUCHY PROBLEM FOR A QUASILINEAR EQUATION OF FIRST ORDER

by

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Consider the quasilinear equation of first order

$$(1) \qquad u_t + f(u)_x = 0$$

with t, x, u being real scalars and $f(u)$ being a given smooth function on $(-\infty, \infty)$. Wanted is a solution $u = u(t, x)$, $t > 0$, that is prescribed for $t = 0$. The solution can be found by means of the classical method of characteristics. Such a smooth solution exists, in general, only in some neighbor-

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hood of the initial line $t = 0$, $u = u_o(x)$. Under wide assumptions about $f(u)$, however, the solution can be continued beyond this neighbourhood. The continued solution has, in general, lines of discontinuity in analogy to the shock waves of compressible fluid flow. It is only a weak solution of the Cauchy problem for (1). The property of being a weak solution does not, however, characterize the function $u(t, x)$. The Cauchy problem for (1) has always more than one weak solution if $f(u)$ is not linear in $(-\infty, \infty)$. It must be remembered that the particular weak solution $u(t, x)$ spoken of above is originally thought of as

$$(2) \qquad u(t, x) = \lim_{\varepsilon \downarrow 0} u(t, x; \varepsilon)$$

where $u(t, x; \varepsilon)$ is the solution of the parabolic second order equation $(\varepsilon > o)$

$$(3) \qquad u_t + f(u)_x = \varepsilon u_{xx}$$

with the same initial data $u_o(x)$ at $t = 0$. This is, of course, a complicated description of the particular weak solution $u(x, t)$, and it is an important and apparently not completely solved problem to characterize this solution $u(x, t)$ in substantially simpler ways. The present note contains another approach to this question.

In order to justify the subsequent procedure we begin by quoting known results about the Cauchy problem for (3) and about the limit (2). The solution $u(t, x; \epsilon)$ of (3) with given initial values $u_o(x)$ at $t = 0$ exists if $f(u)$ is "sufficiently smooth" and if $u_o(x)$ is bounded, $|u_o(x)| \leqq M$, and smooth, $u_o'(x)$ bounded; the solution $u(t, x)$ has the same bound M (see [2], end of page 277). If $u_o(x)$ is bounded and monotone the functions $u(t, x; \epsilon)$ converge in the mean, as $\epsilon \to 0$, to a function $u(t, x)$ in any finite interval of any straight line $t = t_o \geqq 0$ (see [2], theorem 1).

Let $h(u) \in C'(-\infty, \infty)$ and let

$$I(u) = \int h(u)du ,$$

$$F(u) = \int h(u)df(u) = \int h(u)f'(u)du . \tag{4}$$

On multiplying the differential equation (3) on both sides by $h(u)$ we may write the resulting equation

$$I(u)_t + F(u)_x = \varepsilon(u_x h(u))_x - \varepsilon h'(u)u_x^2$$

(5)

$$= \varepsilon I(u)_{xx} - \varepsilon h'(u)u_x^2 \,.$$

In this equation $u = u(t, x; \varepsilon)$ is supposed to be substituted. On multiplying (5) on both sides by a test function $\varphi(t, x)$ in the open half plane $t > 0$ (function with compact support in this half plane and of class C') and on integrating over this half plane we obtain (after some integrations by part)

$$\iint_{t>0} \left[I(u)\varphi_t + F(u)\varphi_x\right]dtdx = -\varepsilon \iint_{t>0} I(u)\varphi_{xx}dtdx$$

(6)

$$+\varepsilon \iint_{t>0} h'(u)u_x^2 \varphi dtdx$$

with $u = u(t, x; \varepsilon)$. Now let $\varepsilon > 0$ go to zero. From the convergence $u(t, x; \varepsilon) \to u(t, x)$ as explained above and from $|u(t, x; \varepsilon)| \leqq M$ it is easily inferred that

(7) $$I(u(t, x; \varepsilon)) \to I(u(t, x))$$

holds also in the mean on every finite interval on any line $t = \text{const}$; the same holds for F. The integral on the left of (6) stays uniformly bounded as $\epsilon \to 0$. Consequently, the integral on the left, $u = u(t,x;\epsilon)$, tends toward the same integral with $u = u(t,x)$. The first term on the right is easily seen to go to zero. The second term is always $\geqq 0$ as long as $h'(u)\varphi(t,x) \geqq 0$ holds for all t, x, u. Hence we conclude:

The limit function $u(t,x)$ satisfies the inequality

$$(8) \qquad \iint_{t>0} [\, I(u)\varphi_t + F(u)\varphi_x] dt dx \geqq 0$$

whenever

$$(8') \qquad h'(u)\varphi(t,x) \geqq 0$$

holds for all t, x, u.

Condition (8′) i~~f~~ satisfied ~~o~~f $h = 1$ or /s /i
$h = -1$ and if φ is an arbitrary test function in the open half plane $t > 0$. In this case there holds

$$(9) \qquad I(u) = \pm u \,, \quad F(u) = \pm f(u) \,,$$

respectively. The result is that the integral (8) with $I = u$, $F = f$ is both $\geqq 0$ and $\leqq 0$. Therefore the statement made above includes the statement that $u(t,x)$ is a weak solution of (1):

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x] dtdx = 0 \tag{10}$$

holds for every test function $\varphi(t,x)$ in the half plane $t > 0$.

In order to see more clearly what the property (8), (8′) means we consider the case where the solution $u(t,x)$ is of class C' in the t-x-plane, except on a finite number of smooth lines of discontinuity along each of which it has limits u^-, u^+ on the two sides ~~/ the minus side is the one which is reached first by a parallel to the axis of x/~~. We say in this case that u is of class PC'. Let the test function φ be of the form

$$\varphi(t, x) = \Psi(t)\chi(\xi), \quad \xi = x - x(t),$$

where $x = x(t)$ is the equation of the discontinuity line and where χ, with $\epsilon > 0$, is given by

$$(11)\qquad \chi(\xi) = \begin{cases} 1 - |\xi|/\epsilon \ , & |\xi| < \epsilon \\ 0 \ , & |\xi| \geqq \epsilon \ . \end{cases}$$

$\Psi(t)$ is a nonnegative test function which is first kept fixed and later made to approach a δ-function around a given value of t. The integral (8) now becomes the sum of the two integrals

$$(12)\qquad \iint \left[-I(u)\dot{x} + F(u)\right]\chi'(\xi)\,\Psi(t)\,dt\,d\xi$$

and

$$(13)\qquad \iint I(u)\,\chi(\xi)\,\dot{\Psi}(t)\,dt\,d\xi \ .$$

By virtue of (8) the sum is ≥ 0. From (11) it follows easily that (13) $\to 0$ as $\epsilon \to 0$. (12) equals

$$(14)\quad \int_{(t)}\left\{\frac{1}{\epsilon}\int_{-\epsilon}^{0}\Big(-I(u)\dot{x} + F(u)\Big)d\xi - \frac{1}{\epsilon}\int_{0}^{\epsilon}\Big(-I(u)\dot{x} + F(u)\Big)d\xi\right\}\Psi(t)\,dt \ ,$$

and it is clear that (14) converges to

$$(15)\quad \int_{(t)}\left\{\Big(I(u^+) - I(u^-)\Big)\dot{x} - \Big(F(u^+) - F(u^-)\Big)\right\}\Psi(t)\,dt \ \geqq 0 \ .$$

(15) holds (see (8′)) if

$$(15') \qquad h'(u) \geqq 0 \quad \text{and} \quad \Psi(t) \geqq 0 \; .$$

If now $\Psi(t)$ is made to concentrate upon a single value t (15) becomes simply

$$(16) \quad \big(I(u^+) - I(u^-)\big)\dot{x} - \big(F(u^+) - F(u^-)\big) \geqq 0 \; ,$$

and this holds whenever

$$(16') \qquad h(u) \text{ non-decreasing.}$$

(It is easy to see that condition (15′) may be relaxed to (16′).)
If the whole argument is applied to (10) in place of (8) there results the wellknown formula

$$(17) \qquad (u^+ - u^-)\dot{x} - \big(f(u^+) - f(u^-)\big) = 0 \; .$$

Altogether we have the following result:

The inequality

$$(18) \; \big(I(u^+) - I(u^-)\big)\frac{f(u^+)-f(u^-)}{u^+ - u^-} - \big(F(u^+)-F(u^-)\big) \geqq 0$$

holds whenever (16′) holds.

The meaning of this inequality becomes clear when the special increasing functions

$$(19) \qquad h(u) = \begin{cases} 0 \ , & u < u_o \\ 1 \ , & u \geqq u_o \end{cases}$$

are used. In this case there follows from (4)

$$I(u) = \begin{cases} 0 \ , & u < u_o \\ u - u_o \ , & u \geqq u_o \end{cases} \ , \quad F(u) = \begin{cases} 0 \ , & u < u_o \\ f(u) - f(u_o), & u \geqq u_o . \end{cases}$$

We suppose that u_o lies in the interval u^-, u^+ and we distinguish between the two cases

$$\alpha)\ u^- \leqq u_o \leqq u^+ \qquad\qquad \beta)\ u^+ \leqq u_o \leqq u^- .$$

In either case the simple calculations furnish the result that--we write u in place of u_o--

$$(20) \qquad \frac{f(u^+) - f(u)}{u^+ - u} \leqq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

holds. Geometrically this means in the case α) that the curve $v = f(u)$ lies above the chord (the chord subtended between $(u^-, f(u^-))$ and $(u^+, f(u^+))$ in the interval (u^-, u^+), and in the case β) that it lies below the chord in the interval (u^+, u^-).

This is precisely the condition E introduced by O. A. Oleinik [3]. It would be of considerable interest to prove the uniqueness of the weak solution of the Cauchy problem for (1) under the additional condition (8), (8′).

BIBLIOGRAPHY

[1] I. M. Gelfand, Some problems in the theory of quasilinear equations (Russian), Uspehi Mat. Nauk 14 (1959), no. 2(86), 87-158.

[2] O. A. Oleinik, Construction of a generalized solution of the Cauchy problem for a quasilinear equation of first order by the introduction of "vanishing viscosity", American Mathematical Society Translations Series 2, Vol. 33 (1963), 277-283.

[3] O. A. Oleinik, Uniqueness and stability of the generalized solution of the Cauchy problem for a quasilinear equation, American Mathematical Society Translations Series 2, Vol. 33 (1963), 285-290.