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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TRANSMISSION CHARACTERISTICS
OF SPLIT-PHASE PCM CODES
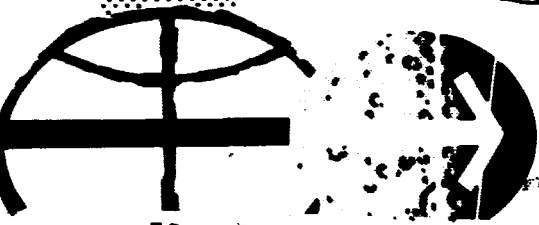
TELE/COMMUNICATIONS SYSTEMS DIVISION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

SC FORM 2026A (REV MAY 68)

TRANSMISSION CHARACTERISTICS

OF

SPLIT-PHASE PCM CODES

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TABLE OF CONTENTS

LIST OF ACRONYMS, ABBREVIATIONS, AND SYMBOLS

ASK	Amplitude Shift Keying
Biϕ-L	Bi-Phase Level (or Split-Phase)
$E(X_{t1})$	Expected value of the random variable X_{t1}
FSK	Frequency Shift Keying
NRZ	Non Return to Zero
PCM	Pulse Code Modulation
PSK	Phase Shift Keying
$P_{X_{t1}}\left(\hat{X}_{t1}\right)$	The probability density function of the random variable X_{t1}
$P_{X_{t1},X_{t2}}\left(\hat{X}_{t1},\hat{X}_{t2}\right)$	Joint probability density function of X_{t1} and X_{t2}
$P\left(X_{t1}=E\right)$	The unconditional probability that the random variable X at time t_1 assumes the value E.
$P\left(X_{t2}=E \mid X_{t1}=E\right)$	The probability that $X_{t2}=E$, *given* that $X_{t1}=E$.
rad	radian
RZ	Return to Zero
$R(\tau)$	Ensemble-average autocorrelation function
SNR	Signal-to-Noise Ratio
$S(\omega)$	Power Spectral Density
T_1	PCM Code Bit Period
t_1, t_2	Times at which the values of the members of an ensemble of random functions are sampled
X_{t1}, X_{t2}	Values which the members of an ensemble of random functions assume at the sampling times t_1 and t_2

$\delta(X)$	Unit impulse function
τ	Time-shift
ϕ_c	Initial carrier phase (relative to the modulating sequence)
ω	Angular frequency ($\omega = 2\pi f$)
ω_c	Carrier angular frequency ($\omega_c = 2\pi f_c$)

LIST OF REFERENCES

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2. Batson, B. H., *An Analysis of the Relative Merits of Various PCM Code Formats*, MSC-EB-R-68-5, NASA Manned Spacecraft Center, Houston, Texas, November 1, 1968.

3. Lee, Y. W., *Statistical Theory of Communications*, John Wiley and Sons, New York, New York, 1960.

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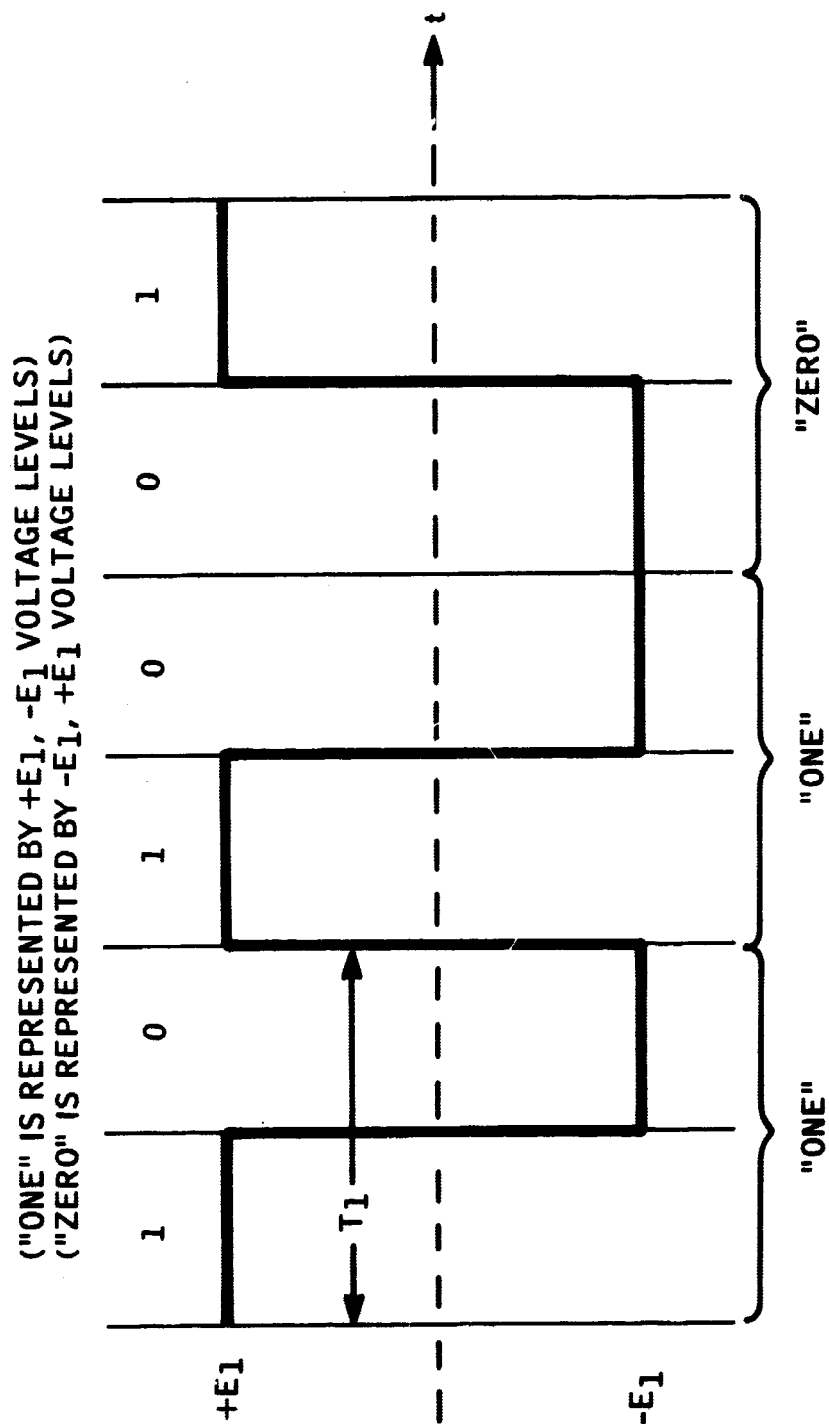
SECTION 1

INTRODUCTION

1.1 GENERAL

Pulse code modulation (PCM) telemetry utilizes a series of binary
digits ("ones" and "zeros") to describe the analog level of a
sample taken from a data channel. As indicated in Figure 1-1
(Reference 1), the bi-phase-level, or *split-phase*, PCM code
utilizes the binary states "10" to represent a "one" and the binary
states "01" to represent a "zero." One advantage offered by split-
phase coding over other types of PCM code formats (such as NRZ and
RZ) is that the transition density for a random bit pattern is
higher for split-phase than for the other formats. *At least* one
binary level transition will occur during each bit period of a
split-phase code, whereas it is possible for the other code formats
to have long groups of consecutive "ones" or "zeros." The greater
bit transition density for the split-phase format generally allows
more efficient bit synchronization (recovery of the bit rate clock
frequency) to be maintained at the receiver.

Prior to transmission, a PCM bit stream is used to modulate some
parameter (phase, frequency, or amplitude) of an RF carrier. In
some systems, the PCM signal first modulates a subcarrier, which
subsequently is used to modulate the main carrier. Such systems
are capable of transmitting other channels of information, such
as voice, in addition to the PCM telemetry channel.

1.2 PURPOSE

The purpose of this document is to determine certain transmission
characteristics (autocorrelation function and power spectral den-
sity) of a carrier which is modulated in some manner by a split-
phase PCM code. These characteristics are potentially useful in
determining transmission bandwidth requirements for channels over
which split-phase codes must be transmitted. In addition, these
characteristics must be known before precise calculations of bit
error rate can be made for various signal-to-noise ratios (SNR's).

Figure 1-1 Bi-Phase-Level or *Split-Phase* PCM Code Format

("ONE" IS REPRESENTED BY +E₁, -E₁ VOLTAGE LEVELS)
("ZERO" IS REPRESENTED BY -E₁, +E₁ VOLTAGE LEVELS)

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1.3   SCOPE

This document is concerned with the determination of transmission
characteristics of a carrier which is modulated in various ways
(amplitude, phase, and frequency) by a split-phase PCM code with
a random bit pattern. It is assumed that "ones" and "zeros" occur
with equal likelihood. In addition, it is assumed that the modu-
lation process is *noncoherent* (i.e., the phase of the PCM modulat-
ing sequence is not related to the phase of the carrier).

The transmission characteristics which are determined include the
(ensemble-average) autocorrelation function and the power spectral
density (which is the Fourier Transform of the autocorrelation
function).

SECTION 2

SUMMARY AND CONCLUSIONS

2.1 GENERAL

The results of the calculations presented in this document provide
some insight into the relative transmission bandwidth requirements
for carriers which are modulated in various ways by split-phase PCM
codes. It is demonstrated that the case of frequency modulation by
a split-phase code is the most general case to be analyzed; the
amplitude modulation and phase modulation cases are actually degen-
erate forms of this case.

2.2 CONCLUSIONS

A comparison of Figures 3-2, 4-1, and 5-2 indicates that the spectral
occupancy (and, therefore, the required transmission bandwidth) is
greater for a carrier which is *frequency*-modulated by a split-phase
code than for carriers which are amplitude- or phase-modulated by
that code. As expected, the spectral occupancy of the frequency-
modulated carrier depends upon the frequency deviation due to the
PCM code or, alternately, on the frequencies which are "keyed"
between by the code.

The spectral occupancy for the amplitude modulation case is the
same as that for the phase modulation case. For either case, the
baseband PCM code spectrum is translated to appear about plus and
minus the carrier frequency. A discrete carrier component is
always present for the amplitude modulation case, however, while
it is possible in the phase modulation case to convert all of the
available channel power into sideband power. Therefore, it may be
concluded that phase modulation is inherently a more efficient
technique than either amplitude or frequency modulation for trans-
mission of a random split-phase PCM code.

SECTION 3

AMPLITUDE-SHIFT KEYING BY SPLIT-PHASE PCM CODES

3.1 GENERAL

For the general case of amplitude modulation of a carrier by a binary sequence, level transitions result in the carrier being switched between two possible amplitude levels. It is common to refer to this modulation process as amplitude-shift keying (ASK). A special case of ASK is "on-off" keying of the carrier. This is the case when one of the carrier amplitude levels is zero.

3.2 AUTOCORRELATION FUNCTION

A generalized expression for a sinusoidal carrier which is amplitude-modulated by a binary sequence is

$$e_{ASK}(t) = A\left[1 + m(t)E_1\beta_c\right]\cos\left(\omega_c t + \phi_c\right) \qquad (1)$$

where   A is the unmodulated carrier amplitude,

$m(t) = \pm 1$ is a random (PCM) switching function,

E_1 represents the absolute voltage level of the binary sequence,

β_c is the carrier modulation sensitivity (volt/volt),

ω_c is the carrier frequency (rad/sec),

and ϕ_c is the initial phase of the carrier.

For the general case indicated above, binary transitions result in the carrier being switched between two possible amplitude levels, $A(1 + E_1\beta_c)$ and $A(1 - E_1\beta_c)$. Note that for $E_1\beta_c=1$, these two levels are 2A and 0, respectively. This is the special case of "on-off" keying of the carrier.

In order to determine the power spectral density of a random process, it is first necessary to determine the ensemble-average autocorrelation function of that process. This function is merely the expected value of the product of two samples of each member of an ensemble of the random process, evaluated for various sampling times. The auto-correlation function of the ASK signal of equation (1) is given by

$$R_{ASK}(\tau) = E\left[e_{ASK}(t_1)\, e_{ASK}(t_1+\tau)\right] \qquad (2)$$

where t_1 and $t_1+\tau$ are the times at which the members of the ensemble are sampled.

Equation (2) may be further expressed as

$$R_{ASK}(\tau) = E\left\{\left\{A\left[1+m\left(t_1\right)E_1\beta_c\right]\cos\left(\omega_c t_1+\phi_c\right)\right\}\right.$$
$$\cdot \left\{A\left[1+m\left(t_1+\tau\right)E_1\beta_c\right]\cos\left(\omega_c t_1+\omega_c\tau+\phi_c\right)\right\}\right\}$$
$$= A^2 E\left\{\left[1+m\left(t_1\right)E_1\beta_c\right]\left[1+m\left(t_1+\tau\right)E_1\beta_c\right]\right.$$
$$\left.\cdot \cos\left(\omega_c t_1+\phi_c\right)\cos\left(\omega_c t_1+\omega_c\tau+\phi_c\right)\right\} \tag{3}$$

But

$$\cos\left(\omega_c t_1+\phi_c\right)\cos\left(\omega_c t_1+\omega_c\tau+\phi_c\right) = \frac{1}{2}\cos\left(\omega_c\tau\right) + \frac{1}{2}\cos\left(2\omega_c t_1+\omega_c\tau+2\phi_c\right) \tag{4}$$

So

$$R_{ASK}(\tau) = A^2 E\left\{\left[1+m\left(t_1\right)E_1\beta_c+m\left(t_1+\tau\right)E_1\beta_c+m\left(t_1\right)m\left(t_1+\tau\right)E_1^2\beta_c^2\right]\frac{1}{2}\cos\left(\omega_c\tau\right)\right\}$$
$$+ A^2 E\left\{\left[1+m\left(t_1\right)E_1\beta_c+m\left(t_1+\tau\right)E_1\beta_c+m\left(t_1\right)m\left(t_1+\tau\right)E_1^2\beta_c^2\right]\right.$$
$$\left.\cdot \frac{1}{2}\cos\left(2\omega_c t_1+\omega_c\tau+2\phi_c\right)\right\} \tag{5}$$

If the modulation process is *non-coherent*, then the PCM signal and the carrier may be assumed to be statistically independent (Reference 2). Since the expected value of the product of two statistically independent random variables is equal to the product of their expected values, then

$$R_{ASK}(\tau) = \frac{A^2}{2}E\left[\cos\left(\omega_c\tau\right)\right] + \frac{A^2 E_1\beta_c}{2}E\left[m\left(t_1\right)\right]E\left[\cos\left(\omega_c\tau\right)\right] + \frac{A^2 E_1\beta_c}{2}E\left[m\left(t_1+\tau\right)\right]E\left[\cos\left(\omega_c\tau\right)\right]$$
$$+ \frac{A^2 E_1^2\beta_c^2}{2}E\left[m\left(t_1\right)m\left(t_1+\tau\right)\right]E\left[\cos\left(\omega_c\tau\right)\right] + \frac{A^2}{2}E\left[1+m\left(t_1\right)E_1\beta_c+m\left(t_1+\tau\right)E_1\beta_c\right.$$
$$\left.+ m\left(t_1\right)m\left(t_1+\tau\right)E_1^2\beta_c^2\right]E\left[\cos\left(2\omega_c t_1+\omega_c\tau+2\phi_c\right)\right] \tag{6}$$

If ϕ_C is assumed to be a random variable, uniformly distributed over the range 0 to 2π, then

$$E\left[\cos\left(2\omega_c t_1 + \omega_c \tau + 2\phi_c\right)\right] = 0 \qquad (7)$$

or the ensemble average of a sinusoid of random phase is the same as the time average of that sinusoid. It can also be noted that the assumption of a random PCM code with equally likely ones and zeros results in

$$E\left[m\left(t_1\right)\right] = E\left[m\left(t_1+\tau\right)\right] = 0 \qquad (8)$$

Since $\omega_c \tau$ is constant for a given value of τ, then

$$E\left[\cos\left(\omega_c \tau\right)\right] = \cos\left(\omega_c \tau\right) \qquad (9)$$

Substitution of equations (7), (8) and (9) into equation (6) yields

$$R_{ASK}(\tau) = \frac{A^2}{2} \cos\left(\omega_c \tau\right) + \frac{A^2 E_1^2 \beta_c^2}{2} \cos\left(\omega_c \tau\right) E\left[m\left(t_1\right) m\left(t_1+\tau\right)\right] \qquad (10)$$

Several observations can be made regarding equation (10). First, the term $(A^2/2)\cos(\omega_c \tau)$ is recognized as being the autocorrelation function of the carrier, $A \cos(\omega_c t + \phi_c)$. Second, the term $E_1^2 E[m(t_1)m(t_1+\tau)]$ is recognized as being an expression for the autocorrelation function of the binary sequence (split-phase PCM code) under consideration. Thus,

$$R_{ASK}(\tau) = R_{CARRIER}(\tau) + \beta_c^2 R_{PCM}(\tau) R_{CARRIER}(\tau) \qquad (11)$$

The autocorrelation function of the split-phase code, $R_{PCM}(\tau)$, is easily determined (Reference 2) and is shown in Figure 3-1.

3.3 POWER SPECTRAL DENSITY

The Wiener-Khintchine theorem (Reference 3) states that the power spectral density and the ensemble-average autocorrelation function are Fourier Transforms of each other, or

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau \qquad (12)$$

$R_{PCM}(\tau)$

E_1^2

$$R_{PCM}(\tau) \begin{cases} 0 & \text{FOR } \tau < -T_1 \\ \dfrac{-E_1^2}{T_1}(\tau + T_1) & \text{FOR } -T_1 \le \tau \le \dfrac{-T_1}{2} \\ \dfrac{E_1^2}{T_1}(3\tau + T_1) & \text{FOR } \dfrac{-T_1}{2} \le \tau \le 0 \\ \dfrac{E_1^2}{T_1}(-3\tau + T_1) & \text{FOR } 0 \le \tau \le \dfrac{T_1}{2} \\ \dfrac{E_1^2}{T_1}(\tau - T_1) & \text{FOR } \dfrac{T_1}{2} \le \tau \le T_1 \\ 0 & \text{FOR } \tau > T_1 \end{cases}$$

$\dfrac{E_1^2}{4}$

$-T_1 \quad \dfrac{-3T_1}{4} \quad \dfrac{-T_1}{2} \quad \dfrac{-T_1}{4} \quad \dfrac{T_1}{4} \quad \dfrac{T_1}{2} \quad \dfrac{3T_1}{4} \quad T_1$

τ

0

$\dfrac{-E_1^2}{4}$

$\dfrac{-E_1^2}{2}$

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$$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} \right)$$
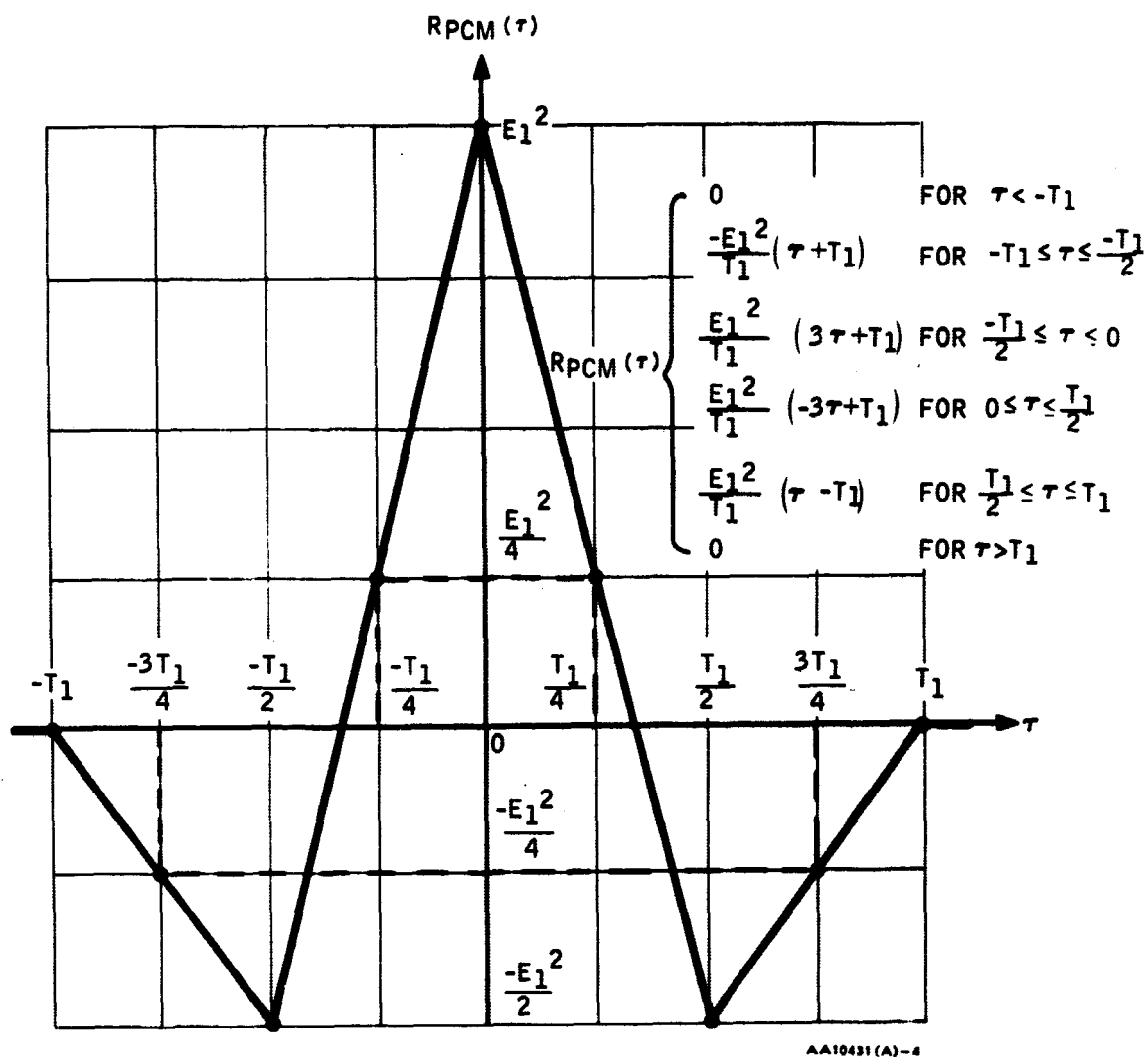
Figure 3-1   Ensemble-Average Autocorrelation Function
for a Random Split-Phase PCM Code

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Substituting equation (11) into Equation (12), it is seen that

$$S_{ASK}(\omega) = S_{CARRIER}(\omega) + \beta_c^2\left[S_{PCM}(\omega) * S_{CARRIER}(\omega)\right] \qquad (13)$$

where * denotes convolution (multiplication of autocorrelation functions results in convolution of power spectra).

Substituting the expression for $R_{CARRIER}(\tau)$ into equation (12), it is easily shown that the power spectrum of the carrier consists of two impulses (of weight $A^2/4$) located at $\omega = \omega_c$ and at $\omega = -\omega_c$, or

$$S_{CARRIER}(\omega) = \frac{A^2}{4}\left[\delta\left(\omega - \omega_c\right) + \delta\left(\omega + \omega_c\right)\right] \qquad (14)$$

Similarly, substitution of the expressions for $R_{PCM}(\tau)$ into equation (12) results in the following expression for the power spectrum of the split-phase code (Reference 2):

$$S_{PCM}(\omega) = \frac{E_1^2 T_1}{2\pi}\left[\frac{\sin^4\left(\frac{\omega T_1}{4}\right)}{\left(\frac{\omega T_1}{4}\right)^2}\right] \qquad (15)$$

The first term of equation (13), then, is given by equation (14), and the second term is given by

$$\beta_c^2\left[S_{PCM}(\omega) * S_{CARRIER}(\omega)\right] = \beta_c^2\int_{-\infty}^{\infty}S_{PCM}(\omega_o)S_{CARRIER}(\omega-\omega_o)d\omega_o$$

$$= \frac{A^2 E_1^2 \beta_c^2 T_1}{8\pi}\int_{-\infty}^{\infty}\left[\frac{\sin^4\left(\frac{\omega_o T_1}{4}\right)}{\left(\frac{\omega_o T_1}{4}\right)^2}\right]$$

$$\cdot \left[\delta\left(\omega-\omega_c-\omega_o\right) + \delta\left(\omega+\omega_c-\omega_o\right)\right]d\omega_o$$

$$= \frac{A^2 E_1^2 \beta_c^2 T_1}{8\pi}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_c)T_1}{4}\right]}{\left[\frac{(\omega+\omega_c)T_1}{4}\right]^2}\right.$$

$$\left. + \frac{\sin^4\left[\frac{(\omega-\omega_c)T_1}{4}\right]}{\left[\frac{(\omega-\omega_c)T_1}{4}\right]^2}\right\} \qquad (16)$$

Therefore, the final expression for the power spectral density of a carrier which is amplitude-shift-keyed by a split-phase PCM code is

$$S_{ASK}(\omega) = \frac{A^2}{4}\left[\delta\left(\omega-\omega_c\right)+\delta\left(\omega+\omega_c\right)\right]$$

$$+ \frac{A^2 E_1^2 \beta_c^2 T_1}{8\pi}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_c)T_1}{4}\right]}{\left[\frac{(\omega+\omega_c)T_1}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_c)T_1}{4}\right]}{\left[\frac{(\omega-\omega_c)T_1}{4}\right]^2}\right\} \quad (17)$$

As indicated in Figure 3-2, this expression clearly consists of discrete carrier components plus sidebands resulting from the split-phase baseband spectrum being translated to appear about plus and minus the carrier frequency. Maximum sideband power occurs for $E_1\beta_c = 1$, and, as noted previously, this corresponds to "on-off" keying of the carrier by the split-phase code. Note that the power contained in the discrete carrier components is unaffected by the modulation level, $E_1\beta_c$.
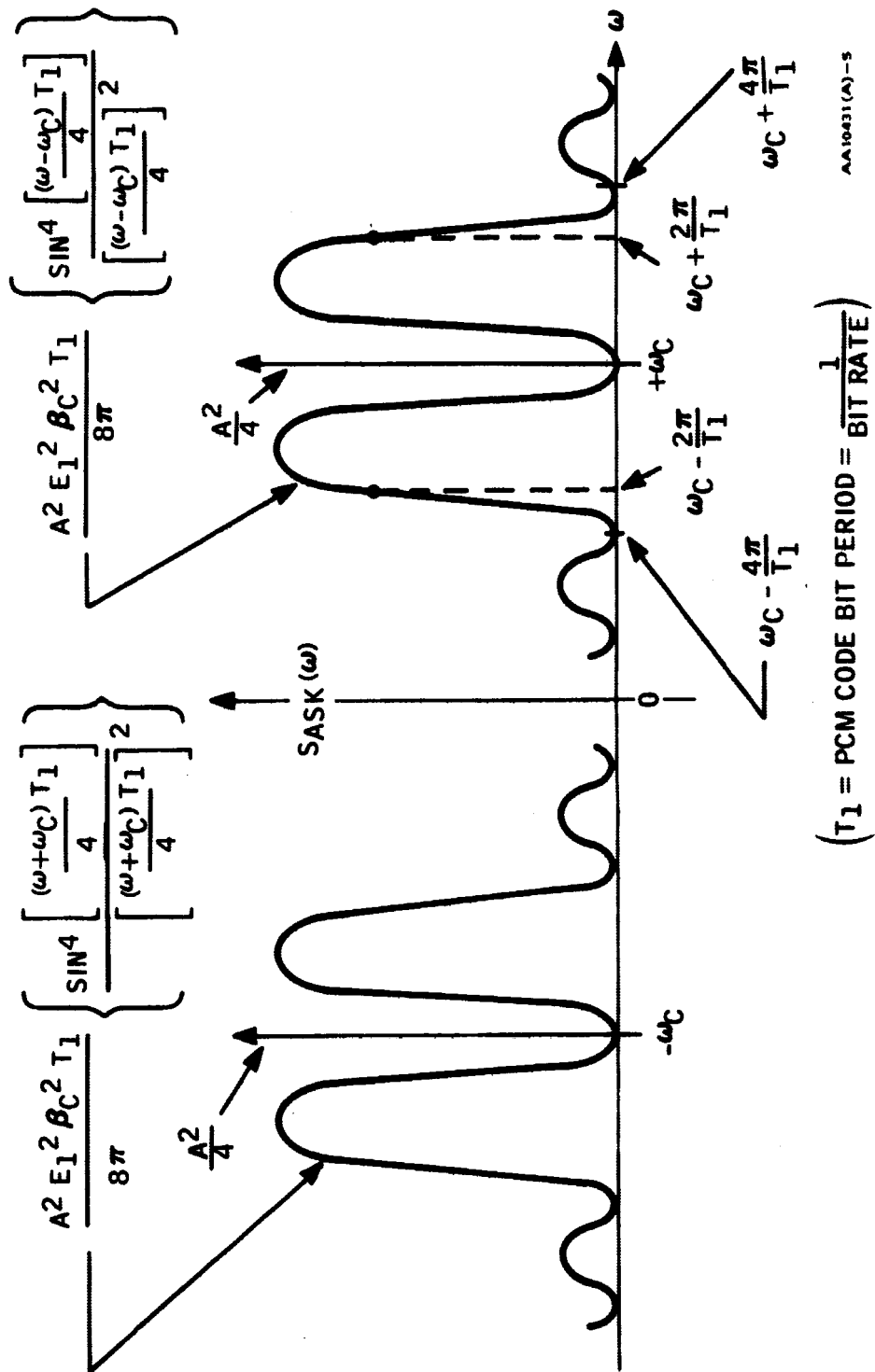
$$\frac{A^2 E_1^2 \beta_C^2 T_1}{8\pi} \left[\frac{SIN^4 \left[\frac{(\omega - \omega_C) T_1}{4} \right]}{\left[\frac{(\omega - \omega_C) T_1}{4} \right]^2} \right]$$

ω

$\omega_C + \frac{4\pi}{T_1}$

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$\omega_C + \frac{2\pi}{T_1}$

$+\omega_C$

$\frac{A^2}{4}$

$\omega_C - \frac{2\pi}{T_1}$

$\omega_C - \frac{4\pi}{T_1}$

$S_{ASK}(\omega)$

0

$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} \right)$

$$\frac{A^2 E_1^2 \beta_C^2 T_1}{8\pi} \left[\frac{SIN^4 \left[\frac{(\omega + \omega_C) T_1}{4} \right]}{\left[\frac{(\omega + \omega_C) T_1}{4} \right]^2} \right]$$

$-\omega_C$

$\frac{A^2}{4}$

Figure 3-2 Power Spectral Density of a Carrier which is
Amplitude-Shift-Keyed by a Split-Phase PCM Code

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SECTION 4

PHASE-SHIFT KEYING BY SPLIT-PHASE PCM CODES

4.1 GENERAL

For the case of phase modulation of a carrier by a binary sequence, a level transition results in the carrier phase being shifted by a discrete amount, either in the positive or negative direction. This modulation process is commonly referred to as phase-shift keying (PSK). It will be shown that a limiting case of PSK is identical to double-sideband (suppressed carrier) modulation.

4.2 AUTOCORRELATION FUNCTION

A generalized expression for a sinusoidal carrier which is phase-modulated by a binary sequence is

$$e_{PSK}(t) = A \cos\left[\omega_c t + m(t)E_1\beta_c + \phi_c\right] \qquad (18)$$

where A, ω_c, $m(t)$, and ϕ_c are as defined in Paragraph 3.2 and
β_c is now the carrier modulation sensitivity in rad/volt.

Expanding equation (18) trigonometrically, the following expression is obtained:

$$e_{PSK}(t) = A \cos\left(\omega_c t + \phi_c\right)\cos\left[m(t)E_1\beta_c\right]$$

$$- A \sin\left(\omega_c t + \phi_c\right)\sin\left[m(t)E_1\beta_c\right] \qquad (19)$$

But, since $m(t) = \pm 1$ and

$$\cos(\pm X) = \cos(X) \qquad (20)$$

$$\sin(\pm X) = \pm\sin(X) \qquad (21)$$

then

$$e_{PSK}(t) = A \cos\left(\omega_c t + \phi_c\right)\cos\left(E_1\beta_c\right)$$

$$- A\, m(t)\sin\left(\omega_c t + \phi_c\right)\sin\left(E_1\beta_c\right) \qquad (22)$$

Since $E_1\beta_c$ is a constant, the second term of the preceding expression represents double-sideband (suppressed-carrier) modulation of the carrier by the binary sequence, while the first term represents a discrete carrier component. Note that for $E_1\beta_c = \pi/2$, the carrier component vanishes and the sideband term is maximized.

Following the same procedure as for the ASK case, the autocorrelation function of the PSK signal of equation (22) is found to be given by

$$
\begin{aligned}
R_{PSK}(\tau) &= \frac{A^2 \cos^2\left(E_1\beta_c\right)}{2} \cos\left(\omega_c\tau\right) \\
&\quad + \frac{A^2 \sin^2\left(E_1\beta_c\right)}{2} \cos\left(\omega_c\tau\right) E\left[m\left(t_1\right)m\left(t_1+\tau\right)\right] \\
&= \cos^2\left(E_1\beta_c\right) R_{CARRIER}(\tau) \\
&\quad + \sin^2\left(E_1\beta_c\right) R_{CARRIER}(\tau) \left[\frac{R_{PCM}(\tau)}{E_1^2}\right]
\end{aligned}
\tag{23}
$$

where $R_{CARRIER}(\tau)$ and $R_{PCM}(\tau)$ are as determined previously for the ASK case.

4.3 POWER SPECTRAL DENSITY

Substituting equation (23) into equation (12), the power spectral density for the PSK case is found to be

$$
\begin{aligned}
S_{PSK}(\omega) &= \cos^2\left(E_1\beta_c\right) S_{CARRIER}(\omega) \\
&\quad + \frac{1}{E_1^2} \sin^2\left(E_1\beta_c\right) S_{CARRIER}(\omega) * S_{PCM}(\omega)
\end{aligned}
\tag{24}
$$

The expressions for $S_{CARRIER}(\omega)$ and $S_{PCM}(\omega)$ are the same as those contained in equations (14) and (15), respectively. After performing the convolution operation indicated in equation (24), $S_{PSK}(\omega)$ reduces to

$$
\begin{aligned}
S_{PSK}(\omega) &= \frac{A^2 \cos^2\left(E_1\beta_c\right)}{4}\left[\delta\left(\omega-\omega_c\right)+\delta\left(\omega+\omega_c\right)\right] + \frac{A^2 T_1 \sin^2\left(E_1\beta_c\right)}{8\pi} \\
&\quad \cdot \left\{ \frac{\sin^4\left[\frac{(\omega+\omega_c)T_1}{4}\right]}{\left[\frac{(\omega+\omega_c)T_1}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_c)T_1}{4}\right]}{\left[\frac{(\omega-\omega_c)T_1}{4}\right]^2} \right\}
\end{aligned}
\tag{25}
$$

Thus, the power spectral density for a carrier which is phase-shift-keyed by a split-phase PCM code is as indicated in Figure 4-1. As for the ASK case, discrete carrier components exist in addition to sidebands resulting from the baseband spectrum of the split-phase code. However, unlike the ASK case, the carrier components vanish as sideband power is maximized. Therefore, it is possible to convert all of the available transmitted power into useable sideband power if binary phase-shift-keying ($\pm\pi/2$) is used.

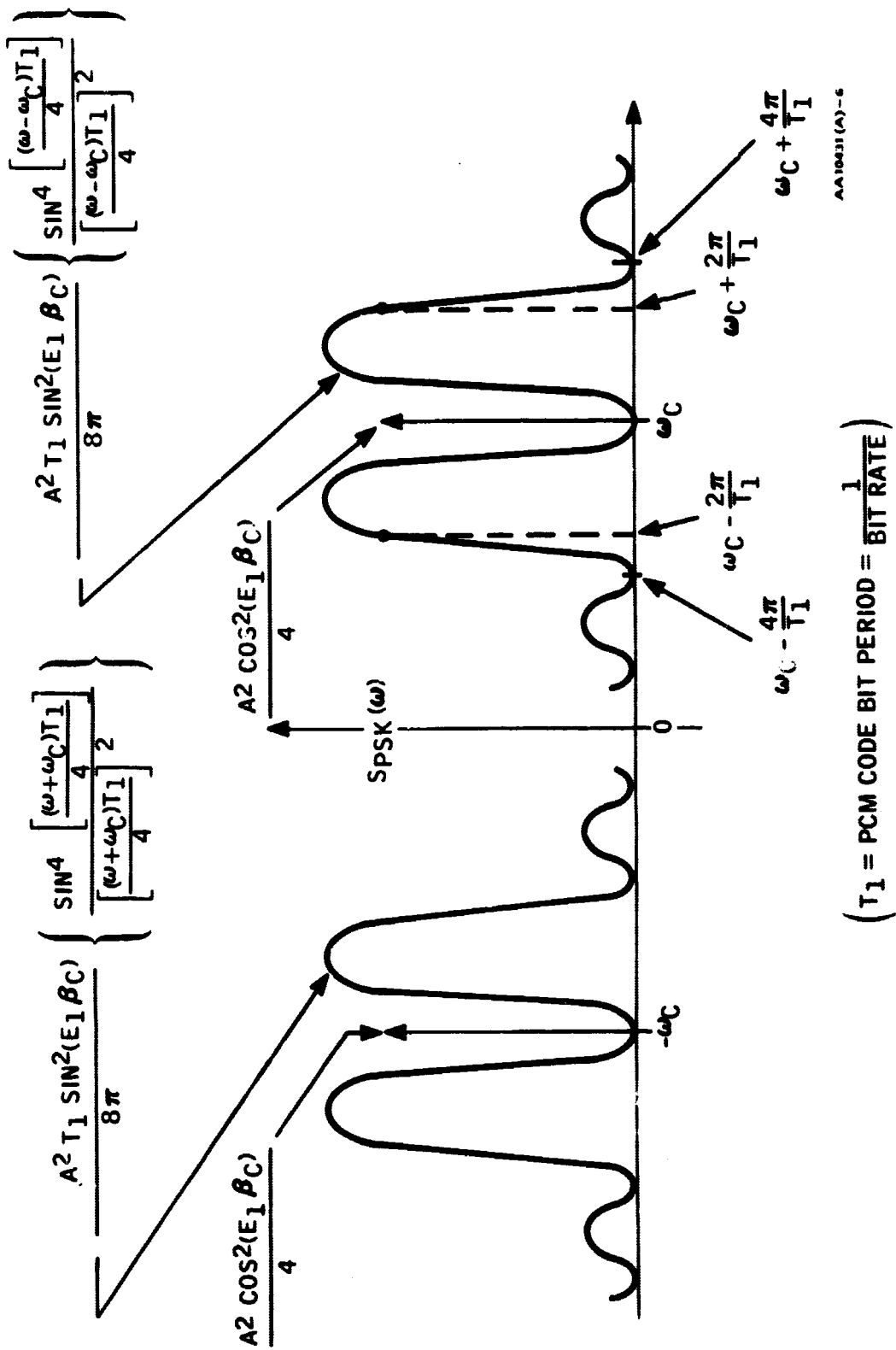Figure 4-1 Power Spectral Density of a Carrier which is Phase-Shift-Keyed by a Split-Phase PCM Code

SECTION 5

FREQUENCY-SHIFT KEYING BY SPLIT-PHASE PCM CODES

5.1 GENERAL

Since this work is concerned only with noncoherent modulation, the case of frequency-shift keying by split-phase PCM codes can be considered as being equivalent to switching between two independent (unsynchronized) oscillators of different frequency. When the PCM code assumes the $+E_1$ voltage level, the modulated signal should be of the form

$$e_{FSK1}(t) = K \cos\left(\omega_1 t + \phi_1\right) \tag{26}$$

and when the code assumes the $-E_1$ voltage level, the output signal should be

$$e_{FSK2}(t) = K \cos\left(\omega_2 t + \phi_2\right) \tag{27}$$

Physically, then, the frequency-shift keying problem consists of turning the first oscillator ON and the second oscillator OFF when a $+E_1$ level is present, and reversing these conditions when a $-E_1$ level is present. One scheme that will allow this switching pattern to be accomplished is shown in Figure 5-1. This scheme requires first that the bipolar ($+E_1$, $-E_1$) split-phase code and its inverse be converted to unipolar ($+E_1$, 0) codes. (Such conversion is easily accomplished using standard digital techniques.) Each of these unipolar codes is used to "on-off" key the output signals from the two oscillators, as indicated in Figure 5-1. These individually keyed outputs are then summed to provide the composite frequency-shift-keyed signal.

Figure 5-1 A Scheme for Using a Split-Phase PCM Code to Key between Two Unsynchronized Oscillators

5.2 AUTOCORRELATION FUNCTION

To determine the autocorrelation function of the composite frequency-shift-keyed signal, it is first noted that the original bipolar split-phase (Biϕ-L) code may be represented by

$$e_{PCM}(t) = m(t)E_1 \tag{28}$$

where $m(t) = \pm 1$ is the random (Biϕ-L PCM) bipolar switching function and E_1 represents the PCM code absolute voltage level.

Then the inverted split-phase code may be expressed as

$$e_{PCM}'(t) = m'(t)E_1 \tag{29}$$

where $m'(t) = \begin{cases} -1 \text{ when } m(t) = +1 \\ +1 \text{ when } m(t) = -1 \end{cases}$

Next, it is observed that the corresponding unipolar codes are

$$e_{PCM1}(t) = m_1(t)E_1 \tag{30}$$

and

$$e_{PCM1}'(t) = m_1'(t)E_1 \tag{31}$$

where $m_1(t) = +1, 0$ is the random (Biϕ-L PCM) unipolar switching function and
$$m_1'(t) = \begin{cases} 0 \text{ when } m_1(t) = +1 \\ +1 \text{ when } m_1(t) = 0 \end{cases}$$

The output signal from the first multiplier is

$$e_{01}(t) = m_1(t)E_1 A \cos\left(\omega_1 t + \phi_1\right) \tag{32}$$

Similarly, the output from the second multiplier is given by

$$e_{02}(t) = m_1'(t)AE_1 \cos\left(\omega_2 t + \phi_2\right) \tag{33}$$

The composite output signal is given by the sum of equations (32) and (33), or

$$e_{FSK}(t) = e_{01}(t) + e_{02}(t)$$

$$= m_1(t)AE_1 \cos\left(\omega_1 t + \phi_1\right) + m_1{'}(t)AE_1 \cos\left(\omega_2 t + \phi_2\right) \quad (34)$$

The autocorrelation function of the FSK signal is found in the same manner as for the ASK and PSK signals:

$$R_{FSK}(\tau) = E\left[e_{FSK}\left(t_1\right)e_{FSK}\left(t_1 + \tau\right)\right] \quad (35)$$

After substituting equation (34) into equation (35) and performing operations analogous to those performed in equations (3) through (9) for the ASK signal, the preceding autocorrelation function becomes

$$R_{FSK}(\tau) = \frac{A^2 E_1{}^2}{2} \cos\left(\omega_1\tau\right)E\left[m_1\left(t_1\right)m_1\left(t_1 + \tau\right)\right]$$

$$+ \frac{A^2 E_1{}^2}{2} \cos\left(\omega_2\tau\right)E\left[m_1{'}\left(t_1\right)m_1{'}\left(t_1 + \tau\right)\right] \quad (36)$$

Further, it can be observed that

$$R_{FSK}(\tau) = R_{PCM1}(\tau)R_{CARRIER1}(\tau) + R_{PCM2}(\tau)R_{CARRIER2}(\tau) \quad (37)$$

It is noted, however, that the above autocorrelation functions, $R_{PCM1}(\tau)$ and $R_{PCM2}(\tau)$, for the unipolar split-phase code are not the same as for the bipolar split-phase code. Since the unipolar codes have no negative voltage levels, it is not possible for their autocorrelation functions to ever be negative. Using the procedure outlined in Reference 2, the autocorrelation function for the unipolar split-phase codes may be found. Detailed calculations are included in Appendix A, and the result of these calculations (note that $R_{PCM1}(\tau) = R_{PCM2}(\tau)$) is shown in Figure A-2. Inspection of this figure reveals that the autocorrelation function of the unipolar split-phase code is merely an attenuated version of that of the bipolar split-phase code (Figure 3-1), displaced vertically by $E_1{}^2/4$. Thus,

$$R_{PCM1}(\tau) = R_{PCM2}(\tau) = \frac{E_1{}^2}{4} + \frac{1}{4} R_{PCM}(\tau) \quad (38)$$

where $R_{PCM}(\tau)$ is the autocorrelation function of the bipolar split-phase code.

The autocorrelation function of the FSK signal is found by substituting equation (38) into equation (37).

$$R_{FSK}(\tau) = \frac{E_1^2}{4} R_{CARRIER1}(\tau) + \frac{E_1^2}{4} R_{CARRIER2}(\tau)$$

$$+ \frac{1}{4} R_{PCM}(\tau)R_{CARRIER1}(\tau) + \frac{1}{4} R_{PCM}(\tau)R_{CARRIER2}(\tau) \quad (39)$$

5.3 POWER SPECTRAL DENSITY

Substitution of equation (39) into equation (12) results in the following expression for the power spectral density for the FSK case:

$$S_{FSK}(\omega) = \frac{E_1^2}{4} S_{CARRIER1}(\omega) + \frac{E_1^2}{4} S_{CARRIER2}(\omega)$$

$$+ \frac{1}{4}\left[S_{PCM}(\omega) * S_{CARRIER1}(\omega)\right] + \frac{1}{4}\left[S_{PCM}(\omega) * S_{CARRIER2}(\omega)\right] \quad (40)$$

The individual terms of the above expression are very similar to those obtained for the ASK and PSK signals. Hence,

$$S_{FSK}(\omega) = \frac{A^2 E_1^2}{16}\left[\delta\left(\omega-\omega_1\right) + \delta\left(\omega+\omega_1\right)\right]$$

$$+ \frac{A^2 E_1^2}{16}\left[\delta\left(\omega-\omega_2\right) + \delta\left(\omega+\omega_2\right)\right]$$

$$+ \frac{A^2 E_1^2 T_1}{32\pi}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_1)T_1}{4}\right]}{\left[\frac{(\omega+\omega_1)T_1}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_1)T_1}{4}\right]}{\left[\frac{(\omega-\omega_1)T_1}{4}\right]^2}\right\}$$

$$+ \frac{A^2 E_1^2 T_1}{32\pi}\left\{\frac{\sin^4\left[\frac{(\omega+\omega_2)T_1}{4}\right]}{\left[\frac{(\omega+\omega_2)T_1}{4}\right]^2} + \frac{\sin^4\left[\frac{(\omega-\omega_2)T_1}{4}\right]}{\left[\frac{(\omega-\omega_2)T_1}{4}\right]^2}\right\} \quad (41)$$

The power spectral density just calculated for the FSK signal is illustrated in Figure 5-2. It is observed that this case is quite similar to the ASK case, and this is intuitively correct, since the FSK signal was shown to be merely the sum of two "on-off" keyed carriers of different frequency.
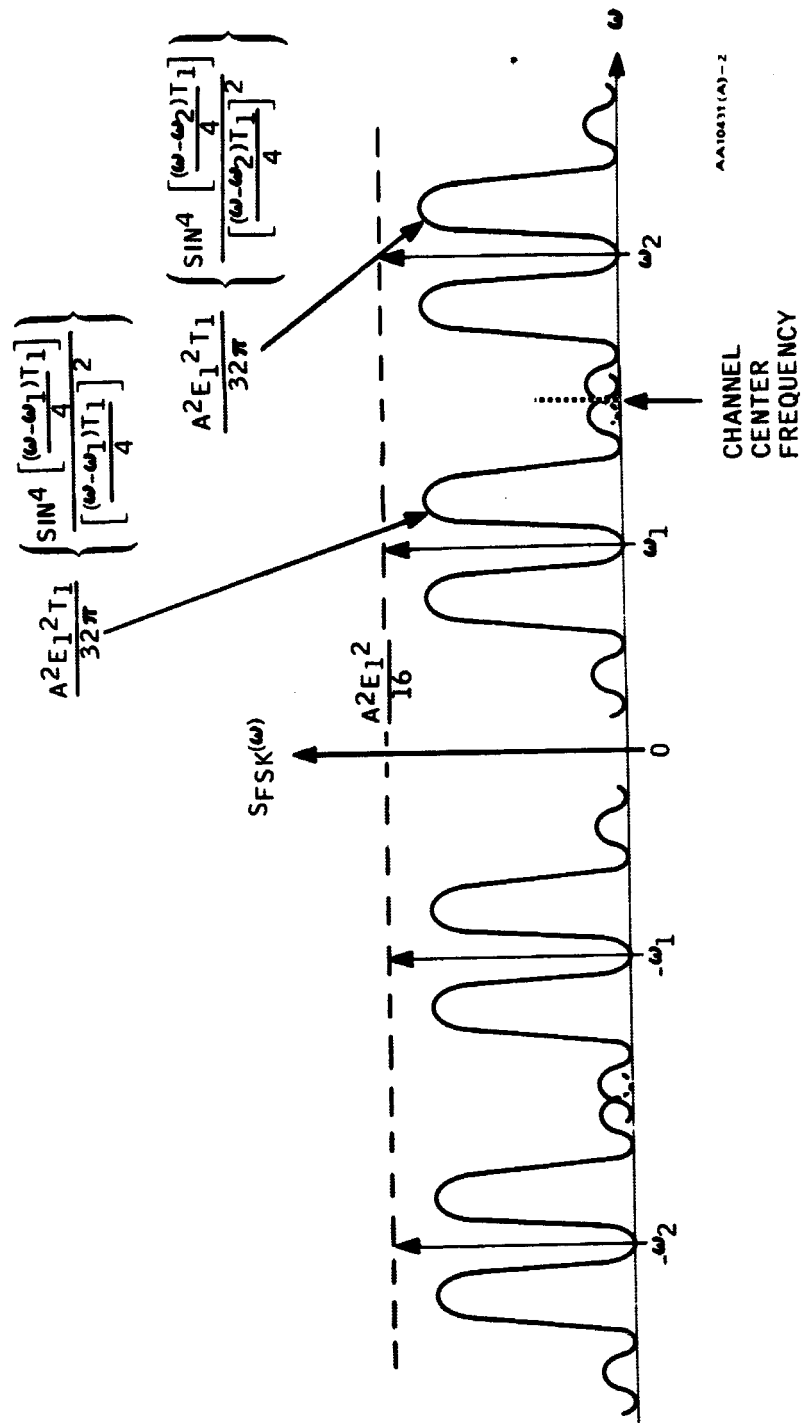
Figure 5-2 Power Spectral Density for Frequency-Shift-Keying
by a Split-Phase PCM Code

APPENDIX A

CALCULATION OF THE ENSEMBLE-AVERAGE AUTOCORRELATION FUNCTION FOR RANDOM (UNIPOLAR) SPLIT-PHASE PCM CODES

The members (sample functions) of an ensemble of random unipolar split-phase PCM codes are illustrated in Figure A-1. In order to calculate the ensemble-average autocorrelation function, the following initial assumptions are made.

A. The process is at least wide-sense stationary (i.e., its autocorrelation function is dependent only on the time, τ, between successive samples and not on the actual sampling times t1 and t2).

B. The probability of occurrence of a "one" is equal to the probability of occurrence of a "zero," or

$$P\left(X_{t1}=E_1\right) = P\left(X_{t1}=0\right) = P\left(X_{t2}=E_1\right) = P\left(X_{t2}=0\right) = 1/2 \qquad \text{(A-1)}$$

The ensemble-average autocorrelation function is equal to the expected value (mean) of the product of the samples X_{t1} and X_{t2} of each member of the ensemble, or

$$R_{PCM1}(\tau) = E\left[X_{t1} \ X_{t2}\right] \qquad \text{(A-2)}$$

In general, the expected value of the product of two random variables X_{t1} and X_{t2} is given by (Reference 4):

$$E\left[X_{t1}X_{t2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_{t1}X_{t2}P_{X_{t1},X_{t2}}\left(X_{t1},X_{t2}\right)dX_1 dX_2 \qquad \text{(A-3)}$$

where $P_{X_{t1},X_{t2}}\left(X_{t1},X_{t2}\right)$ is the joint probability density function of X_{t1} and X_{t2}.

$$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} \right)$$
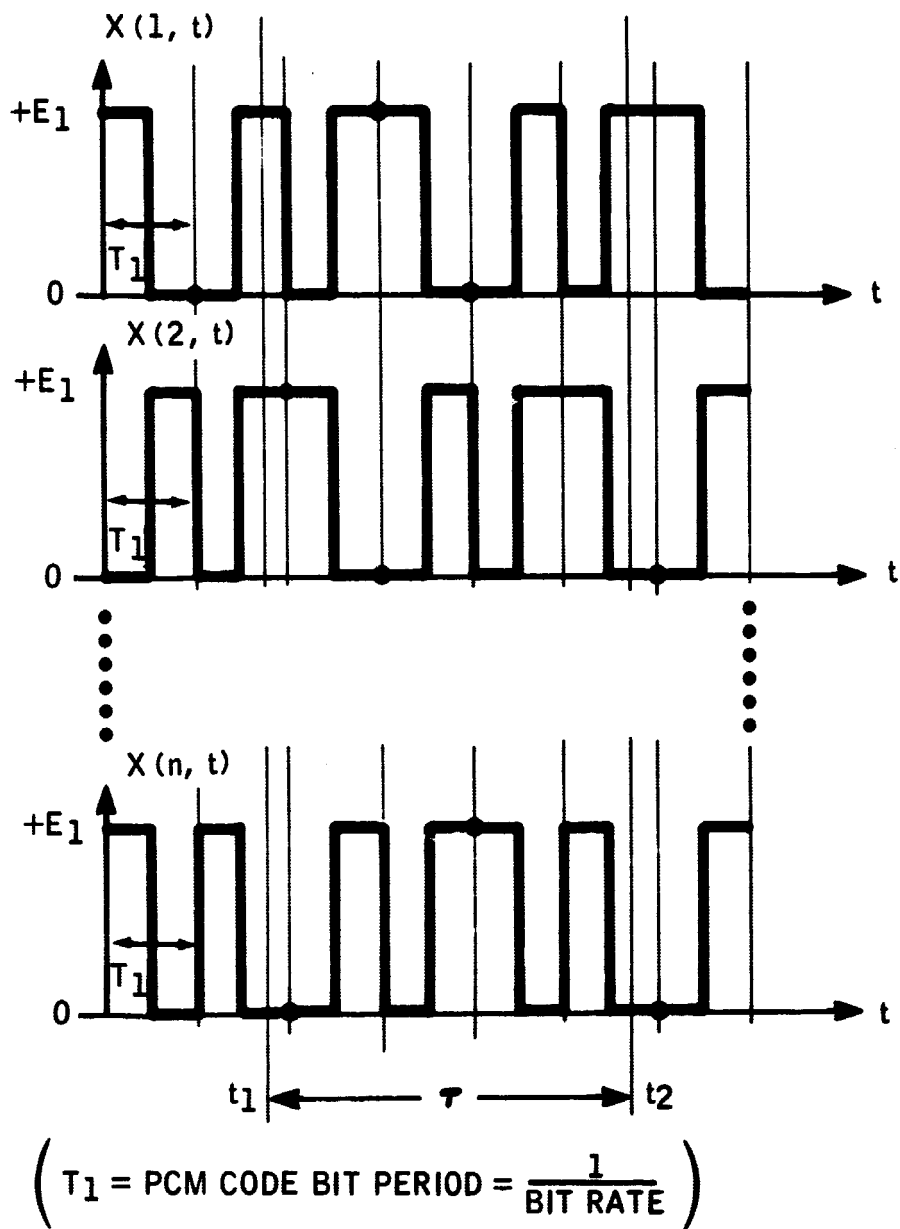
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Figure A-1   Ensemble of Sample Functions of a Random
(Unipolar) Split-Phase Code

For the case of *discrete* random variables (which take on only a finite number of values), the integrals in the previous expression reduce to summations, and the expected value is given by

$$E\left[X_{t1}X_{t2}\right] = \sum_{\text{all } i} \sum_{\text{all } j} \chi_{t1,i}\chi_{t2,j} P\left(X_{t1}=\chi_{t1,i}, X_{t2}=\chi_{t2,j}\right) \quad (A-4)$$

where the $\chi_{t1,i}$ are the possible values that X_{t1} can assume,

the $\chi_{t2,j}$ are the possible values that X_{t2} can assume,

and $P\left(X_{t1}=\chi_{t1,i}, X_{t2}=\chi_{t2,j}\right)$ is the probability of joint occurrence of $\chi_{t1,i}$ and $\chi_{t2,i}$.

The possible values of each of the discrete random variables X_{t1} and X_{t2}, for the case of the random unipolar split-phase PCM code, are $+E_1$ volts and 0 volts. The autocorrelation function, then, is

$$
\begin{aligned}
R_{PCM1}(\tau) &= E\left[X_{t1}X_{t2}\right] \\
&= \left(E_1\right)\left(E_1\right)P\left(X_{t1}=E_1, X_{t2}=E_1\right) + \left(E_1\right)(0)P\left(X_{t1}=E_1, X_{t2}=0\right) \\
&\quad + (0)\left(E_1\right)P\left(X_{t1}=0, X_{t2}=E_1\right) + (0)(0)P\left(X_{t1}=0, X_{t2}=0\right) \\
&= E_1^2 P\left(X_{t1}=E_1, X_{t2}=E_1\right) \quad (A-5)
\end{aligned}
$$

But the probability of the joint occurrence of the events $X_{t1}=E_1$ and $X_{t2}=E_1$ may be expressed as

$$P\left(X_{t1}=E_1, X_{t2}=E_1\right) = P\left(X_{t2}=E_1 \mid X_{t1}=E_1\right)P\left(X_{t1}=E_1\right) \quad (A-6)$$

where $P\left(X_{t2}=E_1 \mid X_{t1}=E_1\right)$ is the probability of occurrence of the event $X_{t2}=E_1$, provided the event $X_{t1}=E_1$ has occurred,

and $P\left(X_{t1}=E_1\right)$ is the probability of occurrence of the event $X_{t1}=E_1$.

Substitution of the conditional probability expression of equation (A-6), along with substitution of equation (A-1), into equation (A-5) yields the following:

$$R_{PCM1}(\tau) = \frac{E_1^2}{2} \, P\left(X_{t2}=E_1 \,|\, X_{t1}=E_1\right) \qquad (A-7)$$

Evaluation of the conditional probability of equation (A-7) is dependent upon the value of τ, the time difference between samples X_{t1} and X_{t2}. For instance, if $\tau = 0$, then $t2 = t1$ and the probability of the event $X_{t2} = E_1$, given that $X_{t1} = E_1$, is unity. Then

$$R_{PCM1}(0) = \frac{E_1^2}{2} \qquad (A-8)$$

$R_{PCM1}(\tau)$ is easily evaluated for $|\tau| \geq T_1$, as the conditional probabilities reduce to simple unconditional probabilities. For $|\tau| \geq T_1$, X_{t2} and X_{t1} are samples of different bit periods, and the value of X_{t2} is *independent* of the value of X_{t1}. Therefore,

$$P\left(X_{t2}=E_1 \,|\, X_{t1}=E_1\right) = P\left(X_{t2}=E_1\right) = \frac{1}{2} \qquad (A-9)$$

For $|\tau| \geq T_1$, then, the autocorrelation function is

$$R_{PCM1}(\tau) = \left(\frac{E_1^2}{2}\right)\left(\frac{1}{2}\right) = \frac{E_1^2}{4} \qquad (A-10)$$

In order to evaluate $R_{PCM1}(\tau)$ for $0 < |\tau| < T_1$, it is necessary to introduce additional conditional probabilities. For instance, if $\tau = T_1/2$, then $P(X_{t2}=E_1 \,|\, X_{t1}=E_1)$ is dependent upon whether X_{t1} is in the first or second half of a bit period. If X_{t1} is in the first half of a bit period, then a sample of the second half of that same bit period must be of opposite polarity. Hence,

$$P\left[\left(X_{t2}=E_1 \,|\, X_{t1}=E_1\right) \,|\, \left(X_{t1} \text{ in first half}\right)\right] = 0 \qquad (A-11)$$

However, if X_{t1} is in the second half of a bit period, then X_{t2} will be a sample function of the next bit period and, therefore, is independent of X_{t1}. Then,

$$P\left[\left(X_{t2}=E_1 | X_{t1}=E_1\right) | \left(X_{t1} \text{ in second half}\right)\right] = P\left(X_{t2}=E_1\right) = \frac{1}{2} \quad \text{(A-12)}$$

The total conditional probabilities for $\tau = T_1/2$ are

$$P\left(X_{t2}=E_1 | X_{t1}=E_1\right) = P\left[\left(X_{t2}=E_1 | X_{t1}=E_1\right) | \left(X_{t1} \text{ in 1st half}\right)\right] P\left(X_{t1} \text{ in 1st half}\right)$$

$$+ P\left[\left(X_{t2}=E_1 | X_{t1}=E_1\right) | \left(X_{t1} \text{ in 2nd half}\right)\right]$$

$$\cdot P\left(X_{t1} \text{ in 2nd half}\right)$$

$$= (0)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \quad \text{(A-13)}$$

$R_{PCM1}(\tau)$, then, for $\tau = T_1/2$ is given by

$$R_{PCM1}\left(\frac{T_1}{2}\right) = \left(\frac{E_1^2}{2}\right)\left(\frac{1}{4}\right) = \frac{E_1^2}{8} \quad \text{(A-14)}$$

Similarly,

$$R_{PCM1}\left(-\frac{T_1}{2}\right) = \frac{E_1^2}{8} \quad \text{(A-15)}$$

If $\tau = T_1/4$, then $P(X_{t2}=E_1 | X_{t1}=E_1)$ is dependent upon which *quarter* of a bit period X_{t1} is located. If X_t is in the first quarter of a bit period, then a sample of the second quarter of that bit period must be of the same polarity. So,

$$P\left[\left(X_{t2}=E_1 | X_{t1}=E_1\right) | \left(X_{t1} \text{ in 1st qtr}\right)\right] = 1 \quad \text{(A-16)}$$

However, if X_{t1} is in the second quarter of a bit period, then a sample of the third quarter of that bit period must be of opposite polarity. Then,

$$P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 2nd qtr}\right)\right] = 0 \qquad (A-17)$$

Similarly,

$$P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 3rd qtr}\right)\right] = 1 \qquad (A-18)$$

and

$$P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 4th qtr}\right)\right] = P\left(X_{t2}=E_1\right) = \frac{1}{2} \qquad (A-19)$$

The total conditional probability for $\tau = T_1/4$, then, is

$$P\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) = P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 1st qtr}\right)\right] P\left(X_{t1} \text{ in 1st qtr}\right)$$

$$+ P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 2nd qtr}\right)\right]$$

$$\cdot P\left(X_{t1} \text{ in 2nd qtr}\right)$$

$$+ P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 3rd qtr}\right)\right]$$

$$\cdot P\left(X_{t1} \text{ in 3rd qtr}\right)$$

$$+ P\left[\left(X_{t2}=E_1 \mid X_{t1}=E_1\right) \mid \left(X_{t1} \text{ in 4th qtr}\right)\right]$$

$$\cdot P\left(X_{t1} \text{ in 4th qtr}\right)$$

$$= (1)\left(\frac{1}{4}\right) + (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{5}{8} \qquad (A-20)$$

Then, $R_{PCM1}(\tau)$, for $\tau = T_1/4$, is given by

$$R_{PCM1}\left(\frac{T_1}{4}\right) = \left(\frac{E_1^{\,2}}{2}\right)\left(\frac{5}{8}\right) = \frac{5}{16} E_1^{\,2} \qquad (A-21)$$

Again,

$$R_{PCM1}\left(-\frac{T_1}{4}\right) = \frac{5}{16}E_1^2 \qquad (A-22)$$

The procedure followed in the preceding may be repeated for various values of τ, thereby allowing a point-by-point calculation of $R_{PCM1}(\tau)$ to be performed. The resultant plot of $R_{PCM1}(\tau)$, for all τ, is contained in Figure A-2.

The entire procedure for the calculation of $R_{PCM1}(\tau)$ can be repeated to determine $R_{PCM2}(\tau)$, the autocorrelation function of the *inverted* unipolar split-phase PCM code of Figure 5-1. The result is that $R_{PCM2}(\tau) = R_{PCM1}(\tau)$ for all τ. This is intuitively correct, since inversion of a random process should not result in a change in its autocorrelation function.
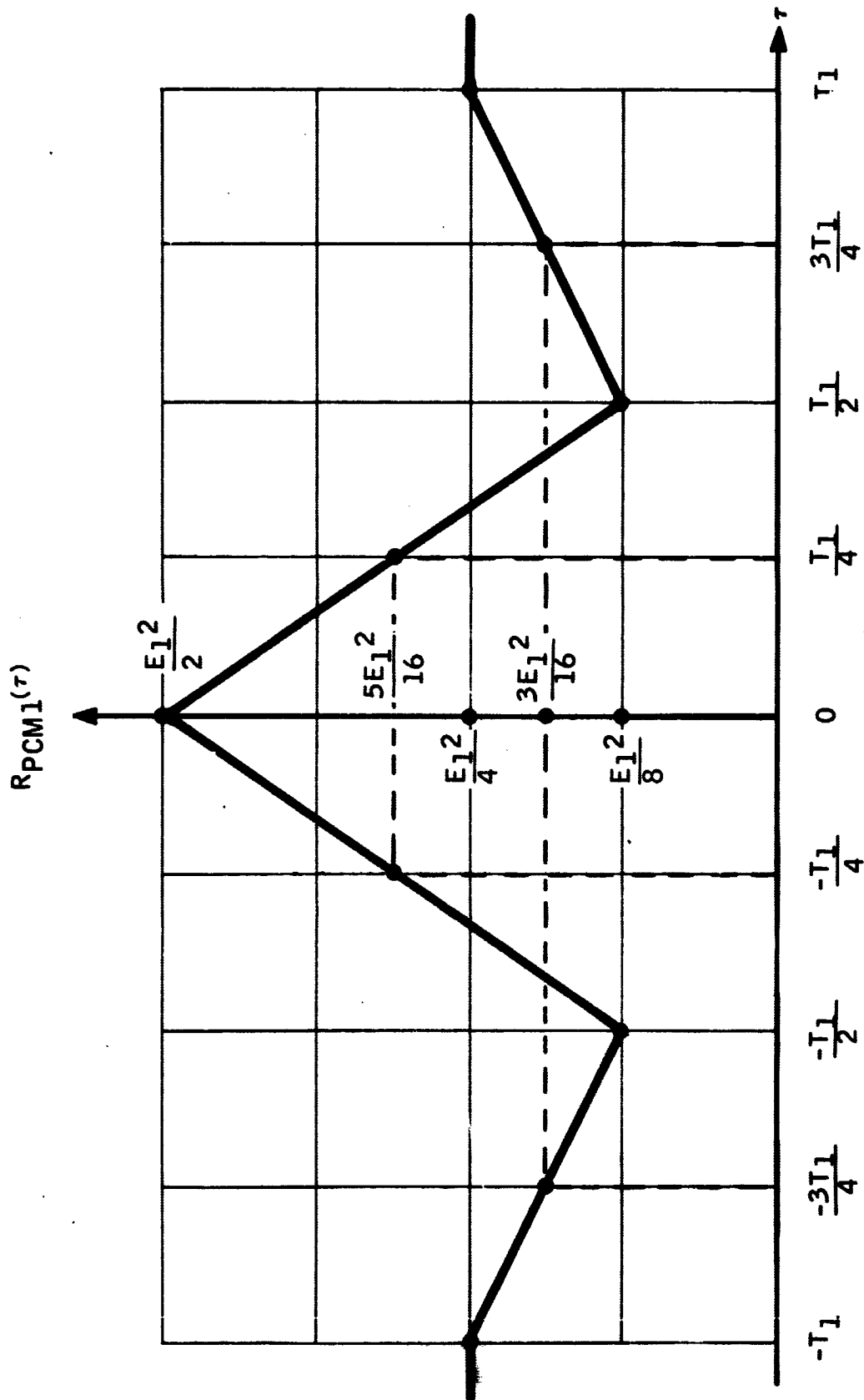
Figure A-2   Ensemble-Average Autocorrelation Function for a
Random (Unipolar) Split-Phase PCM Code

$$\left(\tau_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}}\right)$$

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