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FIRST-ORDER EFFECTS OF PLANE SURFACES ON THE KINETIC BEHAVIOR OF GASES

by Willard E. Meador

Langley Research Center
Hampton, Va. 23365

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16. Abstract

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SUMMARY

A linearized Boltzmann equation is solved exactly for the problem of constant velocity and temperature gradients in a semi-infinite gas of Maxwellian particles. Contrary to similar previous work, the solution satisfies all physical requirements at the boundary without the introduction of complex exponential terms which are nonanalytic in the mean free path. The results apply also to simple Couette flow and are accurate over a range of Knudsen numbers extending well into the transition flow regime. Less rigorous extensions to large flow velocities are investigated in the midst of the gas and are compared with the work of Truesdell.

INTRODUCTION

The accommodation of gases to solid surfaces is a problem of long standing in kinetic theory. (See ref. 1 for a review of the earliest works.) Theoretical expressions for the associated velocity and temperature jumps in the immediate vicinity of a wall were first derived by Maxwell, but not without several assumptions which seemed to lack any rigorous support. In particular, the velocity distribution function was assumed to have the same characteristics close to the wall as in the midst of the gas, the velocity and temperature profiles were not permitted any unusual behavior, and the reflected particles were divided into two groups corresponding to specular and diffuse reflection. Only the last assumption is regarded as obsolete (see refs. 2 and 3) in the present paper.

One of the questions concerning Maxwell's theory is whether the velocity and temperature jumps are real phenomena or simply mathematical consequences of the basic assumptions; for example, more rigorous treatments may yield continuous approaches of the flow velocity and gas temperature to the wall values and still satisfy the boundary conditions. Among the attempts to derive the behavior of gases near walls directly from kinetic theory is a series of three papers (refs. 4 to 6) by Wang Chang and Uhlenbeck which indeed show substantial trends toward the continuous approach to wall properties, but which also predict nonanalytic velocity distribution functions with respect to the mean free path. Such results are disturbing because they contradict both the Chapman-Enskog perturbation procedure (ref. 7) and the Grad moment expansion (ref. 8) in all but the near-continuum flow regime described by Navier-Stokes theory. Breakdowns of the standard techniques may not occur so rapidly with increasing values of the Knudsen number.

A primary purpose of the present research is to show that Wang Chang and Uhlenbeck's nonanalytic distribution functions are direct consequences of the use of Maxwell's hypothesis about the character of the reflected particles. The combination of Boltzmann's kinetic equation with oversimplified boundary conditions inevitably leads to the incorrect appearance of nonanalytic hyperbolic terms. More physical boundary conditions with regard to the conservation of individual macroscopic moments, as opposed to Maxwell's separation of reflected particles into specular and diffuse groups, are shown in the present paper to be entirely consistent with the assumption that the velocity distribution function has the same characteristics close to the wall as in the midst of the gas. Velocity and temperature jumps occur precisely as Maxwell predicted.

These conclusions are derived from calculations based on the exact solution of a linearized Boltzmann equation for the problem of constant velocity and temperature gradients in a semi-infinite gas of Maxwellian particles. The linearization technique is equivalent to that of references 4 to 6; thus the only difference between the general methods is in the expression and use of constraints at the wall. Use also is made of the fundamental premise that exact solutions which satisfy all boundary conditions accurately describe the gas behavior.

The present results apply also to simple Couette flow and are accurate for low Mach numbers over a range of Knudsen numbers extending well into the transition flow regime. This last feature provides a reliable extrapolation of the Navier-Stokes theory for the drag experienced by a flat plate; in addition, similar extrapolations are discussed for the heat transfer between flat plates at different temperatures. Finally, less rigorous extensions to greater flow velocities in the midst of a viscous gas are investigated and compared with Truesdell's work (ref. 9), which has been used in somewhat the same manner as that of Wang Chang and Uhlenbeck in criticizing the standard Chapman-Enskog and Grad approximations. (See ref. 10.)

SYMBOLS

a_1,a_2 distribution function constants

A Wang Chang-Uhlenbeck accommodation coefficient

B constant in Wang Chang-Uhlenbeck boundary condition

$\vec{c}$ particle velocity

d parallel plate separation

D drag force per unit area

f velocity distribution function

f_o Maxwellian distribution function relative to mean gas velocity

$f_o(0)$ Maxwellian distribution function with respect to gas temperature at surface of plate and zero flow

G arbitrary velocity function

h perturbation function relative to plate conditions

ΔH excess random energy striking plate per unit area per second

H_o random energy striking plate per unit area per second

H_w random energy reemitted by plate per unit area per second

$\hat{i},\hat{j},\hat{k}$ unit vectors in x-, y-, and z-directions

k Boltzmann's constant

K,K_0,K_1,K_3 velocity parameters

l mean free path

m particle mass

n particle number density

n_0 particle number density at surface of plate

p scalar pressure

p_{xzz}	third moment defined by equation (59)
$\overset{\leftrightarrow}{\overset{o}{P}}$	traceless pressure tensor
$\vec{q}$	conductive heat flux
t	time
T	temperature of gas
T_0	gas temperature at surface of plate
T_w	plate (or wall) temperature if single plate
T_1,T_2	temperatures of lower and upper plates
$\vec{u}$	dimensionless particle velocity relative to mean gas flow
$\overset{\leftrightarrow}{U}$	unit tensor
$\vec{v}$	mean gas velocity
$\bar{v}$	mean equilibrium particle speed
v_s	slip velocity
v_w	velocity of upper plate in Couette flow
x,y,z	Cartesian coordinates; designate vector and tensor components when used as subscripts
α	accommodation coefficient for energy
β	magnitude of dimensionless flow velocity
β_s	dimensionless slip velocity
β_w	dimensionless velocity (Mach number) of upper plate in Couette flow

$\vec{\gamma}$ dimensionless particle velocity relative to stationary plate

ϵ temperature parameter defined by equation (30)

ϵ_N momentum parameter introduced in equation (24)

η viscosity

θ arbitrary velocity function

λ thermal conductivity

ξ temperature parameter

ρ mass density

σ accommodation coefficient for parallel momentum

σ_N accommodation coefficient for perpendicular momentum

σ' accommodation coefficient for parallel conductive heat flux

ϕ perturbation function

Special notations:

$\left(\frac{\partial f}{\partial t}\right)_c$ collisional time derivative of velocity distribution function

$\langle G \rangle$ velocity average of G based on complete distribution function

$\langle G|h \rangle$ velocity average of G based on perturbation function $f_o(0)h$

$\langle G|1 \rangle$ velocity average of G based on Maxwellian distribution $f_o(0)$

Vector quantities without arrows denote magnitudes.

THE LINEARIZED KINETIC THEORY

The purpose of this section is to develop the linearized kinetic theory for a semi-infinite gas of Maxwellian particles having the velocity and temperature profiles

$$\vec{v} = \hat{i}v = \hat{i}\left(\frac{2kT}{m}\right)^{1/2}\beta = \hat{i}\left(\frac{2kT}{m}\right)^{1/2}\left(\beta_s + Kz\right) \tag{1}$$

and

$$T = T_0(1 + \xi z) \tag{2}$$

where K and ξ are constants specified in the problem definition and T_0 and β_s are the gas temperature and dimensionless flow velocity at the surface (z = 0) of a stationary flat plate. These expressions are assumed to hold throughout the gas.

Equations (1) and (2) are employed directly in the Boltzmann kinetic equation (ref. 7)

$$\vec{c} \cdot \nabla f = (1 + \phi)\vec{c} \cdot \nabla f_0 + f_0\vec{c} \cdot \nabla\phi = \left(\frac{\partial f}{\partial t}\right)_c \tag{3}$$

for the steady-state velocity distribution function

$$f = f_0(1 + \phi) = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-u^2}(1 + \phi) \tag{4}$$

The symbol n refers to the particle number density and

$$\vec{u} = \left(\frac{m}{2kT}\right)^{1/2}(\vec{c} - \vec{v}) \tag{5}$$

is the dimensionless particle velocity. Body forces such as gravity are neglected.

If the macroscopic gas properties vary only with z, the gradient of the Maxwellian function f_0 can be written as

$$\nabla f_0 = \hat{k}\left\{n\left(\frac{m}{2\pi kT}\right)^{3/2} \frac{\partial}{\partial z} \exp\left[-\frac{m(\vec{c} - \vec{v})^2}{2kT}\right] + ne^{-u^2} \frac{d}{dz}\left(\frac{m}{2\pi kT}\right)^{3/2} + \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-u^2} \frac{dn}{dz}\right\}$$

$$= \hat{k}f_0\left[\frac{u^2}{T}\frac{dT}{dz} + 2\left(\frac{m}{2kT}\right)^{1/2} \frac{dv}{dz} u_x - \frac{3}{2T}\frac{dT}{dz} + \frac{1}{n}\frac{dn}{dz}\right]$$

$$= \hat{k}f_0\left[2\left(\frac{m}{2kT}\right)^{1/2} \frac{dv}{dz} u_x + \frac{1}{T}\left(u^2 - \frac{5}{2}\right)\frac{dT}{dz} + \frac{1}{p}\frac{dp}{dz}\right]$$

$$\approx \hat{k}f_0\left[2\frac{d\beta}{dz} u_x + \frac{1}{T}\left(u^2 - \frac{5}{2}\right)\frac{dT}{dz} + \frac{1}{p}\frac{dp}{dz}\right]$$

$$\approx \hat{k}f_0\left[2Ku_x + \xi\left(u^2 - \frac{5}{2}\right) + \frac{1}{p}\frac{dp}{dz}\right] \tag{6}$$

with the aid of equations (1) and (2) and the ideal gas law $p = nkT$. The approximate equalities in equation (6) refer to the arbitrary neglect of quadratic and higher order terms in both β and $(T - T_0)/T$.

This neglect of second and higher order terms, together with the assumption that the largest contributions to ϕ are first order (that is, linear in β or $(T - T_0)/T$), completely defines the linearization process and yields

$$f_0\left(\frac{2kT}{m}\right)^{1/2}\left[2Ku_x + \xi\left(u^2 - \frac{5}{2}\right) + \frac{1}{p}\frac{dp}{dz} + \frac{\partial\phi}{\partial z}\right]u_z = \left(\frac{\partial f}{\partial t}\right)_c \tag{7}$$

from equations (3) and (6). The perpendicular distance from the plate is obviously limited to moderate values by the restrictions on β and T.

Another convenient assumption is that the diagonal elements of the traceless pressure tensor

$$\overset{\circ}{\overleftrightarrow{P}} = 2p\left(\left\langle\vec{u}\vec{u}\right\rangle - \frac{1}{3}\left\langle u^2\right\rangle\overleftrightarrow{U}\right) \tag{8}$$

are zero through first order, which is a familiar result in the midst of gases experiencing simple shear and implies a constant scalar pressure because of the steady-state equation of motion

$$\frac{d}{dz}\left(\overset{\circ}{P}_{zz} + p\right) = 0 \tag{9}$$

The substitution into equation (7) of $dp/dz = 0$ and the trial function

$$\phi = a_1u_xu_z + a_2\left(u^2 - \frac{5}{2}\right)u_z \tag{10}$$

yields

$$f_0\left(\frac{2kT}{m}\right)^{1/2}\left[2Ku_x + \xi\left(u^2 - \frac{5}{2}\right)\right]u_z = \left(\frac{\partial f}{\partial t}\right)_c = -\frac{p}{\eta}f_0\left[a_1u_x + \frac{2a_2}{3}\left(u^2 - \frac{5}{2}\right)\right]u_z \tag{11}$$

for Maxwellian particles (that is, interparticle potentials varying as the inverse fourth power of the separation). Only for such interactions do the Boltzmann collision integrals preserve the spherical harmonic character of equation (10) and permit an exact closed-form solution (ref. 7). The viscosity η in equation (11) is a correct representation of the detailed collision dynamics.

Equations (10) and (11) combine to give

$$\phi = -\frac{\eta}{p}\left(\frac{2kT}{m}\right)^{1/2}\left[2Ku_x + \frac{3\xi}{2}\left(u^2 - \frac{5}{2}\right)\right]u_z \tag{12}$$

which is the exact first-order perturbation function in the midst of the gas and is assumed not to change in character as z approaches zero. However, this expression and the velocity and temperature profiles of equations (1) and (2) also must satisfy the boundary conditions at the surface of the flat plate before they can be termed a precise physical description of the problem. Wang Chang and Uhlenbeck's constraints are not so satisfied. The critical subject of correct boundary conditions is discussed in some detail in the next two sections.

BOUNDARY CONDITIONS

Since the flat plate is stationary relative to the observer and located at the position where the gas temperature is T_0 and the particle number density is n_0, the most appropriate formulation of equations (4) and (12) for the specification of boundary conditions is

$$f = f_0(1 + \phi) \approx n_0\left(\frac{n}{n_0}\right)\left(\frac{m}{2\pi kT_0}\right)^{3/2}\left(\frac{T_0}{T}\right)^{3/2} \exp\left[-\frac{mc^2}{2kT_0}\left(\frac{T_0}{T}\right) + 2\beta\left(\frac{m}{2kT_0}\right)^{1/2} c_x\right](1 + \phi)$$

$$\approx n_0\left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\left(-\frac{mc^2}{2kT_0}\right)(1 - \xi z)\left(1 - \frac{3\xi}{2} z\right)\left[1 + \frac{\xi mc^2}{2kT_0} z + 2\beta\left(\frac{m}{2kT_0}\right)^{1/2} c_x\right](1 + \phi)$$

$$\approx n_0\left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\left(-\frac{mc^2}{2kT_0}\right)\left[1 + 2\beta\left(\frac{m}{2kT_0}\right)^{1/2} c_x + \xi z\left(\frac{mc^2}{2kT_0} - \frac{5}{2}\right) + \phi\right] \tag{13}$$

The expression for n employed in equation (13) is obtained from equation (2) and $dp/dz = 0$ to be

$$n = n_0(1 - \xi z) \tag{14}$$

If the new definitions

$$\vec{\gamma} = \left(\frac{m}{2kT_0}\right)^{1/2} \vec{c} \tag{15}$$

and

$$f_0(0) = n_0\left(\frac{m}{2\pi kT_0}\right)^{3/2} e^{-\gamma^2} \tag{16}$$

are introduced, and if the mean free path l is defined by the viscosity relation

$$\eta = \frac{1}{2}\rho\bar{v}l \tag{17}$$

where ρ is the mass density and

$$\bar{v} = 2\left(\frac{2kT}{\pi m}\right)^{1/2} \tag{18}$$

is the mean equilibrium particle speed, equation (13) becomes

$$f = f_0(0)\ (1 + h) \tag{19}$$

with

$$h = 2\left(\beta_s + Kz\right)\gamma_x + \xi z\left(\gamma^2 - \frac{5}{2}\right) - \frac{4l}{\pi^{1/2}}\left[K\gamma_x + \frac{3\xi}{4}\left(\gamma^2 - \frac{5}{2}\right)\right]\gamma_z \tag{20}$$

The conservation laws to be satisfied by equations (19) and (20) at $z = 0$ are

$$0 = \left\langle \gamma_z \middle| h \right\rangle \tag{21}$$

$$\sigma\left\langle \gamma_x\gamma_z \middle| h \right\rangle_{\gamma_z<0} = \left\langle \gamma_x\gamma_z \middle| h \right\rangle \tag{22}$$

$$\sigma\left\langle \gamma_y\gamma_z \middle| h \right\rangle_{\gamma_z<0} = \left\langle \gamma_y\gamma_z \middle| h \right\rangle \tag{23}$$

$$\sigma_N\left(\left\langle \gamma_z{}^2 \middle| h \right\rangle_{\gamma_z<0} + \epsilon_N\left\langle \gamma_z{}^2 \middle| 1 \right\rangle_{\gamma_z<0}\right) = \left\langle \gamma_z{}^2 \middle| h \right\rangle_{\gamma_z<0} - \left\langle \gamma_z{}^2 \middle| h \right\rangle_{\gamma_z>0} \tag{24}$$

and

$$\alpha\left(\left\langle \gamma^2\gamma_z \middle| h \right\rangle_{\gamma_z<0} - \frac{2\Delta H}{\pi^{1/2}p\bar{v}}\right) = \left\langle \gamma^2\gamma_z \middle| h \right\rangle \tag{25}$$

corresponding to particles, the x-, y-, and z-components of momentum, and energy, respectively, with gas-wall accommodation coefficients 0, σ, σ, σ_N, and α. Special definitions include ΔH for the magnitude of the excess random energy striking unit area of the wall in 1 second, the parameter ϵ_N relating to the z-momentum associated with ΔH, and the symbolic integral

$$\left\langle G \middle| h \right\rangle = \frac{1}{n_0}\int f_0(0)\, Gh\, d\vec{c} = \frac{1}{\pi^{3/2}}\int e^{-\gamma^2} Gh\, d\vec{\gamma} \tag{26}$$

for velocity-averaged quantities.

If each accommodation coefficient refers on the average to the fraction of a particle's corresponding macroscopic property which is lost on impact with the wall, the left-hand sides of equations (21) to (25) represent the total amounts of such properties absorbed in 1 second by a unit area of the wall. The steady-state conservation laws require these absorptions to be balanced by the net property fluxes on the right-hand sides of equations (21) to (25). Only ϵ_N in equation (24) and ΔH in equation (25) remain to be specified in order to utilize these conditions.

Since the random energy of the incident stream is governed entirely by the Maxwellian function $f_0(0)$ at $z = 0$, the magnitude of such energy crossing unit area in 1 second must satisfy

$$H_0 = \frac{m}{2} \frac{2kT_0}{m} \frac{\int_{\gamma_z>0} e^{-\gamma^2} \gamma^2 \gamma_z \, d\vec{\gamma}}{\int_{\gamma_z>0} e^{-\gamma^2} \gamma_z \, d\vec{\gamma}} \left[\frac{n_0}{\pi^{3/2}} \left(\frac{2kT_0}{m} \right)^{1/2} \int_{\gamma_z>0} e^{-\gamma^2} \gamma_z \, d\vec{\gamma} \right] = \left(2kT_0\right)\left(\frac{n_0 \bar{v}}{4}\right) \tag{27}$$

The terms in the two parentheses following the second equality sign in equation (27) correspond, respectively, to the average energy per particle issuing toward the wall from an equilibrium gas at temperature T_0 and the number of such particles per unit area per second. A similar reflected stream at the temperature T_w of the wall (but with the same number of equilibrium particles per unit area per second in order not to imply a net particle flux (see ref. 1, p. 313)) would carry energy away in the amount

$$H_w = \left(2kT_w\right)\left(\frac{n_0 \bar{v}}{4}\right) \tag{28}$$

The substitution into equation (25) of the energy difference

$$\Delta H = H_0 - H_w = \frac{p\bar{v}}{2T_0}\left(T_0 - T_w\right) = \frac{p\bar{v}\epsilon}{2} = -\frac{\pi^{1/2} p\bar{v}\epsilon}{2} \left\langle \gamma^2 \gamma_z \middle| 1 \right\rangle_{\gamma_z<0} \tag{29}$$

obtained from equations (2), (27), and (28), the definition

$$\epsilon = \frac{T_0 - T_w}{T_0} \tag{30}$$

and the identity

$$\left\langle \gamma^2 \gamma_z \middle| 1 \right\rangle_{\gamma_z<0} \equiv \frac{1}{\pi^{3/2}} \int_{\gamma_z<0} e^{-\gamma^2} \gamma^2 \gamma_z \, d\vec{\gamma} \equiv -\pi^{-1/2} \tag{31}$$

yields the final energy condition

$$\alpha\left(\left\langle \gamma^2 \gamma_z \middle| h \right\rangle_{\gamma_z<0} + \epsilon \left\langle \gamma^2 \gamma_z \middle| 1 \right\rangle_{\gamma_z<0} \right) = \left\langle \gamma^2 \gamma_z \middle| h \right\rangle \tag{32}$$

This particular form is chosen for subsequent comparisons with Wang Chang and Uhlenbeck's constraints.

Of the five boundary conditions listed in equations (21) to (24) and (32), only the conservation of energy and the x-component of momentum are not satisfied automatically by the h of equation (20) if

$$\epsilon_N = \frac{3\xi l}{\pi}\left(\frac{2 - \sigma_N}{\sigma_N}\right) \tag{33}$$

Equation (33) is obtained by using equation (20) to evaluate the integrals in equation (24) and solving the resulting expression for ϵ_N. The proportionality to ξ confirms the interpretation that ϵ_N is related to ΔH and vanishes in the absence of temperature gradients.

Accordingly, equations (22) and (32) are sufficient to determine the two remaining unknown parameters β_s and ϵ and thereby complete the first-order description of a semi-infinite gas with given velocity and temperature gradients and mean free path and with given wall and gas-wall parameters T_w, α, σ, and σ_N. Each given quantity either contributes to the basic problem definition or must be determined from the details of gas-wall interactions; hence, the role of pure kinetic theory is finished with the application of equations (22) and (32) to equation (20).

Pertinent integrations for this purpose are given by the expressions

$$\left\langle \gamma_x \gamma_z \middle| h \right\rangle_{\gamma_z<0} = -\frac{1}{2\pi^{1/2}}\left(\beta_s + Kl\right) \tag{34}$$

$$\left\langle \gamma_x \gamma_z \middle| h \right\rangle = -\frac{Kl}{\pi^{1/2}} \tag{35}$$

$$\left\langle \gamma^2 \gamma_z \middle| h \right\rangle_{\gamma_z<0} = -\frac{15\xi l}{8\pi^{1/2}} \tag{36}$$

$$\left\langle \gamma^2 \gamma_z \middle| 1 \right\rangle_{\gamma_z<0} = -\pi^{-1/2} \tag{37}$$

and

$$\left\langle \gamma^2 \gamma_z \middle| h \right\rangle = -\frac{15\xi l}{4\pi^{1/2}} \tag{38}$$

the appropriate combinations of which are

$$\frac{\sigma}{2}\left(\beta_s + Kl\right) = Kl \tag{39}$$

and

$$\alpha\left(\epsilon + \frac{15\xi l}{8}\right) = \frac{15\xi l}{4} \tag{40}$$

according to equations (22) and (32).

Hence, the velocity and temperature jumps, v_s and ΔT, satisfy

$$v_s = \left(\frac{2kT}{m}\right)^{1/2}\beta_s = \frac{(2 - \sigma)l}{\sigma}\left(\frac{2kT}{m}\right)^{1/2}K = \frac{(2 - \sigma)l}{\sigma}\frac{dv}{dz} \tag{41}$$

and

$$\Delta T = T_0 - T_W = T_0\epsilon = \frac{15(2 - \alpha)\xi l T_0}{8\alpha} = \frac{15(2 - \alpha)l}{8\alpha}\frac{dT}{dz} \tag{42}$$

with the aid of equations (1), (2), and (30).

These results are precisely those of Maxwell (see ref. 1, pp. 296 and 314) but with the distinction that they are now supported by a rigorous first-order kinetic theory of Maxwellian particles. This conclusion depends, of course, on the completeness of the boundary conditions of equations (21) to (24) and (32). The question might be asked, for example, whether other particle properties should be constructed from the basic ones of particle identity, momentum, and energy, and then forced by steady-state arguments to satisfy their own conservation laws at the wall.

A typical such constructed property is the x-component of a particle's heat flux which in the midst of a gas seems to infer a net flow of this property in the z-direction according to the integral $\left\langle \gamma^2\gamma_x\gamma_z \middle| h \right\rangle$. Should a new accommodation coefficient σ' be introduced so that

$$\sigma' \left\langle \gamma^2\gamma_x\gamma_z \middle| h \right\rangle_{\gamma_z<0} = \left\langle \gamma^2\gamma_x\gamma_z \middle| h \right\rangle \tag{43}$$

represents an additional constraint on the solution of the kinetic equation at $z = 0$? Or should σ' be defined exclusively by equation (43) without any independent existence from detailed gas-wall physics? The most likely interpretation is the following: Since the conservation of energy and momentum completely describes the dynamics of gas-wall collisions if the accommodation coefficients α, σ, and σ_N are known, equation (43) serves only to express σ' in terms of α, σ, and σ_N and thus should not be regarded as an additional restriction on the velocity distribution function. More precisely,

$$\sigma' = \frac{14\sigma}{\sigma + 12} \tag{44}$$

from the application of equation (20) to equation (43) and the subsequent use of equations (1) and (41).

Although this distinction between the conservation of basic and constructed particle properties may seem trivial from the standpoint of a straightforward treatment of the present problem, it is absolutely essential for the critical analysis of Wang Chang and Uhlenbeck's boundary conditions given in the next section. Their application of equation (43) with σ' equal to σ instead of the value in equation (44) is directly responsible for the previously mentioned nonanalytic exponential contributions to h in references 5 and 6.

WANG CHANG AND UHLENBECK'S THEORY

Wang Chang and Uhlenbeck's theory (refs. 4 to 6) of simple shear in a semi-infinite gas of Maxwellian particles differs from that of the preceding sections only in the replacement of the conservation laws of equations (21) to (24) and (32) by the single microscopic condition

$$h(\gamma_z) = AB + (1 - A)h(-\gamma_z) \qquad (\gamma_z > 0) \tag{45}$$

at $z = 0$. The accommodation coefficient A is said to represent the fraction of particles striking the plate which is reemitted with the Maxwellian distribution corresponding to T_W and zero flow velocity – that is, which is deprived by the plate of all properties characteristic of the incident stream. The remaining particles are specularly reflected, whereas the unknown constant B is related to the gas density near the plate. These concepts are reiterated in a later survey by Uhlenbeck and Ford (ref. 11).

But equation (45) must be statistically equivalent to the definition of A as the average fraction of a particle's total nonequilibrium properties (plus excess random energy and associated z-momentum) absorbed by the plate; hence, the question must be asked whether a single accommodation coefficient is meaningful. In particular, the $\theta(\gamma^2,\gamma_x,\gamma_y)\gamma_z$-moment of equation (45) yields

$$\begin{aligned}\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle_{\gamma_z>0} &= AB\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|1\right\rangle_{\gamma_z>0} + (1 - A)\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h(-\gamma_z)\right\rangle_{\gamma_z>0} \\ &= -AB\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|1\right\rangle_{\gamma_z<0} + (A - 1)\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle_{\gamma_z<0}\end{aligned} \tag{46}$$

which combines with the identity

$$\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle \equiv \left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle_{\gamma_z<0} + \left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle_{\gamma_z>0} \tag{47}$$

to give the following conservation law for the quantity $\theta\left(\gamma^2,\gamma_x,\gamma_y\right)$:

$$A\left[\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle_{\gamma_z<0} - B\left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|1\right\rangle_{\gamma_z<0}\right] = \left\langle\theta\left(\gamma^2,\gamma_x,\gamma_y\right)\gamma_z\middle|h\right\rangle \tag{48}$$

The coefficient A clearly refers in this expression to the average fraction of a particle's $\theta\left(\gamma^2,\gamma_x,\gamma_y\right)$-property absorbed by the plate and h is the perturbation function of Wang Chang and Uhlenbeck. But equation (48) also raises the obvious and critical question of whether Wang Chang and Uhlenbeck's constants A and B are adequate to describe the infinite variety of absorptions associated with different choices of θ.

Similarly, the γ_z^2-moment of equation (45) gives

$$A\left(\left\langle \gamma_z^2 \middle| h \right\rangle_{\gamma_z<0} - B\left\langle \gamma_z^2 \middle| 1 \right\rangle_{\gamma_z<0}\right) = \left\langle \gamma_z^2 \middle| h \right\rangle_{\gamma_z<0} - \left\langle \gamma_z^2 \middle| h \right\rangle_{\gamma_z>0} \tag{49}$$

for the conservation of the z-component of momentum. Again the dependence of A on the moment taken is obvious: Unless the same average absorption fraction applies to every particle property, Wang Chang and Uhlenbeck's microscopic boundary condition is incorrect.

However, the error is not as serious in some applications as might be inferred from the preceding discussion. A detailed comparison between equations (48) and (49) and equations (21) to (24) and (32) is given in table I for the present problem and also for the problems of zero shear ($dv/dz = 0$) and no temperature gradient. The h of equation (20) is used in the calculations of A and B, which thus correspond to the exact first-order solution of the present research.

TABLE I.- CORRELATIONS OF WANG CHANG AND UHLENBECK'S BOUNDARY CONDITIONS WITH THE CONSERVATION LAWS FOR THREE DIFFERENT PROBLEMS

Property conserved	$\frac{dv}{dz} \neq 0;\ \frac{dT}{dz} = 0$		$\frac{dv}{dz} = 0;\ \frac{dT}{dz} \neq 0$		$\frac{dv}{dz} \neq 0;\ \frac{dT}{dz} \neq 0$	
	A	B	A	B	A	B
Particles	Arbitrary	0	Arbitrary	0	Arbitrary	0
x-momentum	σ	Arbitrary	Arbitrary	Arbitrary	σ	Arbitrary
y-momentum	Arbitrary	Arbitrary	Arbitrary	Arbitrary	Arbitrary	Arbitrary
z-momentum	Arbitrary	0	σ_N	$-\epsilon_N$	σ_N	$-\epsilon_N$
Energy	Arbitrary	0	α	$-\epsilon$	α	$-\epsilon$

The most outstanding feature of table I is the liberal sprinkling of arbitrary values of A and B for each problem. Although two of the problems require more than one value of A and B and thereby invalidate Wang Chang and Uhlenbeck's boundary condition of equation (45), the one corresponding to simple shear with no temperature gradient clearly shows the constants $A = \sigma$ and $B = 0$ to be consistent with all five conservation laws. Why then do not Wang Chang and Uhlenbeck (refs. 5 and 6) obtain the results of this paper when T is constant? What is the origin of their nonanalytic exponential contributions to the velocity distribution function and the velocity profile?

Answers to these questions are found in Wang Chang and Uhlenbeck's special applications of equation (45) rather than in the properties listed in table I. Just as in the preceding sections, the boundary conditions of references 4 to 6 are applied only in the integrated forms of equations (48) and (49), but with no restrictions on the function $\theta\left(\gamma^2,\gamma_x,\gamma_y\right)$. Consequently, to take one example, the function $\theta = \gamma^2\gamma_x$ is both permitted and actually used by Wang Chang and Uhlenbeck to obtain the constraint

$$\sigma\left\langle\gamma^2\gamma_x\gamma_z\middle|h\right\rangle_{\gamma_z<0} = \left\langle\gamma^2\gamma_x\gamma_z\middle|h\right\rangle \tag{50}$$

for $A = \sigma$ and $B = 0$ in equation (48). Except for the nonphysical values $\sigma = 0$ and $\sigma = 2$, this expression is clearly inconsistent with equations (43) and (44) and thus imposes an improper condition on the solution of the kinetic equation. The single accommodation coefficient A in equation (45) is never sufficient to describe the absorptions by the wall of all gas properties or conceivable moments.

One of the principal objections to standard kinetic theories is removed by this analysis of Wang Chang and Uhlenbeck's research. No longer can the basic assumptions underlying the Burnett or 13-moment formulations be dismissed on the grounds that they disagree with the first-order behavior of a viscous gas near a wall. (See ref. 10.) Other difficulties do exist (ref. 12) and some criticisms are justified, especially with regard to the higher Chapman-Enskog approximations, but such aspects are irrelevant to the simple problems considered here. The exact first-order distribution function of equations (4) and (12) is precisely Grad's 13-moment solution.

More recent work in the same problem area includes two papers by Scharf (refs. 13 and 14), in which boundaries are regarded as sources of energy and momentum in the gaseous kinetic equations. The source terms appear as combinations of Dirac delta functions centered on the walls and correspond to gas properties at such positions having average values between those of the wall and those of the gas in the absence of the wall. These additions of explicit mixtures of gas and wall properties to the gaseous kinetic equations are neither required nor implied by kinetic theory and boundary conditions.

COUETTE FLOW

The results of the preceding sections are easily extended to the problem of simple Couette flow between two parallel plates, one of which is stationary at $z = 0$ while the other at $z = d$ moves in the positive x-direction with the constant velocity v_W. Equation (41) and the linear velocity profile

$$v = v_S + z\,\frac{dv}{dz} = \left[z + \frac{(2 - \sigma)l}{\sigma}\right]\frac{dv}{dz} \tag{51}$$

are rigorously valid through first order, so that

$$\frac{dv}{dz} = \frac{(v_w - v_s) - v_s}{d} = \frac{v_w}{d}\left[1 + \frac{2(2 - \sigma)l}{\sigma d}\right]^{-1} \tag{52}$$

Finally, since the traceless pressure tensor element $\overset{o}{P}_{xz}$ is $-\eta\, dv/dz$ from equations (8) and (12), the drag force D exerted on unit area of the lower plate must satisfy

$$D = -\overset{o}{P}_{xz} = \frac{\eta v_w}{d}\left[1 + \frac{2(2 - \sigma)l}{\sigma d}\right]^{-1} \tag{53}$$

according to equation (52).

If the temperature gradient is zero, the only restriction imposed on equations (51) to (53) is that the Mach number β_w of the upper plate be small – or more precisely, from equation (12), that the parameter $l\beta_w/d$ be small. Hence, the Knudsen number l/d can extend almost to unity in many applications, which is well above the lower limit usually assigned to the transition flow regime, and equation (53) represents a correct generalization of the Navier-Stokes formula

$$D = \frac{\eta v_w}{d} \tag{54}$$

A similar study of the conductive heat flux between two stationary parallel plates, one at $z = 0$ with temperature T_1 and the other at $z = d$ with temperature T_2, shows

$$\vec{q} = p\left(\frac{2kT}{m}\right)^{1/2}\left\langle u^2\vec{u}\right\rangle = -\hat{k}\,\frac{15k\eta}{4m}\frac{dT}{dz} = -\hat{k}\lambda\,\frac{dT}{dz} = -\hat{k}\,\frac{\lambda\left(T_2 - T_1\right)}{d}\left[1 + \frac{15(2 - \alpha)l}{4\alpha d}\right]^{-1} \tag{55}$$

to be restricted only by small values of $l\left(T_2 - T_1\right)/Td$. The successive steps in equation (55) are obtained by using equations (2), (12), and (42).

HIGHER MACH NUMBERS

In addition to the criticism based on Wang Chang and Uhlenbeck's theory, Schaaf and Chambré (ref. 10) state that the Burnett and 13-moment representations do not adequately approximate the results found by Truesdell (ref. 9) for the less difficult problem of simple shear in an infinite medium of Maxwellian particles. Since Truesdell claims an exact solution of the complete Boltzmann equation, at least a partial analysis of his work is appropriate to the present research.

Truesdell's method begins with the complete set of macroscopic equations of change obtained by taking all moments of the rigorous Boltzmann kinetic equation. However, instead of following Grad's example of using an approximate distribution function to break the chain of relations after a specified number of moments, he accomplishes the same purpose by making arbitrary assumptions about the spatial or time variations of all but a few of the infinite number of moments. In the particular problem of simple shear, Truesdell assumes all second and higher moments to be spatially homogeneous; as a result, only the equations of continuity, motion, and energy balance are nontrivial. The additional assumptions of constant number density and a linear velocity profile then force the problem to be time dependent, a fact which Truesdell uses to criticize previous steady-state solutions based on Burnett's equations.

The contention of the present paper is that Truesdell's criticism is invalid for two reasons: first, the higher moments are not subject to direct laboratory control (or accurate theoretical speculation) and may behave very differently from his assumptions; secondly, Truesdell's criticism fails to distinguish between the problem conditions arbitrarily imposed by his assumptions and the somewhat different conditions underlying the Burnett solutions. If, for example, the simple shear is maintained by the relative motion of two widely separated parallel plates to establish a definite plane of symmetry, the gas temperature will eventually peak at this plane and cause a steady-state conduction of friction-generated heat toward each plate. Truesdell's restriction to spatially homogeneous second moments (including the temperature) precludes this heat conduction and thus corresponds to a steadily increasing temperature with time.

A better and certainly more physical approach is to insist upon a steady state and deduce from the kinetic theory what the velocity profile must be. If the expansion parameter $l\, d\beta/dz$ is small enough, the linearized theory of the preceding sections is applicable and predicts a constant velocity gradient. For somewhat larger values of $l\, d\beta/dz$ (or the maximum Mach number if the flow is Couette), a convenient procedure starts with the following macroscopic equations for spatial variations in the z-direction and a mean particle velocity in the x-direction:

Conservation of x-momentum:

$$\overset{o}{P}_{xz} = \text{Constant} \tag{56}$$

Conservation of z-momentum:

$$\overset{o}{P}_{zz} + p = \text{Constant} \tag{57}$$

and the $\overset{o}{P}_{xz}$ equation:

$$\overset{o}{P}_{xz} + \frac{\eta}{p}\left(\overset{o}{P}_{zz} + p\right)\frac{dv}{dz} + \frac{2\eta}{p}\frac{dp_{xzz}}{dz} = 0 \tag{58}$$

where

$$p_{xzz} \equiv p\left(\frac{2kT}{m}\right)^{1/2} \left\langle u_x u_z{}^2 \right\rangle \tag{59}$$

Each of equations (56) to (58) is obtained from a moment of the Boltzmann equation.

Unless higher moment equations are considered, some assumption must be made about the character of p_{xzz} before equation (58) can be utilized effectively. If the subsequent analysis is restricted to third order, which at most is the order of p_{xzz} in this problem, and if the leading contribution to any gas property is assumed to be constant or proportional to z, then p_{xzz} satisfies

$$\frac{d^2 p_{xzz}}{dz^2} = 0 \tag{60}$$

Although the assumption underlying this restriction is far from established, it at least conforms to the previous first-order results for the flow velocity, pressure tensor, and heat flux.

The second and final assumption is that the third-order flow velocity must satisfy

$$v = K_0 + K_1 z + K_3 z^3 \tag{61}$$

where K_1 and K_3 are first- and third-order constants, respectively.

Since K_1 is input data controlled by whatever mechanism maintains the flow, the determination of K_3 as a function of K_1 will complete the present description. Accordingly, the differentiation of equation (58) and use of equations (56), (57), (60), and (61) yield the third-order expression

$$\frac{d}{dz}\left(\eta \frac{dv}{dz}\right) = 6\eta K_3 z + K_1 \frac{d\eta}{dz} = 0 \tag{62}$$

and thus

$$K_3 = -\frac{K_1}{6\eta z} \frac{d\eta}{dz} \tag{63}$$

for constant p and second-order $\overset{o}{P}_{zz}$. The coefficient K_3 is obviously a third-order constant because η is proportional to some power of the temperature and the temperature undergoes a second-order parabolic decay from its peak value at the symmetry plane. No temperature gradient was induced at the first-order level in the previous sections.

The two fundamental conclusions of this brief study are summarized as follows: (1) induced temperature gradients cause spatially dependent viscosities which lead to modifications of the velocity profiles in the simplest kinds of viscous flow, and (2) these modifications are at most third order in the expansion parameter β or $l\, d\beta/dz$. Hence, the Burnett prediction of simple shear (that is, a linear velocity profile) in a steady-state infinite medium is entirely consistent with the second-order limitations of that approximation.

CONCLUDING REMARKS

The simple problems of viscous flow near a flat plate, Couette flow between parallel plates, and heat transfer between parallel plates are solved exactly through linear terms in typical Chapman-Enskog expansion parameters. Accurate corrections for Knudsen numbers extending almost to unity are applied to the Navier-Stokes theory of Couette flow with low Mach numbers. None of these examples displays the nonanalytic exponential character of velocity distribution functions and calculated results found near the walls by Wang Chang and Uhlenbeck. Differences between the two theories are explained in terms of boundary conditions, some of which are nonphysical in Wang Chang and Uhlenbeck's procedure. Less rigorous extensions of the present theory to higher Mach numbers show that Truesdell's disagreements with the standard Chapman-Enskog and Grad approximations are caused by nonphysical assumptions about the behavior of higher moments.

In summary, much of Maxwell's original theory of velocity and temperature jumps is supported by the present exact solutions of linearized Boltzmann equations for Maxwellian molecules; more specifically, his assumptions about the distribution function and the nature of velocity and temperature profiles in the immediate vicinity of a wall are accurate through first order. Only the hypothesis that the reflected particles can be divided into diffuse and specular groups is obsolete.

Extensions to more general interparticle interaction potentials and system geometries can be very difficult, but no major changes are expected in the basic conclusions.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., June 25, 1970.

REFERENCES

1. Kennard, Earle H.: Kinetic Theory of Gases. McGraw-Hill Book Co., Inc., 1938.

2. French, J. B.: Molecular Beams for Rarefied Gasdynamic Research. AGARDograph 112, Apr. 1966.

3. Jackson, D. P.: A Theory of Gas-Surface Interactions at Satellite Velocities. UTIAS Rep. No. 134, Inst. Aerosp. Studies, Univ. of Toronto, Nov. 1968.

4. Wang Chang, C. S.; and Uhlenbeck, G. E.: The Heat Transport Between Two Parallel Plates as Functions of the Knudsen Number. M999 (Contract N6-onr-23222), Eng. Res. Inst., Univ. of Michigan, Sept. 1953.

5. Wang Chang, C. S.; and Uhlenbeck, G. E.: The Couette Flow Between Two Parallel Plates as a Function of the Knudsen Number. 1999-1-T (Contract No. N6onr-23222), Eng. Res. Inst., Univ. of Michigan, June 1954. (Available from DDC as AD 36271.)

6. Wang Chang, C. S.; and Uhlenbeck, G. E.: On the Behavior of a Gas Near a Wall. 2457-2-T (Contract No. Nonr-1224-(15)), Eng. Res. Inst., Univ. of Michigan, Aug. 1956. (Available from DDC as AD 111875.)

7. Chapman, Sydney; and Cowling, T. G.: The Mathematical Theory of Non-Uniform Gases. Second ed., Cambridge Univ. Press, 1952. (Reprinted 1960.)

8. Hirschfelder, Joseph O.; Curtiss, Charles F.; and Bird, R. Byron: Molecular Theory of Gases and Liquids. John Wiley & Sons, Inc., c.1954. (Reprinted with corrections 1964.)

9. Truesdell, C.: On the Pressures and the Flux of Energy in a Gas According to Maxwell's Kinetic Theory, II. J. Rational Mech. Anal., vol. 5, no. 1, 1956, pp. 55-128.

10. Schaaf, S. A.; and Chambré, P. L.: Flow of Rarefied Gases. Fundamentals of Gas Dynamics, Howard W. Emmons, ed., Princeton Univ. Press, 1958, pp. 687-739.

11. Uhlenbeck, G. E.; and Ford, G. W.: Lectures in Statistical Mechanics. Amer. Math. Soc., c.1963.

12. Meador, Willard E.: Modifications to Fourier's Law of Heat Conduction Through the Use of a Simple Collision Model. NASA TN D-5883, 1970.

13. Scharf, G.: Normal Solutions of the Linearized Boltzmann Equation. Helvetica Physica Acta, vol. 42, fasc. I, 1969, pp. 5-22.

14. Scharf, G.: Stokes' Problems in Kinetic Theory. Phys. Fluids, vol. 13, no. 4, Apr. 1970, pp. 848-857.

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