375

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

INTERNAL NOTE MSC-ED-IN-68-79

COORDINATE SYSTEM INFLUENCE ON THE REGULARIZED TRAJECTORY OPTIMIZATION PROBLEM

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

March 1969

INTERNAL NOTE MSC-ED-IN-68-79

COORDINATE SYSTEM INFLUENCE ON THE REGULARIZED TRAJECTORY OPTIMIZATION PROBLEM

PREPARED BY

J. M. Lewallen
Chief, Theory and Analysis Office
Computation and Analysis Division, NASA

O. A. Schwausch
Scientific Programmer, Senior
Lockheed Electronics Company

APPROVED BY

Eugene H. Brock
Chief, Computation and Analysis Division, NASA

J. A. Barnes
Supervisor, Theory and Analysis Group
Lockheed Electronics Company

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS
March 1969

CONTENTS

TABLES

FIGURES

COORDINATE SYSTEM INFLUENCE ON THE REGULARIZED TRAJECTORY OPTIMIZATION PROBLEM

By J. M. Lewallen, Manned Spacecraft Center,
and O. A. Schwausch, Lockheed Electronics Company

SUMMARY

This investigation studies the effect of using regularized variables to enhance the numerical integration process associated with the optimal trajectory of a continuously thrusting space vehicle. The integration characteristics of both the rectangular Cartesian and polar cylindrical coordinates are considered for an optimal, low-thrust, Earth-escape, spiral trajectory. The numerical accuracy achieved and the computer time required are compared for various numerical integration error bounds, by using both the unregularized and regularized equations. The results obtained indicate that for space vehicles which experience wide variations in the gravitational force magnitude, significant reductions in computing time can be obtained by using the regularized trajectory optimization equations. In some cases, the computing time is reduced by a factor of three if regularized variables are used. Furthermore, for the problem considered here, use of the polar coordinates consistently results in more favorable computer times than when rectangular coordinates are used. In addition, if the numerically evaluated Hamiltonian, which is theoretically constant, is used as an indication of integration error generation, the trade-off between integration time and integration error becomes apparent. Finally, it is shown that the polar coordinates are less sensitive than the rectangular coordinates to errors in the initial Lagrange multipliers.

INTRODUCTION

During the past decade, considerable effort has been directed toward determining numerical methods for optimization of nonlinear, dynamic systems. A comparison of the characteristics of several of the more popular direct and indirect numerical optimization methods is given in Ref. 1. Further investigations dealing with the procedures for accelerating convergence of the indirect optimization methods are discussed in Ref. 2. The primary consideration in evaluating an optimization method is the computing time required for convergence to a sufficiently accurate solution. These characteristics may be influenced by the functional form of the equations of motion as well as the choice of the coordinate system in which the motion is computed.

Regularizing transformations have been used in celestial mechanics to eliminate singularities associated with gravitational force centers. Results reported in Ref. 3 indicate that the numerical integration characteristics can be enhanced considerably when a regularized set of differential equations are used for trajectories that experience close primary body approaches. This conclusion has been reached also in Ref. 4 for a wide range of problems in celestial mechanics. Based on these conclusions, a study was made of the applicability of using regularizing transformations to the problem of improving the computational characteristics of numerical optimization procedures. The results described in Ref. 5 indicate significant numerical advantages in terms of computational time and accuracy of terminal condition satisfaction if regular variables are used.

The effect of the regularizing transformation is obviously dependent on the choice of the coordinate system for the unregularized variables. The influence of the coordinate system on numerical error generation in the two-body problem has been studied in Ref. 6 and in the unregularized trajectory optimization problem in Refs. 7 and 8. These investigations indicate that the coordinate sytem used can have a significant effect on computation time and the accuracy of the resulting numerical solution. In particular, these investigations revealed that the polar coordinates were computationally superior to the rectangular coordinates for the continuously powered escape spiral.

In the investigation discussed in the following section, the effect of using both rectangular Cartesian and polar cylindrical coordinate systems is studied for a minimum time, low-thrust, Earth escape spiral. The numerical accuracy, the computation time and the convergence characteristics are compared by using both the regularized and unregularized equations for various bounds on the integration error.

FORMULATION

If the transfer trajectory for a continuously powered, low-thrust, space vehicle is to be time optimal, the following equations must be satisfied in the interval $t_0 \leq t \leq t_f$:

$$\ddot{\bar{r}} = -\mu \frac{\bar{r}}{r^3} - \frac{T\bar{\lambda}}{m\lambda} , \qquad \dot{m} = -\beta \tag{1}$$

$$\ddot{\bar{\lambda}} = -\mu \frac{\bar{\lambda}}{r^3} + 3\mu \frac{(\bar{\lambda} \cdot \bar{r})}{r^5} \bar{r} , \qquad \dot{\lambda} = -\frac{T\lambda}{m^2} \tag{2}$$

The quantity $m = m_0 - \beta t$ where β is a constant mass flow rate and $\bar{\lambda}$ and $\bar{\omega}$ are Lagrange multiplier vectors. The boundary conditions that must be satisfied are

$$\bar{r}(t_0) = \bar{r}_0 \qquad \bar{v}(t_0) = \bar{v}_0 \qquad m(t_0) = m_0 \tag{3}$$

$$\bar{r}(t_f) = \bar{r}_f \qquad \bar{v}(t_f) = \bar{v}_f \qquad \lambda_m(t_f) = 0 \tag{4}$$

$$1 + \bar{\lambda} \cdot \left[-\mu \frac{\bar{r}}{r^3} - \frac{T\bar{\lambda}}{m\lambda} \right] + \bar{\omega} \cdot \bar{v} - \lambda_m \beta \Big|_{t_f} = 0 \tag{5}$$

By using a generalization of the classical Sundman regularizing transformation discussed in Ref. 9, i.e.,

$$d\tau = r^{-3/2} dt \tag{6}$$

a set of regularized equations for the optimal trajectory can be obtained as follows:

$$\bar{r}'' = \frac{3/2(\bar{r} \cdot \bar{r}')\bar{r}'}{r^2} - \mu\bar{r} - \frac{Tr^3\bar{\lambda}}{m\lambda} , \quad m' = -\beta r^{3/2} \tag{7}$$

$$\bar{\lambda}'' = \frac{3/2(\bar{r} \cdot \bar{r}')\bar{\lambda}'}{2} - \mu\bar{\lambda} + 3\mu \frac{(\bar{\lambda} \cdot \bar{r})\bar{r}}{r^2} , \quad \lambda_m' = -\frac{Tr^{3/2}\lambda}{m^2} \tag{8}$$

where the primes indicate derivatives with respect to the pseudo time variable τ rather than the real time t. This transformation is discussed in Ref. 5 where it is shown that Eqs. (7) and (8) are mathematically regular. This vector form of the regularized equations is invariant with the choice of coordinate system. Hence, Eqs. (1) and (2) describe the optimal process in the unregularized rectangular and polar coordinates, while Eqs. (7) and (8) describe the regularized equations associated with each of the coordinate systems. Either set of equations represents the usual, nonlinear, two-point boundary value problem.

DISCUSSION OF RESULTS

From the preceding section, it is seen that the solution to the optimal trajectory problem involves the solution of a nonlinear two-point boundary value problem. Usually, efforts are made to obtain a numerical solution to Eqs. (1) and (2) which satisfies the boundary conditions given by Eqs. (3), (4), and (5). Since Eqs. (3) specify only half the necessary initial conditions, values for the remaining unknown initial conditions, usually Lagrange multipliers and the unknown time, must be assumed before a numerical solution can be determined. Inasmuch as the values of the unknown initial boundary conditions are arbitrarily selected, the terminal constraints given by Eqs. (4) and (5) will not be satisfied. These arbitrarily selected initial conditions are changed systematically on subsequent iterations until the terminal constraints are satisfied more exactly. There are numerous procedures for obtaining the corrections to the unknown conditions. Several of the currently popular iteration procedures are discussed in Ref. 1.

Adequate satisfaction of the specified terminal constraints as well as sufficient numerical accuracy, must be achieved if an acceptable numerical solution is to be obtained. Adequate terminal constraint satisfaction is obtained by requiring the norm of the terminal constraint error to be less than 10^{-7}. Sufficient numerical accuracy is obtained by using full-double precision arithmetic on the UNIVAC 1108 at the NASA Manned Spacecraft Center and by performing the integrations with a variable step-size integration scheme, thereby maintaining the single-step error within certain desired tolerances. The integration scheme employed is a modified version of the scheme discussed in Ref. 10. This scheme uses a fourth-order Runge-Kutta starter and a fourth-order Adams-Bashford predictor corrector.

In order to determine the individual effects of the coordinate system and regularization, the same problem must be solved in both coordinate systems and in both unregularized and regularized form. The optimal Earth escape spiral for a low-thrust space vehicle is an excellent example problem for regularization investigations since the gravitational force magnitude varies by approximately 10^2, and hence it is expected that a wide range of numerical integration step sizes will be required to maintain certain specified error bounds.

Figure 1 shows the optimal escape spiral. Initially, the spacecraft is in a circular near-Earth orbit with a radius equal to 1.05 times the Earth radius. For a constant low-thrust space vehicle subjected to a thrust to mass ratio of 0.1, the spacecraft acquires escape energy in approximately 70 normalized time units (approximately 15.7 hours) and reaches an orbit of radius equal to 8.5 times the Earth radius. Although this thrust to mass ratio is relatively large, it was selected to compromise between a computationally expensive realistic trajectory and an inexpensive unrealistic one. The trend of the results is probably unaltered. Figure 1 also shows the optimal control programs for both the rectangular and polar coordinate systems. Figure 2 shows the relationship between the real and regularized time for the optimal trajectory.

Tables 1 through 3 compare the integration characteristics of the regularized and unregularized polar and rectangular coordinate systems for various absolute single-step integration error bounds. The error-bound separations in Tables 1, 2, and 3 are 10^6, 10^4, and 10^2, respectively. The numerical integration characteristics which are compared include the amount of computer time needed to perform all integrations for the final converged iteration, the average amount of computer time required per integration step, the number of integration steps required, the number of step size changes made, and the norm of the terminal constraint error.

The integration time shown in Tables 1 through 3 represents the computation time needed to integrate the state equations, the Euler-Lagrange equations, and the perturbation equations from the initial time to the final time. The values shown also include the time required to monitor the single-step integration error and determine the appropriate integration step size. The appropriate step size is determined by comparing the single-step error with the desired accuracy limits. If either the maximum or minimum error limit is encountered, the step size is either halved or doubled. If, by doubling the step size, the maximum bound is violated, then the step size remains unchanged. The total number of integration steps taken in the interval and the number of step-size changes necessary to maintain the desired accuracy are recorded also. No distinction is made in the Tables between step-size changes associated with doubling and halving. The average computer time per integration step is recorded to indicate the degree of complexity of the equations for each case. Finally, in order to indicate the degree to which the terminal constraints are satisfied, the norm of the constraint error is recorded. This quantity should be considered with some reservation since the routine simply requires that the norm be less than 10^{-7}. The extent to which this criterion is exceeded is not controlled and is an indication of the convergence rate. However, it also depends on how close the terminal norm for the previous iteration was to the required value of 10^{-7}.

The results presented in Table 1 are for the relatively large error-bound separation of 10^{6}. It is seen that the regularized variables, in either coordinate system, require considerably less computation time per iteration than the unregularized variables. In some cases, the time is reduced by a factor of three. The reason for the large saving in time is readily apparent when the combination of time per iteration step and the total number of steps is examined. Although the regularized equations are more time consuming to evaluate, as indicated by the time required per step, the large number of steps taken by the unregularized system of equations quickly causes the total time to exceed that of the regularized systems. Table 1 also indicates that the polar coordinates generally require less computer time than the rectangular coordinates.

The results shown in Table 2 for an error-bound separation of 10^4 agree with those presented in Table 1 and substantiate the previous conclusions. Again, the regularized variables require less total computer time than the unregularized variables, and the polar coordinate systems exhibit shorter integration times than the rectangular coordinate systems. However, for this error-bound separation, the computation time advantage of the regularized systems has been reduced slightly. Note also that the difference in the total number of integration steps between the regularized and unregularized variables has been reduced. In addition, the number of step-size changes for the regularized variables is less than the number of changes required by the unregularized variables. This is in keeping with the regularization theory which predicts that regularized variables will undergo fewer step-size changes than unregularized variables, provided a certain integration accuracy is to be maintained. (For the previous error-bound separation of 10^6, a comparison of the number of step-size changes is invalid since, in some instances, the lower error bound was never encountered.)

The results presented in Table 3 for the error-bound separation of 10^2 generally agree with the results of Tables 1 and 2. As in the previous tables, the polar coordinate system requires shorter integration times than the rectangular system. However, for this magnitude of error-bound separation, the integration times for the regularized and unregularized variables are essentially the same. The departures from the previously indicated trend can be explained by examining Table 4.

Shown in Table 4 are the error-bound encounters for certain integration error tolerances. The top line in each set of four lines represents the upper or maximum allowable error bound. Each succeeding line represents the minimum allowable error for a particular error-bound separation. Thus, the first set of four lines represents the integration error bounds of 10^{-4} and 10^{-6}, 10^{-4} and 10^{-8}, and 10^{-4} and 10^{-10}. The boundary encounters are plotted as a function of the normalized trajectory time. One of the appropriate symbols, keyed in Table 4, records the encounter of the numerical error magnitude with either of the boundaries. An encounter with the lower bound means the step size will be doubled; an encounter with the upper bound means the step size will be halved.

Table 4 indicates that by maintaining the small integration error-bound separation of 10^2, the error in the unregularized rectangular variables is such that the step size is doubled three times during the escape trajectory for the 10^{-4} to 10^{-6} accuracy limits. Upon increasing the error separation to 10^4, to give error bounds 10^{-4} to 10^{-8}, the unregularized rectangular error becomes less than the minimum acceptable error only twice, with the first boundary encounter coming after the 10^{-6} bound in the previous case had already been crossed twice. By doubling the step size early in the trajectory flight time in the 10^{-4} to 10^{-6} case, 7 seconds of computer time were saved per iteration. This time saving was increased to approximately 10 seconds when comparing with the 10^{-4} to 10^{-10} accuracy level since the lower boundary for this case was never encountered. Thus, by requiring the rectangular error to be within the 10^{-4} – 10^{-6} accuracy level rather than the 10^{-4} – 10^{-8}

accuracy level, 253 integration steps were eliminated. Elimination of these 253 steps, each consuming approximately .0276 seconds of computer time, resulted in saving 7 seconds of computer time per iteration. Likewise, by requiring the integration error to be within the $10^{-4} - 10^{-6}$ accuracy level rather than the $10^{-4} - 10^{-10}$ interval, a 10-second saving in computer time per iteration was realized. This same trend appeared in both the rectangular and polar coordinates, for the other error bounds shown. By maintaining the integration error within the smaller error bounds, the total integration time was reduced and made comparable to that for the regularized system.

From examination of Table 4, it becomes evident that integration errors in the regularized coordinate systems propagate differently than do errors in the unregularized systems. Since a feature of regularization is the automatic scaling of integration step size, an increasing radius vector magnitude will automatically increase the step size whereas a decreasing radius vector magnitude will automatically decrease the integration step size. Thus, due to the nature of the Earth escape spiral trajectory, the radius vector is continually increasing, and it is conceivable that the step size will have to be reduced in order to maintain the desired accuracy. From examination of Table 4, it is evident that with only one exception, the integration step size for the regularized variables is always halved. The exception occurs for the 10^{-4} to 10^{-6} error limits using the polar coordinates. In this case, the error is such that the 10^{-6} boundary is just crossed thereby doubling the step size. With further integration, the error becomes large and the step size is halved again. In all other instances, the lower boundaries are never encountered. Since the lower boundaries are not encountered, increasing the error-bound separation limit does not affect the regularized systems and only penalizes the unregularized system by increasing the integration times.

An alternative approach to regularization is suggested by the lack of encounters at the lower boundaries for the regularized variables. Since only the upper boundary is encountered, a value of $n < 3/2$, in the transformation $d\tau = r^{-n}dt$, could be selected. This would keep the step size from increasing so rapidly with increasing values of the radius and thus eliminate the decrease in step size associated with an encounter with the upper boundary. Such a value of n would not eliminate the mathematical singularities; however, in most normal cases the singularities are never encountered anyway. This concept presents an interesting possibility for numerical integration step size control.

All information presented thus far has been associated with the characteristics of the last trajectory generated by an iteration process, that is the converged trajectory. It is of interest to know how the four different cases studied are affected by making certain errors in the initial assumption for boundary conditions (the Lagrange multipliers and terminal time). Table 5 presents information on the number of iterations required and the computer time expended in converging from certain specified initial error percentages in the Lagrange multipliers. Since all possible combinations of the four multipliers and percentage errors represent too many cases to examine efficiently, all multipliers were considered to be in error by the same percentage for each case studied.

The results presented in Table 5 indicate that the polar coordinates are less sensitive than the rectangular coordinates to errors in the initial Lagrange multipliers. Table 5 also indicates that regularized variables are less sensitive than the unregularized variables to erroneous initial conditions. Although the number of iterations required to achieve convergence is essentially the same for all cases, the computer time requirements are not. The reason that the regularized variables require less computer time than the unregularized variables may be seen readily by examining Figure 3.

Figure 3 shows that the convergence rate of the regularized variables for initial multiplier errors of 8 percent is greater than the respective rate of the unregularized variables. The trend presented in Figure 3 is considered to be representative of all cases given in Table 5. Had Table 5 been expanded to include errors greater than ±20 percent, the computer time savings of the regularized variables would probably have been more significant. Note that for results presented in Figure 3 and Table 5, the value of the terminal time was not perturbed. This, in general, is not realistic. If the problem is such that the radius vector increases with time and regularized variables are being used, care must be taken in the initial assumption for the terminal time. The sensitivity of the terminal pseudo time τ to errors in the terminal time t in seen in Fig. 2. One solution involves continuously monitoring the terminal norm and selecting the terminal time which corresponds to the minimum norm for the first assumption.

Although for some cases the regularized and unregularized systems may exhibit nearly equal integration times, the integration accuracy of each system may differ. Since a closed-form solution to the problem considered here does not exist, the error generated by the numerical integration process is unknown. However, there does exist a constant of motion which may be considered in evaluating the accuracy of the numerical integration procedure. This constant of motion, evaluated at the final time, is given by Equation 5. For the example discussed, this constant, referred to as 1+H, must be zero throughout the trajectory. Thus, the deviation of 1+H from zero is one indication of the inaccuracy of the numerical integration process. It should be noted, however, that the satisfaction of $1+H = 0$ is necessary but is not sufficient to insure numerical integration accuracy. Since some of the terms in the expression for 1+H contain combinations of the integrated variables, large error generation in two separate terms could cancel, leaving the impression that numerical accuracy had been achieved.

The relative values of 1+H for converged iterations using the regularized and unregularized systems may be seen by comparing Figures 4 and 5. Figure 4 shows that the error in 1+H for the unregularized polar system is less than the error in 1+H for the rectangular system. Figure 5 indicates that the error in 1+H for the regularized polar system is larger than the error in 1+H for the regularized rectangular system. However, at the terminal time, the polar coordinate error is less than the rectangular coordinate error. Note also that the error in 1+H for the regularized polar system is quite constant during most of the integration interval; hence the automatic step-size adjustment associated with the regularized variables tends to control the numerical error. Figure 4 illustrates that, for the unregularized variables, the error passes from a relatively large value to a relatively small value during the course of the trajectory.

CONCLUSIONS

Based on the results obtained in this study, the following general conclusion can be drawn. Care in the selection of the coordinate system used to describe an optimal trajectory can lead to increased accuracy and reduced computation time. In addition, for space vehicles subjected to a continuous thrust force which undergo wide variations in the gravitational force magnitude, significant reductions in computing time can be achieved by using a regularized form for the equations regardless of the error-bound magnitude employed. In this study, reductions in computing time by a factor of three are obtained in some cases by using regularized variables. In addition, if the Hamiltonian is used as an indication of numerical accuracy, the trade-off between integration time and integration accuracy is apparent. It is shown that regularizing results in an automatic step-size change that produces relatively constant numerical error over the trajectory interval. These results indicate the importance of obtaining more definitive methods for selecting regularization schemes.

REFERENCES

1. Tapley, B. D. and Lewallen, J. M., "Comparison of Several Numerical Optimization Methods," Journal of Optimization Theory and Applications, Vol. 1, No. 1, July 1967.

2. Lewallen, J. M., Tapley, B. D. and Williams, S. D., "Iteration Procedures for Indirect Trajectory Optimization Methods," Journal of Spacecraft and Rockets, Vol. 5, No. 3, March 1968.

3. Szebehely, V., Pierce, D.A. and Standish, S.M., "A Group of Earth to Moon Trajectories with Consecutive Collisions," Progress in Astronautics: Vol. 14, Academic Press, New York, 1964.

4. Stiefel, E., Rössler, M., Waldvogel, J. and Burdet, C. A., "Methods of Regularization for Computing Orbits in Celestial Mechanics," Swiss Federal Institute of Technology, NASA Contractor Report, DR-769, June 1967.

5. Tapley, B. D., Szebehely, V. and Lewallen, J. M., "Trajectory Optimization Using Regularized Variables," AAS/AIAA Astrodynamic Specialists Conference AAS Paper No. 68-099, Jackson, Wyoming, September 1968.

6. Schwausch, O. A., "Numerical Error Comparisons for Integration of Near Earth Orbits in Various Coordinate Systems," Engineering Mechanics Research Laboratory, The University of Texas at Austin, EMRL RM 1054, January 1968.

7. Rainbolt, M. R., "Coordinate System Influence on Numerical Solution of the Trajectory Optimization Problem," Masters Thesis, Mechanical Engineering Department, The University of Houston, Houston, Texas, May 1968.

8. McDermott, Make Jr., "Comparison of Coordinate Systems for Numerical Computation of Optimal Trajectories," Lockheed Technical Report, TR-23, Houston, Texas, April 1967.

9. Sundman, K. F., "Mémoire sur le Probléme des Trois Corps," Acta Math., Vol. 36, 1912.

10. Fowler, W. T. and Lastman, G. J., "FORTRAN Subroutines for the Numerical Integration of First Order Ordinary Differential Equations," Engineering Mechanics Research Laboratory, The University of Texas at Austin, EMRL RM 1024, March 1967.

TABLE 1.- NUMERICAL INTEGRATION CHARACTERISTICS FOR ERROR BOUND SEPARATION OF 10^6 FOR THE OPTIMAL LOW THRUST EARTH ESCAPE SPIRAL

	Allowable Error (Absolute)	Unregularized		Regularized	
		Rectangular	Polar	Rectangular	Polar
Computation time for integration of state and perturbation equations (Seconds)	10^{-4} - 10^{-10}	19.5	20.6	8.3	7.7
	10^{-5} - 10^{-11}	38.0	21.0	15.2	8.1
	10^{-6} - 10^{-12}	71.1	42.5	29.4	15.6
Mean computation time per integration step (Seconds)		.0275	.0300	.0304	.0307
Number of integration steps	10^{-4} - 10^{-10}	702	685	272	251
	10^{-5} - 10^{-11}	1381	702	497	261
	10^{-6} - 10^{-12}	2594	1403	971	508
Number of step size changes	10^{-4} - 10^{-10}	0	1	1	1
	10^{-5} - 10^{-11}	2	0	2	2
	10^{-6} - 10^{-12}	3	1	2	2
Terminal error norm	10^{-4} - 10^{-10}	.1375 E -10	.4365 E -13	.6228 E -11	.9087 E -12
	10^{-5} - 10^{-11}	.1524 E -11	.3681 E -13	.9438 E -09	.8325 E -12
	10^{-6} - 10^{-12}	.2010 E -11	.5336 E -09	.1330 E -08	.2150 E -11

TABLE 2.- NUMERICAL INTEGRATION CHARACTERISTICS FOR ERROR BOUND SEPARATION OF 10^4 FOR THE OPTIMAL LOW THRUST EARTH ESCAPE SPIRAL

	Allowable Error (Absolute)	Unregularized		Regularized	
		Rectangular	Polar	Rectangular	Polar
Computation time for integration of state and perturbation equations (Seconds)	10^{-4} - 10^{-8}	16.4	13.9	8.4	7.7
	10^{-5} - 10^{-9}	27.8	18.2	15.2	8.1
	10^{-6} - 10^{-10}	51.2	31.8	30.1	15.7
	10^{-7} - 10^{-11}	64.0	37.7	34.0	21.7
	10^{-8} - 10^{-12}	108.6	72.4	60.1	32.1
Mean computation time per integration step (Seconds)		.0276	.0299	.0307	.0310
Number of integration steps	10^{-4} - 10^{-8}	585	460	272	251
	10^{-5} - 10^{-9}	993	606	497	261
	10^{-6} - 10^{-10}	1862	1080	971	508
	10^{-7} - 10^{-11}	2327	1254	1088	709
	10^{-8} - 10^{-12}	3957	2417	1991	1049

TABLE 2.- NUMERICAL INTEGRATION CHARACTERISTICS FOR ERROR BOUND SEPARATION OF 10^4 FOR THE OPTIMAL LOW THRUST EARTH ESCAPE SPIRAL (Concluded)

	Allowable Error (Absolute)	Unregularized		Regularized	
		Rectangular	Polar	Rectangular	Polar
Number of step size changes	10^{-4} - 10^{-8}	2	2	1	1
	10^{-5} - 10^{-9}	3	1	2	2
	10^{-6} - 10^{-10}	4	3	2	2
	10^{-7} - 10^{-11}	4	2	3	3
	10^{-8} - 10^{-12}	5	3	4	4
Terminal error norm	10^{-4} - 10^{-8}	.5603 E -10	.1265 E -10	.6228 E -11	.9087 E -12
	10^{-5} - 10^{-9}	.1849 E -11	.5304 E -13	.9438 E -09	.8325 E -12
	10^{-6} - 10^{-10}	.1766 E -11	.5328 E -09	.1330 E -08	.2510 E -11
	10^{-7} - 10^{-11}	.1413 E -11	.5336 E -09	.1244 E -08	.2406 E -11
	10^{-8} - 10^{-12}	.1378 E -11	.6035 E -09	.1258 E -08	.2042 E -11

TABLE 3.- NUMERICAL INTEGRATION CHARACTERISTICS FOR ERROR BOUND SEPARATION OF 10^2 FOR THE OPTIMAL LOW THRUST EARTH ESCAPE SPIRAL

	Allowable Error (Absolute)	Unregularized		Regularized	
		Rectangular	Polar	Rectangular	Polar
Computation time for integration of state and perturbation equations (Seconds)	10^{-4} - 10^{-6}	9.4	7.5	8.3	6.1
	10^{-5} - 10^{-7}	17.3	10.6	15.4	8.1
	10^{-6} - 10^{-8}	26.6	15.5	30.1	15.7
	10^{-7} - 10^{-9}	36.4	26.3	33.8	21.7
	10^{-8} - 10^{-10}	66.8	40.6	61.6	32.6
	10^{-9} - 10^{-11}	105.5	60.7	119.1	61.2
	10^{-10} - 10^{-12}	147.1	102.5	132.7	77.8
Mean computation time per integration step (Seconds)		.0279	.0301	.0307	.0307
Number of integration steps	10^{-4} - 10^{-6}	332	241	272	193
	10^{-5} - 10^{-7}	611	345	497	261
	10^{-6} - 10^{-8}	954	514	971	508
	10^{-7} - 10^{-9}	1314	869	1088	709
	10^{-8} - 10^{-10}	2423	1363	1991	1049
	10^{-9} - 10^{-11}	3757	2039	3884	2038
	10^{-10} - 10^{-12}	5235	3467	4355	2582

TABLE 3.- NUMERICAL INTEGRATION CHARACTERISTICS FOR ERROR BOUND SEPARATION OF 10^2 FOR THE OPTIMAL LOW THRUST EARTH ESCAPE SPIRAL (Concluded)

	Allowable Error (Absolute)	Unregularized		Regularized	
		Rectangular	Polar	Rectangular	Polar
Number of step size changes	10^{-4} - 10^{-6}	3	3	1	3
	10^{-5} - 10^{-7}	4	3	2	2
	10^{-6} - 10^{-8}	6	4	2	2
	10^{-7} - 10^{-9}	5	3	3	3
	10^{-8} - 10^{-10}	6	5	4	4
	10^{-9} - 10^{-11}	8	6	4	5
	10^{-10} - 10^{-12}	7	5	5	5
Terminal error norm	10^{-4} - 10^{-6}	.2197 E -08	.9750 E -13	.6228 E -11	.1527 E -13
	10^{-5} - 10^{-7}	.1515 E -10	.1676 E -08	.9438 E -09	.8325 E -12
	10^{-6} - 10^{-8}	.1826 E -10	.2231 E -09	.1329 E -09	.2150 E -11
	10^{-7} - 10^{-9}	.2580 E -11	.5122 E -09	.1244 E -08	.2406 E -11
	10^{-8} - 10^{-10}	.1133 E -11	.5962 E -09	.1258 E -08	.2042 E -11
	10^{-9} - 10^{-11}	.1624 E -11	.6061 E -09	.1260 E -08	.2054 E -11
	10^{-10} - 10^{-12}	.1560 E -10	.6081 E -09	.1259 E -08	.2005 E -11

TABLE 4.-INTEGRATION ERROR BOUNDARY ENCOUNTERS FOR VARIOUS ERROR BOUND SEPARATIONS FOR THE OPTIMAL LOW THRUST EARTH ESCAPE SPIRAL

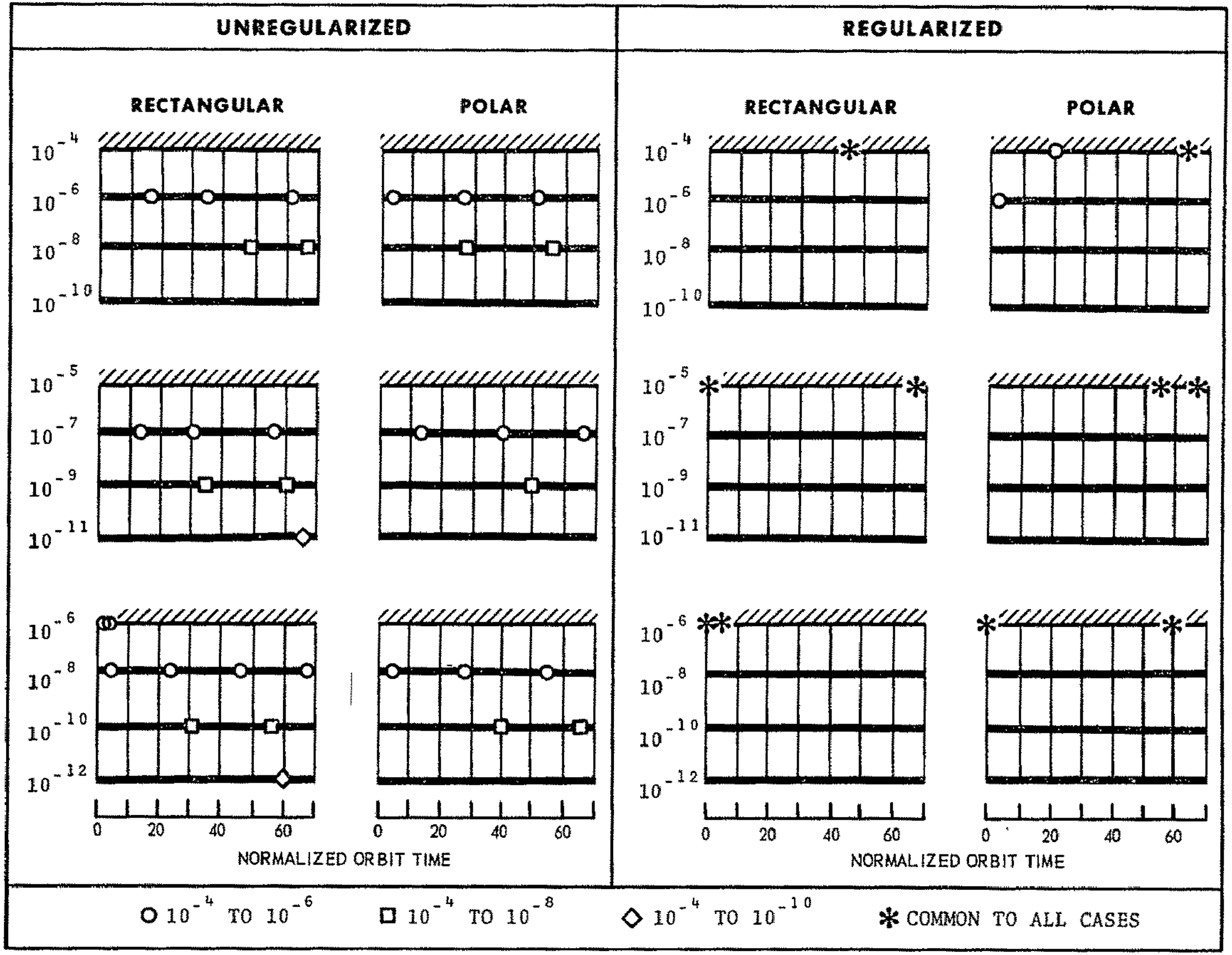

TABLE 5.– INITIAL ERROR INFLUENCE ON THE CONVERGENCE CHARACTERISTICS FOR UNREGULARIZED AND REGULARIZED RECTANGULAR AND POLAR COORDINATES FOR INTEGRATION ERROR BOUNDS OF 10^{-5} TO 10^{-9}

Initial Error In λ	Unregularized				Regularized			
	Rectangular		Polar		Rectangular		Polar	
	Iterations Required For Convergence	Computation Time (min)	Iterations Required For Convergence	Computation Time (min)	Iterations Required For Convergence	Computation Time (min)	Iterations Required For Convergence	Computation Time (min)
+20	6	2.9	5	1.5	6	1.7	5	0.8
+16	5	2.3	5	1.5	6	1.7	5	0.8
+12	5	2.4	4	1.1	5	1.4	4	0.6
+ 8	5	2.4	4	1.1	5	1.4	4	0.6
+ 4	4	1.8	4	1.1	5	1.4	4	0.6
0	0	0.06	0	0.04	0	0.04	0	0.03
- 4	5	2.3	4	1.2	5	1.7	4	0.6
- 8	6	2.9	4	1.2	6	1.7	4	0.6
-12	9	4.7	4	1.2	13	4.2	4	0.6
-16	7	3.5	4	1.1	6	1.7	4	0.6
-20	7	3.5	4	1.1	6	1.7	5	0.7

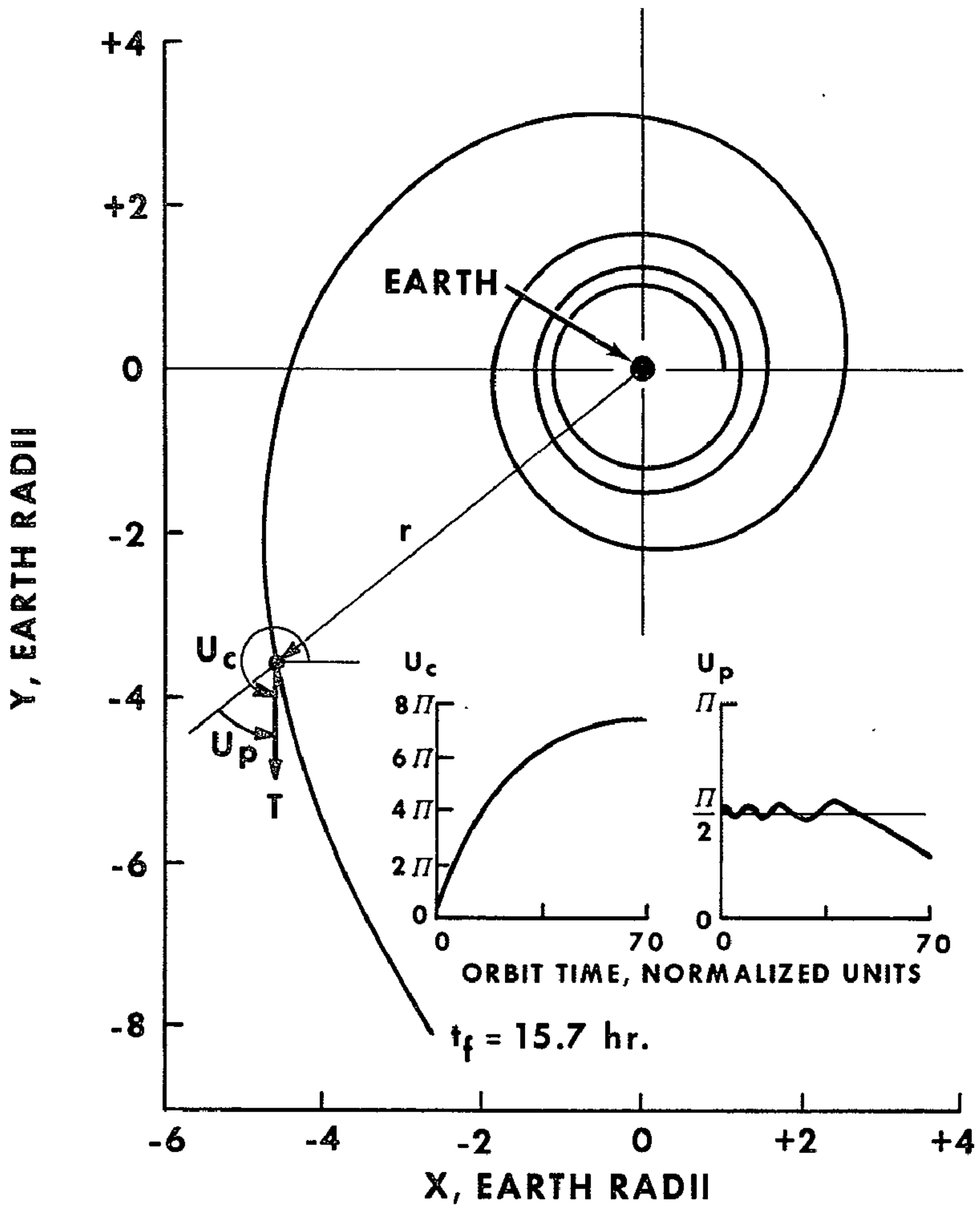

Figure 1. - Optimal low thrust Earth escape spiral trajectory for T/M = 0.1

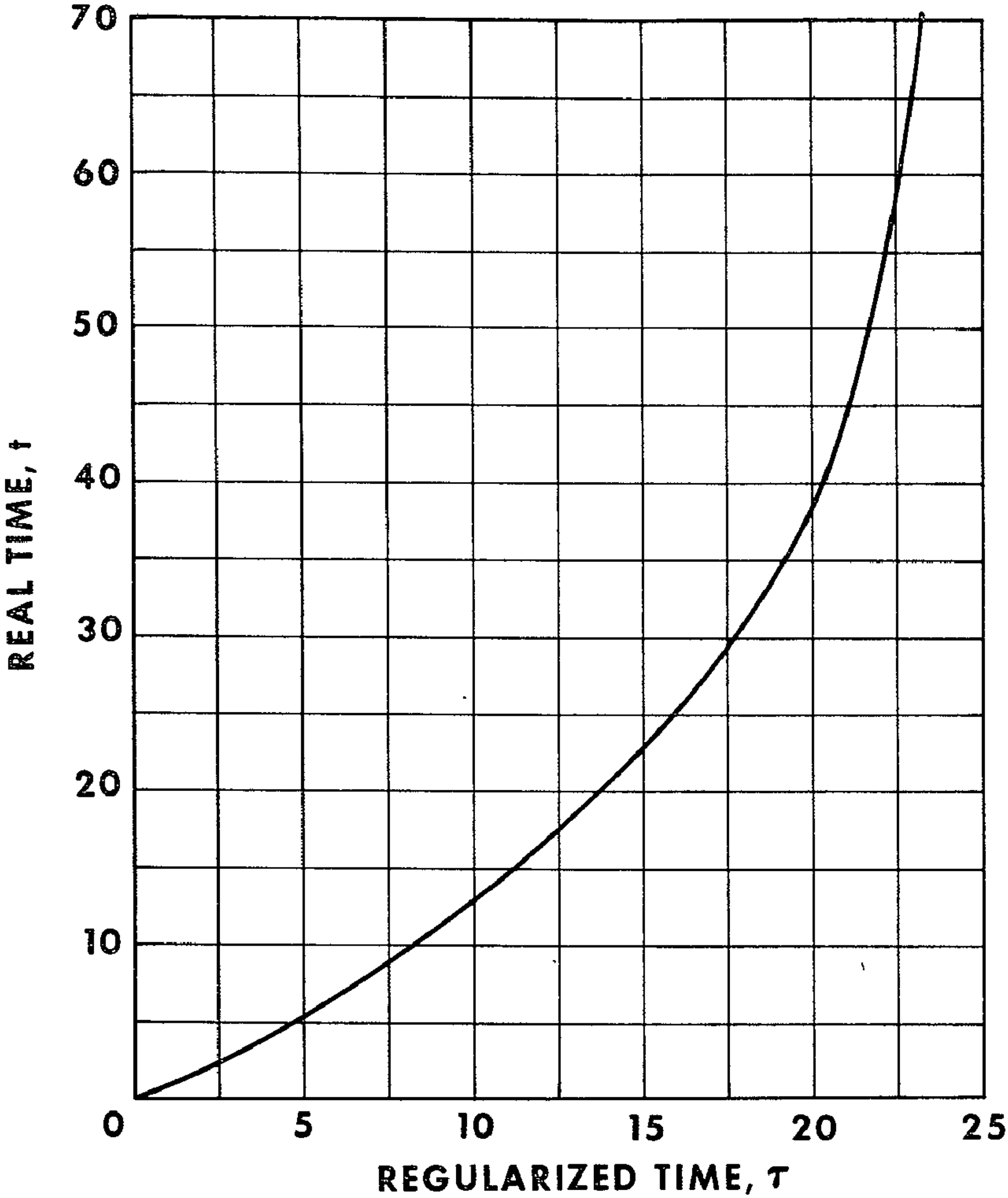

Figure 2. - Real time vs regularized time for the optimal low thrust Earth escape spiral trajectory

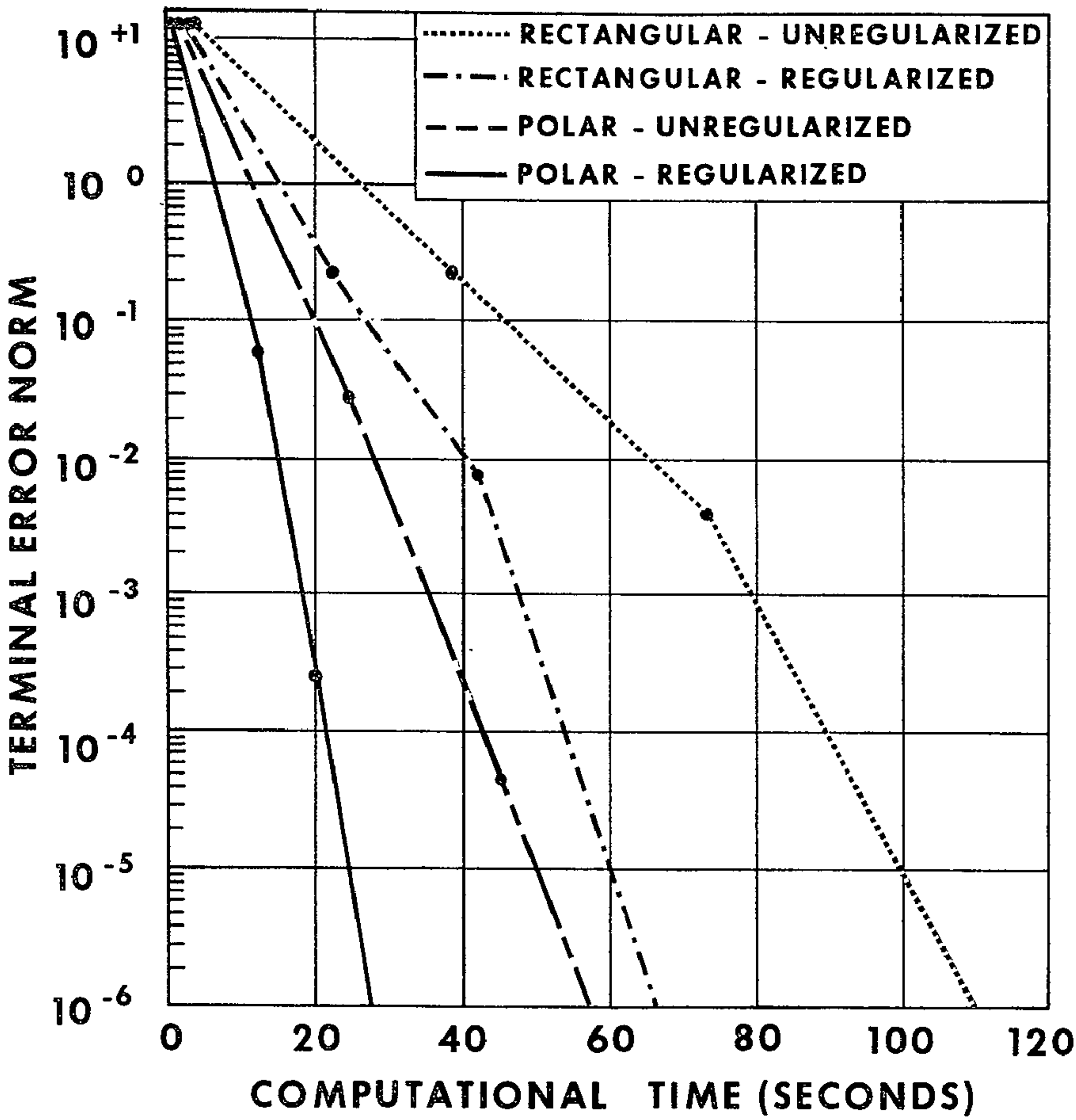

Figure 3. - Terminal error norm vs computational time for a $\delta\lambda_o = +8\%$ and $dt_f = 0$

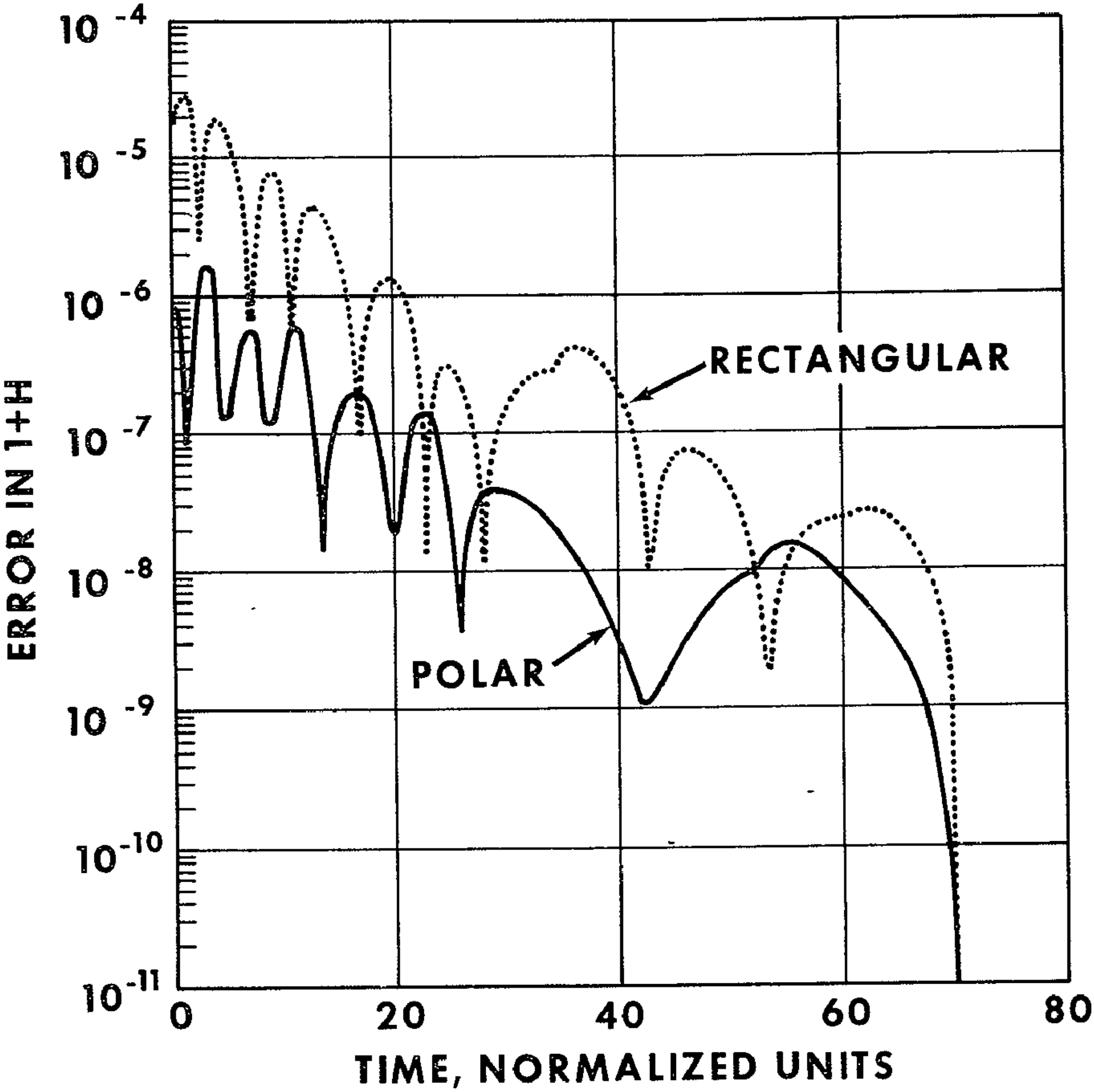

Figure 4. - Error in 1+H for the unregularized rectangular and polar coordinates for an error bound of 10^{-5} to 10^{-9} (rectangulars took 993 steps and polars took 606 steps)

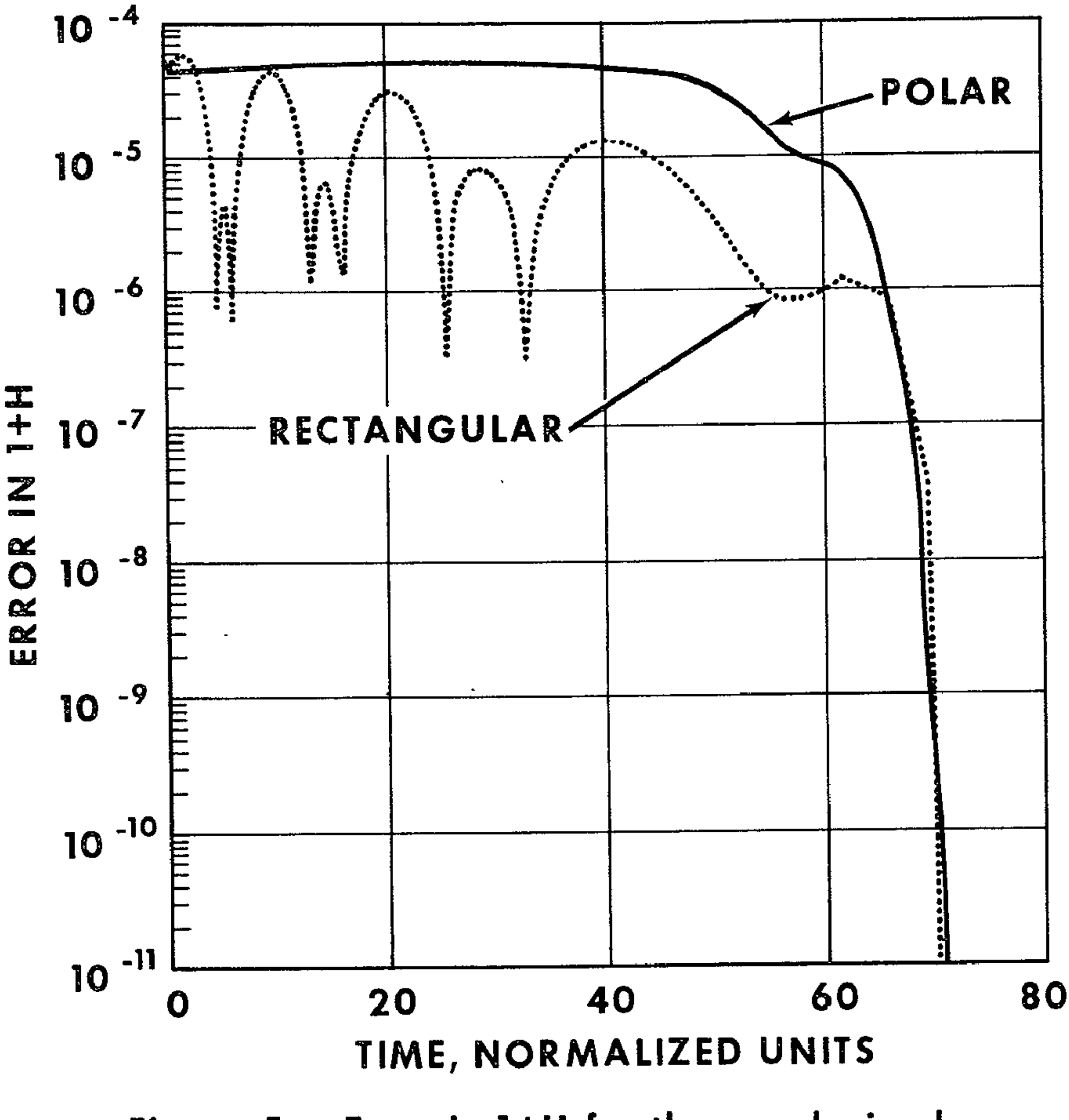

Figure 5. - Error in 1+H for the regularized rectangular and polar coordinates for an error bound of 10^{-5} to 10^{-9} (rectangulars took 497 steps and polars took 261 steps)

APPENDIX A

RECTANGULAR COORDINATES — UNREGULARIZED

RECTANGULAR COORDINATES — UNREGULARIZED

The equations of motion for the unregularized rectangular coordinates are:

$$\dot{u} = -\frac{\mu x}{r^3} - \frac{T\lambda_u}{M\lambda}$$

$$\dot{v} = -\frac{\mu y}{r^3} - \frac{T\lambda_v}{M\lambda}$$

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{M} = -\beta$$

where

$$r = \sqrt{x^2 + y^2}$$

$$\lambda = \sqrt{\lambda_u^2 + \lambda_v^2}$$

μ = gravitational constant

T = thrust

β = mass flow rate

The Euler-Lagrange equations are:

$$\dot{\lambda}_u = -\omega_u$$

$$\dot{\lambda}_v = -\omega_v$$

$$\dot{\omega}_u = \frac{\mu\lambda_u}{r^3} - \frac{3\mu(x\lambda_u + y\lambda_v)x}{r^5}$$

$$\dot{\omega}_v = \frac{\mu\lambda_v}{r^3} - \frac{3\mu(x\lambda_u + y\lambda_v)y}{r^5}$$

$$\dot{\lambda}_M = -\frac{T\lambda}{M^2}$$

The coefficients for the perturbation equations (nonzero terms) are:

$$\frac{\partial \dot{u}}{\partial x} = -\frac{\mu}{r^3} + \frac{3\mu x^2}{r^5}$$

$$\frac{\partial \dot{u}}{\partial y} = \frac{3\mu xy}{r^5}$$

$$\frac{\partial \dot{u}}{\partial M} = \frac{T\lambda_u}{M^2\lambda}$$

$$\frac{\partial \dot{u}}{\partial \lambda_u} = \frac{T}{M\lambda}\left[\frac{\lambda_u^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial \dot{u}}{\partial \lambda_v} = \frac{T\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial \dot{v}}{\partial x} = \frac{3\mu xy}{r^5}$$

$$\frac{\partial \dot{v}}{\partial y} = -\frac{\mu}{r^3} + \frac{3\mu y^2}{r^5}$$

$$\frac{\partial \dot{v}}{\partial M} = \frac{T\lambda_v}{M^2\lambda}$$

$$\frac{\partial \dot{v}}{\partial \lambda_u} = \frac{T\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial \dot{v}}{\partial \lambda_v} = \frac{T}{M\lambda}\left[\frac{\lambda_v^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial \dot{x}}{\partial u} = 1.0$$

$$\frac{\partial \dot{y}}{\partial v} = 1.0$$

$$\frac{\partial \dot{\lambda}_u}{\partial \omega_u} = -1.0$$

$$\frac{\partial \dot{\lambda}_v}{\partial \omega_v} = -1.0$$

$$\frac{\partial \dot{\omega}_u}{\partial x} = -\frac{6\mu x\lambda_u}{r^5} - \frac{3\mu(x\lambda_u + y\lambda_v)}{r^5} + \frac{15\mu(x\lambda_u + y\lambda_v)x^2}{r^7}$$

$$\frac{\partial \dot{\omega}_u}{\partial y} = -\frac{3\mu y\lambda_u}{r^5} - \frac{3\mu x\lambda_v}{r^5} + \frac{15\mu(x\lambda_u + y\lambda_v)xy}{r^7}$$

$$\frac{\partial \dot{\omega}_u}{\partial \lambda_u} = \frac{\mu}{r^3} - \frac{3\mu x^2}{r^5}$$

$$\frac{\partial \dot{\omega}_u}{\partial \lambda_v} = -\frac{3\mu xy}{r^5}$$

$$\frac{\partial \dot{\omega}_v}{\partial x} = -\frac{3\mu x\lambda_v}{r^5} - \frac{3\mu y\lambda_u}{r^5} + \frac{15\mu(x\lambda_u + y\lambda_v)xy}{r^7}$$

$$\frac{\partial \dot{\omega}_v}{\partial y} = -\frac{6\mu y\lambda_v}{r^5} - \frac{3\mu(x\lambda_u + y\lambda_v)}{r^5} + \frac{15\mu(x\lambda_u + y\lambda_v)y^2}{r^7}$$

$$\frac{\partial \dot{\omega}_v}{\partial \lambda_u} = -\frac{3\mu xy}{r^5}$$

$$\frac{\partial \dot{\omega}_v}{\partial \lambda_v} = \frac{\mu}{r^3} - \frac{3\mu y^2}{r^5}$$

$$\frac{\partial \dot{\lambda}_M}{\partial M} = \frac{2T\lambda}{M^3}$$

$$\frac{\partial \dot{\lambda}_M}{\partial \lambda_u} = -\frac{T\lambda_u}{M^2\lambda}$$

$$\frac{\partial \dot{\lambda}_M}{\partial \lambda_v} = -\frac{T\lambda_v}{M^2\lambda}$$

The terminal boundary conditions in the unregularized rectangular coordinates are:

$$H_1 = 0.5(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{r}$$

$$H_2 = \lambda_u - \frac{r^3 \omega_u u}{\mu x}$$

$$H_3 = \lambda_v - \frac{r^3 \omega_u v}{\mu x}$$

$$H_4 = \omega_v - \frac{\omega_u y}{x}$$

$$H_5 = \lambda_M$$

$$H_6 = 1.0 - \frac{\mu}{r^3} (x\lambda_u + y\lambda_v) - \frac{T\lambda}{M} + u\omega_u + v\omega_v - \lambda_M \beta$$

The time derivatives of the terminal constraints are:

$$\dot{H}_1 = u\dot{u} + v\dot{v} + \frac{\mu}{r^3}(ux + vy)$$

$$\dot{H}_2 = \dot{\lambda}_u - \frac{3ru\omega_u(ux + vy)}{\mu x} - \frac{r^3 u\dot{\omega}_u}{\mu x} - \frac{r^3\omega_u\dot{u}}{\mu x} + \frac{r^3\omega_u u^2}{\mu x^2}$$

$$\dot{H}_3 = \dot{\lambda}_v - \frac{3rv\omega_u(ux + vy)}{\mu x} - \frac{r^3 v\dot{\omega}_u}{\mu x} - \frac{r^3\omega_u\dot{v}}{\mu x} + \frac{r^3\omega_u uv}{\mu x^2}$$

$$\dot{H}_4 = \dot{\omega}_v - \frac{\dot{\omega}_u y}{x} - \frac{\omega_u v}{x} + \frac{\omega_u yu}{x^2}$$

$$\dot{H}_5 = \dot{\lambda}_M$$

$$\dot{H}_6 = 0$$

The nonzero elements of the $\frac{\partial H}{\partial z}$ matrix are:

$$\frac{\partial H_1}{\partial u} = u$$

$$\frac{\partial H_1}{\partial v} = v$$

$$\frac{\partial H_1}{\partial x} = \frac{\mu x}{r^3}$$

$$\frac{\partial H_1}{\partial y} = \frac{\mu y}{r^3}$$

$$\frac{\partial H_2}{\partial u} = -\frac{r^3 \omega_u}{\mu x}$$

$$\frac{\partial H_2}{\partial x} = -\frac{3r\omega_u u}{\mu} + \frac{r^3 \omega_u u}{\mu x^2}$$

$$\frac{\partial H_2}{\partial y} = -\frac{3r\omega_u yu}{\mu x}$$

$$\frac{\partial H_2}{\partial \lambda_u} = 1.0$$

$$\frac{\partial H_2}{\partial \omega_u} = -\frac{r^3 u}{\mu x}$$

$$\frac{\partial H_3}{\partial v} = -\frac{r^3\omega_u}{\mu x}$$

$$\frac{\partial H_3}{\partial x} = -\frac{3r\omega_u xv}{\mu x} + \frac{r^3\omega_u v}{\mu x^2}$$

$$\frac{\partial H_3}{\partial y} = -\frac{3r\omega_u yv}{\mu x}$$

$$\frac{\partial H_3}{\partial \lambda_v} = 1.0$$

$$\frac{\partial H_3}{\partial \omega_u} = -\frac{r^3 v}{\mu x}$$

$$\frac{\partial H_4}{\partial x} = \frac{\omega_u y}{x^2}$$

$$\frac{\partial H_4}{\partial y} = -\frac{\omega_u}{x}$$

$$\frac{\partial H_4}{\partial \omega_u} = -\frac{y}{x}$$

$$\frac{\partial H_4}{\partial \omega_v} = 1.0$$

$$\frac{\partial H_5}{\partial \lambda_M} = 1.0$$

$$\frac{\partial H_6}{\partial u} = \omega_u$$

$$\frac{\partial H_6}{\partial v} = \omega_v$$

$$\frac{\partial H_6}{\partial x} = \frac{3\mu(x\lambda_u + y\lambda_v)x}{r^5} - \frac{\mu\lambda_u}{r^3}$$

$$\frac{\partial H_6}{\partial y} = \frac{3\mu(x\lambda_u + y\lambda_v)y}{r^5} - \frac{\mu\lambda_v}{r^3}$$

$$\frac{\partial H_6}{\partial M} = \frac{T\lambda}{M^2}$$

$$\frac{\partial H_6}{\partial \lambda_u} = -\frac{\mu x}{r^3} - \frac{T\lambda_u}{M\lambda}$$

$$\frac{\partial H_6}{\partial \lambda_v} = -\frac{\mu y}{r^3} - \frac{T\lambda_v}{\dot{M}\lambda}$$

$$\frac{\partial H_6}{\partial \omega} = u$$

$$\frac{\partial H_6}{\partial \omega_v} = v$$

$$\frac{\partial H_6}{\partial \lambda_M} = -\beta$$

APPENDIX B

RECTANGULAR COORDINATES — REGULARIZED

RECTANGULAR COORDINATES — REGULARIZED

The equations of motion for the regularized rectangular coordinates are:

$$u' = -\mu x + \frac{3(ux + vy)u}{2r^2} - \frac{Tr^3\lambda_u}{M\lambda}$$

$$v' = -\mu y + \frac{3(ux + vy)v}{2r^2} - \frac{Tr^3\lambda_v}{M\lambda}$$

$$x' = u$$

$$y' = v$$

$$M' = -\beta r^{\frac{3}{2}}$$

where

$$r = \sqrt{x^2 + y^2}$$

$$\lambda = \sqrt{\lambda_u^2 + \lambda_v^2}$$

μ = gravitational constant

T = thrust

β = mass flow rate

The Euler-Lagrange equations are:

$$\lambda_u' = -\omega_u$$

$$\lambda_v' = -\omega_v$$

$$\omega_u' = \mu\lambda_u + \frac{3(ux + vy)\omega_u}{2r^2} - \frac{3\mu(x\lambda_u + y\lambda_v)x}{r^2}$$

$$\omega_v' = \mu\lambda_v + \frac{3(ux + vy)\omega_v}{2r^2} - \frac{3\mu(x\lambda_u + y\lambda_v)y}{r^2}$$

$$\lambda_M' = -\frac{Tr^{3/2}\lambda}{M^2}$$

The coefficients for the perturbation equations (nonzero elements) are:

$$\frac{\partial u'}{\partial u} = \frac{3ux}{2r^2} + \frac{3(ux + vy)}{2r^2}$$

$$\frac{\partial u'}{\partial v} = \frac{3uy}{2r^2}$$

$$\frac{\partial u'}{\partial x} = \frac{3u^2}{2r^2} - \frac{3(ux + vy)ux}{r^4} - \mu - \frac{3Trx\lambda_u}{M\lambda}$$

$$\frac{\partial u'}{\partial y} = \frac{3uv}{2r^2} - \frac{3(ux + vy)uy}{r^4} - \frac{3Try\lambda_u}{M\lambda}$$

$$\frac{\partial u'}{\partial M} = \frac{Tr^3\lambda_u}{M^2\lambda}$$

$$\frac{\partial u'}{\mu\lambda_u} = \frac{Tr^3}{M\lambda}\left[\frac{\lambda_u^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial u'}{\partial \lambda_v} = \frac{Tr^3\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial v'}{\partial u} = \frac{3vx}{2r^2}$$

$$\frac{\partial v'}{\partial v} = \frac{3vy}{2r^2} + \frac{3(ux + vy)}{2r^2}$$

$$\frac{\partial v'}{\partial x} = \frac{3uv}{2r^2} - \frac{3(ux + vy)vx}{r^4} - \frac{3Trx\lambda_v}{M\lambda}$$

$$\frac{\partial v'}{\partial y} = \frac{3v^2}{2r^2} - \frac{3(ux + vy)vy}{r^4} - \frac{3Try\lambda_v}{M\lambda} - \mu$$

$$\frac{\partial v'}{\partial M} = \frac{Tr^3\lambda_v}{M^2\lambda}$$

$$\frac{\partial v'}{\partial \lambda_u} = \frac{Tr^3\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial v'}{\partial \lambda_v} = \frac{Tr^3}{M\lambda}\left[\frac{\lambda_v^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial x'}{\partial u} = 1.0$$

$$\frac{\partial y'}{\partial v} = 1.0$$

$$\frac{\partial M'}{\partial x} = - \frac{3\beta x}{2r^{1/2}}$$

$$\frac{\partial M'}{\partial y} = - \frac{3\beta y}{2r^{1/2}}$$

$$\frac{\partial \lambda_u'}{\partial \omega_u} = -1.0$$

$$\frac{\partial \lambda_v'}{\partial \omega_v} = -1.0$$

$$\frac{\partial \omega_u'}{\partial u} = \frac{3x\omega_u}{2r^2}$$

$$\frac{\partial \omega_u'}{\partial v} = \frac{3y\omega_u}{2r^2}$$

$$\frac{\partial \omega_u'}{\partial x} = \frac{3u\omega_u}{2r^2} - \frac{3(ux + vy)x\omega_u}{r^4} - \frac{3\mu x\lambda_u}{r^2} + \frac{6\mu(x\lambda_u + y\lambda_v)x^2}{r^4} - \frac{3\mu(x\lambda_u + y\lambda_v)}{r^2}$$

$$\frac{\partial \omega_u'}{\partial y} = \frac{3v\omega_u}{2r^2} - \frac{3(ux + vy)y\omega_u}{r^4} - \frac{3\mu x\lambda_v}{r^2} + \frac{6\mu(x\lambda_u + y\lambda_v)xy}{r^4}$$

$$\frac{\partial \omega_u'}{\partial \lambda_u} = \mu - \frac{3\mu x^2}{r^2}$$

$$\frac{\partial \omega_u'}{\partial \lambda_v} = -\frac{3\mu xy}{r^2}$$

$$\frac{\partial \omega_u'}{\partial \omega_u} = \frac{3(ux + vy)}{2r^2}$$

$$\frac{\partial \omega_v'}{\partial u} = \frac{3\omega_v x}{2r^2}$$

$$\frac{\partial \omega_v'}{\partial u} = \frac{3\omega_v y}{2r^2}$$

$$\frac{\partial \omega_v'}{\partial x} = \frac{3u\omega_v}{2r^2} - \frac{3(ux + vy)x\omega_v}{r^4} - \frac{3\mu y\lambda_u}{r^2} + \frac{6\mu(x\lambda_u + y\lambda_v)xy}{r^4}$$

$$\frac{\partial \omega_v'}{\partial y} = \frac{3v\omega_v}{2r^2} - \frac{3(ux + vy)y\omega_v}{r^4} - \frac{3\mu y\lambda_v}{r^2} + \frac{6\mu(x\lambda_u + y\lambda_v)y^2}{r^4}$$

$$- \frac{3\mu(x\lambda_u + y\lambda_v)}{r^2}$$

$$\frac{\partial \omega_v'}{\partial \lambda_u} = - \frac{3\mu xy}{r^2}$$

$$\frac{\partial \omega_v'}{\partial \lambda_v} = - \frac{3\mu y^2}{r^2} + \mu$$

$$\frac{\partial \omega_v'}{\partial \omega_v} = \frac{3(ux + vy)}{2r^2}$$

$$\frac{\partial \lambda_M'}{\partial x} = - \frac{3Tx\lambda}{2M^2 r^{1/2}}$$

$$\frac{\partial \lambda_M'}{\partial y} = - \frac{3Ty\lambda}{2M^2 r^{1/2}}$$

$$\frac{\partial \lambda_M'}{\partial M} = \frac{2Tr^{3/2}\lambda}{M^3}$$

$$\frac{\partial \lambda_M'}{\partial \lambda_u} = - \frac{Tr^{3/2}\lambda_u}{M^2\lambda}$$

$$\frac{\partial \lambda_M'}{\partial \lambda_v} = - \frac{Tr^{3/2}\lambda_v}{M^2\lambda}$$

The terminal boundary conditions in the regularized rectangular coordinates are:

$$H_1 = \frac{0.5(u^2 + v^2)}{r^3} - \frac{\mu}{r}$$

$$H_2 = \lambda_u - \frac{u\omega_u}{\mu x}$$

$$H_3 = \lambda_v - \frac{v\omega_u}{\mu x}$$

$$H_4 = \frac{(\omega_v - \omega_u y/x)}{r^{3/2}}$$

$$H_5 = \dot{\lambda}_M$$

$$H_6 = 1.0 - \frac{\mu(x\lambda_u + y\lambda_v)}{r^3} - \frac{T\lambda}{M} + \frac{(u\omega_u + v\omega_v)}{r^3} - \lambda_M \beta$$

The time derivatives of the terminal constraints are:

$$H_1' = \frac{(uu' + vv')}{r^3} - \frac{1.5(u^2 + v^2)(ux + vy)}{r^5} + \frac{\mu}{r^3}(ux + vy)$$

$$H_2' = \lambda_u' - \frac{u'\omega_u}{\mu x} - \frac{u\omega_u'}{\mu x} + \frac{u^2\omega_u}{\mu x^2}$$

$$H_3' = \lambda_v' - \frac{v'\omega_u}{\mu x} - \frac{v\omega_u'}{\mu x} + \frac{uv\omega_u}{\mu x^2}$$

$$H_4' = \frac{\omega_v'}{r^{3/2}} - \frac{\omega_u' y}{x r^{3/2}} - \frac{\omega_u v}{x r^{3/2}} + \frac{\omega_u y u}{x^2 r^{3/2}} - 1.5\left(\omega_v - \frac{\omega_u y}{x}\right)\frac{(ux + vy)}{r^{7/2}}$$

$$H_5' = \lambda_M'$$

$$H_6' = 0$$

The nonzero elements of the $\frac{\partial H}{\partial z}$ matrix are:

$$\frac{\partial H_1}{\partial u} = \frac{u}{r^3}$$

$$\frac{\partial H_1}{\partial v} = \frac{v}{r^3}$$

$$\frac{\partial H_1}{\partial x} = \frac{\mu x}{r^3} + \frac{3(u^2 + v^2)x}{2r^5}$$

$$\frac{\partial H_1}{\partial y} = \frac{\mu y}{r^3} + \frac{3(u^2 + v^2)y}{2r^5}$$

$$\frac{\partial H_2}{\partial u} = - \frac{\omega_u}{\mu x}$$

$$\frac{\partial H_2}{\partial x} = \frac{u\omega_u}{\mu x^2}$$

$$\frac{\partial H_2}{\partial \lambda_u} = 1.0$$

$$\frac{\partial H_2}{\partial \omega_u} = - \frac{u}{\mu x}$$

$$\frac{\partial H_3}{\partial v} = - \frac{\omega_u}{\mu x}$$

$$\frac{\partial H_3}{\partial x} = \frac{v\omega_u}{\mu x^2}$$

$$\frac{\partial H_3}{\partial \lambda_v} = 1.0$$

$$\frac{\partial H_3}{\partial \omega_u} = -\frac{v}{\mu x}$$

$$\frac{\partial H_4}{\partial x} = \frac{y\omega_u}{x^2 r^{3/2}} - \frac{3}{2}\,(\omega_v - \omega_u y/x)\left(\frac{x}{r^{7/2}}\right)$$

$$\frac{\partial H_4}{\partial y} = -\frac{\omega_u}{x r^{3/2}} - \frac{3}{2}\,(\omega_v - \omega_u y/x)\left(\frac{y}{r^{7/2}}\right)$$

$$\frac{\partial H_4}{\partial \omega_u} = \frac{y}{x r^{3/2}}$$

$$\frac{\partial H_4}{\partial \omega_v} = \frac{1}{r^{3/2}}$$

$$\frac{\partial H_5}{\partial \lambda_M} = 1.0$$

$$\frac{\partial H_6}{\partial u} = \frac{\omega_u}{r^3}$$

$$\frac{\partial H_6}{\partial v} = \frac{\omega_v}{r^3}$$

$$\frac{\partial H_6}{\partial x} = - \frac{\mu\lambda_u}{r^3} + \frac{3\mu(x\lambda_u + y\lambda_v)x}{r^5} - \frac{3(u\omega_u + v\omega_v)x}{r^5}$$

$$\frac{\partial H_6}{\partial y} = - \frac{\mu\lambda_v}{r^3} + \frac{3\mu(x\lambda_u + y\lambda_v)y}{r^5} - \frac{3(u\omega_u + v\omega_v)y}{r^5}$$

$$\frac{\partial H_6}{\partial M} = \frac{T\lambda}{M^2}$$

$$\frac{\partial H_6}{\partial \lambda_u} = - \frac{\mu x}{r^3} - \frac{T\lambda_u}{M\lambda}$$

$$\frac{\partial H_6}{\partial \lambda_v} = \frac{\mu y}{r^3} - \frac{T\lambda_v}{M\lambda}$$

$$\frac{\partial H_6}{\partial \omega_u} = \frac{u}{r^3}$$

$$\frac{\partial H_6}{\partial \omega_v} = \frac{v}{r^3}$$

$$\frac{\partial H_6}{\partial \lambda_M} = - \beta$$

APPENDIX C

POLAR COORDINATES — UNREGULARIZED

POLAR COORDINATES — UNREGULARIZED

The equations of motion for the unregularized polar coordinates are:

$$\dot{u} = \frac{v^2}{\rho} - \frac{\mu}{\rho^2} - \frac{T\lambda_u}{M\lambda}$$

$$\dot{v} = -\frac{uv}{\rho} - \frac{T\lambda_v}{M\lambda}$$

$$\dot{\rho} = u$$

$$\dot{\theta} = \frac{v}{\rho}$$

$$\dot{M} = -\beta$$

where

ρ = radius

$\lambda = \sqrt{\lambda_u^2 + \lambda_v^2}$

μ = gravitational constan

T = thrust

β = mass flow rate

The Euler-Lagrange equations are:

$$\dot{\lambda}_u = \frac{V}{\rho} \lambda_v - \omega_u$$

$$\dot{\lambda}_v = -\frac{V}{\rho} \lambda_u - \omega_v$$

$$\dot{\omega}_u = \frac{V}{\rho} \omega_v - \frac{2\mu\lambda_u}{\rho^3}$$

$$\dot{\omega}_v = -\frac{V}{\rho} \omega_u + \frac{\mu\lambda_v}{\rho^3}$$

$$\dot{\lambda}_M = -\frac{T\lambda}{M^2}$$

The coefficients for the perturbation equations (nonzero terms) are:

$$\frac{\partial \dot{u}}{\partial v} = \frac{2v}{\rho}$$

$$\frac{\partial \dot{u}}{\partial \rho} = -\frac{v^2}{\rho^2} + \frac{2\mu}{\rho^3}$$

$$\frac{\partial \dot{u}}{\partial M} = \frac{T\lambda_u}{M^2\lambda}$$

$$\frac{\partial \dot{u}}{\partial \lambda_u} = \frac{T}{M\lambda}\left[\frac{\lambda_u^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial \dot{u}}{\partial \lambda_v} = \frac{T\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial \dot{v}}{\partial u} = -\frac{v}{\rho}$$

$$\frac{\partial \dot{v}}{\partial v} = -\frac{u}{\rho}$$

$$\frac{\partial \dot{v}}{\partial \rho} = \frac{uv}{\rho^2}$$

$$\frac{\partial \dot{v}}{\partial M} = \frac{T\lambda_v}{M^2\lambda}$$

$$\frac{\partial \dot{v}}{\partial \lambda_u} = \frac{T\lambda_u \lambda_v}{M\lambda^3}$$

$$\frac{\partial \dot{v}}{\partial \lambda_v} = \frac{T}{M\lambda}\left[\frac{\lambda_v^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial \dot{\rho}}{\partial u} = 1.0$$

$$\frac{\partial \dot{\theta}}{\partial v} = \frac{1}{\rho}$$

$$\frac{\partial \dot{\theta}}{\partial \rho} = -\frac{v}{\rho^2}$$

$$\frac{\partial \dot{\lambda}_u}{\partial v} = \frac{\lambda_v}{\rho}$$

$$\frac{\partial \dot{\lambda}_u}{\partial \rho} = -\frac{v\lambda_v}{\rho^2}$$

$$\frac{\partial \dot{\lambda}_u}{\partial \lambda_v} = \frac{v}{\rho}$$

$$\frac{\partial \dot{\lambda}_u}{\partial \omega_u} = -1.0$$

$$\frac{\partial \dot{\lambda}_v}{\partial v} = -\frac{\lambda_u}{\rho}$$

$$\frac{\partial \dot{\lambda}_v}{\partial \rho} = - \frac{v}{\rho^2} \lambda_u$$

$$\frac{\partial \dot{\lambda}_v}{\partial \lambda_u} = - \frac{v}{\rho}$$

$$\frac{\partial \dot{\lambda}_v}{\partial \omega_v} = - 1.0$$

$$\frac{\partial \dot{\omega}_u}{\partial v} = \frac{\omega_v}{\rho}$$

$$\frac{\partial \dot{\omega}_u}{\partial \rho} = - \frac{v\omega_v}{\rho^2} + \frac{6\mu\lambda_u}{\rho^4}$$

$$\frac{\partial \dot{\omega}_u}{\partial \lambda_u} = - \frac{2\mu}{\rho^3}$$

$$\frac{\partial \dot{\omega}_u}{\partial \omega_v} - \frac{v}{\rho}$$

$$\frac{\partial \dot{\omega}_v}{\partial v} = - \frac{\omega_u}{\rho}$$

$$\frac{\partial \dot{\omega}_v}{\partial \rho} = \frac{v\omega_u}{\rho^2} - \frac{3\mu\lambda_v}{\rho^4}$$

$$\frac{\partial \dot{\omega}_v}{\partial \lambda_v} = \frac{\mu}{\rho^3}$$

$$\frac{\partial \dot{\omega}_v}{\partial \omega_u} = - \frac{v}{\rho}$$

$$\frac{\partial \dot{\lambda}_M}{\partial M} = \frac{2T\lambda}{M^3}$$

$$\frac{\partial \dot{\lambda}_M}{\partial \lambda_u} = - \frac{T\lambda_u}{M^2\lambda}$$

$$\frac{\partial \dot{\lambda}_M}{\partial \lambda_v} = - \frac{T\lambda_v}{M^2\lambda}$$

The terminal boundary conditions in unregularized polar coordinates are:

$$H_1 = 0.5(u^2 + v^2) - \frac{\mu}{\rho}$$

$$H_2 = \lambda_u - \frac{u\rho^2\omega_u}{\mu}$$

$$H_3 = \lambda_v - \frac{v\rho^2\omega_u}{\mu}$$

$$H_4 = \omega_v$$

$$H_5 = \lambda_M$$

$$H_6 = 1.0 + u\omega_u + v\omega_v - \frac{\mu\lambda_u}{\rho^2} - \lambda_M\beta - \frac{T\lambda}{M}$$

The time derivatives of the terminal constraints are:

$$\dot{H}_1 = u\dot{u} + v\dot{v} + \frac{\mu u}{\rho^2}$$

$$\dot{H}_2 = \dot{\lambda}_u - \frac{\dot{u}\rho^2\omega_u}{\mu} - \frac{2u^2\rho\omega_u}{\mu} - \frac{u\rho^2\dot{\omega}_u}{\mu}$$

$$\dot{H}_3 = \dot{\lambda}_v - \frac{\dot{v}\rho^2\omega_u}{\mu} - \frac{2uv\rho\omega_u}{\mu} - \frac{v\rho^2\dot{\omega}_u}{\mu}$$

$$\dot{H}_4 = \dot{\omega}_v$$

$$\dot{H}_5 = \dot{\lambda}_M$$

$$\dot{H}_6 = 0$$

The nonzero elements of the $\frac{\partial H}{\partial z}$ matrix are:

$$\frac{\partial H_1}{\partial u} = u$$

$$\frac{\partial H_1}{\partial v} = v$$

$$\frac{\partial H_1}{\partial \rho} = \frac{\mu}{\rho^2}$$

$$\frac{\partial H_2}{\partial u} = -\frac{\rho^2 \omega_u}{\mu}$$

$$\frac{\partial H_2}{\partial \rho} = -\frac{2u\rho\omega_u}{\mu}$$

$$\frac{\partial H_2}{\partial \lambda_u} = 1.0$$

$$\frac{\partial H_2}{\partial \omega_u} = -\frac{u\rho^2}{\mu}$$

$$\frac{\partial H_3}{\partial v} = -\frac{\rho^2 \omega_u}{\mu}$$

$$\frac{\partial H_3}{\partial \rho} = -\frac{2v\rho\lambda_3}{\mu}$$

$$\frac{\partial H_3}{\partial \lambda_v} = 1.0$$

$$\frac{\partial H_3}{\partial \omega_u} = -\frac{v\rho^2}{\mu}$$

$$\frac{\partial H_4}{\partial \omega_v} = 1.0$$

$$\frac{\partial H_5}{\partial \lambda_M} = 1.0$$

$$\frac{\partial H_6}{\partial u} = \omega_u$$

$$\frac{\partial H_6}{\partial v} = \omega_v$$

$$\frac{\partial H_6}{\partial \rho} = \frac{2\mu\lambda_1}{\rho^3}$$

$$\frac{\partial H_6}{\partial M} = \frac{T\lambda}{M^2}$$

$$\frac{\partial H_6}{\partial \lambda_u} = -\frac{u}{\rho^2} - \frac{T\lambda_u}{M\lambda}$$

$$\frac{\partial H_6}{\partial \lambda_v} = -\frac{T\lambda_v}{M\lambda}$$

$$\frac{\partial H_6}{\partial \omega_u} = u$$

$$\frac{\partial H_6}{\partial \omega_v} = v$$

$$\frac{\partial H_6}{\partial \lambda_M} = -\beta$$

APPENDIX D

POLAR COORDINATES — REGULARIZED

POLAR COORDINATES — REGULARIZED

The equations of motion for the regularized polar coordinates are:

$$u' = \frac{v^2}{\rho} + \frac{3u^2}{2\rho} - \mu\rho - \frac{T\rho^3\lambda_u}{M\lambda}$$

$$v' = \frac{1}{2}\frac{uv}{\rho} - \frac{T\rho^3\lambda_v}{M\lambda}$$

$$\rho' = u$$

$$\theta' = \frac{v}{\rho}$$

$$M' = \beta\rho^{3/2}$$

where

ρ = radius

$\lambda = \sqrt{\lambda_u^2 + \lambda_v^2}$

μ = gravitational constant

T = thrust

β = mass flow rate

The Euler-Lagrange equations are:

$$\lambda_u' = \frac{v}{\rho} \lambda_v - \omega_u$$

$$\lambda_v' = - \frac{v}{\rho} \lambda_u - \omega_v$$

$$\omega_u' = \frac{v}{\rho} \omega_v + \frac{3}{2} \frac{u\omega_u}{\rho} - 2\mu\lambda_u$$

$$\omega_v' = - \frac{v}{\rho} \omega_u + \frac{3u\omega_v}{2\rho} + \mu\lambda_v$$

$$\lambda_M' = - \frac{T\rho^{3/2}\lambda}{M^2}$$

The coefficients for the perturbation equations (nonzero terms) are:

$$\frac{\partial u'}{\partial u} = \frac{3u}{\rho}$$

$$\frac{\partial u'}{\partial v} = \frac{2v}{\rho}$$

$$\frac{\partial u'}{\partial \rho} = -\frac{v^2}{\rho^2} - \frac{3u^2}{2\rho^2} - \mu - \frac{3T\rho^2\lambda_u}{M\lambda}$$

$$\frac{\partial u'}{\partial M} = \frac{T\rho^3\lambda_u}{M^2\lambda}$$

$$\frac{\partial u'}{\partial \lambda_u} = \frac{T\rho^3}{M\lambda}\left[\frac{\lambda_u^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial u'}{\partial \lambda_v} = \frac{T\rho^3\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial v'}{\partial u} = \frac{v}{2\rho}$$

$$\frac{\partial v'}{\partial v} = \frac{u}{2\rho}$$

$$\frac{\partial v'}{\partial \rho} = -\frac{uv}{2\rho^2} - \frac{3T\rho^2\lambda_v}{M\lambda}$$

$$\frac{\partial v'}{\partial M} = \frac{T\rho^3\lambda_v}{M^2\lambda}$$

$$\frac{\partial v'}{\partial \lambda_u} = \frac{T\rho^3\lambda_u\lambda_v}{M\lambda^3}$$

$$\frac{\partial v'}{\partial \lambda_v} = \frac{T\rho^3}{M\lambda}\left[\frac{\lambda_v^2}{\lambda^2} - 1.0\right]$$

$$\frac{\partial \rho'}{\partial u} = 1.0$$

$$\frac{\partial \theta'}{\partial v} = \frac{1}{\rho}$$

$$\frac{\partial \theta'}{\partial \rho} = -\frac{v}{\rho^2}$$

$$\frac{\partial M'}{\partial \rho} = -\frac{3}{2}\beta\rho^{1/2}$$

$$\frac{\partial \lambda_u'}{\partial v} = \frac{\lambda_v}{\rho}$$

$$\frac{\partial \lambda_u'}{\partial \rho} = -\frac{\lambda_v v}{\rho^2}$$

$$\frac{\partial \lambda_u'}{\partial \lambda_v} = \frac{v}{\rho}$$

$$\frac{\partial \lambda_u'}{\partial \omega_u} = -1.0$$

$$\frac{\partial \lambda_v'}{\partial v} = -\frac{\lambda_u}{\rho}$$

$$\frac{\partial \lambda_v'}{\partial \rho} = \frac{v\lambda_u}{\rho^2}$$

$$\frac{\partial \lambda_v'}{\partial \lambda_u} = -\frac{v}{\rho}$$

$$\frac{\partial \lambda_v'}{\partial \omega_v} = -1.0$$

$$\frac{\partial \omega_u'}{\partial u} = \frac{3\omega_u}{2\rho}$$

$$\frac{\partial \omega_u'}{\partial v} = \frac{\omega_v}{\rho}$$

$$\frac{\partial \omega_u'}{\partial \rho} = -\frac{v\omega_v}{\rho^2} - \frac{3u\omega_u}{2\rho^2}$$

$$\frac{\partial \omega_u'}{\partial \lambda_u} = -2\mu$$

$$\frac{\partial \omega_u'}{\partial \omega_u} = \frac{3u}{2\rho}$$

$$\frac{\partial \omega_u'}{\partial \omega_v} = \frac{v}{\rho}$$

$$\frac{\partial \omega_v'}{\partial u} = \frac{3\omega_v}{2\rho}$$

$$\frac{\partial \omega_v'}{\partial v} = -\frac{\omega_u}{\rho}$$

$$\frac{\partial \omega_v'}{\partial \rho} = \frac{v\omega_u}{\rho^2} - \frac{3u\omega_v}{2\rho^2}$$

$$\frac{\partial \omega_v'}{\partial \lambda_v} = \mu$$

$$\frac{\partial \omega_v'}{\partial \omega_u} = -\frac{v}{\rho}$$

$$\frac{\partial \omega_v'}{\partial \omega_v} = \frac{3u}{2\rho}$$

$$\frac{\partial \lambda_M'}{\partial \rho} = -\frac{3T\rho^{1/2}\lambda}{2M^2}$$

$$\frac{\partial \lambda_M'}{\partial M} = \frac{2T\rho^{3/2}\lambda}{M^3}$$

$$\frac{\partial \lambda_M'}{\partial \lambda_u} = -\frac{T\rho^{3/2}\lambda_u}{M^2\lambda}$$

$$\frac{\partial \lambda_M'}{\partial \lambda_v} = - \frac{T\rho^{3/2}\lambda_v}{M^2\lambda}$$

The terminal boundary conditions in the regularized polar coordinates are:

$$H_1 = 0.5 \frac{(u^2 + v^2)}{\rho^3} - \frac{\mu}{\rho}$$

$$H_2 = \lambda_u - \frac{u\omega_u}{\mu\rho}$$

$$H_3 = \lambda_v - \frac{v\omega_u}{\mu\rho}$$

$$H_4 = \frac{\omega_v}{\rho^{3/2}}$$

$$H_5 = \lambda_M$$

$$H_6 = 1.0 + \frac{(u\omega_u + v\omega_v)}{\rho^3} - \frac{\mu\lambda_u}{\rho^2} - \frac{T\lambda}{M} - \lambda_M\beta$$

The time derivatives of the terminal constraints are:

$$H_1' = \frac{uu' + vv'}{\rho^3} - \frac{3(u^2 + v^2)u}{2\rho^4} + \frac{\mu u}{\rho^2}$$

$$H_2' = \lambda_u' - \frac{u'\omega_u}{\mu\rho} - \frac{u\omega_u'}{\mu\rho} + \frac{u^2\omega_u}{\mu\rho^2}$$

$$H_3' = \lambda_v' - \frac{v'\omega_u}{\mu\rho} - \frac{v\omega_u'}{\mu\rho} + \frac{uv\omega_u}{\mu\rho^2}$$

$$H_4' = \frac{\omega_v'}{\rho^{3/2}} - \frac{3u\omega_v}{2\rho^{5/2}}$$

$$H_5' = \lambda_M'$$

$$H_6' = 0$$

The nonzero elements of the $\frac{\partial H}{\partial z}$ matrix are

$$\frac{\partial H_1}{\partial u} = \frac{u}{\rho^3}$$

$$\frac{\partial H_1}{\partial v} = \frac{v}{\rho^3}$$

$$\frac{\partial H_1}{\partial \rho} = - \frac{3(u^2 + v^2)}{2\rho^4} + \frac{\mu}{\rho^2}$$

$$\frac{\partial H_2}{\partial u} = - \frac{\omega_u}{\mu\rho}$$

$$\frac{\partial H_2}{\partial \rho} = \frac{u\omega_u}{\mu\rho^2}$$

$$\frac{\partial H_2}{\partial \lambda_u} = 1.0$$

$$\frac{\partial H_2}{\partial \omega_u} = - \frac{u}{\mu\rho}$$

$$\frac{\partial H_3}{\partial v} = - \frac{\omega_u}{\mu\rho}$$

$$\frac{\partial H_3}{\partial \rho} = \frac{v\omega_u}{\mu\rho^2}$$

$$\frac{\partial H_3}{\partial \lambda_v} = 1.0$$

$$\frac{\partial H_3}{\partial \omega_u} = -\frac{v}{\mu\rho}$$

$$\frac{\partial H_4}{\partial \rho} = \frac{3\omega_v}{2\rho^{5/2}}$$

$$\frac{\partial H_4}{\partial \omega_v} = \frac{1}{\rho^{3/2}}$$

$$\frac{\partial H_5}{\partial \lambda_M} = 1.0$$

$$\frac{\partial H_6}{\partial u} = \frac{\omega_u}{\rho^3}$$

$$\frac{\partial H_6}{\partial v} = \frac{\omega_v}{\rho^3}$$

$$\frac{\partial H_6}{\partial \rho} = -\frac{3(u\omega_u + v\omega_v)}{\rho^4} + \frac{2\mu\lambda_u}{\rho^3}$$

$$\frac{\partial H_6}{\partial M} = \frac{T\lambda}{M^2}$$

$$\frac{\partial H_6}{\partial \lambda_u} = - \frac{\mu}{\rho^2} - \frac{T\lambda_u}{M\lambda}$$

$$\frac{\partial H_6}{\partial \lambda_v} = - \frac{T\lambda_v}{M\lambda}$$

$$\frac{\partial H_6}{\partial \omega_u} = \frac{u}{\rho^3}$$

$$\frac{\partial H_6}{\partial \omega_v} = \frac{v}{\rho^3}$$

$$\frac{\partial H_6}{\partial \lambda_M} = - \beta$$

APPENDIX E

NORMALIZED VALUES

NORMALIZED VALUES

In order to enhance the numerical integration accuracy, all numerical calculations were made in a normalized system. The units of normalization are given in Table E-1. The unit of length corresponds to one Earth radius and the unit of velocity to the circular velocity at one Earth radius. The unit of mass was chosen to be 5000 kg. The remaining normalization units are such that consistent dimensional properties are maintained.

Table E-2 gives the normalized values of the constants common to all of the coordinate systems investigated. Since these constants are normalized, the units are indicated by the general notation of L for length, T for time, and M for mass.

Tables E-3 and E-4 present, respectively, the normalized values of the initial and terminal states for all coordinate systems investigated. Again, the dimensions are indicated by the general notation.

TABLE E-1. — NORMALIZATION UNITS

Unit	Value
Length	0.63781450×10^7 m
Velocity	$.79053881 \times 10^4$ m/sec
Time	$.80680985 \times 10^3$ sec
Mass	$.5000 \times 10^4$ kg
Force	$.48991644 \times 10^5$ (kg-m)/sec^2

TABLE E-2. — NORMALIZED VALUES OF CONSTANTS

Constant	Value
Thrust	$0.10205822 \times 10^{-1}$ ML/T^2
Mass flow rate	$.16336057 \times 10^{-5}$ M/T
Gravitation	$.10 \times 10^1$ L^3/T^2

TABLE E-3. — NORMALIZED INITIAL CONDITIONS

Variable	Rectangular		Polar	
	Unregularized	Regularized	Unregularized	Regularized
TIME(T)	0.0	0.0	0.0	0.0
u(L/T)	0.0	0.0	0.0	0.0
v(L/T)	0.97728258	0.10470436×10^{1}	0.97728258	0.10470436×10^{1}
x(L)	0.10470395×10^{1}	0.10470395×10^{1}	0.10470395×10^{1}	0.10470395×10^{1}
y(L)	0.0	0.0	0.0	0.0
M(M)	1.0	1.0	1.0	1.0
$\lambda_u(T^2/L)$	0.29606237×10^{1}	0.2960491×10^{1}	0.29608441×10^{1}	0.29601179×10^{1}
$\lambda_v(T^2/L)$	-0.97928073×10^{2}	-0.9791739×10^{2}	-0.97927892×10^{2}	-0.97975524×10^{2}
$\omega_u(T/L)$	-0.95538761×10^{2}	-0.10234806×10^{3}	-0.95538506×10^{2}	-0.10240578×10^{3}
$\omega_v(T/L)$	0.27633966×10^{1}	0.29604389×10^{1}	0.27635833×10^{1}	0.29607177×10^{1}
$\lambda_M(T/M)$	0.78700772×10^{2}	0.78697428×10^{2}	0.78700659×10^{2}	0.78709925×10^{2}

TABLE E-4. — NORMALIZED TERMINAL CONDITIONS

Variable	Rectangular		Polar	
	Unregularized	Regularized	Unregularized	Regularized
TIME(T)	0.70145389×10^{2}	0.23063345×10^{2}	0.70145336×10^{2}	0.23063301×10^{2}
u(L/T)	0.26064303	0.64876389×10^{1}	0.30879017	0.76866563×10^{1}
v(L/T)	-0.40823787	-0.10162287×10^{2}	0.37315096	0.92887282×10^{1}
x(L)	-0.26111336×10^{1}	-0.26114617×10^{1}	0.85254035×10^{1}	0.85254079×10^{1}
y(L)	-0.81156958×10^{1}	-0.81154810×10^{1}	0.23250630×10^{2}	0.23250559×10^{2}
M(M)	0.99988541	0.99988541	0.99988541	0.99988541
$\lambda_u(T^2/L)$	-0.52721878×10^{2}	-0.52718636×10^{2}	-0.62460890×10^{2}	-0.62461087×10^{2}
$\lambda_v(T^2/L)$	0.82576800×10^{2}	0.82578870×10^{2}	-0.75479544×10^{2}	-0.75479381×10^{2}
$\omega_u(T/L)$	0.85237112	0.21220771×10^{2}	-0.27830104×10^{1}	-0.69276707×10^{2}
$\omega_v(T/L)$	0.26492650×10^{1}	0.65946501×10^{2}	$-0.18643186\times10^{-14}$	$-0.35507188\times10^{-12}$
$\lambda_M(T/M)$	0.22423833×10^{-12}	0.49770030×10^{-10}	0.14723466×10^{-13}	$-0.16084963\times10^{-12}$