MASSACHUSETTS INSTITUTE OF TECHNOLOGY

TE-40
MIXED PROPULSION MODE ORBITAL TRANSFERS

by

C. Julian Vahlberg, Jr.

MEASUREMENT SYSTEMS LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE 39, MASSACHUSETTS

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Measurement Systems Laboratory

Massachusetts Institute of Technology

Cambridge, Massachusetts

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ABSTRACT

Transfers between coplanar coaxial ellipses in a strong inverse square gravitational field are investigated. The control acceleration is provided by a combination of ideal velocity limited and ideal power limited rockets. Since the acceleration of the power limited rocket is assumed to be much smaller than gravity, that phase of the transfer is a slow spiral over many orbits. The velocity limited rocket is assumed capable of impulsive thrust. The combination of impulsive with continuous control accelerations introduces complexities in the optimal control problem which are not present in the independent study of either one. A general cost function for ideal engines and necessary conditions for transfers in an arbitrary gravitational field are derived in a form for application to orbital transfers. The solutions for field free space transfers are given. The necessary conditions for optimal transfers between coplanar coaxial elliptic orbits are derived and then solved on a digital computer. The number and timing of impulses are numerically determined. A substantial improvement in payload is obtained by the optimum combination of propulsion systems.

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The computational work for this thesis was done on the M.I.T. Information Processing Center's IBM 360/65 computer. The payload and trajectory curves were generated on the computer and plotted on the Stromberg-Carlson SC-4020 plotter.

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TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

LIST OF FIGURES

LIST OF FIGURES (cont.)

LIST OF TABLES

Chapter 1

INTRODUCTION

As the scientific goals for earth orbital and planetary missions are becoming more ambitious, there is the continued desire for higher payload through better use of available propulsion systems. Low thrust rockets with a higher specific impulse (pounds of thrust per pound of fuel per second) have been tested and will soon be available for operational use. High thrust chemical rockets with a lower specific impulse have been used exclusively for all space missions carried out to date and are necessary for initial insertion into orbit for any mission. Although transfers using either pure high or pure low thrust have been studied extensively in the past, the combination has received little attention. The possibility of improved efficiency using the combination of both types of rockets is an important area to be explored. Payload improvements are obtained for a class of orbital transfers using the combination of ideal velocity limited (high thrust) and ideal power limited (low thrust) rockets.

The two types of engines considered are limited by their basic operating characteristics. For velocity limited rockets, the propellant is the product of a combustion which also provides the propulsive energy. The exhaust velocity of the propellant and thus the specific impulse of the engine is limited by the attainable chamber temperatures and the molecular weight of the propellant. However, the power available is proportional to the fuel flow rate and thus the possible thrust level is very large. In the payload optimization using such an engine, an impulse is theoretically optimal, and with respect to orbital transfer times, can be achieved. Only the efficiency of the engine (specific impulse) and not the thrust level is limited by the exhaust velocity.

Ion engines are limited by the propulsive power available. An independent electrical power supply is used to generate a field which accelerates the ionized propellant. For any mission, the power available is assumed proportional to the power supply mass which is fixed and independent of the propellant. This mass, which is not normally considered as part of the payload, must be optimally chosen to provide enough power with the minimum extra mass. With limited power, a high exhaust velocity is possible only for low acceleration levels. Optimal pure low thrust transfers have a continuous, but not necessarily constant, thrusting at the lowest acceleration level which will accomplish the transfer in the specified time of flight. The low thrust payload increases with longer times of flight since the necessary acceleration can be lower. The advantage of the high specific impulse which is provided by a power limited engine is obtained at the expense of longer times of flight and a separate power supply.

The combination of rockets has the promise of an improvement in payload over either type used independently. If a long time of flight is possible, the use of low thrust can certainly improve the payload over pure high thrust. But even when low thrust can be used effectively, a high thrust rocket is required for initial insertion into orbit and can improve the payload by its use at opportune times during a transfer when the low thrust rocket has too low an efficiency due to the high acceleration required. Missions which require a power supply in the final orbit for communications or experimental purposes provide further motivation for the consideration of the combination. The mass of this power supply, m_e, is dead weight for high thrust transfers, but can be used to good advantage for a low thrust phase. If the optimum power plant mass is larger than m_e, the undesired portion will be dropped at the final time. Otherwise, use of m_e as the power plant mass maximizes the payload. For non-zero m_e or for a long time of flight, the combination of high and low thrust always has a higher payload than pure high thrust. If the time of flight is very long, the high thrust will only be used for initial insertion into orbit and low thrust will complete the transfer.

Optimal mixed thrust transfers between coplanar coaxial elliptic orbits in a central inverse square gravitational field are studied. Within the assumptions used here, the pure high and pure low thrust transfers have complete closed form analytic solutions. For the combination of the propulsion systems, the dynamic optimization problem is analytically reduced to the maximization of a payload expression over several free parameters. These maxima are numerically found for a large variety of specific transfers in this general class.

1.1 The assumptions used

Since some assumptions are necessary to obtain the pure high and pure low thrust analytic solutions, similar assumptions are required for the analysis of the more complicated combination of propulsion modes. The ideal engine assumptions allow an explicit integration of the mass flow differential equation for the combination of engines. The resultant payload expression is then maximized for particular transfers. An assumption on the magnitude of the low thrust acceleration in comparison to the gravitational acceleration has many ramifications which specialize the nature of the orbital transfers being considered. As a result of all of these assumptions, the complicated dynamic optimization problem for orbital transfers is reduced to the maximization of a payload expression over a few constant parameters.

The three masses considered in this ideal engine formulation are the the fuel, power supply, and payload. For real engines, there would also be the masses of the engines themselves, the fuel tanks, and structures. In addition, there are specific assumptions made about the operational capabilities of each class of engines. The high thrust is assumed capable of producing an impulsive thrust. The low thrust is assumed capable of operating at a variable exhaust velocity. Although all of these assumptions are only valid for preliminary mission studies, their use is justified because of the resultant analytic simplifications and their relationship to the fundamental characteristics of real engines.

Spacecraft propelled by pure high thrust engines are assumed to have only the masses of the fuel and the payload. The efficiency of the engine is completely determined by the propellant exhaust velocity which is limited by the attainable chamber temperatures and the molecular weight of the propellant. The propellant is the product of the combustion which provides the propulsive energy. With this assumption, the mass flow differential equation can be easily integrated to give the fuel required for any specific maneuver. Since the high thrust levels possible in real engines are applied for a small fraction of the period of the orbits being considered, impulsive thrusts are analytically allowed without loss of significant accuracy.

Spacecraft propelled by pure low thrust engines are assumed to have the masses of the propellant, power supply, and payload. The capabilities of these engines are determined by the mass and power capacity of the power supply. The propulsive power is separate from the propellant and fixed for a particular mission. The size of the power supply can be optimally chosen and a portion (or all) of it dropped at the final time if solar cell panels provide the power. These assumptions allow an analytic solution of the mass differential equation and the explicit choice of the optimum power supply size. Since no constraints are assumed on the allowable propellant exhaust velocity, it will be variable, depending on the desired acceleration. Although a variable exhaust velocity will probably not be feasible in a real engine, the resultant optimal control problem formulation is simpler. The optimal payload and trajectory provide a good initial condition for a numerical iteration on the more complicated problem for real engines.

Orbital transfers in a strong, central, inverse squared gravitational field are considered. The low thrust acceleration is assumed to be much smaller than the acceleration of gravity. Typical low thrust accelerations (10^{-4} g's) will be less than one percent of the earth's gravity out to orbital distances of ten earth radii. Thus, by assumption, only variations in radii of a ratio of ten will be considered. With this restriction, optimal high thrust transfers have two impulses at the opposite apses of a coasting orbit. For a low thrust phase, there will only be a small change in the orbit during any one orbital period. For significant transfers to be possible, the result will be a slow spiral over a long time of flight and only the secular variation in orbital elements are of significance. Since the time of half an orbital period is assumed to be a small fraction of the total flight time, any high thrust transfer will be considered as having an open time of flight. For the mixed thrust problem, two impulses applied at the opposite apses of an orbit can be assumed to occur at the same time, with respect to the low thrust phase. Also the effect of the low thrust during this half orbit can be neglected.

1.2 History of the problem

Two distinct classes of propulsion systems are considered for orbital transfers. In the past, independent investigations of each class have determined their relative capabilities. The general theories for transfers, analytic solutions, and results for specific missions have been developed. Edelbaum[6] surveyed the results and compared the nature of some specific optimal transfers using each of the two modes. If the change in radius is less than a factor of 11.94, the Hohmann[11] transfer between coplanar circular orbits using high thrust engines was shown to be optimal by Hoelker and Silber.[10] The transfer has two impulses at the opposite apses of an elliptic coasting orbit. By assumption, only changes in radius of less than a factor of ten are being considered. Lawden[12] summarized these results and extended them to transfers between coplanar coaxial ellipses. Edelbaum[2-5] presented analytic solutions for a variety of low thrust transfers. For coplanar coaxial transfers, he integrated the equations of motion and also solved for the boundary conditions on the costate variables. These results are expanded in Appendix D.

Edelbaum[1], Fimple[7], Hazelrigg[9], and Grodzovsky[8] considered the general optimal combination of propulsion modes and demonstrated their feasibility. They restricted their analysis to elementary transfers and did not investigate the general nature of such trajectories. Other studies have considered the combination, but the mission assumptions constrained the analysis and few theoretical principles resulted. Grodzovsky[8] presented a cost function for the combination which did not consider retention of any or all of the power plant. His results are expanded in Appendix B to include this possibility. In the past, the combination of propulsion modes has not been considered for any general class of transfers, except in field free space.

1.3 The approach used

Before considering the problem of optimal orbital transfers, some preliminary derivations are carried out. A payload expression is derived and optimal transfers in general gravitational fields are considered. The dynamics of the payload differential equations are completely solved. The results are applied to the simple example of transfers in field free space. Then the dynamics of an orbital transfer problem in a central inverse square gravitational field are solved, resulting in a payload expression which must be maximized over several free parameters. The results of the numerical maximization are presented to complete the treatment of coplanar coaxial transfers.

Each chapter contains only those equations which are necessary for the discussion of results. The required derivations appear in the appendices. In general each chapter has an appendix which parallels, but does not duplicate, the chapter. In the following discussion of each chapter, the corresponding appendices are indicated.

In Chapter 2 (and Appendix B) a cost function is derived for ideal engines from the basic principles of momentum and power. The resultant complicated mass flow equations are simplified by the introduction of some intermediate differential equations which are analytically solvable. Using the new mass differential equations to define the payload, the general optimization problem for transfers in an arbitrary gravitational field is formed. Some of the resultant set of necessary conditions are analytically solved to yield a simplified set of necessary conditions for this general problem. Necessary conditions across an impulse are also derived. A different problem simply representing a convenient change of state variables is also given, along with its solution. The form and characteristics of the final necessary conditions are discussed.

The problems of transfers in field free space are considered in Chapter 3 (and Appendix C) in order to obtain as much insight as possible into the basic characteristics of the problem. First the general form of the solution is given and discussed and the specific example of a change in velocity is considered. The optimum payload for this problem can be obtained analytically for all cases except one. A table of these optimum payloads in given along with plots of the payload improvements. The results and characteristics of the problem are discussed.

The analysis for transfers between coplanar coaxial ellipses is presented in Chapter 4 (and Appendix D). The differential equations of motion, with their boundary conditions, are completely solved within the assumptions used here. The possible timing of impulses with respect to the low thrust phase is assumed, since the timing during any one orbit is known. For this assumed mode, expressions are derived for the payload and the conditions which must be satisfied to maximize the payload. Further conditions are presented which can be tested to verify the validity of the assumptions on the timing of the impulses.

In Chapter 5 (and Appendices D and E) the numerical solutions of the necessary conditions obtained in the previous chapter are discussed for a large number of circle to circle transfers and some ellipse to ellipse transfers. The equations employed do not have any special properties for the set of transfers studied and should yield similar results for all coplanar coaxial transfers. The numerical techniques employed and the difficulties encountered are discussed. Plotted results are given both of the improvements in payload and of the orbital paths followed in the transfers.

The results are summarized and recommendations for further research are given in Chapter 6.

Chapter 2

TRANSFERS IN AN ARBITRARY GRAVITATIONAL FIELD

An optimal control problem for the maximization of payload is derived and presented for transfers in an arbitrary gravitational field. After a partial solution of the complete problem, a simplified set of necessary conditions for the controls is presented as a function of a "primer vector". The primer vector is identified for two different forms of differential equations which might represent the vehicle's motion in a position dependent gravitational field.

The ideal engine assumptions described in Chapter 1 are used to derive fuel flow differential equations from basic physics. These equations can be completely solved in terms of some integrals of the control accelerations. The final mass, after all fuel has been burned and any undesired power plant has been discarded, is defined as the payload. A more convenient set of differential equations are then derived which can be used to define the payload.

This expression for payload is used as the "cost function" to be maximized for a general optimal control problem. The payload and vehicle differential equations are adjoined together with appropriate costate multipliers to form the Hamiltonian. Since the vehicle dynamics do not directly enter the cost function, the problem can be logically divided into two parts, the payload and vehicle dynamics. Due to this separation, the payload costate differential equations are independent of the state and only a function of the controls, $\underline{A}$. These payload costate differential equations with their boundary conditions can be completely solved in terms of the control integrals which define the payload. After eliminating these costates, the resultant Hamiltonian is only a function of the vehicle dynamics and the appropriate controls.

In payload optimizations for transfers which can be described by the two possible sets of vehicle dynamics presented here, the payload differential equations must always be solved in the manner of this chapter. For clarity of discussion of specific problems, all of the payload equations are solved once in this chapter and the resultant necessary conditions applied without reference to the total problem which is implied by those conditions. The analytic steps involved for this chapter are described in Appendix B. The results are presented and discussed in this chapter.

2.1 The Payload for ideal engines

The fuel flow rate for any ideal engine is related to the thrust provided, $m\underline{A}_i$, and the propellant exhaust velocity, c_i, by the conservation of momentum

$$\dot{m}\, c_i = -\dot{m}|\underline{A}_i| \tag{2.1}$$

which assumes

$$\dot{m} \leq 0$$
$$c_i \geq 0$$

Fuel can only be expended! For a given desired thrust, the fuel used can be minimized by maximizing c_i. The two classes of engines considered here limit c_i by their basic operating characteristics. For any engine the power required to accelerate the propellant is given by

$$\text{Power} = -\frac{1}{2}\,\dot{m}\, c_i^{\,2}$$

In a chemical engine, the fuel provides both the propulsive energy and the propellant. The attainable chamber temperatures and molecular weight of the propellant limit c_i explicitly. Ion engines are limited by the propulsive power which the independent power supply can provide. For chemical engines

$$\dot{m} = -\frac{m}{c}|\underline{A}_i|$$

and for ion engines

$$\dot{m} = -\frac{\frac{1}{2}(m\ A)^2}{\text{Power}}$$

Since the power plant size is fixed and independent of the engine for ion propulsion systems, c_i and $\dot{m}$ are related and cannot be independently chosen. A high efficiency (c_i) is obtainable only for a low fuel flow rate, and thus low acceleration. In order to get this high efficiency, the thrust must be at a low level, applied for a long time of flight. The acceleration from chemical engines is limited only by the mass flow rate. With respect to orbital transfer times, the thrust can be high enough to be considered as impulsive. Thus the analysis is for the combination of continuous with impulsive controllers.

Only two classes of engines are considered, but for analytic ease, three engines are assumed. The first engine (A_1) operates impulsively with an exhaust velocity c. Although it can be used at any time during the flight except the final time, it is optimal to use it only at the initial time for most transfers. The second engine ($\underline{A}_\ell$) operates continuously during the entire mission. The power is assumed proportional to the mass of the power supply, m_p.

$$\text{Power} = \frac{1}{\alpha} m_p \tag{2.2}$$

The size of m_p remains to be chosen to optimize the payload. As described in Chapter 1 , it may be desirable to save a portion or all of the power supply, and drop the rest at the final time. The third engine ($\underline{A}_3$) is used after any of the power supply is dropped. It has characteristics identical to those of the first engine. The separation of

$\underline{A}_1$ and $\underline{A}_3$ is for notational convenience and the reason will become clear later in the derivation.

Appendix B combines all expressions for the various engines to form a total mass flow differential equation. From these the mass at the final time (the payload) is found to be

$$m_\pi = \begin{cases} J_1^{\,2}\,[(K_1 - L_1)^2 + m_e] & \text{if } m_e \le m_p \\ \\ \dfrac{J_1^{\,2}\,K_1^{\,2}}{1 + \frac{1}{m_e}\,L_1^{\,2}} & \text{if } m_e \ge m_p \end{cases} \tag{2.3}$$

where

m_e = the desired final power plant mass

$$m_p = L_1(K_1 - L_1) = \text{the optimum power plant mass} \tag{2.4}$$

$$J_1 = \exp\left[-\frac{1}{2c}\int_0^{t_f} |\underline{A}_3|\,dt\right] \tag{2.5}$$

$$K_1 = \exp\left[-\frac{1}{2c}\int_0^{t_f} |\underline{A}_1|\,dt\right] \tag{2.6}$$

$$L_1^{\,2} = \frac{\alpha}{2}\int_0^{t_f} K^2(t)\,\underline{A}_\ell^{\,2}(t)\,dt \tag{2.7}$$

The notation $\underline{A}_\ell^{\,2}$ is used to denote $\underline{A}_\ell^{\,T}\,\underline{A}_\ell$, the square of the magnitude of the vector acceleration. It is noted that J_1, K_1, and L_1 are the final values of the solutions to the differential equations

$$\dot{J} = -\frac{1}{2c} J|\underline{A}_3|$$

$$\dot{K} = -\frac{1}{2c} K|\underline{A}_1|$$

$$\dot{L} = -\frac{\alpha}{4} \frac{K^2}{L} \underline{A}_\ell^2$$

with the initial conditions

$$J(0) = K(0) = 1.$$

$$L(0) = 0^+$$

such that $L(t) \geq 0$. The positive root of $L(t)$ is always used. These differential equations for J, K, and L are used in later sections to define J_1, K_1, and L_1, in the expressions for m_π, which is to be maximized.

Before proceeding to the complete optimization problem, it is interesting to note the limits of pure high or pure low thrust. The resultant cost function which could be used if only one class of engines were used is presented. For pure low thrust we have the conditions

$$\underline{A}_1 = 0 \qquad K(t) = 1$$

$$\underline{A}_3 = 0 \qquad J(t) = 1$$

For pure high thrust, only the first engine needs to be used since it is identical to the third. Thus

$$\underline{A}_3 = 0 \qquad\qquad J(t) = 1$$

$$\underline{A}_\ell = 0 \qquad\qquad L(t) = 0$$

$$m_e = m_p = 0$$

for the pure high thrust.

The payloads for these limits follows easily. For the high thrust

$$m_\pi = K_1^{\,2}$$

Maximizing this expression for the payload is clearly equivalent to minimizing the integral

$$\int_0^{t_f} |\underline{A}| \; dt = \sum |\Delta V| \tag{2.8}$$

which is a sum of velocity changes since it is used impulsively. For the pure low thrust

$$m_\pi = \begin{cases} (1-L_1)^2 + m_e & m_e \le m_p \\ \dfrac{1}{1 + \dfrac{1}{m_e} L_1^{\,2}} & m_e \ge m_p \end{cases}$$

$$m_p = L_1 \, (1 - L_1)$$

Minimizing the integral

$$\frac{1}{2} \int_0^{t_f} \underline{A}_\ell^{\,2} \; dt \tag{2.9}$$

is clearly equivalent to maximizing m_π, regardless of m_e.

Although payload maximization is the goal of most trajectory optimizations, the problem is generally stated as a fuel minimization. In the combination, however, due to the interplay of the two modes, the problem must be explicitly stated as a payload maximization. We can also anticipate greater difficulty in solving the problem due to the interplay of the square root (for L_1) and the exponentials (in J_1 and K_1).

2.2 Necessary conditions for transfers in an arbitrary gravitational field.

The necessary conditions for the maximization of payload for transfers in an arbitrary position dependent gravitational field are derived in Appendix B. The general optimal problem is stated and the results discussed in this section. Using the payload differential equations of the previous section, the complete optimal control problem for transfers in a position dependent gravitational field is given below. The payload as given by

$$m_\pi = \begin{cases} J_1^2 \, [(K_1 - L_1)^2 + m_e] & \text{if } m_e \le m_p \\ \\ \dfrac{J_1^2 \, K_1^2}{1 + \frac{1}{m_e} \, L_1^2} & \text{if } m_e \ge m_p \end{cases}$$

with

$$m_p = L_1 \, (K_1 - L_1)$$

is to be maximized subject to the differential equation constraints

$$\underline{\ddot{x}} = \underline{R}(\underline{x}) + \underline{A}_1 + \underline{A}_\ell + \underline{A}_3 \qquad (2.10)$$

$$\dot{J} = -\frac{1}{2c} J|\underline{A}_3|$$

$$\dot{K} = -\frac{1}{2c} K|\underline{A}_1|$$

$$\dot{L} = -\frac{\alpha}{4} \frac{K^2}{L} \underline{A}_\ell{}^2$$

with the boundary conditions on the differential equations given by

$$\underline{x}(t_0) = \underline{x}_0 \qquad \dot{\underline{x}}(t_0) = \dot{\underline{x}}_0$$

$$\underline{x}(t_f) = \underline{x}_f \qquad \dot{\underline{x}}(t_f) = \dot{\underline{x}}_f$$

$$J(t_0) = 1 \qquad K(t_0) = 1$$

$$L(t_0) = 0^+$$

The parameters $\underline{x}_0$, $\underline{x}_f$, $\dot{\underline{x}}_0$, $\dot{\underline{x}}_f$, α, c, m_e and t_f completely specify the desired transfer. The state vector, $\underline{x}$, is the position vector of the spacecraft for these differential equations. The accelerations $\underline{A}$ are chosen to maximize the Hamiltonian

$$H = \underline{\lambda}^T [\underline{R}(\underline{x}) + \underline{A}_1 + \underline{A}_\ell + \underline{A}_3] + \dot{\underline{\lambda}}^T \dot{\underline{x}}$$

$$- \lambda_J \frac{1}{2c} J|A_3| - \lambda_K \frac{1}{2c} K|A_1| + \lambda_L \frac{\alpha}{4} \frac{K^2}{L} \underline{A}_\ell^2$$

where the costate, $\underline{\lambda}$ (t), satisfies the differential equation

$$\ddot{\underline{\lambda}} = \frac{\partial \underline{R}(\underline{x})}{\partial \underline{x}}^T \underline{\lambda} \qquad (2.11)$$

The boundary conditions on $\underline{\lambda}$ are free since they are completely specified for $\underline{x}$. For this state differential equation, the costate, $\underline{\lambda}$, is classically called the "primer vector" and has some properties of interest. The primer, often identified as $\underline{p}$ ($\underline{\lambda}$ here), and its first two derivatives are continuous when any impulsive control is used, since the differential equation for $\underline{\lambda}$ is affected only by $\underline{x}$ (which also is continuous across an impulse).

The optimal accelerations which maximize H are

$$\underline{A}_{\ell}\,(t) = Q(t)\,\underline{\lambda}\,(t)$$

$$\underline{A}_1 = \begin{cases} \infty\underline{\lambda} & \text{if } |\underline{\lambda}(t)| = \delta_1\,(t) \\ 0 & \text{if } |\underline{\lambda}(t)| < \delta_1\,(t) \end{cases}$$

$$\underline{A}_3 = \begin{cases} \infty\underline{\lambda}\,(t_f) & \text{if } |\underline{\lambda}\,(t_f)| = \delta_3 \\ 0 & \text{if } |\underline{\lambda}\,(t_f)| < \delta_3 \end{cases}$$

where

$$Q(t) = \frac{1}{\alpha K^2(t)} \begin{cases} \dfrac{L_1}{J_1{}^2(K_1 - L_1)} & \text{if } m_e \le m_p \\[2ex] \dfrac{m_e + L_1{}^2}{m_\pi} & \text{if } m_e \ge m_p \end{cases}$$

$$\delta_1(t) = \frac{1}{c} \begin{cases} J_1{}^2(K_1 - L_1)^2\left[1 + \dfrac{1}{m_p}\,L^2(t)\right] & \text{if } m_e \le m_p \\[2ex] \dfrac{m_\pi}{1 + \dfrac{1}{m_e}\,L_1{}^2}\left[1 + \dfrac{1}{m_e}\,L^2(t)\right] & \text{if } m_e \ge m_p \end{cases}$$

$$\delta_3 = \frac{1}{c} m_\pi$$

If $\underline{x}$ is a vector of n state variables, $\underline{\lambda}$ will also be a vector of n variables. The 2 n differential equations in $\underline{x}$ and $\underline{\lambda}$ have the 2 n boundary conditions consisting of $\underline{x}_0$ and $\underline{x}_f$. The low thrust acceleration is explicitly related to $\underline{\lambda}$. Although the magnitude of the impulsive controls $\underline{A}_1$ and $\underline{A}_3$ are not explicitly given, they are constrained implicitly by the condition on the primer at the times of the impulses. If the primer is less than the threshold, the impulse explicitly has a zero magnitude. Thus the differential equations have the correct number of boundary conditions and there is an equation which implicitly specifies the magnitude of each impulse. Further, the primer can never be greater than the appropriate threshold on an optimal trajectory, or more impulses would be least locally optimal.

For the pure low thrust problem, if the integral of equation (2.9) is minimized, the term equivalent to $Q(t) = 1$. The boundary conditions on $\underline{\lambda}$ are chosen so that $\underline{A}_\ell$ is adequate to accomplish the desired transfer. Pure low thrust is the optimal limit for the mixed thrust problem when δ_1 and δ_3 are always larger than $|\underline{\lambda}(t)|$. The scaling of the $Q(t)$ is then incorporated in the initial conditions on $\underline{\lambda}$.

For a pure high thrust transfer, if the integral of equation (2.8) is minimized, the equivalent thresholds

$$\delta_1(t) = \delta_3 = 1$$

and the boundary conditions on $|\underline{\lambda}|$ are chosen such that

$$|\underline{\lambda}| = 1$$

at the times of all impulses and is less than 1 for all other times. Mixed thrust transfers become pure high thrust when the low thrust and its cost, L_1 go to zero (m_e must be zero or some low thrust will be used). It is shown in Appendix B that this limit occurs at a discontinuity in the equations for this formulation of the payload. Numerically those difficulties can be avoided by specifying a small, but non-zero m_e. Whereas pure low thrust is an easy extension from the mixed thrust, pure high thrust evolves from the mixed thrust at a discontinuity.

When the combination of engines is being used, $\underline{\lambda}$ will still carry a necessary scaling for the low thrust phase, but it must also match, or remain below a threshold. The threshold for the combination, instead of being constant, is an increasing function of time. The final threshold, δ_3, can be lower than $\delta_1(t_f)$ if a part of the optimum power supply is dropped. With such a decrease in mass, the high thrust engine is, of course, more efficient. As the low thrust engine burns its fuel, its efficiency (c_i) can be higher for the same acceleration. This increased efficiency of the low thrust is represented by the increasing threshold $\delta_1(t)$.

There are some special cases which reflect the nature of the final threshold. When $m_e = 0$

$$\delta_1(t_0) = \delta_3$$

the initial and final thresholds are equal. In the other extreme if $m_e \geq m_p$

$$\delta_1(t_f) = \delta_3$$

there is no change in the threshold at the final time. Thus for intermediate values of m_e

$$\delta_1(t_0) \leq \delta_3 \leq \delta_1(t_f)$$

This change in the threshold and the difference in the manner J_1 enters the payload expressions are the reasons $\underline{A}_3$ is considered different from $\underline{A}_1$, although they might physically represent the same engine.

As long as $m_e \leq m_p$, an increase in m_e only raises the final threshold. This will diminish the final impulse, since it is then less efficient. If there is no final impulse, the only effect will be an increase in the payload equal to the increase in m_e. However if $m_e \geq m_p$ an increase in m_e also increases the efficiency of the low thrust phase. Both the initial and the final impulses will diminish due to the increased low thrust efficiency. There will also be an increase in payload, but now it will be less than the increase in m_e.

For increasing times of flight, L_1 will become smaller as the low thrust is operating at a lower acceleration (higher efficiency). Not only will the payload increase, but both the initial and final thresholds will increase. This is best seen by observing that for $m_e = 0$

$$\delta_1(t_0) = \delta_3 = \frac{1}{c}\, m_\pi$$

The threshold increase is proportional to the increase in payload. Both the initial and final impulse will diminish in size. For non-zero m_e, $\delta_1(t)$ will increase at a lower rate and reach a smaller value, since the time varying portion of δ_1 is $L^2(t)$. Although both impulses will still be used less, the increase in δ_3 will be less than the increase in δ_1 and the final impulse will diminish less than the initial impulse.

Thus there are two aspects of this formulation which favor the low thrust: an increase in either m_e, or the time of flight. As is shown here, the effect of the two on the relative sizes of the initial and final impulses is different. These general observations are verified in the application of these equations to specific problems.

2.3 A convenient change of variables

Certain constants of motion have evolved in the classical study of transfers between orbits in a central inverse square gravitational field. For the very small accelerations of a low thrust phase, these parameters vary slowly enough to permit additional simplifications. The previously stated optimal control problem is implied in total, except for the state differential equations. This new state could be a vector made up of orbital elements. The appropriate necessary conditions for the optimal controls, $\underline{A}$, are presented without the derivation which would closely parallel that of the previous section. The different state formulation is used in order to take advantage of the classical notation for the orbital transfers considered in future chapters. These equations simply represent a restriction on the admissible gravitational field and then a nonlinear change in variables. As such they do not change the nature of the optimal solution.

For the optimization of the payload, m_π, for the state differential equation

$$\dot{\underline{x}} = \underline{B}(\underline{x},t)\ (\underline{A}_1 + \underline{A}_\ell + \underline{A}_3)$$

the Hamiltonian is

$$H = \underline{\lambda}^T \underline{B}(\underline{A}_1 + \underline{A}_\ell + \underline{A}_3) - \delta_1 |\underline{A}_1| - \delta_3 |\underline{A}_3| - \frac{1}{2Q} \underline{A}_\ell^{\,2}$$

where

$\underline{x}$ is a vector of orbital elements

$\underline{B}(\underline{x},t)$ is a matrix function of the state $\underline{x}$ and time. Note that the state, $\underline{x}$, and the costate, $\underline{\lambda}$, are different for this formulation than for the previous section. However the accelerations, $\underline{A}$, are the same. The present costate obeys the differential equation

$$\dot{\underline{\lambda}}^T = -\frac{\partial H}{\partial \underline{x}} = -\lambda^T \frac{\partial \underline{B}(\underline{x}_1 t)}{\partial \underline{x}} (\underline{A}_1 + A_\ell + \underline{A}_3)$$

where care must be taken with the third order tensor notation for $\frac{\partial \underline{\underline{B}}}{\partial \underline{x}}$. Identifying the coefficient of the acceleration vector in the Hamiltonian we see that the primer* for this problem is

$$\underline{p} = \underline{B}^T \underline{\lambda}$$

Although the costate is not constant during an impulse, $\underline{B}$ will also vary such that the primer $\underline{p}$ and its first two derivatives are still continuous across any impulse in the control. Having made this observation, the optimal controls are specified the same as before, having properly adjusted the nomenclature. This should be the case since this new formulation only represents a change of variables. The Hamiltonian is maximized, as before, by the controls

$$\underline{A}_\ell = Q(t)\, \underline{p}$$

$$\underline{A}_1 = \begin{cases} \infty \underline{p} & \text{if } |\underline{p}| = \delta_1 \\ 0 & \text{if } |\underline{p}| < \delta_1 \end{cases}$$

$$\underline{A}_3 = \begin{cases} \infty \underline{p} & \text{if } |\underline{p}|^{(t_f)} = \delta_3 \\ 0 & \text{if } |\underline{p}| < \delta_3 \end{cases}$$

*Lawden[12] uses this expression for the primer in discussing a problem similar to the formulation of Chapter 4.

with Q and the thresholds δ the same as before.

Chapter 3

TRANSFERS IN A POSITION INDEPENDENT GRAVITATIONAL FIELD

The general results of Chapter 2 are applied to problems with two classes of expressions for the gravitational acceleration $\underline{R}(\underline{x})$. Problems with $\underline{R}(\underline{x}) = 0$ are considered first, although, as shown below, these results are applicable to problems with $\underline{R}(\underline{x}) = \underline{g}$ = constant. Chapter 4 considers a second form for $\underline{R}(\underline{x})$. Since the state differential equation for this problem is independent of the state, the primer differential equation will also be independent of the state. The primer, and then the state differential equations can be easily integrated during the low thrust phase. An iteration may be required to determine the boundary conditions on the primer. Since the vector direction of any high thrust impulses are specified by the primer, only the magnitudes of the impulses remain to be chosen. If the desired power plant mass, m_e, is small enough, there can be only initial and final impulses. This may also be true for all transfers in field free space. After the assumption that only initial and final impulses may be used, the general transfer is analytically solved as far as possible and the results applied to the simpler problem of a change in position with no change in velocity. Finally the problem for a change in velocity, with the final position unconstrained, is solved.

The most general transfer in a position independent gravitational field is represented by the state differential equation

$$\ddot{\underline{x}} = \underline{g} + \underline{A}_1 + \underline{A}_\ell + \underline{A}_3 \tag{3.1}$$

with the boundary conditions

$$\underline{x}(0) = \underline{x}_0 \qquad \underline{x}(t_f) = \underline{x}_0 + \frac{1}{2}\,\underline{g}\,t_f^{\,2} + \underline{V}_0\,t_f + \Delta\underline{x}$$

$$\dot{\underline{x}}(0) = \underline{V}_0 \qquad \dot{\underline{x}}(t_f) = \underline{V}_0 + \underline{g}\,t_f + \Delta\underline{V}$$

The components $\Delta\underline{x}$ and $\Delta\underline{V}$ of the final conditions represent the effect of the control accelerations. The equivalent problem for changes in those variables in field free space can be used without loss of generality. For notational convenience, the Δ's are dropped for the remaining discussion. Use

$$\ddot{\underline{x}} = \underline{A}_1 + \underline{A}_\ell + \underline{A}_3$$

$$\underline{x}(0) = \underline{0} \qquad \underline{x}(t_f) = \alpha c^3\,\underline{s} = \frac{ct_f}{\tau}\,\underline{s} \tag{3.2}$$

$$\dot{\underline{x}}(0) = \underline{0} \qquad \dot{\underline{x}}(t_f) = c\,\underline{V}$$

where

$\underline{s}$ = the dimensionless vector change in position

$\underline{V}$ = the dimensionless vector change in velocity

$$\tau = \frac{t_f}{\alpha c^2} = \text{the dimensionless time of flight} \tag{3.3}$$

and as before α and c are the parameters which define the low and high thrust propulsion systems.

3.1 General Transfers in field free space

A complete statement of the optimal control problem is given for the maximization of payload using the combination of propulsion modes for an arbitarary, fixed time transfer in field free space. The results of Chapter 2 are applied to yield a set of necessary conditions.

Assuming that only initial and final impulses are possible, the low thrust equations of motion can be integrated. The results of the integration are summarized and the resultant payload expressions for a change in position with $m_e = 0$ are given. For the maximization of the payload

$$m_\pi = \begin{cases} J_1^{\,2}\left[(K_1-L_1)^2 + m_e\right] & \text{if } m_e \leq m_p \\[2ex] \dfrac{J_1^{\,2}\,K_1^{\,2}}{1 + \frac{1}{m_e}\,L_1^{\,2}} & \text{if } m_e \geq m_p \end{cases}$$

where

$$m_p = L_1(K_1-L_1)$$

governed by the differential equations

$$\ddot{\underline{x}} = \underline{A}_1 + \underline{A}_\ell + \underline{A}_3$$

$$\dot{J} = -\frac{1}{2c}\,J\,|\underline{A}_3|$$

$$\dot{K} = -\frac{1}{2c}\,K\,|\underline{A}_1|$$

$$\dot{L} = \frac{\alpha}{4}\,\frac{K^2}{L}\,\underline{A}_\ell^{\,2}$$

subject to the boundary conditions

$$\underline{x}(0) = \underline{0} \qquad\qquad \underline{x}(t_f) = \frac{ct_f}{\tau}\,\underline{s}$$

$$\dot{\underline{x}}(0) = \underline{0} \qquad\qquad \dot{\underline{x}}(t_f) = c\,\underline{V}$$

$$J(0) = 1 \qquad K(0) = 1$$

$$L(0) = 0^+$$

the Hamiltonian

$$H = -\underline{\dot{\lambda}}^T \underline{\dot{x}} + \underline{\lambda}^T \underline{A}_\ell - \frac{1}{2Q} \underline{A}_\ell^{\,2} + \underline{\lambda}^T \underline{A}_1 - \delta_1 |\underline{A}_1| + \underline{\lambda}^T \underline{A}_3 - \delta_3 |\underline{A}_3|$$

must be maximized by the choice of the primer, $\underline{\lambda}$, and the controls, $\underline{A}$. The primer must satisfy the differential equation

$$\underline{\ddot{\lambda}} = \frac{\partial H}{\partial \underline{x}} = 0$$

and the optimal controls are

$$\underline{A}_\ell = Q(t)\ \underline{\lambda}(t)$$

$$\underline{A}_1 = \begin{cases} \infty\ \underline{\lambda}(t) & \text{if } |\underline{\lambda}(t)| = \delta_1(t) \\ 0 & \text{if } |\underline{\lambda}(t)| < \delta_1(t) \end{cases}$$

$$\underline{A}_3 = \begin{cases} \infty\ \underline{\lambda}(t_f) & \text{if } |\underline{\lambda}(t_f)| = \delta_3 \\ 0 & \text{if } |\underline{\lambda}(t_f)| < \delta_3 \end{cases}$$

where $Q(t)$, $\delta_1(t)$, and δ_3 are given in Chapter 2. If there are no intermediate impulses, $Q(t) = Q_1$, and we can compare $|\underline{A}_\ell(t)|$ with the thresholds

$$\delta_A(t) = Q_1\ \delta_1(t) \qquad (3.4)$$

$$\delta_A = \frac{1}{\alpha c} \begin{cases} \frac{L_1}{K_1}\left(1 - \frac{L_1}{K_1}\right)\left[1 + \frac{1}{m_p} L^2(t)\right] & \text{if } m_e \leq m_p \\ \frac{m_e}{K_1^{\,2}}\left[1 + \frac{1}{m_e} L^2(t)\right] & \text{if } m_e \geq m_p \end{cases}$$

$$\begin{aligned} \delta_B &= Q_1 \delta_3 \qquad (3.5) \\ &= \delta_A(0) + \frac{1}{\delta_A(0)}\left(\frac{L_1}{K_1}\right)^2 \frac{m_e}{K_1^{\,2}} \end{aligned}$$

to verify the local optimality of no intermediate impulses. Since $\underline{A}_\ell$, δ_A, and δ_B are related to the original variables by the same $Q(t)$, we can use Q_1 without loss of information.

The magnitude of $\underline{\lambda}$ is of particular importance since it must be equal to a threshold at the time of any impulse and less than the threshold during the remainder of the transfer. A possible solution for the primer differential equation is

$$\underline{\lambda} = \underline{a}\, t + \underline{b}$$

where $\underline{a}$ and $\underline{b}$ are constant vectors which remain to be chosen. The square of $\underline{\lambda}$

$$\underline{\lambda}^2 = \underline{a}^T\underline{a}\, t^2 + 2\, \underline{a}^T\underline{b}\, t + \underline{b}^T\underline{b}$$

can only have a minimum since the first derivative with respect to time

$$\frac{\partial \underline{\lambda}^2}{\partial t} = 2\underline{a}^T (\underline{a}\, t + \underline{b})$$

has only one possible zero and the second derivative

$$\frac{\partial^2 \underline{\lambda}^2}{\partial t^2} = 2\ \underline{a}^T \underline{a} \geq 0$$

is positive at that zero. Thus $\underline{\lambda}^2$ can have a maximum only at the initial and final times. If $m_e = 0$, since $\delta_1(0) = \delta_3$ and $\delta_1(t) \geq \delta_1(0)$, there can never be an intermediate impulse. As m_e increases, δ_3 increases. An extension of the previous logic indicates that if m_e is small enough, δ_3 will still be small enough so that $|\lambda(t)| < \delta_1(t)$ and there will be no intermediate impulses. It is believed that this conclusion will be true in general, but it has not been proven as yet.

If, by assumption, only initial and final impulses are allowed, the low thrust equations can be analytically integrated and the problem solved except for the specification of the magnitudes of the impulses. Those steps are carried out in Appendix C. The maximum payload is

$$m_\pi = \begin{cases} J_1^{\ 2} \left[(K_1 - L_1)^2 + m_e \right] & \text{if } m_e \leq m_p \\ \dfrac{J_1^{\ 2}\ K_1^{\ 2}}{1 + \dfrac{1}{m_e}\ L_1^{\ 2}} & \text{if } m_e \geq m_p \end{cases}$$

where

$$J_1 = \exp\left(-\frac{1}{2}\ v_2 \right)$$

$$K_1 = \exp\left(-\frac{1}{2}\ v_1 \right)$$

$$L_1 = \frac{K_1}{\sqrt{6\tau}} \sqrt{u_0^{\ 2} + w_0^{\ 2} + u_0\ w_0\ \underline{u}^T \underline{w}}$$

and the positive constants u_0, w_0, v_1, and v_2 are determined by the simultaneous solution of the four relations

$$u_0^{\,2} = \frac{1}{\det^2}\left[\frac{1}{36}\underline{V}^T\underline{V} + \left(\frac{1}{2} + \frac{v_2}{w_0}\right)^2 \frac{\underline{s}^T\underline{s}}{\tau^2} - \frac{1}{3\tau}\left(\frac{1}{2} + \frac{v_2}{w_0}\right)\underline{V}^T\underline{s}\right] \tag{3.6}$$

$$w_0^{\,2} = \frac{1}{\det^2}\left[\left(\frac{1}{3} + \frac{v_1}{u_0}\right)^2 \underline{V}^T\underline{V} + \left(\frac{1}{2} + \frac{v_1}{u_0}\right)^2 \frac{\underline{s}^T\underline{s}}{\tau^2}\right.$$

$$\left. - \frac{2}{\tau}\left(\frac{1}{3} + \frac{v_1}{u_0}\right)\left(\frac{1}{2} + \frac{v_1}{u_0}\right)\underline{s}^T\underline{V}\right] \tag{3.7}$$

$$\frac{u_0}{\tau} = \alpha c\,\delta_A \ \underline{\text{or}}\ (\ v_1 = 0\ \underline{\text{and}}\ \frac{u_0}{\tau} < \alpha c\delta_A) \tag{3.8}$$

$$\frac{w_0}{\tau} = \alpha c\delta_B \ \underline{\text{or}}\ (\ v_2 = 0\ \underline{\text{and}}\ \frac{w_0}{\tau} < \alpha c\delta_B) \tag{3.9}$$

where

$$\det = -\left(\frac{1}{12} + \frac{1}{3}\left(\frac{v_1}{u_0} + \frac{v_2}{w_0}\right) + \frac{v_1}{u_0}\frac{v_2}{w_0}\right)$$

From the nature of these four equations it is clear that their analytic solution will, in general, not be easy. If a final impulse is used, an analytic solution is not possible when $m_e \neq 0$ due to the existence of the exponential term, $K_1^{\,2}$, in δ_B (L_1/K_1 is an expression free of K_1). Analytic solutions will only be possible for pure high thrust, pure low thrust, or mixed thrust with $m_e = 0$, plus a few other special cases of minor importance.

<u>Change in position</u>

The transfer with $\underline{V} = 0$ is solved in Appendix C for $m_e = 0$. The four expressions for the payload and their regions of use are given below. Pure high thrust

$$m_\pi = \exp\left(-2\,\frac{s}{\tau}\right)$$

is used if $\tau \leq 6$. Mixed thrust

$$m_\pi = \frac{6}{\tau}\exp\left[-2\,\frac{s}{\tau} + 1 - \sqrt{\frac{\tau}{6}}\right]$$

is used for $6 \leq \tau \leq 6\left(1 + \frac{s}{\tau}\right)^2$. Pure low thrust is used if $\tau \geq 6\left(1 + \frac{s}{\tau}\right)^2$ and has two payload expressions dependent upon the size of m_e:

$$m_\pi = \begin{cases} \left(1 - \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}\right)^2 + m_e & \text{if } m_e \leq m_p \\ \dfrac{1}{1 + \frac{1}{m_e}\,\frac{6s^2}{\tau^3}} & \text{if } m_e \geq m_p \end{cases}$$

where

$$m_p = \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}\left(1 - \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}\right)$$

It is not possible to get the analytic solution for mixed thrust transfers for a non-zero m_e.

General transfers in field free space have a complete analytic solution only if $m_e = 0$. Otherwise the matching of $|\underline{A}_\ell|$ with the thresholds, δ_A and δ_B, must be done numerically. Since retention of a part, or all, of the power supply is of particular interest, study of problems which require a numerical solution will be deferred until more interesting transfers are considered.

3.2 A change in velocity in field free space

If the final position is not specified, intermediate or final impulses can never increase the payload. There is not a problem in matching the final threshold, δ_B, for non-zero m_e. Thus a more complete analytic solution is possible. However if $m_e \geq m_p$, δ_A can not be matched analytically. All cases except this one have an analytic expression for the payload. The coordinates for this problem can be chosen so that vector notation can be dropped, since all velocities and accelerations will be along the same coordinate. We wish to maximize the payload, m_π, subject to the previous differential equations for J, K, and L and also

$$\ddot{x} = A_1 + A_\ell + A_3$$

with the boundary conditions

$$\dot{x}(0) = 0 \text{ and } \dot{x}(t_f) = c\ V$$

and by assumtion V>0. The Hamiltonian

$$H = \lambda A_\ell - \frac{1}{2Q} A_\ell^{\ 2} + \lambda A_1 - \frac{\delta_A}{Q} |A_1| + \lambda A_3 - \frac{\delta_B}{Q} |A_3|$$

is maximized if λ satisfies the differential equation

$$\dot{\lambda} = - \frac{\partial H}{\partial x} = 0$$

and the optimal controls

$$A_\ell = Q_1\ \lambda\ (t)$$

$$A_1 = \begin{cases} \infty A_\ell(t) & \text{if } |A_\ell(t)| = \delta_A\ (t) \\ 0 & \text{if } |A_\ell(t)| < \delta_A(t) \end{cases}$$

$$A_3 = \begin{cases} \infty A_\ell(t_f) & \text{if } |A_\ell(t_f)| = \delta_B \\ 0 & \text{if } |A_\ell(t_f)| < \delta_B \end{cases}$$

are used, where Q, δ_A, and δ_B are given earlier.

If $|A_\ell|$ is equal to δ_A at the initial time, it can never be equal to $\delta_A(t) \geq \delta_A$ unless $A_\ell = L_1 = 0$. Further, $|A_\ell|$ can only be equal to δ_B if $m_e = 0$. In that case, a final impulse has the same effect upon the payload as an initial impulse. Thus for the mixed thrust problem, intermediate impulses are never optimal, and no approximation or compromise is made by allowing only initial impulses for this problem. The pure high thrust problem ($A_\ell = 0$) has a unique value for the payload, even though the optimal control is not unique. An initial impulse produces this optimal payload. For a change in velocity in field free space, intermediate or final impulses can never increase the payload, if initial impulses are allowed.

The derivations of the five appropriate payload expressions are in Appendix C. There is one expression for the pure high thrust payload, and two expressions for both mixed thrust and pure low thrust, depending upon the relative sizes of m_e and m_p. Table 3.1 gives the optimal expressions for m_π along with the regions of applicability based upon τ, m_e, and V. Figure 3.1 shows the payload as a function of τ for m_e = .05 (5% of the initial mass). Since the pure high thrust is applied impulsively, its payload is not a function of τ. However, the payload for the pure low thrust transfer increases with τ, since the required acceleration is lower. The mixed thrust payload is larger than either of the others. Note an improvement in payload even for times of flight for which the low thrust is not competitive ($\tau \leq 3.$) Figure 3.2 is a composite plot of the payload for different values of m_e. Similar results are obtained for other changes in velocity.

Some conclusions can be drawn about general mixed thrust transfers from the results obtained in this chapter. If the low thrust differential equations can be analytically integrated, the problem of mixed thrust transfers can be reduced to the relatively simple determination of the magnitude and timing of any impulses. Except for $m_e = 0$, and certain other special cases, the mixed thrust problem can not have a complete analytic solution due to the existence of transcendental functions in the necessary conditions. The largest improvement of the combination over the pure high thrust is for larger τ and m_e. The largest improvement over pure low thrust is for shorter τ and the largest improvement over either is for intermediate τ when the two modes are competitive.

m_π = payload	for these conditions on τ	on m_e
$\exp(-V)$	$0 \le \frac{\tau}{2} \le 1$	$m_e = 0$
$m_e + \frac{2}{\tau}\exp(-V + \sqrt{2\tau} - 2)$	$1 \le \frac{\tau}{2} \le \left(1 + \frac{V}{2}\right)^2$	$0 \le m_e \le \frac{2}{\tau}\left(\sqrt{\frac{\tau}{2}} - 1\right)\exp\left(-V + \sqrt{2\tau} - 2\right)$
$\dfrac{K_1^2}{1 + \frac{m_e}{K_1^2}\frac{\tau}{2}}$ where K_1 is the solution of $\frac{m_e \tau}{K_1^2} - \ln K_1^2 = V$	$1 \le \frac{\tau}{2} \le \left(1 + \frac{V}{2}\right)^2$	$\frac{2}{\tau}\left(\sqrt{\frac{\tau}{2}} - 1\right)\exp\left(-V + \sqrt{2\tau} - 2\right) \le m_e \le \frac{V}{\tau}$
	$0 < \frac{\tau}{2} \le 1$	$0 \le m_e \le \frac{V}{\tau}$
$\dfrac{1}{1 + \frac{1}{2m_e\tau}V^2}$	$0 \le \frac{\tau}{2} \le \left(1 + \frac{V}{2}\right)^2$	$\frac{V}{\tau} \le m_e$
	$\left(1 + \frac{V}{2}\right)^2 \le \frac{\tau}{2} < \infty$	$\frac{V}{\tau}\left(\sqrt{\frac{\tau}{2}} - \frac{V}{2}\right) \le m_e$
$m_e + \left(1 - \frac{V}{\sqrt{2\tau}}\right)^2$	$\left(1 + \frac{V}{2}\right)^2 \le \frac{\tau}{2} < \infty$	$0 \le m_e \le \frac{V}{\tau}\left(\sqrt{\frac{\tau}{2}} - \frac{V}{2}\right)$

Table 3.1 The optimum payload for a change in velocity in field free space

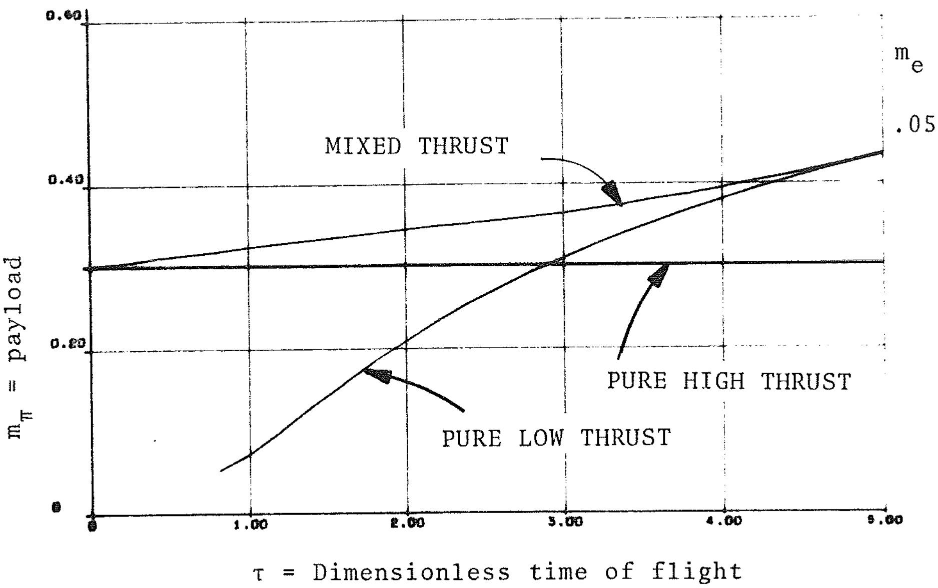


Figure 3.1 Payload vs. τ for a change in velocity in field free space (m_e = .05)

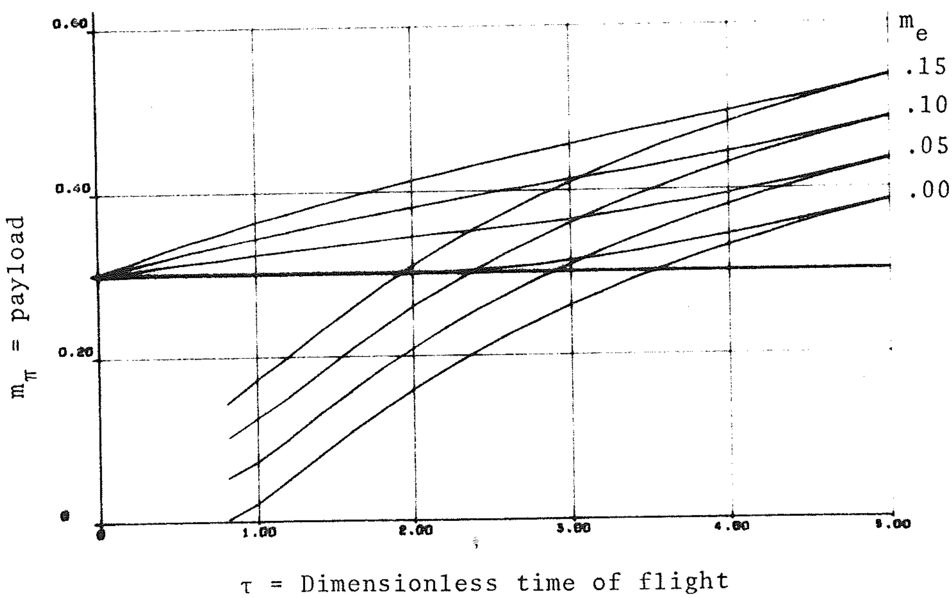


Figure 3.2 Payload vs. τ for a change in velocity in field free space (variable m_e)

Chapter 4

NECESSARY CONDITIONS FOR OPTIMAL TRANSFERS BETWEEN COPLANAR COAXIAL ELLIPTIC ORBITS

The nature of mixed thrust transfers between coplanar coaxial ellipses is approached by investigating the separate properties of pure high and pure low thrust transfers. Some special classes of pure low thrust transfers are discussed to enhance the understanding of the combination. For each class, optimum transfers and payloads are analytically obtained. The characteristics which contribute to the combination are discussed. The mixed thrust problem does not have an analytic solution due to the interplay of transcendental functions. The complete necessary conditions as derived in Appendix D are presented and discussed.

A set of six parameters which could describe a general three dimensional orbit in a central inverse square gravitational field are the size, shape, and orientation (three Euler angles) of the orbit, along with the position on the orbit. Of these six parameters, this class represents a change in only two, the size and shape of the orbit. Of particular importance is the fact that optimal three-dimensional transfers which only change these two parameters do not affect the Euler angles. Rendezvous, which specifies the position on the orbit, is another class of transfers not considered here.

Some practical assumptions are employed which constrain the transfers and lend the analysis to simplification. The gravitational attraction is assumed to be much stronger (by a factor of 100) than the low thrust acceleration. Thus during the low thrust phase of a transfer, the orbital parameters, which would be constant for no control, vary slowly. The transfers will be a slow spiral over many orbital periods. Typical ion engines are assumed to have a thrust on the order of 10^{-4} g's. The gravitational attraction is 100 times larger than that for transfers in earth orbit out to about 10 earth radii. Within this constraint imposed by the low thrust, pure high thrust transfers have a particularly easy solution.

Implicit in the assumption of a long low-thrust spiral over many orbits is the further assumption that half an orbital period is a negligible interval. The fixed time of flight for this problem formulation will be applied only to the low thrust phase. Initial or final impulses can occur at any time during the one orbital period before or after the low thrust phase. This makes these initial and final high thrust phases the equivalent of open time of flight transfers. For such transfers, impulses must be applied only at maxima of the primer. Otherwise, the primer would be larger than the threshold, which cannot occur on an optimal trajectory. It is shown in Appendix D that the primer for this class of transfers can have maxima only at the apses of the orbit. Thus optimal impulses can only occur at the apses.

Pure high thrust transfers have two impulses applied at opposite apses of a coasting orbit and pure low thrust transfers are a slow spiral over many orbital periods. However the form of the combination is not clear. Any impulses must be at an apse, but the number and timing during the transfer of these impulses must be determined, along with the parameters which define the intermediate low thrust phases. The timing of the impulses are assumed for this chapter and verified in the numerical analysis described in the next chapter. Two initial impulses are assumed at opposite apses of a coasting orbit prior to a single low thrust phase. After part of the low thrust power supply is dropped at the end of the transfer, another impulse is allowed. No intermediate impulses are allowed in this formulation, although the necessary conditions to verify the local optimality of this assumption are presented.

In order to analyze the problem, a convenient set of differential equations is necessary which applies both to high and low thrust accelerations. A classical set of perturbation equations is used. The properties of the corresponding set of costate differential equations are discussed in Appendix D, before using them in the solution for the mixed thrust necessary conditions. The pure high and low thrust solutions discussed in this chapter closely follow from the high and low thrust phases of the mixed thrust solution presented in Appendix D.

4.1 The Differential Equations of Motion

Classical perturbation differential equations as defined by Edelbaum[3] and others can be used to define the dynamics for the problem described in the previous section. These equations describe the change in orbital elements due to a small enough perturbing acceleration. In this case we will specify that acceleration in order to accomplish a desired transfer. Although these equations cannot be used for general impulsive accelerations, they are appropriate for any impulse which does not change the eccentric anomaly, E, of the orbit. As will be shown, this special case is indeed optimal, and these equations apply. The optimal control for the low thrust and necessary conditions for the high thrust are both related to the primer vector p. This vector will be specified for this problem and its properties discussed to complete the statement of the basic differential equations of motion for this problem.

Classical perturbation equations use semimajor axis, a, and eccentricity, e. For convenience of notation in further sections θ as defined by

$$\cos 2\theta = e$$

will be used instead of eccentricity. The derivatives of θ and e are simply related. The accelerations used are coordinatized for convenience in a rotating frame such that

$$\underline{A}_i = \begin{Bmatrix} A_{T_i} \\ \\ A_{R_i} \end{Bmatrix}$$

where A_R is in the radial direction from the center of the force field and A_T is perpendicular to it, in the plane of motion.

For the state vector

$$\underline{x} = \begin{Bmatrix} a \\ \\ \theta \end{Bmatrix} \tag{4.1}$$

and the general acceleration $\underline{A}$, the state differential equation is given by

$$\dot{\underline{x}} = \underline{B}(\underline{x},t)\underline{A}$$

where

$$\underline{B}(\underline{x},t) = \frac{\sqrt{\frac{a}{\mu}}}{1-e \cos E} \begin{bmatrix} 2a \sin 2\theta & 2 a e \sin E \\ \\ \frac{1}{2}\left(e+e \cos^2 E-2 \cos E\right) & -\frac{1}{2} \sin 2\theta \sin E \end{bmatrix} \tag{4.2}$$

and the differential equation for E during any one orbit is

$$\frac{dE}{dt} = \frac{\sqrt{\frac{\mu}{a^3}}}{1-e \cos E} \tag{4.3}$$

which gives

$$E - e \sin E = \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$$

if a and e are sufficiently constant. The transfers being considered have only a small low-thrust acceleration, or an impulse which does not change E. Thus a and e change slowly enough, or E is not changed, by assumption.

The optimal control for this state is related to the primer vector given by

$$\underline{p} = \underline{B}^T \underline{\lambda}$$

for this state differential equation. The time history of $\underline{\lambda}$ will be discussed in future sections. In terms of the true anomaly, f, which is related to E by

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \tag{4.4}$$

the primer can be written as

$$\underline{p} = \begin{Bmatrix} C(1+e \cos f) + \dfrac{D}{1+e \cos f} \\ \\ C\, e \sin f \end{Bmatrix} \tag{4.5}$$

where

$$C = 2\sqrt{\frac{a}{\mu}} \left[\frac{\lambda_1 a}{\sin 2\theta} + \frac{\lambda_2}{2 \cos 2\theta} \right] \tag{4.6}$$

$$D = -\sqrt{\frac{a}{\mu}}\ \lambda_2 \tan 2\theta \sin 2\theta \tag{4.7}$$

The primer, like the acceleration, is coordinatized in a rotating frame. Taking this into account, the derivative of the primer is

$$\frac{d\underline{p}}{dt} = -\sqrt{\frac{\mu}{R^3}} \begin{Bmatrix} \dfrac{D\ e\ \sin f}{(1+e\cos f)^{3/2}} \\ \\ C\sqrt{1+e\cos f} + \dfrac{D}{\sqrt{1+e\cos f}} \end{Bmatrix}$$

For this derivative, all parameters except f are assumed to be constant or very slowly varying.

The costates λ_1, and λ_2 must then be chosen so that the desired transfer is accomplished and any necessary conditions are satisfied. The following analysis assumes C and D are slowly varying or constant and is not concerned with their value. Only the general properties of the primer on an optimal trajectory are desired.

Impulses are optimally applied when the magnitude of the primer is a maximum and equal to a threshold δ. Also at an impulse, the primer and its first two derivatives are continuous. Assuming C and D are approximately constant over the period of any one orbit, the magnitude of the primer

$$\underline{p}^2 = \underline{p}^T\underline{p} = [C(1+e^2) + 2C\ D] + 2C^2x + \frac{D^2}{(1+x)^2}$$

where

$$x = e \cos f$$

can be a maximum only when its first derivative is zero

$$\frac{\partial \underline{p}^2}{\partial f} = 0 = 2\left[C^2 - \frac{D^2}{(1+x)^3}\right](-e \sin f)$$

and the second derivative is negative.

$$\frac{\partial^2 \underline{p}^2}{\partial f^2} = 2\left[C^2 - \frac{D^2}{(1+x)^3}\right](-x) + \frac{6D^2}{(1+x)^4}(-e \sin f)^2$$

One possible maximum occurs when the bracketed term equals zero. This solution is

$$x_0 = \left(\frac{D}{C}\right)^{2/3} - 1$$

or

$$f = \cos^{-1}\left(\frac{1}{e} x_0\right)$$

For this solution, the second derivative is always positive and therefore it represents a minimum. The existence of the inverse cosine is thus of no interest here.

Two other possible solutions occur when

$$\sin f = 0$$

The second derivative can be negative for these solutions and they thus represent possible maxima. Note that E and f are identical at these possible maxima

$$f = E = 0$$

or

$$f = E = \pi$$

When the eccentricity, e, is zero, the primer has a constant magnitude since all of its time derivatives are zero.

For convenience of notation define a signum function T such that

$$T = \begin{cases} +1 \text{ at apoapse} \\ \\ -1 \text{ at periapse} \end{cases} \tag{4.8}$$

Note that $T = -\cos f = -\cos E$ at the possible maxima of the primer. Thus

$$\underline{p} = \begin{Bmatrix} p \\ \\ 0 \end{Bmatrix} \qquad \frac{d\underline{p}}{dt} = -\sqrt{\frac{\mu}{R^3}} \begin{Bmatrix} 0 \\ \\ p' \end{Bmatrix}$$

with

$$p = C(1-Te) + \frac{D}{1-Te} \tag{4.9}$$

$$p' = C\sqrt{1-Te} + \frac{D}{\sqrt{1-Te}} \tag{4.10}$$

when the magnitude of the primer is possibly a maximum.

If p is a maximum and equal to a threshold δ, an impulse is optimal at that apse. The impulse is applied in the direction of the primer which in this case is tangential. Since the orbital velocity is also tangential at an apse, such an optimal impulse only changes the altitude of the other apse and not E. Thus the earlier differential equations are indeed applicable for optimal impulses at an apse. A further necessary condition across an impulse is that the primer and its first two derivatives are continuous. In this notation, p and p' will be constant. For an impulse at an apse, C and D will change in a predictable manner dependent only upon T and the change in e.

Having established the importance and characteristics of the primer for this class of transfers, it is now possible to determine the nature of optimal low, high and mixed thrust transfers. Much of the algebraic gaps of the following sections are covered in the complete derivations of Appendix D.

4.2 Pure low thrust transfers

The complete solution for optimal low thrust transfers between coplanar coaxial ellipses is presented in this section. Basic to the problem is the assumption on the magnitude of the control acceleration, $\underline{A}_\ell$. $\underline{A}_\ell$ is so small that the secular change in the elements a and θ during any one orbit is small. The transfers will thus spiral over a long time of flight from the initial to the final orbit. Since the periodic changes in a and θ are also small and of little interest, their variation can be eliminated by an averaging process. The differential equation for the remaining secular variations in elements can then be solved. The general time history of the state, primer and payload for this class of transfers is given and discussed, especially in its relationship to the mixed thrust transfers.

The classical optimization problem for the maximization of payload during a low thrust transfer maximizes the integral

$$- \frac{1}{2} \int_0^{t_f} \underline{A}_\ell{}^2 dt$$

For the differential equation

$$\underline{\dot{x}} = \underline{B}(\underline{x},t)\underline{A}_\ell \qquad \begin{matrix} \underline{x}(0) = \underline{x}_0 \\ \underline{x}(t_f) = \underline{x}_f \end{matrix}$$

the Hamiltonian as given by

$$H = \underline{\lambda}^T \underline{B}\ \underline{A}_\ell - \frac{1}{2} \underline{A}_\ell{}^2$$

carries the full information about the dynamics on the optimum trajectory and maximization of H maximizes the integral given above. $\underline{A}_\ell$ is chosen to maximize H by

$$\underline{A}_\ell = \underline{B}^T \underline{\lambda} = \underline{p}$$

for this problem the low thrust acceleration is identically equal to the primer vector. Thus we have

$$H = \frac{1}{2} \underline{p}^T \underline{p}$$

The dynamics for this problem are given by the canonical differential equations

$$\dot{\underline{x}} = \frac{\partial H^T}{\partial \underline{\lambda}} = \underline{B}\ \underline{B}^T \underline{\lambda}$$

$$\dot{\underline{\lambda}}^T = -\frac{\partial H}{\partial \underline{x}} = -\underline{\lambda}^T \frac{\partial \underline{B}}{\partial \underline{x}} \underline{B}^T \underline{\lambda}$$

where care must be taken with the third order tensor notation in the term $\partial \underline{B}/\partial \underline{x}$. Since the initial and final conditions on $\underline{x}$ are specified, the boundary conditions on $\underline{\lambda}$ must be chosen to accomplish the desired transfer.

These equations contain the complicating periodic variations in elements which are assumed to be small. Taking the time average of H over the period of any arbitrary orbit will eliminate all such periodic terms in H and also in the differential equations implied by H. Let H_1 be the averaged Hamiltonian obtained by

$$H_1 = \frac{1}{t_p} \int_t^{t+t_p} H \, dt$$

$$H_1 = \frac{1}{2\pi} \int_E^{E+2\pi} H(1-e \cos E)\, dE$$

As shown in Appendix D, the result of this operation is

$$H_1 = \frac{1}{2} \frac{a}{\mu} [4(\lambda_1 a)^2 + \frac{5}{8} \lambda_2^2]$$

The parameters $\underline{\lambda}$ and $\underline{x}$ were assumed constant during any one orbit and they thus now represent the slow secular variation in $\underline{\lambda}$ and $\underline{x}$. From general optimal control theory we also know that

$$\frac{dH_1}{dt} = \frac{\partial H_1}{\partial t} = 0$$

Thus on a low thrust trajectory, the averaged Hamiltonian is a constant. Thus we also know that the average of the acceleration squared is constant.

The differential equations for the secular variation in parameters can be obtained, as before, but now from the averaged Hamiltonian

$$\frac{da}{dt} = \frac{\partial H_1}{\partial \lambda_1} = \frac{4}{\mu} \lambda_1 a^3$$

$$\frac{d\theta}{dt} = \frac{\partial H_1}{\partial \lambda_2} = \frac{5}{8\mu} \lambda_2 a$$

$$\frac{d\lambda_1}{dt} = - \frac{\partial H_1}{\partial a} = - \frac{1}{a} H_1 - \frac{4}{\mu} (\lambda_1 a)^2$$

$$\frac{d\lambda_2}{dt} = - \frac{\partial H_1}{\partial \theta} = 0$$

As shown in Appendix D, for the similar low thrust phase of a mixed thrust transfer, the equations can be completely integrated analytically. The results are given for the transfer between

$$a(0) = a_2 \qquad a(t_f) = a_3$$

$$\theta(0) = \theta_2 \qquad \theta(t_f) = \theta_3$$

in terms of the parameters

$$\gamma = \sqrt{\frac{a_2}{a_3}}$$

$$\psi = \sqrt{\frac{8}{5}}\,(\theta_3 - \theta_2)$$

$$h = 1 - 2\,\gamma\,\cos\psi + \gamma^2$$

$$\beta_2 = \sqrt{\frac{\mu}{4c^2 a_2}} = \text{dimensionless gravitational constant}$$

$$\tau = \frac{t_f}{\alpha c^2} = \text{dimensionless time of flight} \tag{4.11}$$

The maximum payload is

$$m_\pi = \begin{cases} (1-L_1)^2 + m_e & m_e \leq m_p \\[2ex] \dfrac{1}{1 + \dfrac{1}{m_e} L_1^2} & m_e \geq m_p \end{cases}$$

with

$$m_p = L_1(1-L_1)$$

$$L_1^2 = \beta_2^2 \frac{2h}{\tau}$$

During the thrusting phase

$$L^2(t) = L_1^2 \frac{t}{t_f}$$

$$\frac{a_2}{a(t)} = \left[1 - (1-\gamma \cos \psi) \frac{t}{t_f}\right]^2 + \left[\gamma \sin \psi \frac{t}{t_f}\right]^2 \tag{4.12}$$

$$\theta(t) = \theta_2 + \sqrt{\frac{5}{8}} \sin^{-1}\left[\sin \psi \, \gamma \sqrt{\frac{a(t)}{a_2}} \frac{t}{t_f}\right] \tag{4.13}$$

$$\underline{p}(t) = \begin{Bmatrix} C(t)(1+e \cos f) + \dfrac{D(t)}{1+e \cos f} \\ \\ C(t)\, e \sin f \end{Bmatrix}$$

where

$$C(t) = \frac{1}{t_f} \sqrt{\frac{\mu}{a_2}} \sqrt{\frac{a(t)}{a_2}} \left\{ \frac{(1-\gamma \cos \psi) - h \dfrac{t}{t_f}}{\sin 2\theta} + \sqrt{\frac{8}{5}} \frac{\gamma \sin \psi}{\cos 2\theta} \right\} \tag{4.14}$$

$$D(t) = -\frac{1}{t_f} \sqrt{\frac{\mu}{a_2}} \sqrt{\frac{a(t)}{a_2}} \gamma \sin \psi \tan 2\theta \sin 2\theta \tag{4.15}$$

From the differential equation in θ, it is clear that θ progresses monotonically from its initial to its final value. The semi-major axis, a, will progress monotonically from its initial to its final value only if the change in eccentricity is small enough. Otherwise there can be a maximum value reached for a. The second derivative of a is always negative.

Of course, the earlier generalizations about the maxima of the primer on any one orbit still hold. But, unfortunately, the time history of these maxima of the primer is too complex for any generalizations which might lead to the understanding of the mixed thrust transfer. Although it seldom forms a part of a mixed thrust transfer, there is one special case which has a significant simplification. When the initial and final eccentricities are equal, $\psi = 0$, and the eccentricity is constant during the entire transfer. For this case, a(t) simplifies to

$$\sqrt{\frac{a_2}{a(t)}} = 1 - (1-\gamma)\ \frac{t}{t_f}$$

The average orbital velocity, $\sqrt{\frac{\mu}{a}}$, progresses linearly from its initial to its final level.

Also for this case, the coefficients which define the primer are constant during the entire transfer

$$C = \frac{1}{t_f}\ \sqrt{\frac{\mu}{a_2}}\ \frac{1 - \gamma}{\sin 2\theta}$$

$$D = 0$$

Since D is zero, the primer has a minimum at apoapse ($f = \pi$) and a maximum only at periapse ($f = 0$). Also this maximum is constant from orbit to orbit, since C is constant. Further, when the initial and final orbits are circular, the radial component of the primer is always zero. Thus the primer is a constant in the tangential direction. Unfortunately these results are not generally applicable since the magnitude of the primer is very sensitive to small changes in θ and ψ.

4.3 Pure high thrust transfers

Within the constraints imposed by the low thrust assumptions, optimal pure high thrust transfers have an easy solution which is well documented. The level of the low thrust with respect to gravity restricts the transfers to a maximum change in radius of 10. Further, since the low thrust transfer is a slow spiral, the time of one orbit is a small fraction of the transfer times being considered. Thus high thrust transfers can be approached as if the time of flight were free. For this case a Hohmann transfer is optimal, with two impulses applied at the opposite apses of a coasting orbit. For short time rendezvous, or changes in radius $R_f/R_0 > 11.94$, three or more impulses can be optimal, but these cases are excluded by the low thrust assumptions.

Pure high thrust transfers will be analyzed from an optimal control point of view. The conditions on the primer which contribut to the optimality of the transfer will be discussed. Many of the conditions described are also appropriate for mixed thrust transfers. The timing (at an apse) and direction (tangential) of any impulses is defined by the discussion of the primer in a previous section. With this knowledge, high thrust transfers can be analyzed from the basis of velocity changes. That approach is taken in Appendix D for the discussion of the high thrust phase of a mixed thrust transfer. This section will use the primer directly to determine the optimal high thrust transfer between coplanar coaxial orbits. The primer, payload and optimal trajectory will all be defined.

The classical payload maximization problem for high thrust transfers maximizes the integral

$$-\int_0^{t_f} |\underline{A}_1| dt$$

for the state differential equation

$$\dot{\underline{x}} = \underline{B}(\underline{x},t)\underline{A}_1$$

The Hamiltonian as given by

$$H = \underline{\lambda}^T\underline{B}\ \underline{A}_1 - |\underline{A}_1|$$

is maximized by the choice

$$\underline{A}_1 = \begin{cases} \infty\underline{p} & \text{if } |\underline{p}| = 1 \\ 0 & \text{if } |\underline{p}| < 1 \end{cases}$$

Note that for this transfer

$$H = 0$$

Since A_1 is optimally impulsive at the maxima of $\underline{p}$ and zero for the rest of the transfer, $\underline{x}$ and $\underline{\lambda}$ are constant except at impulses. In the nomenclature of this chapter C and D which specify the primer are also constant except at an impulse. As shown in Chapter 2, the primer and its first two derivatives are continuous at an impulse. These conditions specify the changes in C and D after an impulse, in terms of C and D before the impulse.

For the transfer between the orbits

$$a(0) = a_0 \qquad a(t_f) = a_2$$

$$\theta(0) = \theta_0 \qquad \theta(t_f) = \theta_2$$

the optimum transfer has two impulses applied at the opposite apses of the coasting orbit specified by a_1 and θ_1. The coast orbit will be between the greater apoapse and the periapse of the other orbit. For T_0 such that

$$T_0 = \begin{cases} +1 & \text{if } a_2(1+e_2) > a_0(1+e_0) \\ -1 & \text{if } a_2(1+e_2) < a_0(1+e_0) \end{cases}$$

the radii at each impulse is given by

$$R_1 = a_0(1-T_0e_0) = a_1(1-T_0e_1)$$

$$R_2 = a_2(1+T_0e_2) = a_1(1+T_0e_1)$$

From this condition, the parameters of the coasting orbit are

$$a_1 = \frac{1}{2}(R_1 + R_2)$$

$$T_0e_1 = \frac{R_2 - R_1}{R_2 + R_1}$$

A necessary condition for the application of impulses is that the primer be equal to 1 at each impulse and less than 1 at all other times. As shown in section 4.1, the primer can only have maxima at an apse.

For this transfer, the primer at each impulse is

$$p_1 = 1 = C_1(1+T_0e_1) + \frac{D_1}{1+T_0e_1}$$

$$p_2 = 1 = C_1(1-T_0e_1) + \frac{D_1}{1-T_0e_1}$$

equal to one. C_1 and D_1, chosen to satisfy this set of conditions, are

$$C_1 = \frac{1}{2}$$

$$D_1 = \frac{2R_1R_2}{(R_1+R_2)^2}$$

The value of the primer at the opposite apse before the first impulse and at the opposite apse after the second impulse must each be less than 1 for this transfer to be optimal. As shown in section 4.1, p' as given by equation 4.10 and the primer p must be continuous across each impulse. These two equations can be used to find the appropriate C's and D's on the initial and final orbits, from which the primer at the opposite apses can be determined. After some algebra, the primer at the opposite apse before the first impulse is

$$p_0^- = -\left(\frac{3-T_0e_0}{1-T_0e_f}\right)p_1 + \frac{4}{\sqrt{1+T_0e_0}\,(1-T_0e_0)}\,p_1'$$

and at the opposite apse after the final impulse, the primer is

$$p_2^+ = -\left(\frac{3+T_0e_f}{1+T_0e_f}\right)p_2 + \frac{4}{\sqrt{1-T_0e_f}\,(1+T_0e_f)}\,p_2'$$

where

$$p_1' = \frac{1}{2}\sqrt{\frac{2R_1}{R_1+R_2}}\left[\frac{R_1+3R_2}{R_1+R_2}\right]$$

$$p_2' = \frac{1}{2}\sqrt{\frac{2R_2}{R_1+R_2}}\left[\frac{R_2+2R_1}{R_1+R_2}\right]$$

For this transfer to be optimal, p_0^- and p_2^+ must both be less than 1.

The payload for this transfer

$$m_\pi = K_1{}^2$$

where

$$K_1 = \exp\left[-\frac{1}{2C}\sum |\Delta V|\right]$$

is (locally) maximized when the previously described necessary conditions are satisfied).

4.4 Necessary conditions for mixed thrust transfers

Pure high and pure low thrust transfers have complete analytic solutions within the assumptions of this chapter. Only the necessary conditions are derivable for the combination of engines. Appendix D contains the analytic derivations of these conditions. Those results are presented and discussed in this section.

During the low thrust phase, the scaling on the primer is still chosen so that $\underline{A}$ is adequate to accomplish that portion of the transfer. However the threshold which indicates the time to use the high thrust is no long simply given, nor is the scaling of the primer arbitrary for the high thrust phase. Since the matching of the primer with the thresholds involves square roots, exponentials, and trigonometric functions, it is not possible to analytically satisfy the necessary conditions. The numerical techniques used for that purpose are given in the next chapter.

The necessary conditions presented in this section are for a specific assumed mode for a mixed thrust transfer. There are two high thrust phases separated by a single low thrust phase. The first high thrust phase allows two impulses at opposite apses of a coasting orbit. Since half an orbital period is assumed to be a very short time, the effect of the low thrust, if used during this interval, is ignored. At the final time, only one impulse is allowed. The low thrust engine is used to spiral between the two high thrust phases. Further necessary conditions will be described which verify the local optimality of these assumptions.

Figure 4.1 shows two optimal trajectories for which two impulses are used. The pertinent orbits are labeled by the θ_i for that orbit. The numbers at the apses indicate the subscript on the primer for that apse. Below the figure is a table of the values for the primer and the thresholds after they have been normalized by a convenient constant. For the first transfer, since only one initial impulse is used

$$|p_2| = \delta_1(0)$$

and $|p_1|$ and $|p_2^+|$ are less than $\delta_1(0)$. There is no "first" impulse. Since there is a final impulse

$$|p_3| = \delta_3$$

and $|p_f|$ and $|p_3^-|$ are both less than δ_3. For the second transfer

$$|p_1| = |p_2| = \delta_1(0)$$

$$|p_3| < \delta_3$$

Also $|p_0|$ and $|p_2^+|$ are less than $\delta_1(0)$, and $|p_f|$ is less than δ_3. During the low thrust phase for both transfers

$$|p(t)| < \delta_1(t)$$

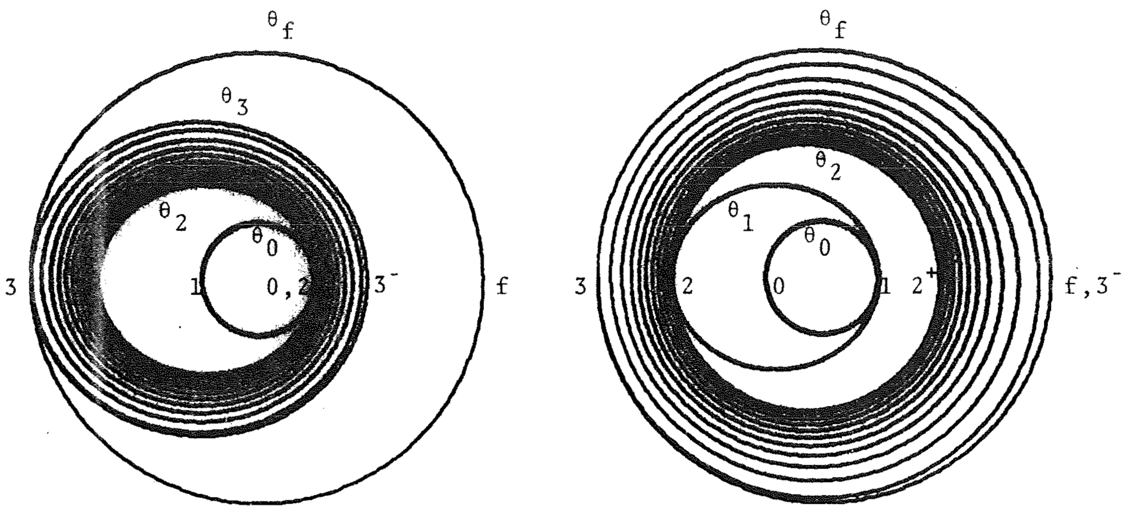


Figure 4.1 The apses of interest on optimal trajectories

For the first transfer		For the second transfer	
$\delta_1 = 1.142$	$\delta_3 = 1.146$	$\delta_1 = 1.186$	$\delta_3 = 1.443$
$p_2 = 1.142$	$p_3 = 1.146$	$p_2 = 1.186$	$p_3 = 1.360$
$p_1 = .669$	$p_f = .436$	$p_1 = 1.186$	$p_f = .529$
$p_0 = 1.142$	$p_3^- = .814$	$p_0 = .846$	$p_3^- = .529$
$p_2^+ = 1.011$		$p_2^+ = .759$	

Table 4.1 The primer and threshold at the apses of interest

Analytically, the expressions for the primer at the appropriate apses are determined by the sequence of calculations

$$C_2 = \frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} (1-\gamma \cos \psi)$$

$$D_2 = -\frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} \left(\sqrt{\frac{8}{5}}\, \gamma \sin \psi \right)$$

$$p_2 = T_0 (C_2 \frac{G_2}{F_2} + D_2)$$

$$p_2^+ = T_0 (C_2 \frac{F_2}{G_2} - D_2)$$

$$p_2' = C_2 \frac{1}{F_2} + D_2 G_2$$

$$p_1 = -(1 + \frac{1}{F_1{}^2}) p_2 + \frac{1}{F_1{}^2 T_0 G_1} p_2'$$

$$p_1' = -\frac{2}{F_1} p_2 + \frac{1+G_1{}^2}{F_1 T_0 G_1} p_2'$$

$$p_0 = -(1 + \frac{1}{G_0{}^2}) p_1 + \frac{1}{G_0{}^2 F_0} p_1'$$

At the final time for

$$C_3 = \frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} (\cos \psi - \gamma)$$

$$D_3 = -\frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} \left(\sqrt{\frac{8}{5}} \sin \psi \right)$$

$$p_3 = T_f (C_3 \frac{G_3}{F_3} + D_3)$$

$$p_3^- = T_f (C_3 \frac{F_3}{G_3} - D_3)$$

$$p_3' = C_3 \frac{1}{F_3} + D_3 G_3$$

$$p_f = -\left(1 + \frac{1}{F_f^{\,2}}\right) p_3 + \frac{1}{F_f^{\,2} T_f G_f}\, p_3'$$

During the low thrust phase, for

$$C(t) = \frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} \sqrt{\frac{a(t)}{a_2}} \left((1 - \gamma \cos \psi) - h \frac{t}{t_f}\right)$$

$$D(t) = \sqrt{\frac{a(t)}{a_2}}\; D_3$$

the primer is

$$p(t) = T\left(C(t) \frac{G(\theta)}{F(\theta)} + D(t)\right)$$

The thresholds are given by

$$\delta_1(t) = \frac{1}{c} \begin{cases} J_1^{\,2} (K_1 - L_1)^2 \left[1 + \frac{1}{m_p} L_1^{\,2} \frac{t}{t_f}\right] & \text{if } m_e \leq m_p \\[2ex] \dfrac{m_\pi}{1 + \frac{1}{m_e} L_1^{\,2}} \left[1 + \frac{1}{m_e} L_1^{\,2} \frac{t}{t_f}\right] & \text{if } m_p \geq m_p \end{cases}$$

$$\delta_3 = \frac{1}{c}\, m_\pi = \frac{1}{c} \begin{cases} J_1^{\,2} \left[(K_1 - L_1)^2 + m_e\right] & \text{if } m_e \leq m_p \\[2ex] \dfrac{J_1^{\,2} K_1^{\,2}}{1 + \frac{1}{m_e} L_1^{\,2}} & \text{if } m_e \geq m_p \end{cases}$$

$$m_p = L_1 (K_1 - L_1)$$

and as derived in Appendix D

$$J_1 = \exp\left[-S_3 \frac{\beta_3}{F_3} (G_f - G_3)\right] \tag{4.16}$$

$$K_1 = \exp\left[-S_1 \frac{\beta_0}{G_0} (F_1 - F_0) - S_2 \frac{\beta_2}{F_2} (G_2 - G_1)\right] \tag{4.17}$$

$$L_1^{\,2} = K_1^{\,2}\ \beta_2^{\,2}\ \frac{2h}{\tau} \tag{4.18}$$

Thus the necessary conditions for the maximization of payload are

$$|p_1| = \delta_1 \qquad \text{or} \qquad \begin{cases} \theta_1 = \theta_0 \\ |p_1| < \delta_1 \end{cases}$$

$$|p_2| = \delta_1 \qquad \text{or} \qquad \begin{cases} \theta_2 = \theta_1 \\ |p_2| < \delta_1 \end{cases}$$

$$|p_3| = \delta_3 \qquad \text{or} \qquad \begin{cases} \theta_3 = \theta_f \\ |p_3| < \delta_3 \end{cases}$$

and

$$|p_0| < \delta_1 \qquad\qquad |p_f| < \delta_3$$

$$|p_2^+| < \delta_1 \qquad\qquad |p_3^-| < \delta_3$$

$$p(t) < \delta_1(t)$$

The first three equations must be satisfied by the choice of θ_1, θ_2, and θ_3. The final set of inequalities must be satisfied to verify the optimality of the assumptions on the timing of the impulses.

If all of these necessary conditions are satisfied, the trajectory will (locally) maximize the payload. A method for the numerical determination of the θ's which satisfy these conditions is presented in the next chapter, along with the numerical results.

Chapter 5

NUMERICALLY DETERMINED OPTIMAL COPLANAR COAXIAL TRANSERS

The necessary conditions derived in Chapter 4 provide a verification of the (local) optimality of a transfer, but do not lead to a convenient numerical iteration. The payload derived in Chapter 4 as a function of the three free parameters

$$\underline{\theta} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \tag{5.1}$$

and its first two derivatives with respect to the $\underline{\theta}$'s as presented in Appendix D are used in the numerical iteration. As discussed in Chapter 4, the iteration assumes that two initial impulses (at the opposite apses of a coasting orbit) and one final impulse can be used. Additional necessary conditions were presented which can be used to numerically verify the local optimality of these assumptions. The timing of the impulses was picked using the knowledge of optimal pure high, and pure low thrust transfers and an assumption on the manner of their combination. The numerical properties of the maximum payload are discussed before describing the numerical techniques used to arrive at that optimum. For examples, transfers are considered between seven sets of initial and final orbits. The payload improvements and optimum trajectories are discussed. The assumed timing of the impulses was verified as being locally optimal for all transfers which were studied.

5.1 Numerical definition of the maximum payload

The payload expression of Chapter 4 must be numerically maximized by a choice of the free parameters $\underline{\theta}$. Before describing the numerical iteration used to reach that optimum value, the numerical properties of the maximum are presented. The payload expression

$$m_\pi = \begin{cases} J_1{}^2[(K_1-L_1)^2+m_e] & \text{if } m_e \leq m_p \\ \dfrac{J_1{}^2K_1{}^2}{1+\frac{1}{m_e}L_1{}^2} & \text{if } m_e \geq m_p \end{cases}$$

is explicitly expressed as a function of the free paramters $\underline{\theta}$ using equation 4.16 for J_1, 4.17 for K_1, and 4.18 for L_1. It is noted that J_1 is a function of θ_3 and that K_1 is a function only of θ_1 and θ_2. If

$$\delta\underline{\theta} = \begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{Bmatrix}$$

is an arbitrary small variation from the optimum $\underline{\theta}$, the Taylor series for the payload about the optimum is

$$m_\pi(\underline{\theta}+\delta\underline{\theta}) = m_\pi(\underline{\theta}) + \underline{g}^T\delta\underline{\theta} + \frac{1}{2}\,\delta\underline{\theta}^T\underline{G}\delta\underline{\theta} + \ldots$$

where all derivatives are evaluated at the optimum value for $\underline{\theta}$. The series converges if all derivatives are finite and $\delta\underline{\theta}$ is small enough so that the higher order derivatives can be neglected. The explicit expressions for the first derivative vector

$$\underline{g} = \frac{\partial m_\pi}{\partial \underline{\theta}} = \begin{Bmatrix} \dfrac{\partial m_\pi}{\partial\theta_1} \\ \dfrac{\partial m_\pi}{\partial\theta_2} \\ \dfrac{\partial m_\pi}{\partial\theta_3} \end{Bmatrix} \qquad (5.2)$$

and the second derivative matrix

$$\underline{G} = \frac{\partial \underline{g}^T}{\partial \underline{\theta}} \tag{5.3}$$

are given in Appendix D. For this $\underline{\theta}$ to (locally) maximize the payload, it is necessary that each component of the term

$$\underline{g}^T \delta\underline{\theta} = g_1\delta\theta_1 + g_2\delta\theta_2 + g_3\delta\theta_3$$

be equal to zero. Otherwise a small variation

$$\delta\theta_i = kg_i$$

for k small enough, could increase the payload. If $\underline{g}^T\delta\underline{\theta}$ is zero, it is sufficient that the condition

$$\delta\underline{\theta}^T\underline{G}\ \delta\underline{\theta} < 0$$

be satisfied at a maximum for any non-zero choice of the vector $\delta\underline{\theta}$. when this inequality is satisfied, it is said that $\underline{G}$ is negative definite and is often indicated by

$$\underline{G} < 0$$

From the definition in Appendix D of the signums S_i, we must have

$S_1(\theta_1-\theta_0) > 0$ or $\theta_1 = \theta_0$ (no first impulse)

$S_2(\theta_2-\theta_1) > 0$ or $\theta_2 = \theta_1$ (no second impulse)

$S_3(\theta_f-\theta_3) > 0$ or $\theta_3 = \theta_f$ (no final impulse)

These conditions constrain the possible choices of $\delta\underline{\theta}$. For example if $\theta_3 = \theta_f$, $\delta\theta_3$ must satisfy $S_3\delta\theta_3 < 0$ from the above relationship. But if $S_3 g_3 > 0$, an allowed $\delta\theta_3$ cannot increase the payload. Thus $\delta\theta_3 = 0$. Unless $\delta\theta_1 = 0$, there are problems if we must have $\delta\theta_2 = \delta\theta_1$. The constraint $\theta_1 = \theta_2$ really implies a change in the mode of the transfer. Since the change in mode is covered by other initial conditions, the iteration is terminated for this mode if $\theta_1 \neq \theta_0$, $\theta_2 = \theta_1$, and $S_2 g_2 < 0$. Otherwise if one of the θ's is specified as indicated above on the right hand side, the corresponding $\delta\theta$ must be zero, unless of course, an allowed $\delta\theta$ would satisfy the left hand inequality. Thus the condition $\underline{g}^T\delta\underline{\theta} = 0$ can be satisfied either by

$$g_1 = 0 \qquad \underline{\text{or}} \quad \delta\theta_1 = 0 \quad (S_1 g_1 < 0 \ \underline{\text{and}} \ \theta_1 = \theta_0)$$

$$g_2 = 0 \qquad \underline{\text{or}} \quad \delta\theta_2 = \delta\theta_1 (S_2 g_2 < 0 \ \underline{\text{and}} \ \theta_2 = \theta_1)$$

$$g_3 = 0 \qquad \underline{\text{or}} \quad \delta\theta_3 = 0 \quad (S_3 g_3 > 0 \ \underline{\text{and}} \ \theta_3 = \theta_f)$$

In appendix D, it is shown that if

$$g_2 = 0$$

the primer at the time of the second impulse is equal to its threshold. Thus, either the primer is equal to its threshold, or there is no impulse. The equivalent condition applies if $g_3 = 0$. However, the primer at the time of the first impulse is equal to its threshold only if

$$g_1 = 0 \qquad \underline{\text{and}} \qquad g_2 = 0$$

Otherwise, if $S_2 g_2 < 0$, $\theta_2 = \theta_1$ and a variation in θ_1 requires an equal variation in θ_2. Thus for computational purposes two different modes are assumed. The first mode allows two initial and one final impulse (no artificial constraints). The second mode does not allow a first impulse. Only one initial and one final impulse are allowed. Often each of these modes possesses at least one local maximum, and the optimum transfer is determined by a comparison of the respective payloads.

As is typical with most numerical iterations, the initial condition is important. As there are two assumed modes of operation, there are two different sets of initial conditions used for the iterations. These were determined to work successfully by some initial trials. When both modes have local maxima, some artificial devices are often necessary to force the iteration for one away from the other solution. Thus when only one initial impulse is desired, a second impulse is not allowed. The iteration then proceeds to the maximum. The initial condition used for this mode is

$$\theta_1 = \theta_0 \text{ (fixed)}$$

and either

$$\theta_2 = .3\ \theta_0 + .7\ \theta_H$$

$$\theta_3 = .7\ \theta_f + .3\ \theta_H$$

if the trajectory is going from an inner to an outer orbit or

$$\theta_2 = .7\ \theta_0 + .3\ \theta_H$$

$$\theta_3 = .3\ \theta_f + .7\ \theta_H$$

if going inbound. The variable θ_H corresponds to the eccentricity of the pure high thrust coasting orbit. For transfers between elliptic orbits, the initial condition of a pure low thrust transfer

$$\theta_1 = \theta_2 = \theta_0$$

$$\theta_3 = \theta_f$$

is also useful. This often leads to a third local maximum.

When two initial impulses are desired, the simpler transfer presented in Appendix D provides an excellent initial condition. If θ_s corresponds to the eccentricity of the coasting orbit between the two impulses of this simpler transfer, the initial conditions for the total transfer are given by

$$\theta_1 = \theta_s$$

$$\theta_2 = \theta_3 = \theta_f$$

The constraint $\delta\theta_3 = 0$ is used until $\underline{g}^T\delta\underline{\theta} = 0$ for a choice of θ_1 and θ_2. Then the constraint is removed and the iteration proceeds. Even when this mode is optimal, the iteration tends to diverge to the mode of one initial and one final impulse if the constraint is not used in the initial portion of the iteration.

5.2 The numerical iteration

The numerical properties of the optimum payload expression have been defined. Now a logic to choose among the possible iteration steps is presented. The details of each of these iteration steps are in Appendix E. Earlier a Taylor series of the payload was expressed about the optimum value. For the iteration a Taylor series expressed about the present value of $\underline{\theta}$ is useful. Thus

$$m_\pi(\underline{\theta}+\delta\underline{\theta}) = m_\pi(\underline{\theta}) + \underline{g}^T\delta\underline{\theta} + \frac{1}{2}\,\delta\underline{\theta}^T\,\underline{G}\,\delta\underline{\theta} + \ldots$$

where now the derivatives are evaluated at the present value of $\underline{\theta}$. Three of the iteration steps evolve from this formulation. The gradient step

$$\delta\underline{\theta} = .1\,\underline{g} \tag{5.4}$$

will always increase the payload (if the step is small enough). The scale factor of .1 was found to provide satisfactory performance much of the time. A Newton-Raphson (N-R) step

$$\delta\underline{\theta} = -\,\underline{G}^{-1}\,\underline{g} \tag{5.5}$$

will find the maximum of the payload if the function is quadratic. Near the maximum, with both $\underline{\theta}$ and $\delta\underline{\theta}$ small, the payload expression is sufficiently quadratic that this step goes right to the maximum. If the step

$$\delta\underline{\theta} = \underline{d}$$

has been taken, and found to be unacceptable, step size control is exercised. The unsuccessful step is discarded, and a smaller step in the same direction is taken. The parabolic step

$$\delta\underline{\theta} = k\,\underline{d}$$

where k, as derived in Appendix E, is

$$k = \frac{1}{2} \frac{\underline{g}^T \underline{d}}{[\underline{g}^T \underline{d} - (m_\pi(\underline{\theta}+\underline{d}) - m_\pi(\underline{\theta})]} \tag{5.7}$$

uses the information obtained from the unsuccessful step to estimate the second derivative in the direction $\underline{d}$. If the first parabolic step size reduction is still not acceptable, k is computed again and another smaller step is taken. The previous step $\underline{d}$ can be the result of a gradient, N-R, or parabolic step. In order to ensure successful progress to the optimum, k is constrained to the values

$$.05 \leq k \leq .5$$

For some of the transfers, an adverse region was encountered for which the previous methods did not provide satisfactory convergence. Such regions are best described as a ridge. The payload increases only slightly in one (vector) direction, but decreases drastically if a step is taken in other directions. The acceleration step

$$\delta\underline{\theta} = 1.4\ \Delta\underline{\theta} \tag{5.8}$$

is used, where $\Delta\underline{\theta}$, a vector in the direction of the ridge, is defined in Appendix E. If such a step increases the payload, another larger step is taken in the same direction. The step size is so increased and more steps taken in the direction $\Delta\underline{\theta}$ as long as the payload increases.

Each of these steps are useful at various times in the search for the optimal payload. The following logic used in the numerical iteration was arrived upon after observing the progress of many trial iterations. The gradient step is used if

$$\underline{g}^2 > .03$$

or

$\underline{G}$ is not negative definite

The N-R step is used if

$$\underline{g}^2 < .03 \qquad \text{and } \underline{G} < 0$$

A step is acceptable if the payload increases

$$m_\pi(\underline{\theta}+\underline{d}) > m_\pi(\underline{\theta})$$

or if the payload does not decrease too much and the gradient does not double back upon itself. These conditions are

$$m_\pi(\underline{\theta}+\underline{d}) > m_\pi(\underline{\theta}) - .001$$

and

$$\underline{g}^T(\underline{\theta}+\underline{d})\underline{g}(\underline{\theta}) > -\ .5\underline{g}^2(\underline{\theta})$$

Use the acceleration step if on a ridge as defined in Appendix E. The iteration is successfully terminated if

$$\underline{g}^2 < 10^{-8}$$

Most iterations converged in less than ten steps if the mode being tested is optimal. In a typical iteration, there will be three or four gradient steps, possibly with parabolic step control. Then the iteration terminates with one or two N-R steps. The acceleration step was only used when two initial impulses were being sought and the usual result was a change in the mode to drop the second impulse.

5.3 Some optimal coplanar coaxial transfers

The examples presented are the result of successful numerical iterations. They were chosen because they display the most interesting properties of the many cases tried. Since it is impossible to test all combinations of change in eccentricity and semi-major axis, the conclusions reached by the study of these transfers do not exclude other possibilities. However, no significant deviations from the results presented here have been observed among the many other cases which have been tried. For these numerical iterations, a mode parameter was used to specify the allowed timing and direction of the impulses along with the appropriate iteration initial conditions. Thus the signums S_i and T_i were specified for each iteration and not allowed to change. Additional input parameters used were β_z, ρ^2, e_0, e_f, τ, and m_e. β_z is the gravitational constant on the inner most orbit. β_0 is computed from β_z, depending upon the direction of the transfer. All examples use the same value $\beta_z = 1.8$ which corresponds to the constant in a low earth orbit with a high thrust exhaust velocity of approximately 7500 ft./sec. Other transfers have been tried for the range $1. \leq \beta_z \leq 2.$ Transfers for the value of β_z chosen best demonstrate the widest variety of modes of transfer. The average computational time required per transfer was 150 milliseconds on an IBM 360/65 computer. A detailed description of the results of the first example is given, followed by brief comments about the differences and similarities of the other examples.

The variations in transfers are affected by a number of factors. It was observed in Chapter 2 that increasing τ and m_e favor the low thrust in different ways. In addition to increasing the payload, a large τ increases the final threshold, δ_3, less than the initial threshold, δ_1. As τ increases, the final impulses will either be favored or decrease less than any initial impulses. Increasing m_e will principally increase the final threshold, thereby favoring initial impulses. In addition to these factors, the dynamics of orbital transfers enter in. It is observed that the high thrust is more effective for changing eccentricity than the low thrust. Also the high thrust is often used closer to the center of attraction where the gravity is stronger and the low thrust is less effective.

#	Definition of the Orbits			Trajectories are shown for					
	$\frac{a_f}{a_0}$	e_0	e_f	τ (vbl m_e)	m_e...for these τ (vbl τ)				
1	4.0	.0	.0	3.6	.05	.2	3.6	5.0	7.0
2	.25	.0	.0	4.2	.001	2.6	3.4	5.0	6.2
3	4.0	.0	.3	3.2	.05	2.6	3.8	7.0	8.0
4	.25	.3	.0	3.8	.001	2.6	3.8	6.0	7.0
5	4.0	.2	.5	3.4	.05	2.6	3.8	9.0	10.0
6	2.0	.0	.0	3.2	.001	2.2	2.8	3.4	4.2
7	6.61	.0	.0	3.2	.10	2.6	4.2	6.2	8.0

Table 5.1 The examples used

Transfers between seven pairs of initial and final orbits are considered. Table 5.1 lists the combinations considered, and for brevity, each will be referred to by its number. The ratio of semi-major axes, a_f/a_0, and initial and final eccentricities, e_0 and e_f, define the transfer being considered. The payload is given in graphical form versus the dimensionless time of flight, τ, and for the four values m_e* = 0, .05, .10, .15. In addition eight representations

*In the numerical iteration m_e=.001 was used instead of zero. The payload difference is less than .001 and only a slight change in mode was caused for small τ. Otherwise the results are essentially the same.

of the optimal trajectories are given for each example. Four trajectories (for the four values of m_e) are given for the fixed time of flight in the table. These are indicated by a ▲ on the payload plot. For fixed m_e, four more trajectories are given for the four values of τ in the table. These examples are indicated by a ● on the payload plots.

A transfer between two coplanar circular orbits with a change in radius $R_f/R_0 = 4.$ is used as example 1. The changes in payload are given in Figure 5.1 for $m_e = .05$. Since the time of flight is always assumed greater than one orbital period, the high thrust payload is not a function of τ. The low thrust payload increases with τ and the mixed mode payload is greater than the others. Note the improvement in payload for the combination even for times of flight for which the low thrust mode is not competitive ($\tau \leq 4$). Figure 5.2 is a composite plot of the payloads for the four values of m_e. For all other examples only the composite plot is given. The improvements over the pure high thrust payload are greatest for larger m_e and τ. The improvements over the pure low thrust payload are greatest for short times of flight.

Figures 5.3 and 5.4 are representations of the optimal trajectories which would be followed for the indicated choice of τ and m_e. The exact transfers are not given because the low thrust phases would be all black if the actual number of orbits were shown. Also the small periodic variations in orbital elements are not represented. These plots are computed using the position vector

$$\underline{R} = a \begin{Bmatrix} \cos E - \cos 2\theta \\ \\ \sin 2\theta \sin E \end{Bmatrix}$$

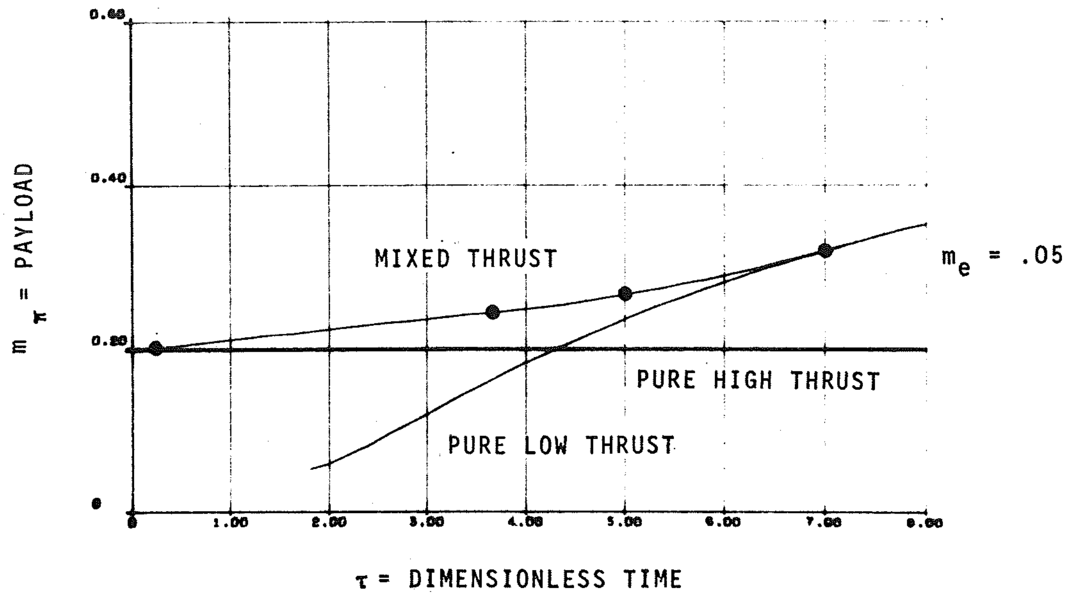


Figure 5.1. Payload vs. τ for transfer #1 (m_e = .05)

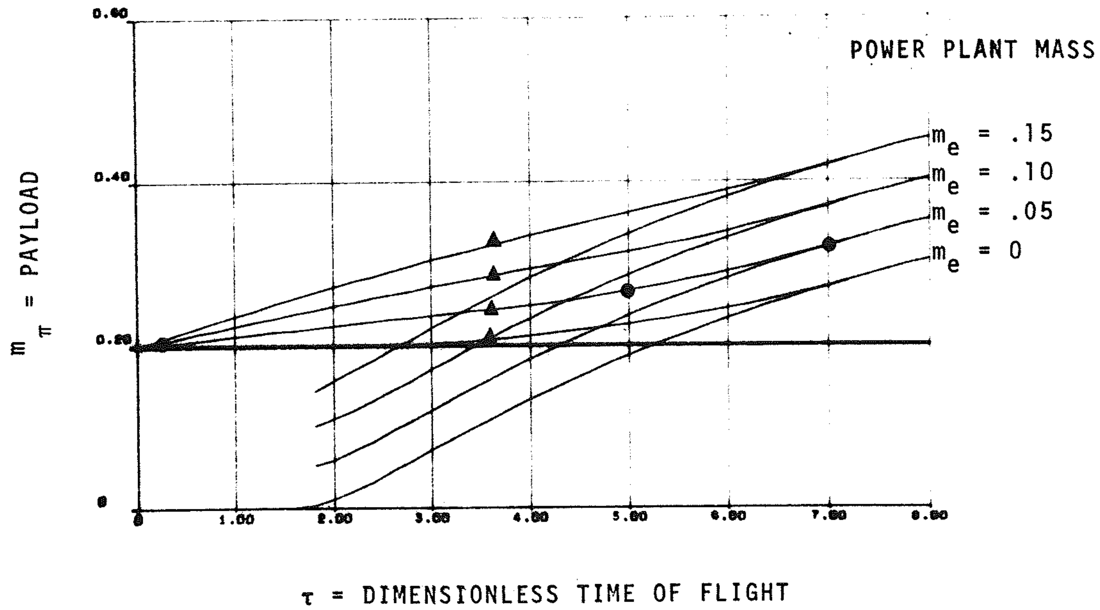


Figure 5.2 Payload vs. τ for transfer #1 (variable m_e)

which is expressed in non-rotating cartesian coordinates. During the initial and final orbits, and the coasting half orbit between the initial impulses, a and θ are constant. During the low thrust spiral, the optimal time history of a and θ are used as given by equations 4.12 and 4.13. E is related to t by the solution of the transcendental integral equation

$$E - e \sin E = \int_0^t \sqrt{\frac{\mu}{a^3(t)}}\, dt$$

For the plots, the number of orbital periods shown is the specified parameter, n. Thus μ for the integral is adjusted so that

$$E(t_f) = 2\pi n$$

The choice of n in the plots bears no relationship to the time of flight. It was chosen so that the spiral would best show the variation in orbital parameters for that transfer. The apse at which impulses are applied are numbered according to the θ which was changed by that impulse. A final impulse is always labeled as #3. Two initial impulses are labeled #1 and #2. If only one initial impulse is used, it is labeled #2 since θ_2 was found numerically and $\theta_1 = \theta_0$.

Figure 5.3 represents four transfers for the same power plant size, $m_e = .05$, and the four different times of flight given in Table 5.1. The first ($\tau = .2$) has two initial impulses followed by a small change during the low thrust spiral. It is nearly pure high thrust. The second ($\tau = 3.6$) has two smaller initial impulses with a much larger low thrust phase. As τ increases, the mode changes. It is more

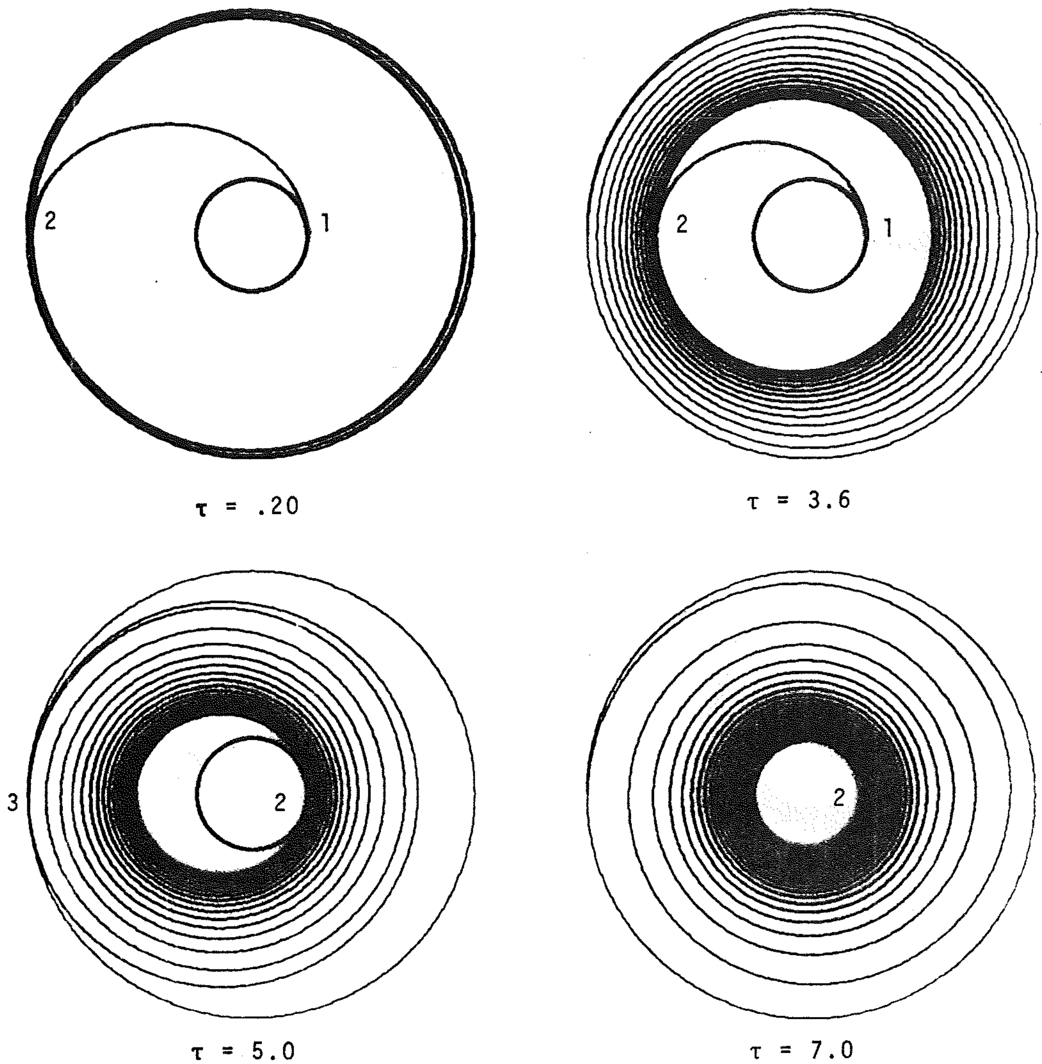


Figure 5.3 Optimal trajectories for transfer #1(vbl. τ, m_e = .05)

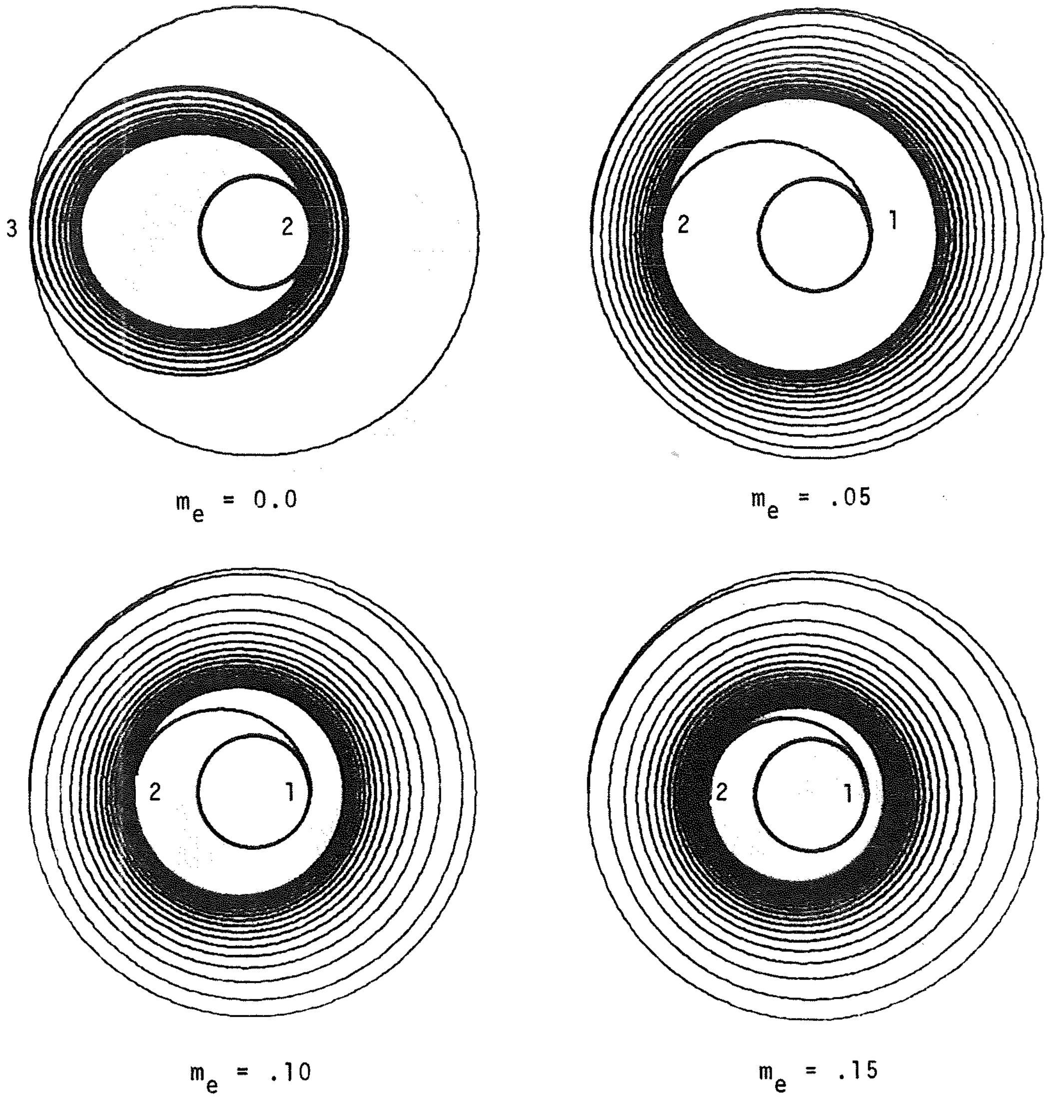


Figure 5.4 Optimal trajectories for transfer #1(vbl. m_e, τ = 3.6)

efficient to use only one initial impulse and add a final impulse. The transfer with two initial impulses still has a local maximum for this τ, but the payload is smaller. The final transfer is nearly pure low thrust with only one small initial impulse. Figure 5.4 shows four plots for the same $\tau = 3.6$. When no power plant is saved, initial and final impulses can be used with equal efficiency relative to the low thrust since $\delta_1 = \delta_3$. For larger m_e, final impulses must accelerate more mass, and thus are not as efficient. Thus the first plot has one initial and one final impulse with an intermediate low thrust spiral. The remaining transfers have two initial and no final impulses, with the low thrust phase increasingly important as m_e increases.

The second example is a transfer between the same orbits as the first example, but in the opposite direction. The initial orbit is the outer circle and the final orbit is the inner circle. The payload (Figure 5.5) and optimum transfers are identical to those of example 1 for the pure high thrust, pure low thrust and mixed thrust for m_e=0 only. Although the payload is virtually the same for increasing m_e,

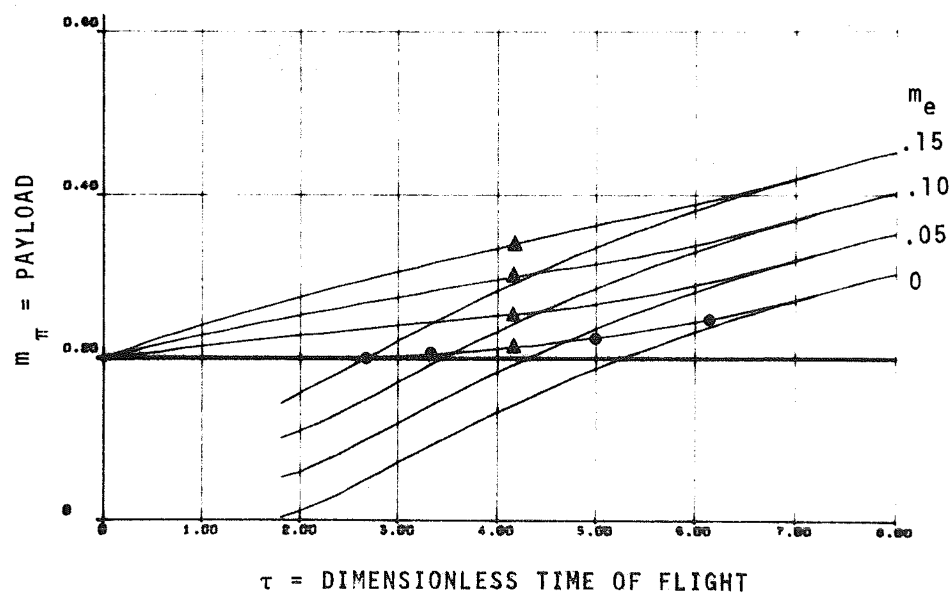


Figure 5.5 Payload vs. τ for transfer #2

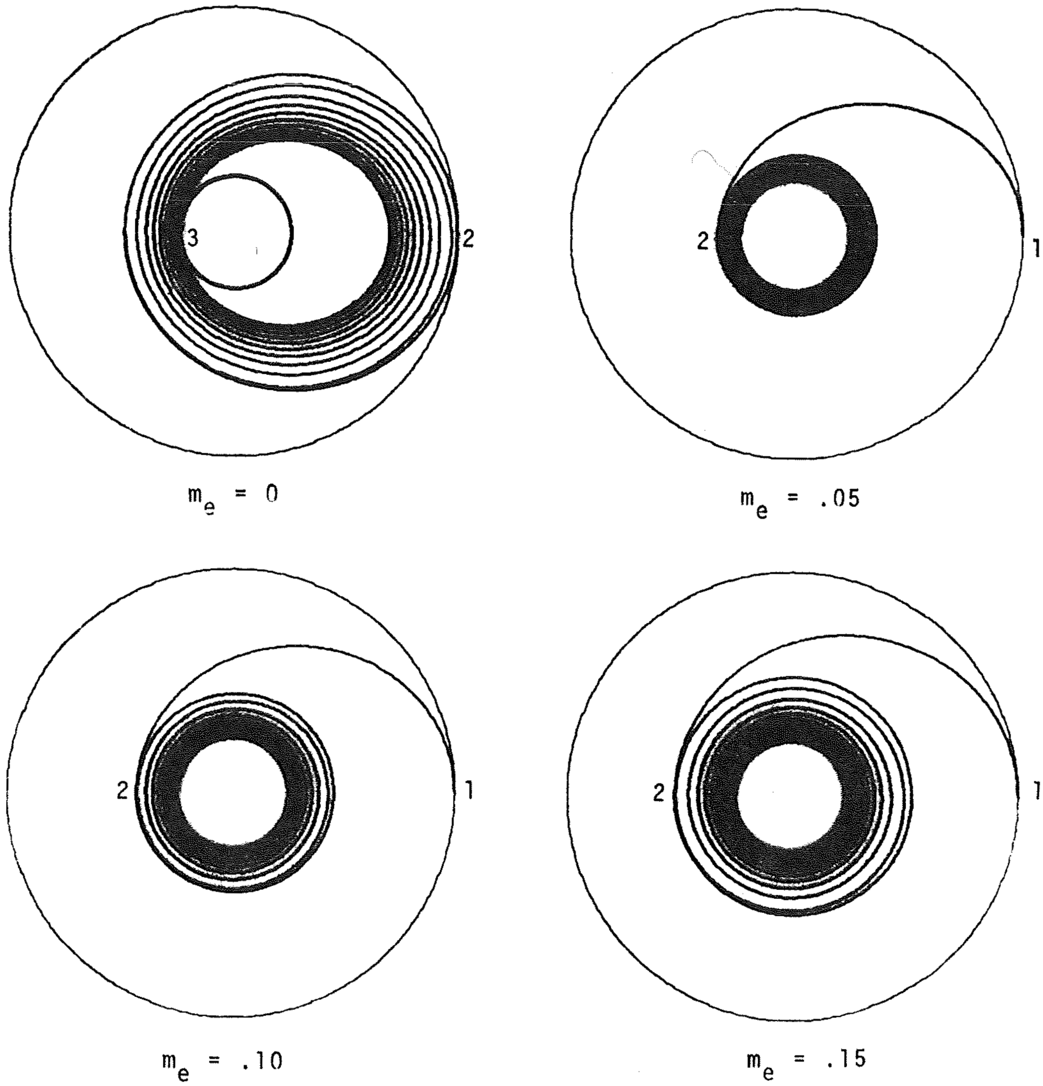


Figure 5.6 Optimal trajectories for transfer #2 (vbl m_e, τ = 4.2)

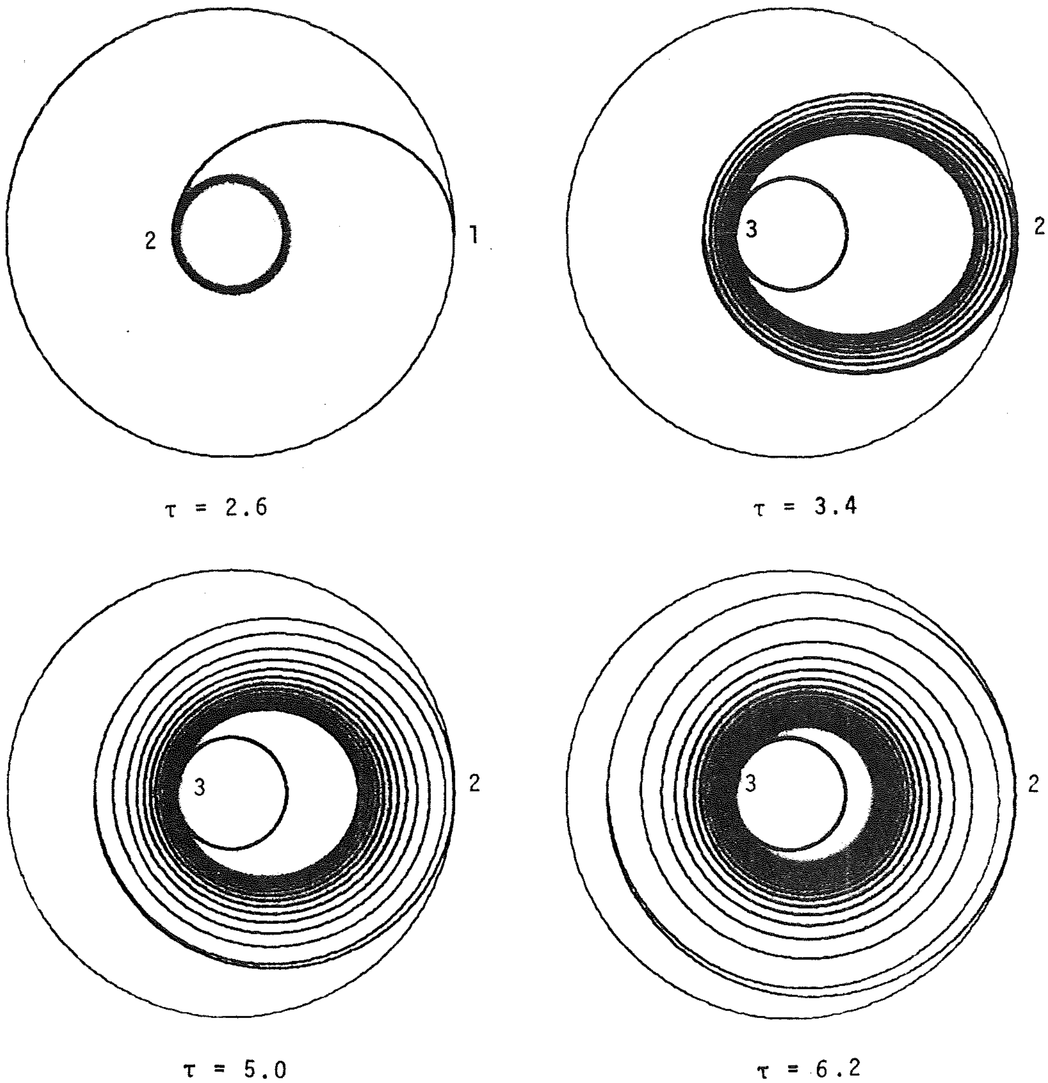


Figure 5.7 Optimal trajectories for transfer #2 (vbl τ, m_e =.001)

the transfers look much different. Figure 5.6 shows four transfers for τ = 4.2. The transfer for m_e = 0 would be the same, whether going in, or out. There is one initial and one final impulse. As m_e increases, the final impulse is dropped and the low thrust accomplishes a larger share of the transfer. The transfer for m_e = .05 does not appear to use more low thrust, but there is a significant increase in payload. With increasing m_e it is clear for the other examples that there is an increased amount of low thrust. Figure 5.7 shows the variation in transfers for increasing times of flight. The observations for the first example also apply to this figure.

The third example is between an inner circular orbit and an outer orbit with a moderate eccentricity (e = .3). The modes of transfer change as the dynamics of the optimal control problem and the transfer itself vary in importance. The changes in mode can be observed by the change in the slope of the payload curve in Figure 5.8. The most noticeable inflections are for m_e = 0. and $\tau \approx 6.8$, m_e = .05 and $\tau \approx 7.8$, and m_e = .15 and $\tau \approx 2.2$. The changes in the transfers for increasing m_e as shown in Figure 5.9 are as before. As m_e increases, first the final impulse is dropped, and then the second impulse is

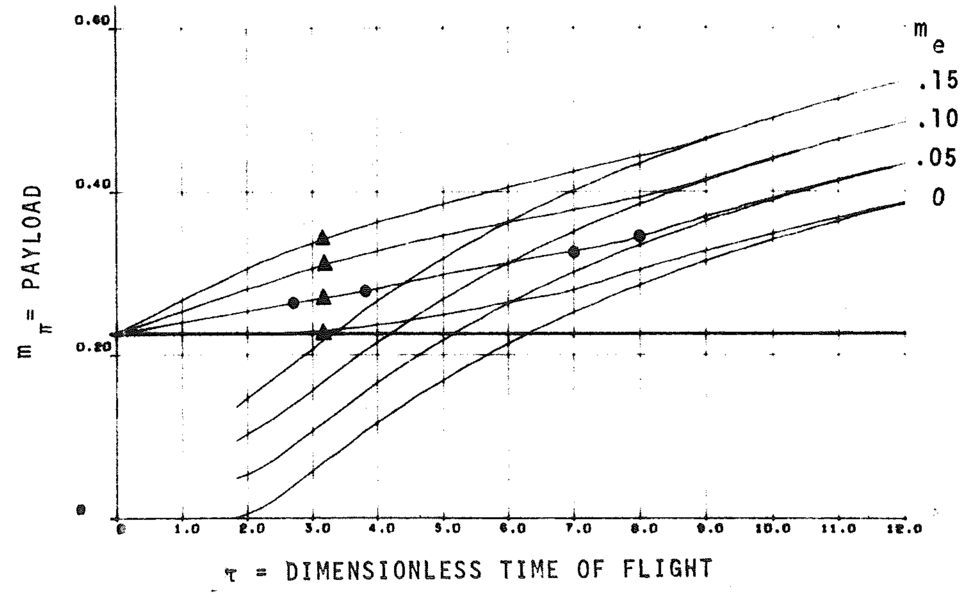
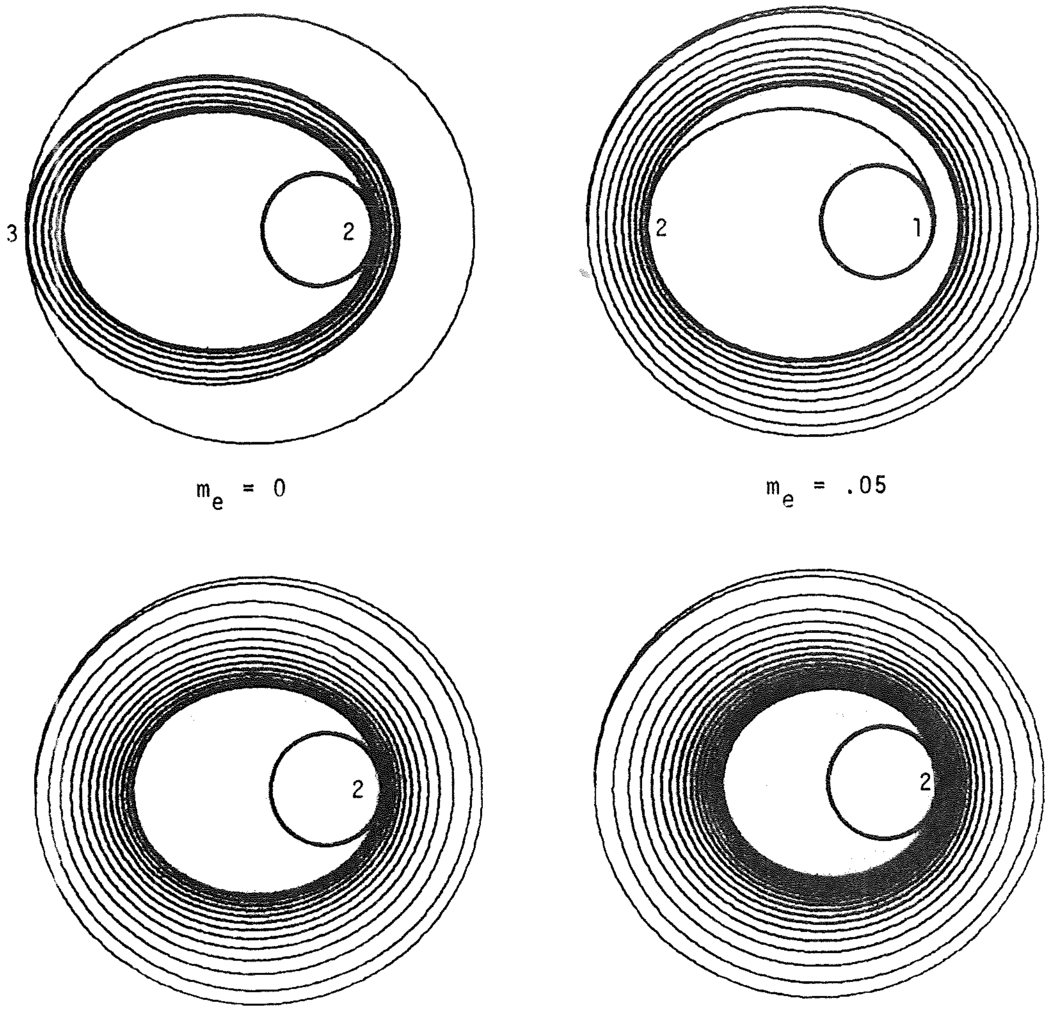


Figure 5.8 Payload vs. τ for transfer #3



m_e = 0 m_e = .05

m_e = .10 m_e = .15

Figure 5.9 Optimal trajectories for transfer #3 (vbl m_e, τ = 3.2)

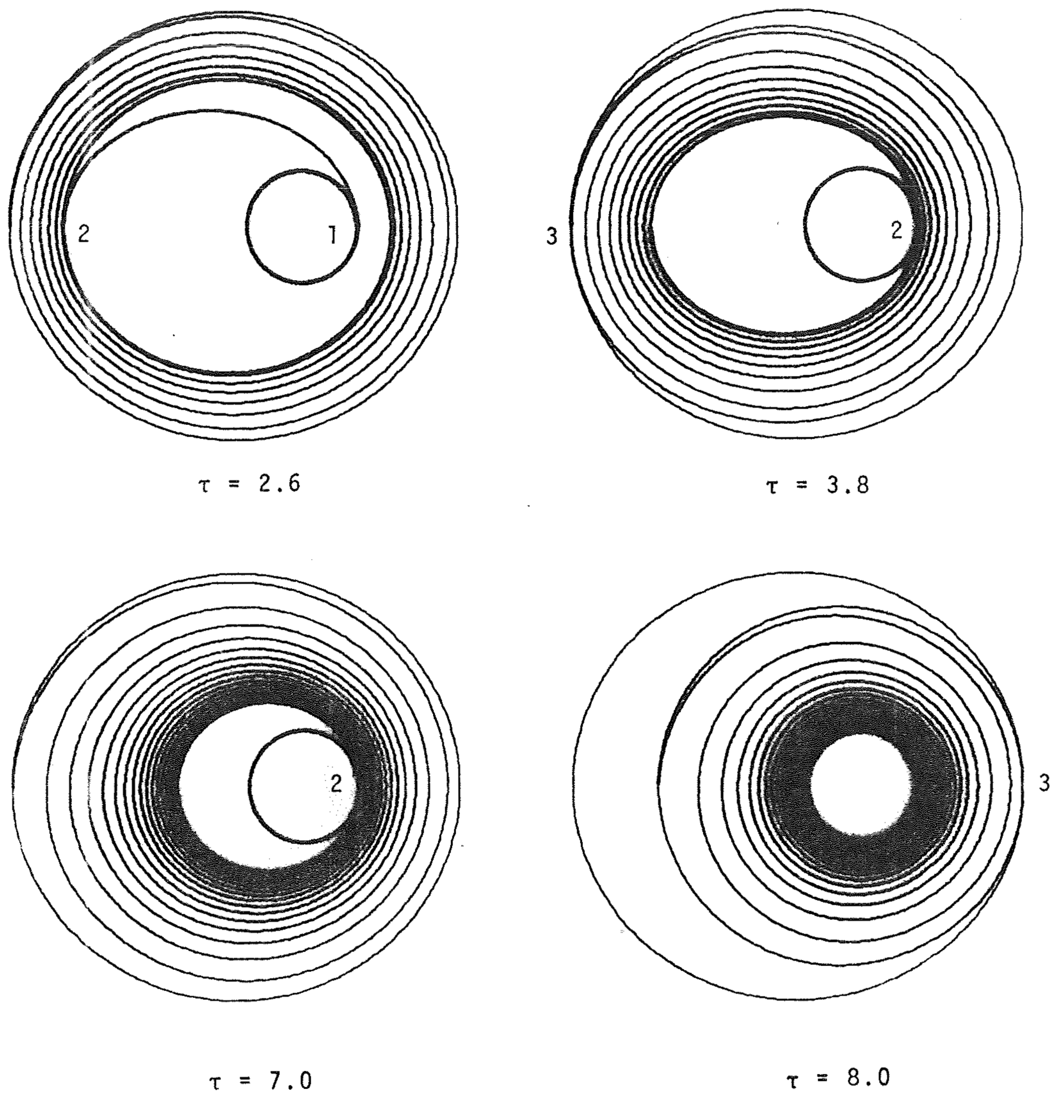


Figure 5.10 Optimal trajectories for transfer #3 (vbl τ, m_e = .05)

dropped. The trajectories for different τ's shown in Figure 5.10 show some dramatic changes in mode. For short times of flight, there are the usual two initial impulses followed by a low thrust spiral. Then the second impulse is no longer used and a small final impulse is added. The final impulse has been dropped for $\tau = 7.$ For $\tau = 8.$, the mode changes. One final impulse is used to increase the eccentricity to its final value after an initial low thrust spiral at low eccentricity.

The fourth example is the inbound version of the third example. Similar to the relationship between example 1 and 2, the payload and transfers are identical for the pure high thrust, pure low thrust, and mixed thrust for $m_e = 0$. However for larger m_e, there is a difference in the payload between examples 3 and 4. For smaller τ, the outbound transfers have a larger payload than the inbound transfers. Although two initial impulses are used for both, going outbound the impulses occur on the inner portion of the transfer, rather than the outer portion. Thus the low thrust phase is more effective since it is used in the region of weaker gravitational force. For larger τ, the inbound transfer has a higher payload since the single impulse which

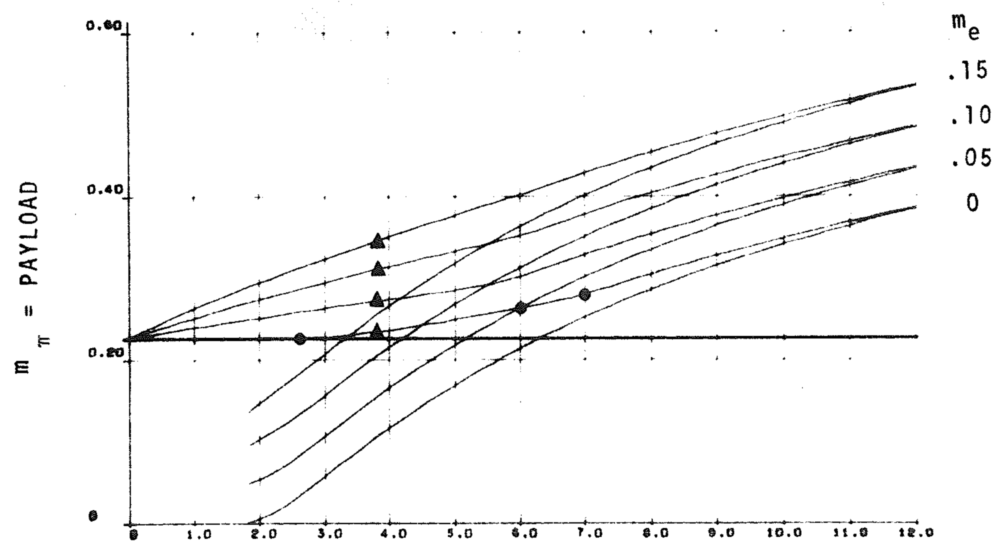
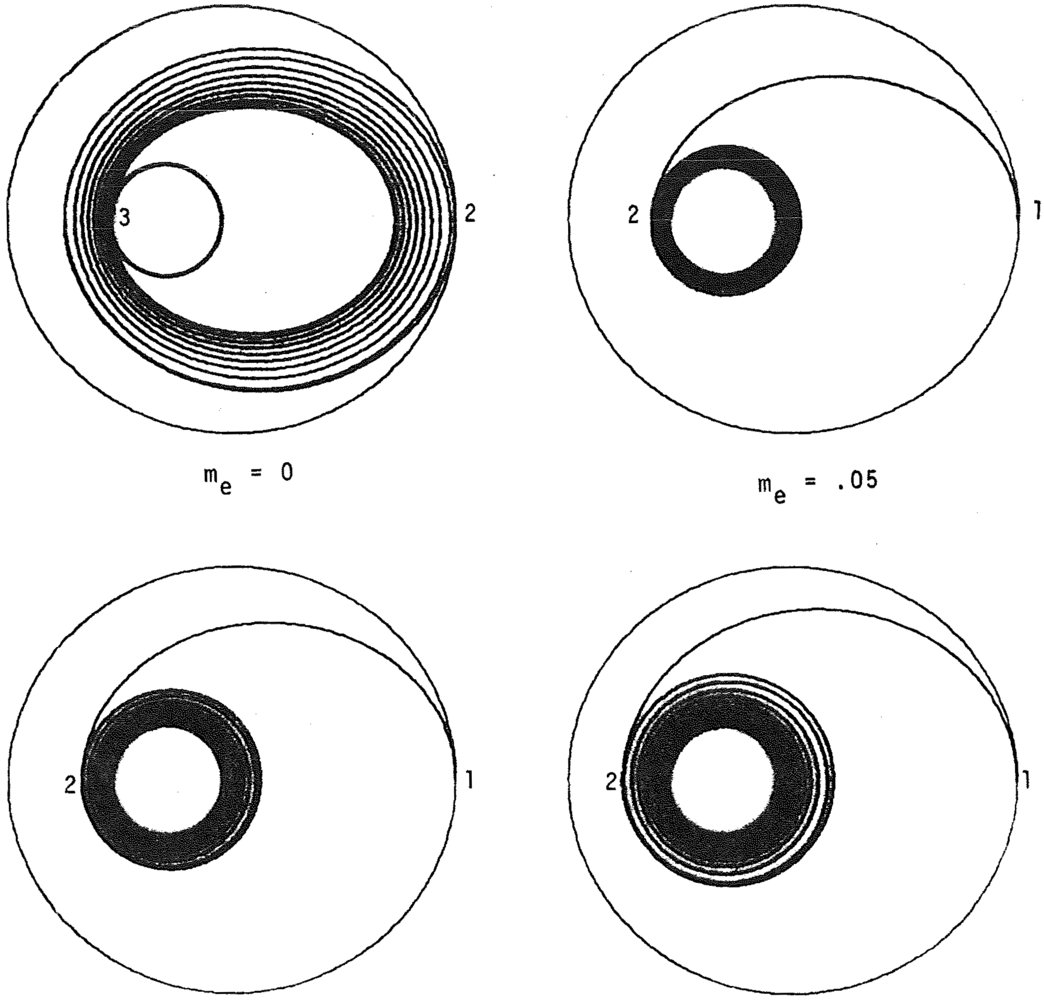


Figure 5.11 Payload vs. τ for transfer #4



$m_e = 0$ $m_e = .05$

$m_e - .10$ $m_e = .15$

Figure 5.12 Optimal trajectories for transfer #4 (vbl m_e, $\tau = 3.8$)

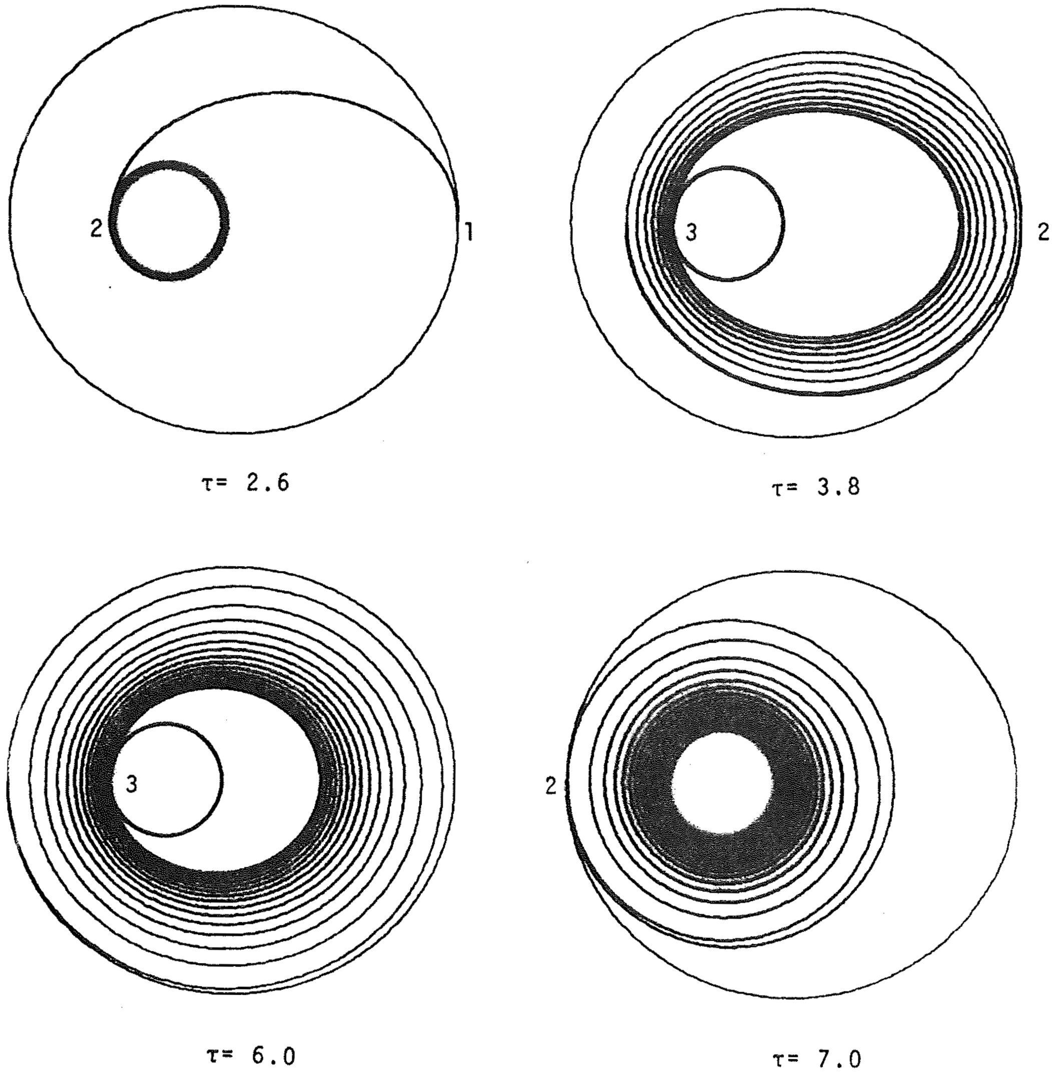


Figure 5.13 Optimal trajectories for transfer #4 (vbl τ, m_e = 0)

changes the eccentricity occurs at the initial, rather than the final time. For the larger m_e, a single impulse has a lower cost at the initial time. The variations in the transfer for increasing m_e shown in Figure 5.12 are similar in nature to those of example 2 in Figure 5.6. The variable τ plots shown in Figure 5.13 are similar to those of Figure 5.10, but with the roles of initial and final impulses reversed.

To demonstrate the similarity of the results for some other transfers, the remaining three examples bear a relationship to the earlier examples. The fifth example has the same change in eccentricity and semi-major axis as the third and fourth examples. However the initial eccentricity is different. The pure low thrust payload for examples 3, 4, and 5 are essentially the same. The high thrust payload is higher for the fifth example. For short times of flight, the improvement over the pure high thrust payload is the same for examples 3 and 5. However for larger τ the improvement over the pure low thrust reflects the greater efficiency of the high thrust at the

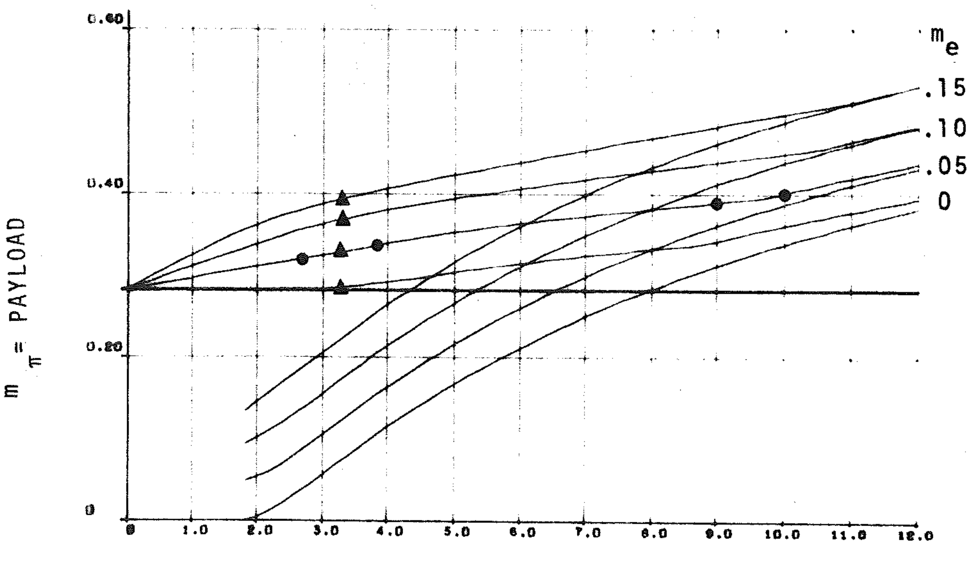


Figure 5.14 Payload vs. τ for transfer #5

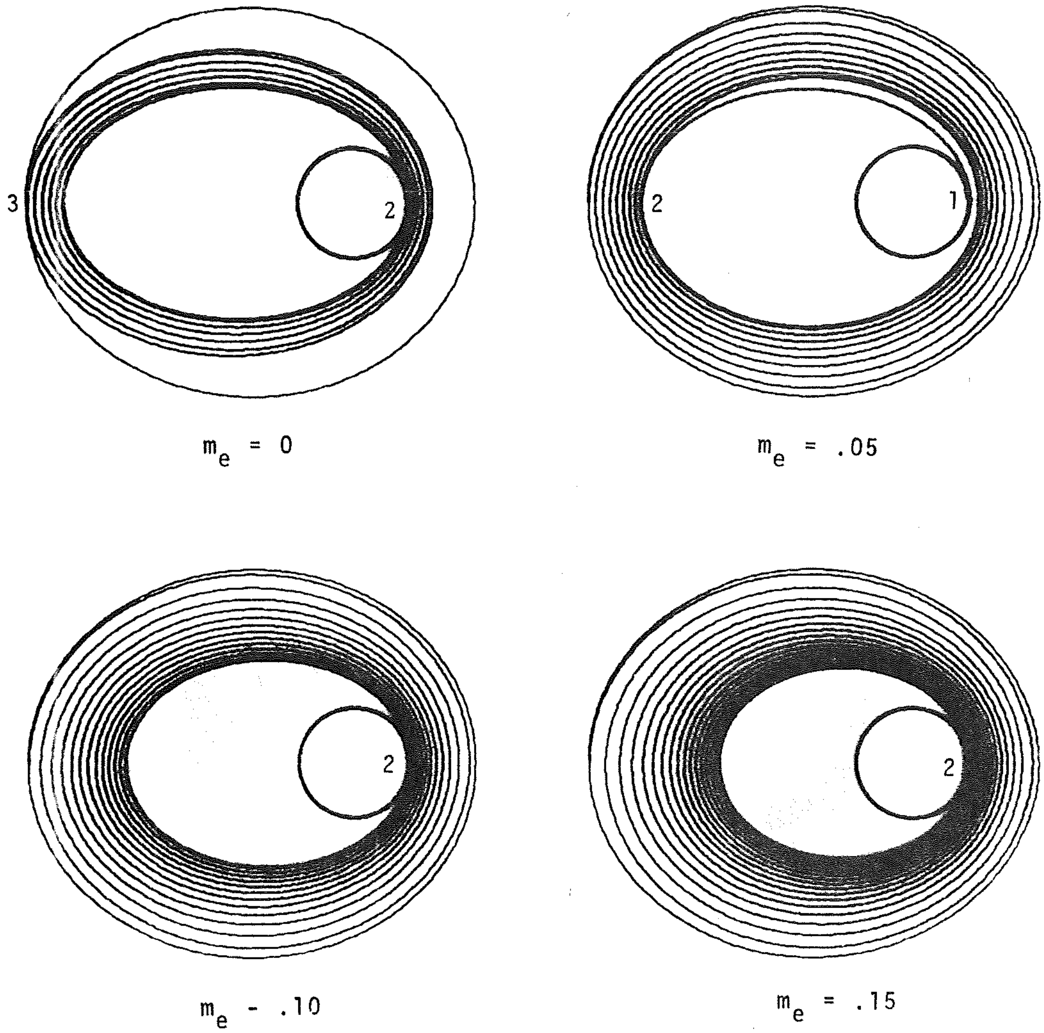
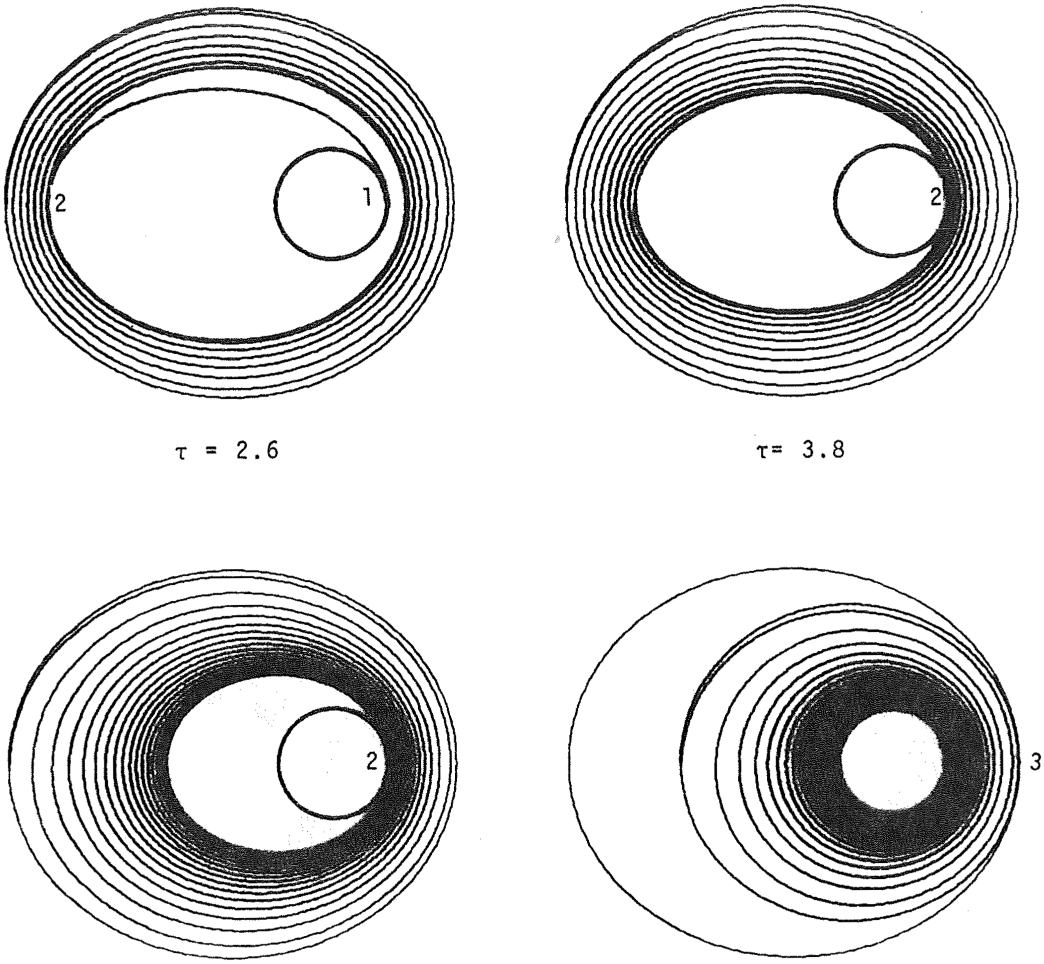


Figure 5.15 Optimal trajectories for transfer #5 (vbl m_e, $\tau = 3.4$)



$\tau = 2.6$ $\tau = 3.8$

$\tau = 9.0$ $\tau = 10.0$

Figure 5.16 Optimal trajectories for transfer #5 (vbl τ, $m_e = .05$)

higher eccentricity. Figures 5.15 and 5.16 represent the optimal trajectories for variable τ and variable m_e. Their variations are similar to those of example 3. Examples 6 and 7 are similar to the first example. They are transfers between coplanar circular orbits with different changes in radius. The relative improvements in payload occur at different τ, but the variations in the transfer are quite similar. The mode of transfer for the combination of propulsion systems has been defined by the number and magnitude of the impulses. The improvement in payload is sufficient to indicate consideration of the combination of propulsion systems on any transfer for which low thrust can be considered.

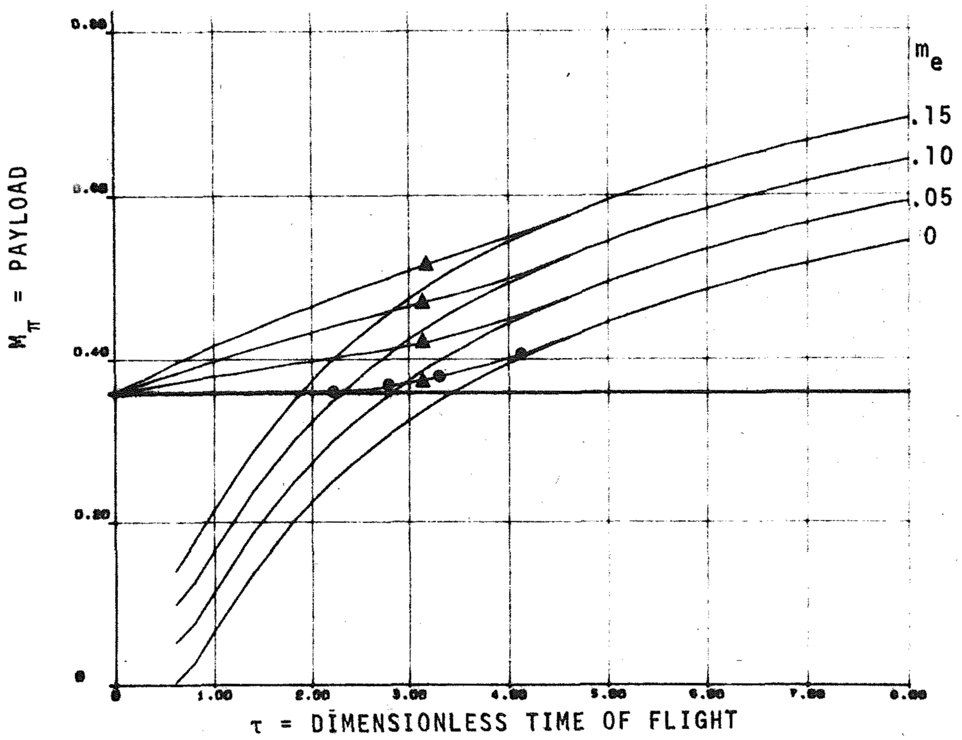


Figure 5.17 Payload vs τ for transfer #6

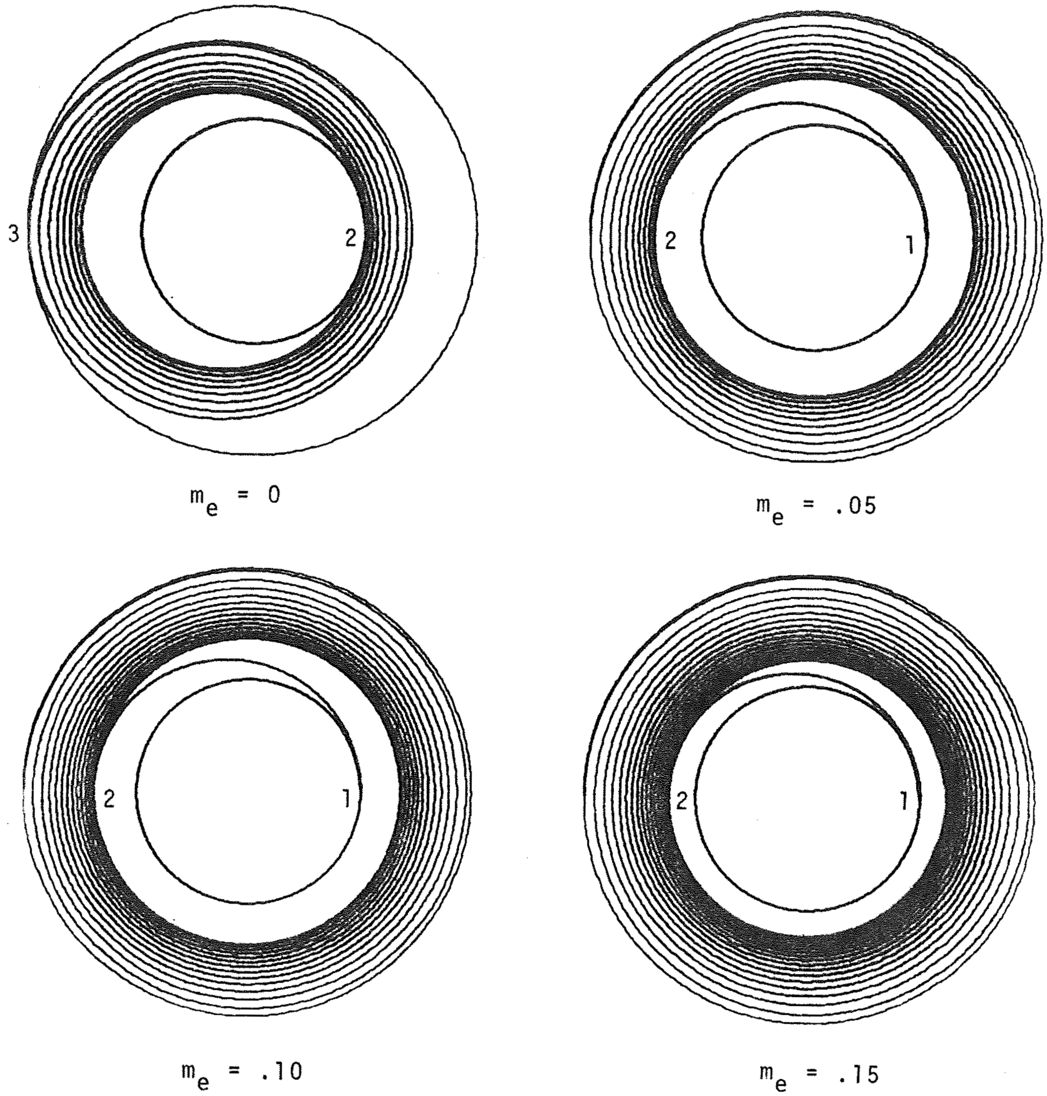


Figure 5.18 Optimal trajectories for transfer #6 (vbl m_e, τ = 3.2)

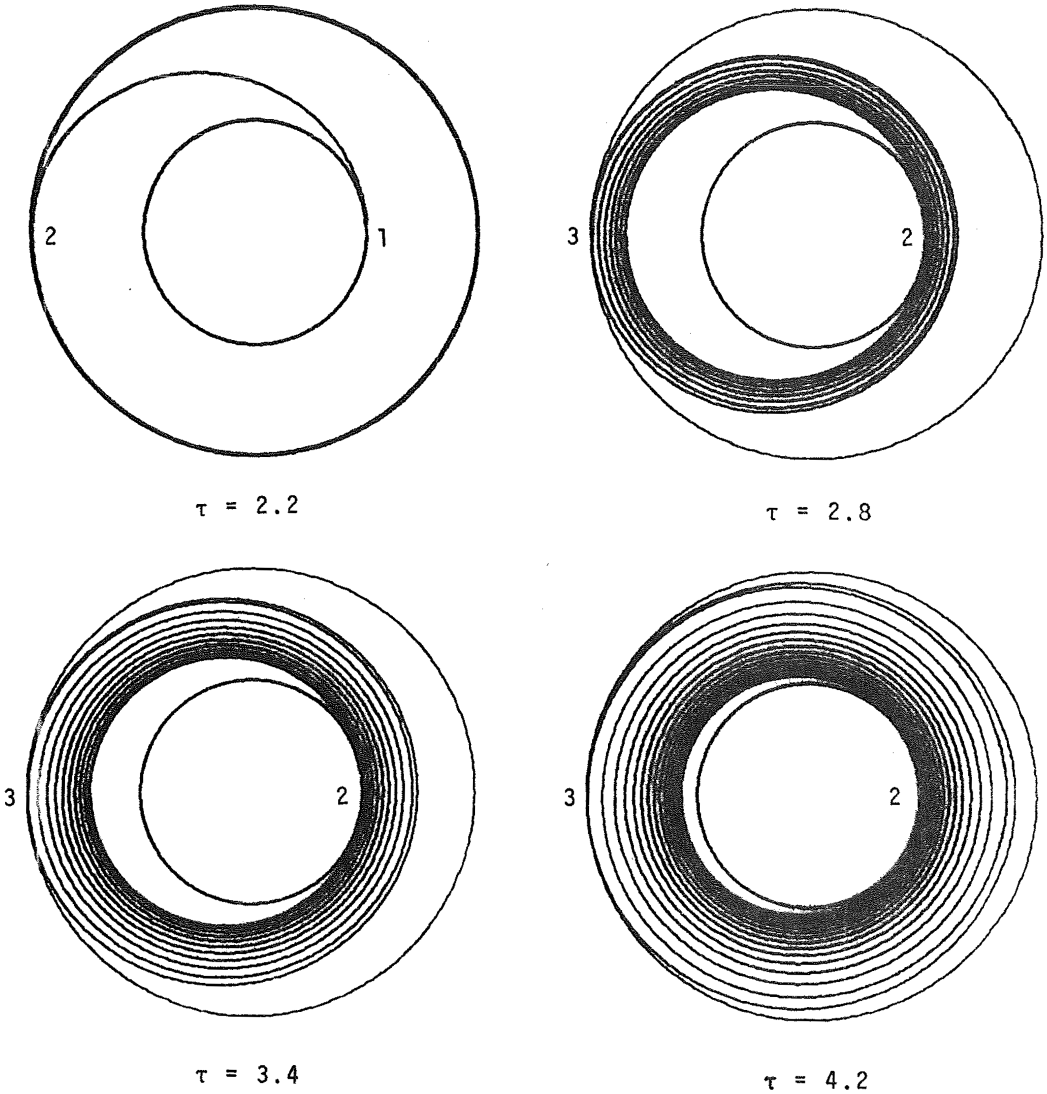


Figure 5.19 Optimal trajectories for transfer #6 (vbl. τ, $m_e = 0$)

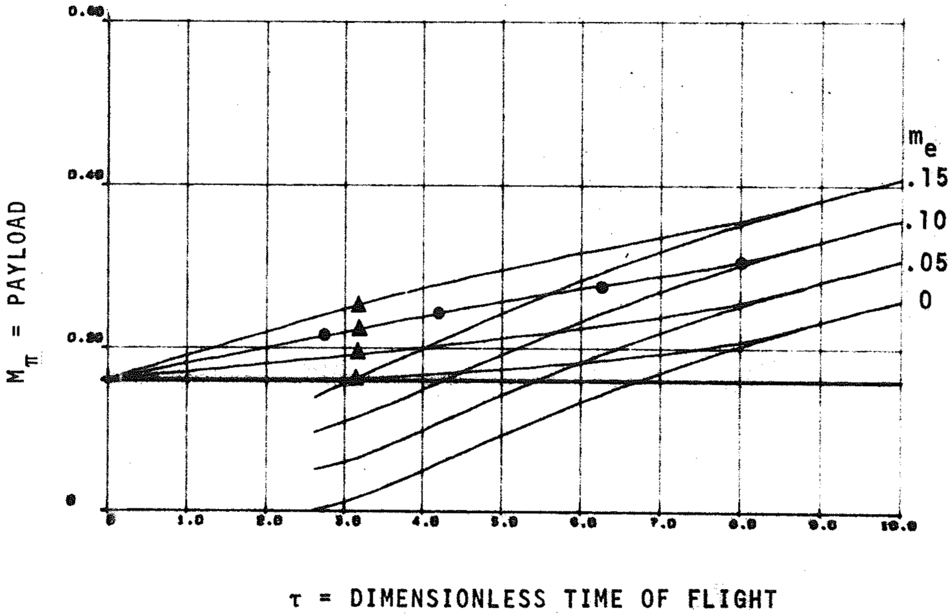


Figure 5.20 Payload vs. τ for transfer #7

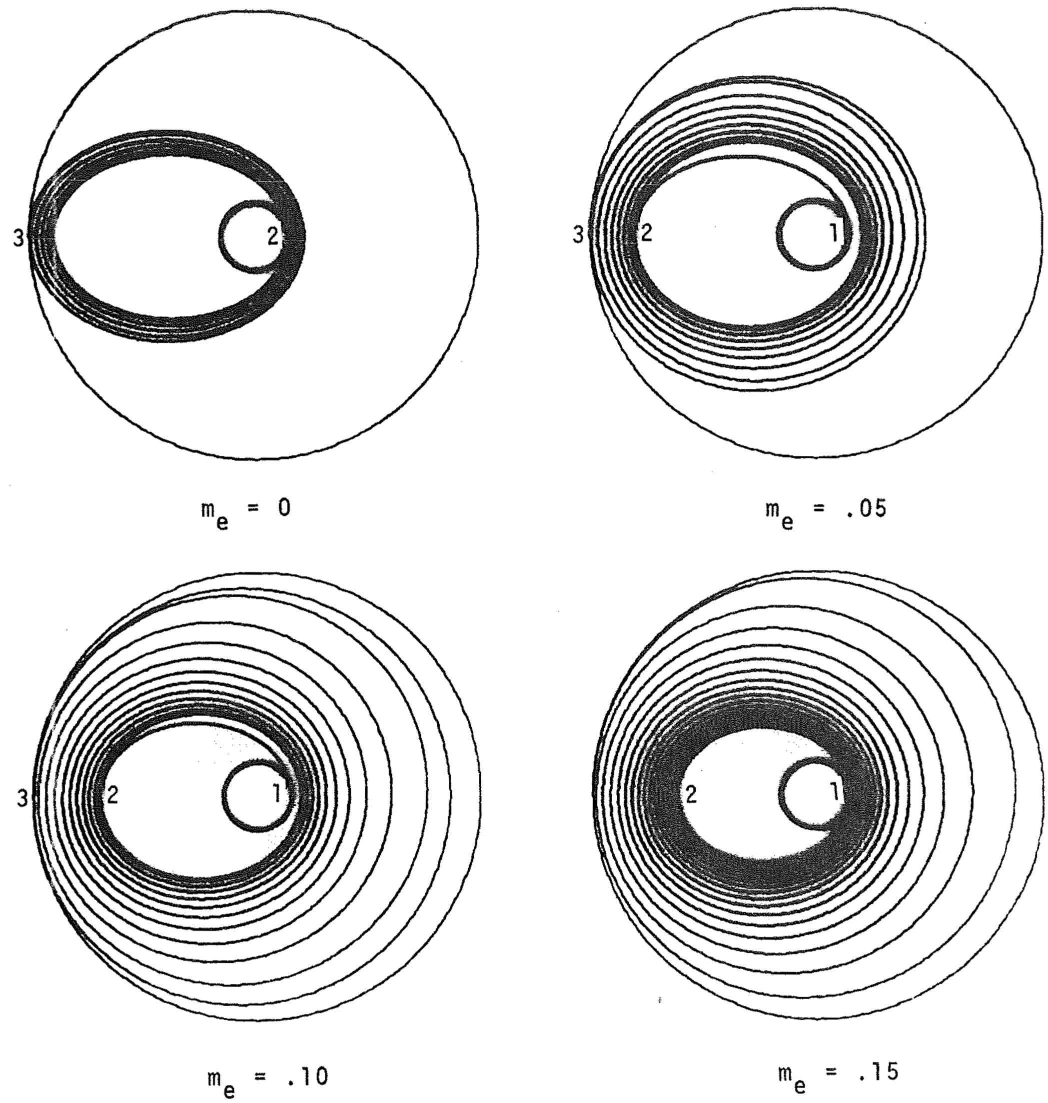


Figure 5.21 Optimal trajectories for transfer #7 (vbl m_e, $\tau = 3.2$)

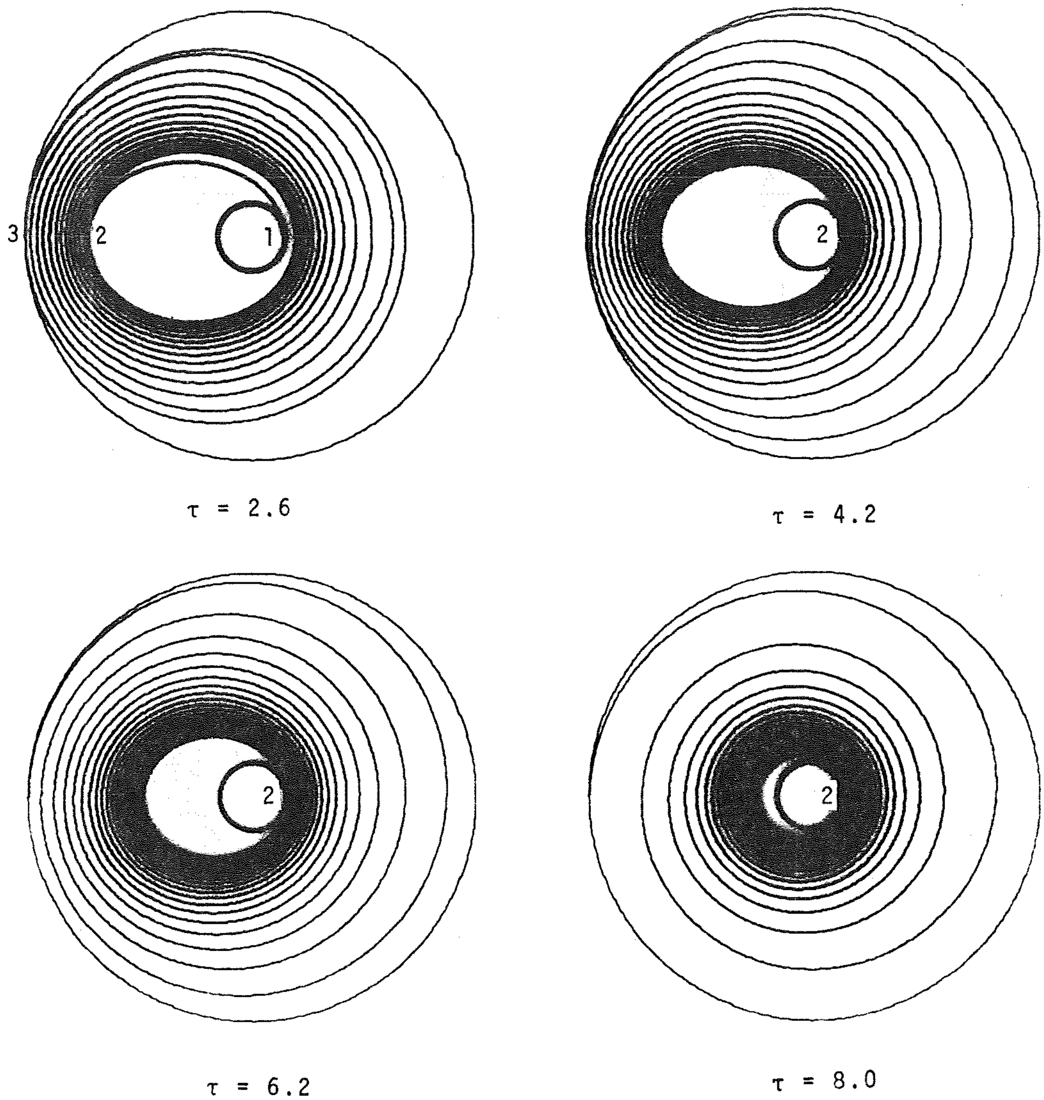


Figure 5.22 Optimal trajectories for transfer #7 (vbl τ, m_e = .10)

Chapter 6

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Significant improvements in payload have been obtained for changes in spacecraft trajectories using the optimal combination of ideal velocity and power limited engines. A payload expression was developed which considers the retention of a portion, or all, of the power supply used for the low thrust propulsion. Retaining the power supply mass, m_e, results in an improvement in payload, regardless of the time of flight. The improvement in payload is typically greater than 80% of m_e for times of flight such that the two modes are independently competitive. The optimal control problem for the combination of engines is much more complex than that for either mode used independently. For transfers between coplanar coaxial ellipses in a strong gravitational field, the dynamics of the transfer have been analytically solved. The resultant payload expression has been numerically maximized by the optimum choice of the remaining free parameters for some specific examples. Similar improvements in payload were found for changes in velocity in field free space. Large enough payload improvements have been obtained to suggest consideration of the combination if it is possible to use a low thrust propulsion system at all.

Some observations made in Chapter 2 regarding the variation in the trajectories due to variations in m_e and τ have been verified for the examples studied. For short times of flight, τ, the mixed thrust payload will always be greater than the high thrust payload if a non-zero power plant, m_e, is desired. If τ is very small, the improvement is insignificant since the low thrust, which must operate at a low acceleration in order to be efficient, does not have enough time to affect the transfer. However, the improvements in payload grow rapidly as τ increases. For intermediate τ, when the high and low thrust are independently competitive, the improvement for the combination is greatest. Increasing m_e will increase the payload by 80% to 100% of the increment in m_e. Thus if the two modes are competitive, and a power supply is desired, it is added "free" payload if the combination is used. For moderately large τ, there will still be some improvement over pure low thrust. But for large τ, the low thrust efficiency is so high that high thrust will be used only for the initial insertion into orbit.

The timing of the high thrust impulses define the mode or character of coplanar coaxial transfers. There were at most two initial impulses (at the opposite apses of a coasting orbit) and one final impulse. Between these high thrust phases, there is a low thrust spiral phase. Although transfers with three impulses were found, typically either two initial and no final, one initial and one final, only one initial, or only one final impulse were used. The numerical initial conditions reflecting these different possible modes frequently resulted in several local maxima. The timing of the impulses is dictated by the interrelationship of the dynamics of the transfer itself and of the optimal control problem (as defined by the thresholds, δ, and the primer). Usually the timing of the impulses was dictated by the relative level of the threshold, while the improvements in payload were affected by the dynamics of the transfer.

For a given transfer between two orbits, the determination of the timing of the impulses is dominated by the optimal control parameters $\underline{p}$, δ_1, and δ_3. These are affected by the time of flight, τ, and the desired power supply mass, m_e. Due to the lower required acceleration level, the low thrust cost will decrease for small increases in τ, even as the low thrust phase accomplishes more of the transfer. This is reflected in the increase in both the initial and final thresholds. However, since the low thrust integral, L_1, is smaller, the increase in δ_3 is not as large as in δ_1. The final impulse loses efficiency less rapidly than the initial impulses. These effects are seen in the plots of trajectories for variable τ. The mode typically changes from two initial impulses to one initial and one final impulse, then to only one initial impulse. Changes in the desired power plant size, m_e, also have a large effect upon the relative sizes of the impulses. If m_e is smaller than the optimum power plant mass, m_p, δ_3 will increase much more than δ_1 (δ_1 increases due to the increase in J_1 as the final impulse diminishes). Thus final impulses will diminish much more than initial impulses. If there is no final impulse, the payload increases by the amount of the increase in m_e, and the transfer modes are the same. However, if m_e is greater than m_p, both δ_1 and δ_3 will increase. The low thrust power supply is now specified at a higher level than the optimum, and the same acceleration can be achieved by a lower exhaust velocity. These effects can be seen in the plots of the trajectories for variable m_e. First the final impulse is dropped, then the initial impulses diminish as the low thrust accomplishes more of the transfer.

The dynamics of the transfer have a larger effect upon the payload than upon the timing of the impulses. Comparing the payloads for like transfers it is observed that high thrust is more efficient (higher payload) for use in the stronger portion of the gravitational field, for changing eccentricity, or operation at higher eccentricity. The low thrust is more efficient at lower eccentricities and in the weaker portion of the gravitational field. Also, there are cases for which the mode changes due to the relative dominance of these effects. For transfers to a final elliptic orbit at a large τ, a final impulse is used to change the ecentricity, even for moderate m_e.

The improvements obtained for this class of transfers are sufficient to warrant further investigations into the use of the combination of propulsion systems. There are basically three areas for further study. Within the engine and strong gravity assumptions used here, other classes of orbital transfers remain to be studied. Removing the assumptions on the strength of the gravitational field would allow an extension of this problem formulation to interplanetary transfers. Before applying these results to a specific mission, the non-spherical character of the earth's gravitational field must be considered and the ideal engine assumptions must be dropped. Real engines involve the additional masses of the engine, structure and fuel tanks, along with non-ideal operating capabilities.

Insertion of a communications satellite into synchronous orbit is an appropriate application of the theory presented here. However, for launches from the United States, the inclination of the orbit must also be changed. Transfers involving a change in all orbital parameters (except position on the orbit) should also be considered. As more complicated transfers are tried, the completeness of the solution for the low thrust phase is no longer possible. Although analytic solutions exist for general low thrust transfers, the boundary conditions on the costate must be numerically obtained. Also the timing of the impulses during the orbit will not be specified for general transfers as it is for the class considered here. Within the assumption of a strong gravitational field, rendezvous is not a particularly interesting class of transfers. Over the many orbits of the spiral which are required, a small perturbation in each orbit is all that is required to achieve rendezvous. Using the combination of propulsion modes, similar payload improvements should be possible for these other classes of transfers. An initial choice on the timing of impulses for other classes of transfers should be motivated by the results found here.

Appendix A

NOTATIONAL CONVENTIONS AND NOMENCLATURE

The variables are divided into categories appropriate to their application for a description of their roles. The notational conventions used are given first, followed by the masses, state variables, defining constants, and the controls. The equation number of the defining equation(s) or the first occurence of each variable is given in parentheses after each description. Since each chapter of derivations has a related appendix, many variables have two equation numbers indicated for the respective first occurrences.

Notational conventions

The vector and other notations are explained using x as a dummy variable. The discussion is concerned with how any such variable is treated.

$\underline{x}$ = underbar, to indicate that the variable is a matrix. An n x 1 matrix is often called a vector

$\underline{x}^T$ = use of the transpose of the matrix

$\underline{x}^2$ = $\underline{x}^T\underline{x}$. The notation for the square of the vector magnitude is shortened

$|\underline{x}|$ = $+\sqrt{\underline{x}^T\underline{x}}$ The scalar magnitude of a vector

$\frac{\partial}{\partial \underline{x}}$ = the vector of first derivatives with respect to each component of $\underline{x}$

$\dot{x}$ = over dot. One (two) overdot indicates the first (second) time derivative of the variable in an inertially fixed coordinate system.

Some subscripts are used which are of general application

0 = at the initial time of the transfer

f = at the final time of the transfer

1 = final time (only for J_1, K_1, L_1, and Q_1)

Nomenclature

All masses are dimensionless, having been normalized by the initial mass of the spacecraft

m_π = payload. This is the mass remaining after the transfer is completed and any undesired power plant has been dropped (2.3, B.11)

m_e = the power plant mass desired at the end of the transfer (a specified parameter)

m_p = optimal power plant size for any transfer (2.4, B.10)

$\dot{m}$ = fuel flow rate (B.6)

Some integrals of the controls are used to describe the payload

$J(t)$ = the change in mass (a ratio) of the spacecraft due to any final ($\underline{A}_3$) controls (B.9)

$K(t)$ = the change in mass (a ratio) of the spacecraft due to any other ($\underline{A}_1$) high thrust controls (B.7)

$L^2(t)$ = the integral of the low thrust acceleration which determines the change in mass (B.8)

By definition, the final value on these three variables is indicated by a subscript 1.

The current position or orbit of a spacecraft is given in general by a state vector $\underline{x}$ which depends upon the problem statement. When

$\underline{x}$ = the position vector (2.10., 3.2)

$\underline{R}(\underline{x})$ = the arbitrary position dependent gravitational acceleration (2.10)

$\underline{g}$ = the gravitational acceleration in position independent space (3.1)

$\underline{V}$ = the dimensionless change in velocity (3.2)

$\underline{s}$ = the dimensionless change in position (3.2)

When

$\underline{x}$ = a vector of orbital elements (4.1)

a = semi-major axis of the orbit

θ = $\frac{1}{2}\cos^{-1}e$ (by definition)

e = eccentricity of the orbit

Subscripts on the orbital elements are given by equations D.7

$\underline{\theta}$ = the vector of the unspecified θ's (5.1, D.18)

$\delta\underline{\theta}$ = the deviation from the optimum, or a step taken trying to reach the optimum. The four possible steps are (5.4, 5.5, 5.6, 5.8)

$\underline{g}$ = the gradient vector of the payload (5.2, D.17)

$\underline{G}$ = the second derivative matrix of the payload (5.3, D.21)

k = parabolic step sizing parameter (5.7)

R = $a(1-e\cos E)$ = the radius on an orbit

Two other classical elements are used to indicate the time variation

f = true anomaly (4.4)

E = eccentric anomaly (4.3)

$\underline{B}(\underline{x},t)$ = the matrix of coefficients which relate the control accelerations to the derivatives of $\underline{x}$. (4.2)

Two signum functions are used for convenience of notation

T = $-\cos f$ when $f = 0$, or $f = \pi$ (4.8)

T_0, T_f = define the apses at which impulses are applied (D.8, D.9)

S_i = T_i signum (p_i) (D.16)

and two functions of the eccentricity are used

$$F = \sqrt{\frac{1+T_e}{2}} \qquad (D.5)$$

$$G = \sqrt{\frac{1-T_e}{2}} \qquad (D.6)$$

Some constants are used to define the transfers which are considered

t_0 = the initial time (usually $t_0 = 0$)

t_f = the final time. If $t_0 = 0$, t_f is the time of flight

α = the reciprocal specific power of the low thrust power supply (2.2, B.3)

c = the high thrust rocket exhaust velocity (2.1, B.5)

μ = the gravitational constant (D.4)

β_i $= \sqrt{\frac{\mu}{4c^2 a_i}}$, a dimensionless gravitational constant (D.1)

τ $= \frac{t_f}{\alpha c^2}$ = dimensionless time of flight (3.3, 4.11, D.2)

ρ $= \sqrt{\frac{a_0}{a_f}}$ = dimensionless change in semi-major axis (D.3)

ψ $= \sqrt{\frac{8}{5}}\,(\theta_3-\theta_2)$ to indicate the low thrust change in eccentricity (D.10)

γ $= \sqrt{\frac{a_2}{a_3}}$ to indicate the low thrust change in semi-major axis (D.11)

The optimal controls are described in terms of

$\underline{p}$ = the primer vector, the velocity costate (2.11, 4.5)

$\underline{\lambda}$ = the costate vector appropriate to the problem formulation

p = the non-zero component of the primer at an apse (4.9)

p' = the non-zero component of the first derivative of the primer at an apse (4.10)

C,D = two constants which define the primer. Either (4.6, 4.7) or (4.14, 4.15)

When the state is a vector of orbital elements

λ_1 = costate corresponding to the averaged semi-major axis (D.13)

λ_2 = costate corresponding to the averaged θ (D.12)

$\lambda_{10}, \lambda_{20}$ = the initial conditions on λ_1, λ_1 (D.14, D.15)

$\lambda_J, \lambda_K, \lambda_L$ = the costates corresponding to the mass flow functions J, K, L. (B.12, B.13, B.14)

The optimal accelerations are

$\underline{A}_1$ = a high thrust acceleration used at any time except the final time (2.13, B.5)

$\underline{A}_\ell$ = the low thrust acceleration used during the entire mission (2.12, B.4)

$\underline{A}_3$ = a high thrust acceleration used only at the final time after dropping any undesired power plant (2.14, B.5)

These accelerations are determined by the primer using the terms:

$Q(t)$ = a coefficient which related $\underline{A}_\ell$ to $\underline{p}$ (2.15, B.17)

$\delta_1(t)$ = the threshold to determine use of $\underline{A}_1$(2.16, B.15)

δ_3 = the threshold to determine use of $\underline{A}_3$ (2.17, B.16)

We sometimes use

δ_A = $Q_1\delta_1$ (3.4, D.19)

δ_B = $Q_1\delta_3$ (3.5, D.20)

and for transfers in field free space

$u_0\underline{u}$ = a vector which partially defines $\underline{A}_\ell$ (3.6, C.1)

$w_0\underline{w}$ = a vector which partially defines $\underline{A}_\ell$ (3.7, C.1)

v_1 = the dimensionless magnitude of an initial impulse (3.8, C.2)

v_2 = the dimensionless magnitude of a final impulse (3.9, C.3)

Appendix B

THE GENERAL OPTIMIZATION PROBLEM

The analytic steps involved in the maximization of payload for transfers in an arbitrary gravitational field are presented in this appendix. Chapters 1 and 2 outline the ideal engine assumptions which form the basis of the derivations. The mass flow differential equations are derived from basic physics and then analytically solved. The equations for propulsive power required by each engine and conservation of momentum will be used. The solution for the final payload is found in terms of some special functions. From these, equivalent, but simpler, differential equations can be stated for the payload. Then the general optimal control problem for the maximization of payload is formulated and a partial solution obtained to yield a simplified set of necessary conditions.

B.1 Performance Functions for Ideal Engines

Two basic equations define the operating characteristics of both velocity and power limited engines. The conservation of momentum for any engine gives

$$\dot{m}\, c_i = -m|\underline{A}_i| \tag{B.1}$$

and the propulsive power required to accelerate the propellant is

$$\text{Power} = -\frac{1}{2}\dot{m}\, c_i^{\,2} \tag{B.2}$$

where

$m(t)$ = the current spacecraft mass (expressed as a fraction of the initial mass

c_i = the rocket exhaust velocity for a particular engine

$\underline{A}_i$ = the acceleration vector for the spacecraft as a result of the effect of a particular engine

There are also the restrictions

$$\dot{m} \leq 0$$

$$c_i \geq 0$$

Three distinct engines are assumed for this formulation. Engine 2 is assumed to be limited by the power supply mass, which has a power output proportional to its mass. Thus

$$\text{Power} = \frac{1}{\alpha} m_p \tag{B.3}$$

where

m_p = the power plant mass

α = the reciprocal specific power of the power supply

Although fixed for a mission, the power plant size remains to be determined. For this fixed power, equations B.1 and B.2 can be combined to eliminate the exhaust velocity

$$c_\ell = \frac{2 \text{ Power}}{m|\underline{A}_\ell|}$$

to give the fuel flow rate

$$\dot{m}_2 = - \frac{\alpha}{2m_p} \; (m \; \underline{A}_\ell)^2 \tag{B.4}$$

Since the propulsive power is fixed, and the accelerations, $\underline{A}_\ell$, are to be optimally chosen to maximize the payload, the exhaust velocity, c, is variable, depending upon $\underline{A}_\ell$.

It is assumed that a portion (or all) of m_p can be dropped at the final time $t = t_f$

$$\dot{m}_p = -(m_p - m_e)\dot{u}(t_f)$$

where

m_e = power plant retained

$m_e \leq m_p$

$\dot{u}(t_f)$ = unit impulse at $t = t_f$

$u(t_f)$ = unit step at $t = t_f$

$$= \begin{cases} 1 & t \geq t_f \\ 0 & t < t_f \end{cases}$$

Engines 1 and 3 have identical characteristics, but are used at different times. These engines are assumed to be limited by their exhaust velocity

$$c_1 = c_3 \leq c$$

For these engines, the mass flow is obviously minimized if the exhaust velocity is the maximum c. Thus

$$\dot{m}_1 = -\frac{1}{c}\, m|\underline{A}_1|$$
$$\dot{m}_3 = -\frac{1}{c}\, m|\underline{A}_3| \tag{B.5}$$

Engine 1 can be used at any time during the flight except the final time. Engine 3 can only be used at the final time, after a portion of the power supply, if any, has been discarded.

The total mass flow differential equation is given by the sum of the individual elements

$$\dot{m} = -\frac{1}{c}\, m(|\underline{A}_1|+|\underline{A}_3|)-(m_p-m_e)\quad \dot{u}\ (t_f)-\frac{\alpha}{2m_p}\, m^2\underline{A}_\ell{}^2 \tag{B.6}$$

Before using $\underline{A}_3$ or dropping any of the power plant, this differential equation is solved by

$$m(t) = \frac{K^2(t)}{1+\frac{1}{m_p}\, L^2(t)}$$

where

$$K(t) = \exp\left[-\frac{1}{2c}\int_0^t |\underline{A}_1|\ dt\right] \tag{B.7}$$

$$K_1 = K(t_f)$$

$$L^2(t) = \frac{\alpha}{2}\int_0^t K^2(t)\ \underline{A}_\ell{}^2(t)\ dt \tag{B.8}$$

$$L_1 = L(t_f)$$

Define

$$L(t) \geq 0$$

After dropping the power plant we have the solution

$$m(t) = J^2(t)\left[\frac{K_1^{\;2}}{1+\frac{1}{m_p}\,L_1^{\;2}} - (m_p - m_e)\right]$$

where

$$J(t) = \exp\left[-\frac{1}{2c}\int_0^t |\underline{A}_3|\,dt\right] \tag{B.9}$$

$$J_1 = J(t_f)$$

Thus the explicit solution for the total mass differential equation is given by

$$m(t) = J^2(t)\left[\frac{K^2(t)}{1+\frac{1}{m_p}\,L^2(t)} - (m_p - m_e)u(t_f)\right]$$

The payload is here assumed to be equal to the final mass

$$m_\pi = J_1^{\;2}\left[\frac{K_1^{\;2}}{1+\frac{1}{m_p}\,L_1^{\;2}} - m_p + m_e\right]$$

which is a maximum over m_p for

$$m_p = L_1(K_1 - L_1) \tag{B.10}$$

If the desired power plant size m_e is larger than the optimum m_p, we are not free to choose the power plant mass. For this case the low thrust power supply will have the size m_e. The notation m_p will be retained so that

m_p = the optimum power plant size

m_e = the desired power plant size

Thus there are two expressions for payload dependent upon the desired power supply size

$$m_\pi = \begin{cases} J_1{}^2[(K_1-L_1)^2 + m_e] & \text{if } m_e \leq m_p \\ \dfrac{J_1{}^2 K_1{}^2}{1 + \frac{1}{m_e} L_1{}^2} & \text{if } m_e \geq m_p \end{cases} \tag{B.11}$$

J, K, and L can be described by the differential equations

$$\dot{J} = -\frac{1}{2c} J|\underline{A}_3|$$

$$\dot{K} = -\frac{1}{2c} K|\underline{A}_1|$$

$$\dot{L} = \frac{\alpha}{4} \frac{K^2}{L} \underline{A}_\ell{}^2$$

with the boundary conditions

$J(0) = 1$ $K(0) = 1$

$L(0) = 0^+$

Thus the maximization of the payload m_π subject to these differential equations is entirely equivalent to the maximization of the final mass, subject to the previously given mass flow differential equations.

B.2 A partial solution of the general payload optimization problem

A general optimal control problem for the maximization of payload for transfers in an arbitrary position dependent gravitational field is presented and partially solved. The payload with its associated differential equations as derived in the previous section is maximized. In addition to the control accelerations, the spacecraft is assumed to have a position dependent acceleration. A partial solution is obtained for ease of application to problems with specific gravitational fields. Two basic forms of the vehicle differential equations are considered.

The payload m_π as given by

$$m_\pi = \begin{cases} J_1{}^2[(K_1-L_1)^2+m_e] & \text{if } m_e \leq m_p \\ \dfrac{J_1{}^2 K_1{}^2}{1+\frac{1}{m_e}K_1{}^2} & \text{if } m_e \geq m_p \end{cases}$$

with

$$m_p = L_1(K_1 - L_1)$$

is to be maximized subject to the differential equation constraints

$$\ddot{\underline{x}} = \underline{R}(\underline{x}) + \underline{A}_1 + \underline{A}_\ell + \underline{A}_3$$

$$\dot{J} = -\frac{1}{2c} J|\underline{A}_3|$$

$$\dot{K} = -\frac{1}{2c} K|\underline{A}_1|$$

$$\dot{L} = \frac{\alpha}{4}\frac{K^2}{L}\underline{A}_\ell{}^2$$

with the boundary conditions

$$\underline{x}(t_0) = \underline{x}_0 \qquad \dot{\underline{x}}(t_0) = \dot{\underline{x}}_0$$

$$\underline{x}(t_f) = \underline{x}_f \qquad \dot{\underline{x}}(t_f) = \dot{\underline{x}}_f$$

$$J(t_0) = 1 \qquad K(t_0) = 1$$

$$L(t_0) = 0^+ \qquad t_f \text{ is fixed}$$

and by definition

$$J_1 = J(t_f) \qquad K_1 = K(t_f)$$

$$L_1 = L(t_f)$$

The remainder of this section is a straightforward application of classical optimal control techniques. The differential equation constraints are adjoined to the cost function integral (none here) to form the Hamiltonian. The costate differential equations and boundary conditions are then presented and partially solved. The resultant form of the Hamiltonian is then maximized by specification of the control accelerations $\underline{A}_i$.

Using λ with an appropriate subscript for the costate variables, the Hamiltonian is given by

$$H = \underline{\lambda}^T(\underline{R}(\underline{x}) + \underline{A}_1+\underline{A}_\ell+\underline{A}_3)+\dot{\underline{\lambda}}^T\dot{\underline{x}}-\lambda_J \frac{1}{2c} J|\underline{A}_3|-\lambda_K \frac{1}{2c} K|\underline{A}_1|+\lambda_L \frac{\alpha}{4} \frac{K^2}{L} \underline{A}_\ell{}^2$$

The costate differential equations are found from H

$$\dot{\lambda}_J = -\frac{\partial H}{\partial J} = +\frac{1}{2c}\lambda_J|\underline{A}_3|$$

$$\dot{\lambda}_K = -\frac{\partial H}{\partial K} = +\frac{1}{2c}\lambda_K|\underline{A}_1| - \frac{\alpha}{2}\lambda_L\frac{K}{L}\underline{A}_\ell^{\,2}$$

$$\dot{\lambda}_L = -\frac{\partial H}{\partial L} = \frac{\alpha}{4}\frac{K^2}{L^2}\underline{A}_\ell^{\,2}\lambda_L$$

and after a couple of substitutions

$$\ddot{\underline{\lambda}} = \left(\frac{\partial \underline{R}(\underline{x})}{\partial \underline{x}}\right)^T \underline{\lambda}$$

The costate $\underline{\lambda}$ is actually the costate corresponding to the velocity, $\dot{\underline{x}}$, if only first derivatives were used.

Since the boundary conditions on $\underline{x}$ are completely specified, the boundary conditions on $\underline{\lambda}$ are free to be chosen (so that $\underline{x}_0$, $\underline{x}_f$, $\dot{\underline{x}}_0$, and $\dot{\underline{x}}_f$ are satisfied). The other costate boundary conditions are found from the payload m_π. The f subscripts indicate a final condition on the variable.

$$\lambda_{J_f} = \frac{\partial m_\pi}{\partial J_1} = \frac{2m_\pi}{J_1}$$

$$\lambda_{K_f} = \frac{\partial m_\pi}{\partial K_1} = \begin{cases} 2J_1^{\,2}(K_1 - L_1) & \text{if } m_e \leq m_p \\ \dfrac{2m_\pi}{K_1} & \text{if } m_e \geq m_p \end{cases}$$

$$\lambda_{L_f} = \frac{\partial m_\pi}{\partial L_1} = \begin{cases} -2J_1^{\,2}(K_1 - L_1) & \text{if } m_e \leq m_p \\ -\dfrac{2m_\pi L_1}{m_e + L_1^{\,2}} & \text{if } m_e \geq m_p \end{cases}$$

The differential equations for λ_J, λ_K, and λ_L can all be solved explicitly. In terms of the solutions J(t), K(t), and L(t) of the previous section, we have

$$J(t)\lambda_J(t) = J_1\lambda_{J_f} = 2m_\pi \tag{B.12}$$

$$K(t)\lambda_K(t) = K_1\lambda_{K_f} + L_1\lambda_{L_f} - L^2(t)\,\frac{\lambda_{L_f}}{L_1} \tag{B.13}$$

$$= \begin{cases} 2J_1^{\,2}(K_1-L_1)^2[1+\frac{1}{m_p}\,L^2(t)] & \text{if } m_e \le m_p \\ \dfrac{2m_\pi}{1+\frac{1}{m_e}\,L_1^{\,2}}\,[1+\frac{1}{m_e}\,L^2(t)] & \text{if } m_e \ge m_p \end{cases}$$

$$\frac{\lambda_L(t)}{L(t)} = \frac{\lambda_{L_f}}{L_1} \tag{B.14}$$

$$= \begin{cases} -2\,\dfrac{J_1^{\,2}}{L_1}(K_1-L_1) & \text{if } m_e \le m_p \\ -\,\dfrac{2m_\pi}{m_e+L_1^{\,2}} & \text{if } m_e \ge m_p \end{cases}$$

In terms of these solutions

$$H = \underline{\lambda}^T(\underline{R}(\underline{x}) + \underline{A}_1+\underline{A}_\ell+\underline{A}_3) + \dot{\underline{\lambda}}^T\dot{\underline{x}} - \delta_1(t)\,|\underline{A}_1| - \delta_3\,|\underline{A}_3| - \frac{1}{2Q}\,\underline{A}_\ell^{\,2}$$

where

$$\delta_1(t) = \frac{1}{c}\begin{cases} J_1^{\,2}(K_1-L_1)^2[1+\frac{1}{m_p}\,L^2(t)] & \text{if } m_e \le m_p \\ \dfrac{m_\pi}{1+\frac{1}{m_e}\,L_1^{\,2}}\,[1+\frac{1}{m_e}\,L^2(t)] & \text{if } m_e \ge m_p \end{cases} \tag{B.15}$$

$$\delta_3 = \frac{1}{c} m_\pi \tag{B.16}$$

$$Q(t) = \frac{1}{\alpha K^2(t)} \begin{cases} \dfrac{L_1}{J_1{}^2 (K_1 - L_1)} & \text{if } m_e \leq m_p \\ \\ \dfrac{m_e + L_1{}^2}{m_\pi} & \text{if } m_e \geq m_p \end{cases} \tag{B.17}$$

The Hamiltonian is maximized by the controls

$$\underline{A}_\ell = Q(t)\ \underline{\lambda}(t)$$

$$\underline{A}_1 = \begin{cases} \infty\ \underline{\lambda} & \text{if } |\underline{\lambda}| = \delta_1(t) \\ \\ 0 & \text{if } |\underline{\lambda}| < \delta_1(t) \end{cases}$$

$$\underline{A}_3 = \begin{cases} \infty\ \underline{\lambda} & \text{if } |\underline{\lambda}| = \delta_3 \\ \\ 0 & \text{if } |\underline{\lambda}| < \delta_3 \end{cases}$$

In summary the Hamiltonian, state and costate equations are

$$H = (|\lambda| - \delta_1(t))|\underline{A}_1| + (|\lambda| - \delta_3)|\underline{A}_3| + \frac{Q}{2}\ \underline{\lambda}^2 + \underline{\lambda}^T \underline{R}(\underline{x}) + \dot{\underline{\lambda}}^T \dot{\underline{x}}$$

$$\ddot{\underline{x}} = \underline{R}(\underline{x}) + \underline{\lambda}(Q(t) + |\underline{A}_1| + |\underline{A}_3|)$$

$$\ddot{\underline{\lambda}} = \left(\frac{\partial \underline{R}(\underline{x})}{\partial \underline{x}}\right)^T \underline{\lambda}$$

The costate vector $\underline{\lambda}$ as defined for this problem is often called the primer vector $\underline{p}$ in the classical literature. From the homogeneous differential equation in $\underline{p}(\underline{\lambda})$ it is clear that $\underline{p}$ and its first two derivatives are continuous when $\underline{A}_1$ or $\underline{A}_3$ are impulsively applied. This property will be used in later sections. In general the primer is the velocity costate, the coefficient of the acceleration vector in the Hamiltonian.

The limits of low thrust for this mixed thrust formulation is easily given. When $\underline{\lambda}$ is scaled such that $\underline{A}_\ell$ provides an adequate acceleration to accomplish the entire transfer and

$$|\underline{\lambda}(t)| < \delta_1(t)$$

$$|\underline{\lambda}(t_f)| < \delta_3$$

no high thrust will be used.

The limit of pure high thrust occurs at a discontinuity in the equations due to the manner in which the low thrust cost integral is defined. When no low thrust is used $\underline{A}_\ell = Q_1 = L_1 = m_p = 0$. There are difficulties in the $\frac{1}{m_p} L^2(t)$ term in $\delta_1(t)$ which can be easily circumvented by never using a zero m_e. If m_e is very small (.001 will do) there will be a negligible increment in the payload and some low thrust will always be used. Analytically, the nature of the high thrust limit is easily shown. The low thrust cost

$$L_1{}^2 = \frac{\alpha}{2} \int_0^{t_f} K^2(t)\underline{A}_\ell{}^2(t)dt$$

can be rewritten using

$$\underline{A}_\ell = Q(t)\underline{\lambda}(t)$$

and the expression for Q when $m_e = 0 < m_p$ to give

$$L_1^{\ 2} = \frac{L_1^{\ 2}}{(K_1 - L_1)^2} \ I_1^{\ 2}$$

where

$$I_1^{\ 2} = \frac{1}{2\alpha J_1^{\ 4}} \int_0^{t_f} \frac{\underline{\lambda}^2(t)}{K^2(t)} \, dt$$

Note that I_1 can take on a non-zero value, even if no low thrust is used!

Solving this equation for L we get

$$L_1 = K_1 \pm I_1$$

There are limits on L_1 implicit in the formulation of the payload

$$0 \leq L_1 \leq K_1$$

We must have

$$L_1 = K_1 - I_1$$

If

$$I_1 > K_1$$

then

$$L_1 = 0$$

which implies that Q, $\underline{A}_\ell$ and m_p are all zero. Thus pure high thrust occurs at a discontinuity in the equations which evolve from this formulation of the problem.

Appendix C

FIELD FREE SPACE TRANSFERS

The necessary conditions for arbitrary transfers in field free space are solved and applied to the specific problem of a change in position. The problem of a change in velocity is also solved. These problems are defined, and the results summarized in Chapter 3.

C.1 General transfers

The dynamic equations of motion for the low thrust phase are analytically determined for arbitrary, fixed time transfers in field free space. Since initial and final impulses are in the direction of the low thrust acceleration which is known, only their magnitudes, along with the boundary conditions on the primer, remain to be determined. As shown in Chapter 3, the primer and thus the low thrust acceleration for this problem are linear functions of time. A possible choice for $\underline{A}_\ell$ which satisfies this constraint is

$$\underline{A}_\ell = \frac{c}{t_f}\left[u_0\underline{u}\left(1 - \frac{t}{t_f}\right) + w_0\underline{w}\left(\frac{t}{t_f}\right)\right] \tag{C.1}$$

where by definition

$|\underline{u}| = |\underline{w}| = 1$

u_0, w_0 are positive dimensionless constants which remain to be determined

$\underline{u}$, $\underline{w}$ are unit vectors which remain to be determined

Optimal impulses are in the direction of the primer (and thus $\underline{A}_\ell$).

A first impulse will result in the change in velocity

$$\underline{V}_1 = c\, v_1 \underline{u} \tag{C.2}$$

where $v_1 \geq 0$ is the dimensionless magnitude of the first velocity change. Similarly, the change in velocity due to a final impulse in the direction of $\underline{A}_\ell(t_f)$ is

$$\underline{V}_2 = c\, v_2 \underline{w} \tag{C.3}$$

where $v_2 \geq 0$ is the dimensionless magnitude of the final velocity change. Intermediate impulses are not allowed. The optimality of this assumption must be verified for any specific transfer by comparing $|\underline{A}_\ell(t)|$ with $\delta_A(t)$.

The change in velocity during the low thrust phase is easily obtained by integrating $\underline{A}_\ell$. The initial condition is $\underline{V}_1$ and the final condition is $c\,\underline{V} - \underline{V}_2$, so that after the final impulse we have the desired final velocity. These are conditions on the boundaries of the low thrust phase, not the total transfer.

$$\begin{aligned}\dot{\underline{x}}(t) &= \underline{V}_1 + \int_0^t \underline{A}_\ell dt \\ &= c\left[v_1\underline{u} + u_0\underline{u}\left(\frac{t}{t_f} - \frac{1}{2}\frac{t^2}{t_f^2}\right) + \frac{1}{2} w_0\underline{w}\left(\frac{t}{t_f}\right)^2\right]\end{aligned}$$

At the end of the low thrust phase

$$\dot{\underline{x}}(t_f) = c(\underline{V} - v_2\underline{w}) = c[(v_1 + \tfrac{1}{2} u_0)\underline{u} + \tfrac{1}{2} w_0\underline{w}]$$

or

$$\underline{V} = \underline{u}(v_1 + \frac{1}{2} u_0) + \underline{w}(v_2 + \frac{1}{2} w_0) \qquad (C.4)$$

The position integral is also easily obtained

$$\underline{x}(t) = \underline{0} + \int_0^t \dot{\underline{x}}(t)\, dt$$

$$= c\, v_1 \underline{u}\, t + c\, u_0 \underline{u} \left(\frac{1}{2} \frac{t^2}{t_f} - \frac{1}{6} \frac{t^3}{t_f^{\,2}}\right) + \frac{1}{6} c\, w_0 \underline{w} \frac{t^3}{t_f^{\,2}}$$

At the final time

$$\underline{x}(t_f) = \frac{ct_f}{\tau} \underline{s} = c\, t_f[(v_1 + \frac{1}{3} u_0)\underline{u} + \frac{1}{6} w_0 \underline{w}] \qquad (C.5)$$

The solutions for $u_0\underline{u}$ and $w_0\underline{w}$ are found by satisfying equations C.4 and C.5.

$$\begin{Bmatrix} u_0\underline{u} \\ \\ w_0\underline{w} \end{Bmatrix} = \frac{1}{\det} \begin{bmatrix} \frac{1}{6} & -\left(\frac{1}{2} + \frac{v_2}{w_0}\right) \\ \\ -\left(\frac{1}{3} + \frac{v_1}{u_0}\right) & \left(\frac{1}{2} + \frac{v_1}{u_0}\right) \end{bmatrix} \begin{Bmatrix} \underline{V} \\ \\ \frac{1}{\tau}\underline{s} \end{Bmatrix}$$

where

$$\det = -\left(\frac{1}{12} + \frac{1}{3}\left(\frac{v_1}{u_0} + \frac{v_2}{w_0}\right) + \frac{v_1}{u_0}\frac{v_2}{w_0}\right)$$

The magnitudes u_0 and w_0 are determined by the solution of

$$u_0^{\,2} = \frac{1}{\det^2}\left[\frac{1}{36}\underline{V}^T\underline{V} + \left(\frac{1}{2} + \frac{v_2}{w_0}\right)^2 \frac{\underline{s}^T\underline{s}}{\tau^2} - \frac{1}{3\tau}\left(\frac{1}{2} + \frac{v_2}{w_0}\right)\underline{V}^T\underline{s}\right]$$

$$w_0{}^2 = \frac{1}{\det^2}\left[\left(\frac{1}{3} + \frac{v_1}{u_0}\right)^2 \underline{V}^T\underline{V} + \left(\frac{1}{2} + \frac{v_1}{u_0}\right)^2 \frac{\underline{s}^T\underline{s}}{\tau^2}\right.$$

$$\left. - \frac{2}{\tau}\left(\frac{1}{3} + \frac{v_1}{u_0}\right)\left(\frac{1}{2} + \frac{v_1}{u_0}\right)\underline{V}^T\underline{s}\right]$$

The only undetermined parameters are v_1 and v_2, the dimensionless magnitudes of the two impulses. The two conditions that the primer be equal to the threshold at the times of each impulse are used to determine v_1 and v_2. The thresholds are in terms of the integrals J_1, K_1, and L_1. By its definition, the integral L_1 as given by

$$L_1{}^2 = \frac{\alpha}{2}\int_0^{t_f} K^2(t)\underline{A}_\ell{}^2(t)\ dt$$

is found to be

$$L_1{}^2 = \frac{K_1{}^2}{6\tau}\left(u_0{}^2 + w_0{}^2 + u_0 w_0 \underline{u}^T\underline{w}\right)$$

$$\frac{L_1}{K_1} = \frac{1}{\sqrt{6\tau}}\sqrt{u_0{}^2 + w_0{}^2 + u_0 w_0 \underline{u}^T\underline{w}}$$

The integrals K_1 and J_1 are simple expressions due to the definitions of v_1 and v_2.

$$K_1 = \exp\left(-\frac{1}{2}v_1\right)$$

$$J_1 = \exp\left(-\frac{1}{2}v_2\right)$$

At the initial time

$$|\underline{A}_\ell(0)| = \frac{c}{t_f} u_0 = \frac{1}{\alpha c} \frac{u_0}{\tau}$$

and at the final time

$$|\underline{A}_\ell(t_f)| = \frac{c}{t_f} w_0 = \frac{1}{\alpha c} \frac{w_0}{\tau}$$

Thus v_1 and v_2 are determined by the conditions at the times of the impulses. Either

$$\frac{u_0}{\tau} = \alpha c\ \delta_A \qquad \underline{\text{or}} \quad (v_1 = 0 \quad \underline{\text{and}} \quad \frac{u_0}{\tau} \leq \alpha c\ \delta_A)$$

and

$$\frac{w_0}{\tau} = \alpha c\ \delta_B \qquad \underline{\text{or}} \quad (v_2 = 0 \quad \underline{\text{and}} \quad \frac{w_0}{\tau} \leq \alpha c\ \delta_B)$$

These are four conditions which can be used to determine the four parameters u_0, w_0, v_1, and v_2. These results are summarized in Chapter 3.

C.2 Change in position

The specialization of the previous general transfer to one which changes only position ($\underline{V} = \underline{0}$) has a complete analytic solution if $m_e=0$. With $\underline{V} = 0$, the coordinates can be chosen so that the problem involves only a single dimension. Thus let

$$\underline{s} = s > 0$$

The pure high thrust solution applies an initial impulse in the desired direction and an equal and opposite final impulse at the final position such that $\underline{V} = 0$. The magnitude of these velocity changes depends upon the time of flight. The pure low thrust solution is shown in Figure C.1. The primer is a linear function of time. Since the integral

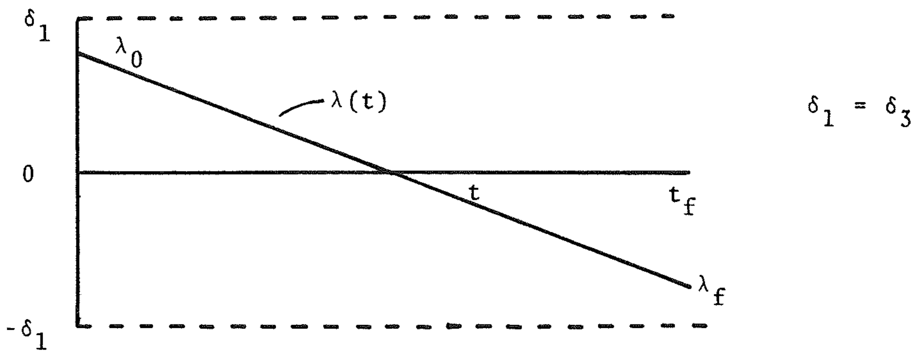


Figure C.1 Relationship of the primer to the thresholds

of $\underline{A}_\ell$ must be zero for $\underline{V} = 0$, the initial and final values of $|\underline{A}_\ell|$ must be equal as shown. For the mixed thrust, when $m_e = 0$, the two thresholds δ_A and δ_B are equal. Thus if $|A_\ell(0)| = \delta_A$, then $|A_\ell(t_f)| = \delta_B$ and vice-versa. If both impulses are used, they must be equal in magnitude, and in opposite directions. For this case, it is not possible to use only one impulse. Any impulse must be in the direction of the primer, and A_ℓ can not have a negative (positive) integral to balance an initial positive (negative) impulse since $|A_\ell(t_f)| < |A_\ell(0)|$. Thus for the mixed thrust with $\delta_A = \delta_B$, we must have

$$\frac{u_0}{\tau} = \frac{w_0}{\tau}$$

Since

$$u_0^{\,2} = \left[\frac{1}{\det}\left(\frac{1}{2} + \frac{v_2}{w_o}\right)\frac{s}{\tau}\right]^2$$

$$w_0^{\,2} = \left[\frac{1}{\det}\left(\frac{1}{2} + \frac{v_1}{u_0}\right)\frac{s}{\tau}\right]^2$$

where

$$\det = -\left(\frac{1}{12} + \frac{1}{3}\left(\frac{v_1}{u_0} + \frac{v_2}{w_0}\right) + \frac{v_1}{u_0}\,\frac{v_2}{w_0}\right)$$

we must also have

$$v_1 = v_2$$

After some algebra we get

$$v_1 = \frac{s}{\tau} - \frac{1}{6}\,u_0$$

Using these values we get

$$\underline{u} = +\,1$$

$$\underline{w} = -\,1$$

for the vector directions which define the direction of the high and low thrust controls. Using these values we get

$$\frac{L_1}{K_1} = \frac{u_0}{\sqrt{6\tau}}$$

Applying the condition at the threshold

$$\frac{u_0}{\tau} = \alpha c \delta_A = \frac{u_0}{\sqrt{6\tau}} \left(1 - \frac{u_0}{\sqrt{6\tau}}\right)$$

we get

$$u_0 = \sqrt{6\tau} - 6$$

and

$$\frac{L_1}{K_1} = 1 - \sqrt{\frac{6}{\tau}}$$

The solution $u_0 = 0$ corresponds to pure high thrust which will be discussed later. To complete the definition of the problem

$$v_1 = v_2 = \frac{s}{\tau} + 1 - \sqrt{\frac{\tau}{6}}$$

and the mixed thrust payload is

$$m_\pi = \frac{6}{\tau} \exp\left[-2\left(\frac{s}{\tau} + 1 - \sqrt{\frac{\tau}{6}}\right)\right]$$

The constants u_0 and v_1 must be positive by their definition. u_0 is positive if

$$\tau \geq 6$$

and v_1 is positive if

$$\tau \leq 6(1 + \frac{s}{\tau})^2$$

If $\tau \leq 6$, pure high thrust should be used ($u_0 = 0$). The resultant payload is

$$m_\pi = \exp\left(-2 \frac{s}{\tau}\right)$$

If $\tau \geq 6\ (1 + \frac{s}{\tau})^2$, pure low thrust should be used. The resultant payload is determined by

$$u_0 = 6 \frac{s}{\tau}$$

and

$$\frac{L_1}{K_1} = \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}$$

Since we are not matching a threshold condition, the pure low thrust payload can be given for all m_e

$$m_\pi = \begin{cases} m_e + \left(1 - \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}\right)^2 & \text{if } m_e \leq m_p \\ \dfrac{1}{1 + \frac{1}{m_e}\,\frac{6s^2}{\tau^3}} & \text{if } m_e \leq m_p \end{cases}$$

where

$$m_p = \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}\left(1 - \sqrt{\frac{6}{\tau}}\,\frac{s}{\tau}\right)$$

This completes the description of the optimal payloads for a change in position in field free space, since other analytic solutions are not possible.

C.3 Change in velocity

The solution is found for a change in velocity in field free space as defined in Chapter 3. Since, by assumption, the dimensionless change in velocity, V, is positive, the acceleration A_ℓ will also be positive. Let

$$A_\ell = \frac{c}{t_f} u = \frac{1}{\alpha c} \frac{u}{\tau}$$

where u is a positive dimensionless constant which must be determined. The change in velocity, V, due to an initial impulse is optimally in the direction of A_ℓ. Let

$$V_1 = c\, v_1$$

where $v_1 \geq 0$ is the dimensionless magnitude of the first (and only) high thrust velocity change. Integrating the equation of motion for the state we get

$$\begin{aligned} \dot{\underline{x}}(t) &= V_1 + \int_0^t A_\ell\,(t)dt \\ &= c\left(v_1 + u\,\frac{t}{t_f}\right) \end{aligned}$$

At the final time

$$\dot{\underline{x}}(t_f) = c\,V = c\,(v_1 + u)$$

Thus v_1 is specified in terms of V and u

$$v_1 = V - u$$

The necessary condition on A_ℓ at the time of the impulse determines u. We must have

$$A_\ell = \frac{1}{\alpha c} \frac{u}{\tau} = \delta_A$$

or

$$v_1 = 0 \qquad \text{and } A_\ell < \delta_A$$

where as before

$$\delta_A = \frac{1}{\alpha c} \begin{cases} \frac{L_1}{K_1}\left(1 - \frac{L_1}{K_1}\right) & \text{if } m_e \leq m_p \\ \frac{m_e}{K_1^{\,2}} & \text{if } m_e \geq m_p \end{cases}$$

and

$$m_p = L_1 \, (K_1 - L_1)$$

Some integrals of the controls are necessary for the specification of the thresholds, Because of the definition of v_1

$$K_1 = \exp\left(-\frac{1}{2} v_1\right)$$

Also

$$L_1^{\,2} = \frac{\alpha}{2} \int_0^{t_f} K^2(t) \, A_\ell^{\,2}(t) \, dt = \frac{1}{2\tau} K_1^{\,2} \, u^2$$

Thus

$$\frac{L_1}{K_1} = \frac{u}{\sqrt{2\tau}}$$

The optimal controls can now be completely determined. If $m_e \leq m_p$, $A_\ell = \delta_A$ if

$$\frac{u}{\tau} = \frac{u}{\sqrt{2\tau}} \left(1 - \frac{u}{\sqrt{2\tau}}\right)$$

which is solved to give

$$u = \sqrt{2\tau} - 2$$

and then

$$v_1 = V - \sqrt{2\tau} + 2$$

These equations are only valid when the constraints $u \geq 0$, $v_1 \geq 0$, and $m_e \leq m_p$ are satisfied by

$$2 \leq \tau \leq 2 \left(1 + \frac{V}{2}\right)^2$$

and

$$m_e \leq \sqrt{\frac{2}{\tau}} \left(1 - \sqrt{\frac{2}{\tau}}\right) \exp \left(- V + \sqrt{2\tau} - 2\right)$$

If $\tau \leq 2$, the solution $u = 0$ must be used to match the threshold. This possibility is discussed later.

The payload for this case is

$$m_\pi = m_e + \quad \exp \left(-V + \sqrt{2\tau} - 2 \right)$$

The second expression for the mixed thrust payload is obtained for $m_e \geq m_p$ when $A_\ell = \delta_A$ as determined by

$$\frac{u}{\tau} = \frac{m_e}{K_1{}^2}$$

and as before

$$v_1 = V - u$$

Since

$$K_1{}^2 = \exp \left(- \quad v_1 \right)$$

it must satisfy the transcendental equation

$$K_1{}^2 = \exp \left(- V + \tau \frac{m_e}{K_1{}^2} \right) \qquad \text{(C.6)}$$

which must be solved numerically. In terms of K_1

$$L_1 = \frac{\tau}{2} \frac{m_e}{K_1}$$

The payload expression

$$m_\pi = \frac{K_1{}^2}{1 + \frac{\tau}{2} \frac{m_e}{K_1{}^2}}$$

where K_1 is the solution of equation C.6, is applicable when $u \geq 0$, $v_1 \geq 0$ $(K_1 \leq 1)$, and $m_e \geq m_p$ as expressed by the relations

$$m_e \geq 0$$

$$m_e \leq \frac{V}{\tau}$$

$$m_e \geq \begin{cases} \frac{2}{\tau}\left(\sqrt{\frac{\tau}{2}} - 1\right) \exp\left(-V + \sqrt{2\tau} - 2\right) & \text{if } \tau \geq 2 \\ 0 & \text{if } \tau \leq 2 \end{cases}$$

Outside the bounds of applicability of the mixed thrust payload expressions, either pure high, or pure low thrust will be used. Use pure low thrust when

$$v_1 = 0 = V - u$$

or

$$u = V$$

for which

$$L_1 = \frac{V}{\sqrt{2\tau}}$$

and the pure low thrust payload is

$$m_\pi = \begin{cases} m_e + \left(1 - \frac{V}{\sqrt{2\tau}}\right)^2 & \text{if } m_e \leq m_p \\ \dfrac{1}{1 + \frac{1}{2\tau m_e} V^2} & \text{if } m_e \geq m_p \end{cases}$$

where

$$m_p = \frac{V}{\sqrt{2\tau}} \left(1 - \frac{V}{\sqrt{2\tau}}\right)$$

If $m_e \leq m_p$, pure low thrust is used when $A_\ell \leq \delta_A$ is satisfied by

$$\tau \geq 2 \left(1 + \frac{V}{2}\right)^2$$

If $m_e \geq m_p$, that relation is given by

$$m_e \geq \frac{V}{\tau}$$

Note that $m_p = \frac{V}{\tau}$ for $\tau = 2 \left(1 + \frac{V}{2}\right)^2$ Pure high thrust is used if $u = 0$, and $v_1 = V$ for which

$$m_\pi = \exp(-V)$$

As shown earlier, $u = 0$ is the only non-negative solution of $A_\ell = \delta_A$ for $\tau \leq 2$ if $m_e \leq m_p$. If $u = 0$, $m_p = 0$. Thus use pure high thrust for

$$\tau \leq 2 \qquad\qquad m_e = 0$$

There are five expressions for the payload. Choice of the one to use is determined by the relative magnitudes of τ, m_e, and V. These results are summarized in Chapter 3.

Appendix D

THE NECESSARY CONDITIONS FOR TRANSFERS BETWEEN COPLANAR COAXIAL ELLIPSES

Necessary conditions for the maximization of payload on transfers between coplanar coaxial ellipses are found. Ideal high and low thrust engines as defined in Chapter 2 are used to provide the propulsion and the differential equations of motion introduced in Chapter 4 define the transfer. Additional notation is introduced in order to simplify many of the expressions. Using the assumptions of Chapters 2 and 4, the equations of motion can be completely integrated for high and low thrust phases. Using these results, the payload can be analytically expressed as a function of the parameters necessary to define the various phases. It is not possible to analytically determine the relative amount of high and low thrust used due to the existence of transcendental functions in the necessary conditions. This appendix contains the analytics which reduce the dynamic optimization problem to a set of necessary conditions for the parametric maximization of an expression for the payload.

For transfers between coplanar coaxial ellipses, analytic expressions are known for ideal high and low thrust transfers. For the combination, we wish to determine the payload improvements and the character of the transfers. Certain assumptions must be made about the mode shapes of the transfers and then verified by checking other relations. The derivations of this appendix assume two high thrust phases separated by a low thrust phase. The first high thrust phase can have two impulses at opposite apses of a coasting orbit. The second high thrust phase allows one impulse at the final apse. The low thrust phase accomplishes the remaining portion of the transfer between these high thrust phases by the application of a small continuous control. Pure high thrust transfers occur when the first high thrust phase accomplishes the entire transfer. If no high thrust impulses are used, the result of course is a pure low thrust transfer.

For this assumed mode of transfer, the general optimization problem can be stated using the results of Chapter 2. After introducing some necessary notation, the problem is divided into the separate analysis of the high and low thrust phases. The necessary equations of each phase are derived prior to their combination as an expression for the mixed thrust payload and the necessary conditions for its optimality. These necessary conditions are then related to a set of derivatives convenient for the numerical maximization of the payload expression. Expressions for the second derivatives of the payload are also derived for use in the numerical procedures. Finally a simpler suboptimal transfer is derived.

D.1 The optimal control problem and some nomenclature

The general optimal control problem of Chapter 2 will be stated for the specific state differential equations of Chapter 4. Under the mode shape assumptions of this appendix, the necessary conditions for this problem will be stated. A simplifying notation is then introduced for use in later sections of this appendix.

As shown in Chapter 2, for the maximization of the payload

$$m_\pi = \begin{cases} J_1^{\,2}[(K_1-L_1)^2 + m_e] & \text{if } m_e \le m_p \\ \dfrac{J_1^{\,2}K_1^{\,2}}{1 + \dfrac{1}{m_e} L_1^{\,2}} & \text{if } m_e \ge m_p \end{cases}$$

where

$$m_p = L_1(K_1-L_1)$$

governed by the differential equations

$$\dot{\underline{x}} = \underline{B}(\underline{x},t)(\underline{A}_{11} + \underline{A}_{12} + \underline{A}_{\ell} + \underline{A}_3) \qquad \underline{x}(0) = \underline{x}_0, \quad \underline{x}(t_f) = \underline{x}_f$$

$$\dot{J} = -\frac{1}{2c} J|\underline{A}_3| \qquad J(0) = 1$$

$$\dot{K} = -\frac{1}{2c} K(|\underline{A}_{11}| + |\underline{A}_{12}|) \qquad K(0) = 1$$

$$\dot{L} = \frac{\alpha}{4}\frac{K^2}{L}\underline{A}_{\ell}^{\,2} \qquad L(0) = 0^+$$

the Hamiltonian

$$H = \underline{p}^T\underline{A}_{\ell} - \frac{1}{2Q}\underline{A}_{\ell}^2 + \underline{p}_1{}^T\underline{A}_{11} - \delta_1|\underline{A}_{11}| + \underline{p}_2{}^T\underline{A}_{12} - \delta_1|\underline{A}_{12}| + \underline{p}_3{}^T\underline{A}_3 - \delta_3|\underline{A}_3|$$

is maximized by the optimal controls

$$\underline{A}_{11} = \begin{cases} \infty\underline{p}_1 & |\underline{p}_1| = \delta_1 \\ 0 & |\underline{p}_1| < \delta_1 \end{cases}$$

$$\underline{A}_{12} = \begin{cases} \infty\underline{p}_2 & |\underline{p}_2| = \delta_1 \\ 0 & |\underline{p}_2| < \delta_1 \end{cases}$$

$$\underline{A}_3 = \begin{cases} \infty\underline{p}_3 & |\underline{p}_3| = \delta_3 \\ 0 & |\underline{p}_3| < \delta_3 \end{cases}$$

$$\underline{A}_{\ell} = Q(t)\underline{p}(t)$$

where

$$\underline{p}(t) = \underline{B}^T(\underline{x},t)\underline{\lambda}(t)$$

$$Q(t) = \frac{1}{\alpha K^2(t)} \begin{cases} \dfrac{L_1}{J_1{}^2(K_1-L_1)} & \text{if } m_e \leq m_p \\[2ex] \dfrac{m_e+L_1{}^2}{m_\pi} & \text{if } m_e \geq m_p \end{cases}$$

and $\underline{\lambda}$ is governed by the differential equation

$$\dot{\underline{\lambda}}^T = - \frac{\partial H}{\partial \underline{x}} = - \underline{\lambda}^T \frac{\partial \underline{B}(\underline{x},t)}{\partial \underline{x}} (\underline{A}_{11} + \underline{A}_{12} + \underline{A}_\ell + \underline{A}_3)$$

with the boundary conditions on $\underline{\lambda}$ free, but implicitly specified by the other necessary conditions. The subscripts on $\underline{p}$ indicate the evaluation of $\underline{p}$ at the apse and time appropriate for each of the impulses. The thresholds δ are given by

$$\delta_1 = \frac{1}{c} \begin{cases} J_1{}^2(K_1-L_1)^2 & \text{if } m_e \leq m_p \\[2ex] \dfrac{m_\pi}{1+\dfrac{1}{m_e} L_1{}^2} & \text{if } m_e \geq m_p \end{cases}$$

$$\delta_3 = \frac{1}{c} m_\pi$$

To verify the validity of the timing assumptions for the impulses, the primer, $\underline{p}$, at the other initial and final apses must be smaller in magnitude than the primer at the apses of the impulses. The optimality of the assumed mode requires

$$|\underline{p}(t)| < \delta_1(t)$$

during the entire low thrust phase. Otherwise, a small intermediate impulse would locally increase the payload. This appendix derives explicit expressions for the various parameters stated here in the payload and necessary conditions.

Before the problem of mixed thrust transfers is solved, some notation needs to be established. There are three dimensionless constants which are used

$$\beta_i = \sqrt{\frac{\mu}{4c^2 a_i}} = \text{dimensionless gravitational constant} \tag{D.1}$$

$$\tau = \frac{t_f}{\alpha c^2} = \text{dimensionless time of flight} \tag{D.2}$$

$$\rho = \sqrt{\frac{a_0}{a_f}} = \text{dimensionless change in semi-major axis} \tag{D.3}$$

where

μ = the gravitational constant for the basic orbital differential equation (D.4)

$$\ddot{\underline{R}} = - \frac{\mu}{|\underline{R}|^3} \underline{R}$$

c = exhaust velocity of the high thrust engine

α = the reciprocal specific power of the low thrust power supply

a_i = the semi-major axis on the same orbit as indicated by θ_i. (The subscript will be explained later)

Note that the same definition of τ was used for field free space transfers. The constants β_0, ρ, θ_0, θ_f, and τ completely specify the orbital transfers being considered here.

Transfers are discussed in terms of the change in semi-major axis, a, and a variable θ which is related to eccentricity by

$$\cos 2\theta_i = e_i$$

This was chosen because the terms $\sqrt{1 \pm e}$ are frequently encountered. Note that

$$\cos \theta = \sqrt{\frac{1+e}{2}}$$

$$\sin \theta = \sqrt{\frac{1-e}{2}}$$

In discussing the transfers we are interested in similar equations for the primer at the opposite apses of an ellipse. Choosing

$$T = - \cos f = - \cos E$$

when $E = f = 0$ or π we have

$$T = \begin{cases} +1 & \text{at apoapse} \\ -1 & \text{at periapse} \end{cases}$$

Define two functions of θ such that

$$F = \sqrt{\frac{1 + T e}{2}} \tag{D.5}$$

$$G = T \sqrt{\frac{1 - T e}{2}} \tag{D.6}$$

From this definition

$$\frac{dF}{d\theta} = -G \qquad \frac{dG}{d\theta} = F$$

and

$$\frac{d(\frac{G}{F})}{d\theta} = 1 + (\frac{G}{F})^2$$

of course we also have the identity

$$F^2 + G^2 = 1$$

There are two parameters of interest at an apse, the radius and velocity. These are given by

$$R = a(1+T\ e)$$

$$v^2 = \frac{\mu}{R}(1 - T\ e)$$

The different phases of the transfer can be identified by appropriately defined θ's. Let

$$\begin{array}{ll} \theta_0 & \text{be on the initial orbit before any thrusting} \\ \theta_1 & \text{be on a coasting orbit after the first impulse} \\ \theta_2 & \text{be at the beginning of the low thrust phase} \\ & \text{(after the second impulse)} \\ \theta_3 & \text{be at the end of the low thrust phase} \\ & \text{(before the final impulse)} \\ \theta_f & \text{be on the final orbit after all thrusting is completed} \end{array} \tag{D.7}$$

The intermediate semi-major axes are all related to these θ's.

In order to simplify notation only two T's will be used to define the apses. Let

$$T_0 = +1 \tag{D.8}$$

when the first impulse is at periapse and the second impulse is at apoapse and $T_0 = -1$ if the opposite. Then let

$$F_0 = \sqrt{\frac{1+T_0 e_0}{2}} \qquad G_0 = T_0\sqrt{\frac{1-T_0 e_0}{2}}$$

$$F_1 = \sqrt{\frac{1+T_0 e_1}{2}} \qquad G_1 = T_0\sqrt{\frac{1-T_0 e_1}{2}}$$

$$F_2 = \sqrt{\frac{1+T_0 e_2}{2}} \qquad G_2 = T_0\sqrt{\frac{1-T_0 e_2}{2}}$$

With this definition, the radii at the first two impulses are

$$R_1 = 2a_0 G_0{}^2 = 2a_1 G_1{}^2$$

$$R_2 = 2a_1 F_1{}^2 = 2a_2 F_2{}^2$$

Thus we have

$$a_1 = a_0 \left(\frac{G_0}{G_1}\right)^2$$

$$a_2 = a_0 \left(\frac{G_0}{F_2} \; \frac{F_1}{G_1}\right)^2$$

From this relation we get

$$\beta_2 = \beta_0 \frac{G_1 F_2}{F_1 G_0}$$

At the final time, let

$$T_f = +1 \tag{D.9}$$

if the final impulse is at apoapse, and $T_f = -1$ if at periapse. Thus

$$F_3 = \sqrt{\frac{1+T_f e_3}{2}} \qquad , \qquad G_3 = T_f \sqrt{\frac{1-T_f e_3}{2}}$$

$$F_f = \sqrt{\frac{1+T_f e_f}{2}} \qquad\qquad G_f = T_f \sqrt{\frac{1-T_f e_f}{2}}$$

and the radius at the final impulse is

$$R_3 = 2 a_f F_f^{\,2} = 2 a_3 F_3^{\,2}$$

Thus

$$a_f = \frac{a_0}{\rho^2}$$

$$a_3 = \frac{a_0}{\rho^2} \left(\frac{F_f}{F_3}\right)^2$$

Two parameters are useful in the description of the low thrust phase. Let

$$\psi = \sqrt{\frac{8}{5}} \, (\theta_3 - \theta_2) \tag{D.10}$$

and

$$\gamma = \sqrt{\frac{a_2}{a_3}}$$

by definition. From the previous relations we have

$$\gamma = \rho\left(\frac{F_3 G_0 F_1}{F_f F_2 G_1}\right) \tag{D.11}$$

Using these definitions, the parameters which are used in the necessary conditions can be derived in a more concise notation.

D.2 The low thrust phase

During the low thrust phase of a mixed mode transfer, the high thrust is zero ($\underline{A}_{11} = \underline{A}_{12} = \underline{A}_3 = 0$) and certain aspects of the equations simplify. By the assumption of Chapter 4, there is only a small change in the orbital parameters during any one orbit. The periodic and secular variations in the orbital parameters can be separated by averaging the equations of motion to eliminate the periodic terms. Then the differential equations for the secular variation in parameters can be completely integrated and the free boundary conditions on the costate $\underline{\lambda}$ completely determined. Finally the analytic expressions for the payload, and time history of the primer and state are given to complete the solution for the low thrust phase.

The Hamiltonian can be explicitly given using the expression for $\underline{B}(\underline{x},t)$ in equation 4.2.

$$H = \frac{1}{2} Q_1 \underline{p}(t)^T \underline{p}(t) = \frac{1}{2} Q_1 \underline{\lambda}^T \underline{B}\ \underline{B}^T \underline{\lambda}$$

$$= \frac{\frac{1}{2} Q_1 \frac{a}{\mu}}{1-e \cos E} [4(\lambda_1 a)^2 + \frac{5}{8} \lambda_2{}^2 + \frac{3}{8} \lambda_2{}^2 \cos 2E$$

$$- \frac{1}{4e} \cos^3 E + \cos E(4(\lambda_1 a)^2 e - 4\lambda_1 a\lambda_2 \sin 2\theta - \frac{3}{4} \lambda_2{}^2 e)]$$

The complete dynamics of the low thrust phase are given by the Hamiltonian since the state and costate can be obtained by the canonical differential equations

$$\dot{\underline{x}} = \frac{\partial H}{\partial \underline{\lambda}}^T \qquad \dot{\underline{\lambda}}^T = - \frac{\partial H}{\partial \underline{x}}$$

The periodic terms in H (and implicitly in the other equations) can be eliminated by taking the time average of H over any orbital period.

$$H_1 = \frac{1}{2\pi} \int_E^{E+2\pi} H(1-e \cos E)dE$$

$$= \frac{1}{2} Q_1 \frac{a}{\mu} [4(\lambda_1 a)^2 + \frac{5}{8} \lambda_2{}^2]$$

Since time does not appear as an explicit variable, the averaged Hamiltonian, H_1, is a constant. The state and costate variables in H_1 must now be regarded as averaged variables, since the information about their periodic variation has been lost. The differential equations for the averaged state and costate are still related to H_1 by the differential equations given above.

Carrying out the indicated partial differentiation, we get the set of differential equations.

$$\frac{da}{dt} = \frac{4Q_1}{\mu} \lambda_1 a^3$$

$$\frac{d\theta}{dt} = \frac{5}{8} \frac{Q_1}{\mu} \lambda_2 a$$

$$\frac{d\lambda_1}{dt} = - \frac{1}{a} H_1 - \frac{4Q_1}{\mu} (\lambda_1 a)^2$$

$$\frac{d\lambda_2}{dt} = 0$$

with the boundary conditions

$$a(0) = a_2 \qquad a(t_f) = a_3$$

$$\theta(0) = \theta_2 \qquad \theta(t_f) = \theta_3$$

by the earlier definition of the subscripts.

The following steps are used to analytically integrate these equations. Of course we have

$$\lambda_2 = \lambda_{20} \tag{D.12}$$

The differential equations for λ_1 and a can be combined to give

$$\frac{d(\lambda_1 a)}{dt} = a \frac{d\lambda_1}{dt} + \lambda_1 \frac{da}{dt}$$

$$= -H_1$$

which can be integrated easily

$$\lambda_1 a = \lambda_{10} a_2 - H_1 t \tag{D.13}$$

Now a can be determined from

$$\frac{1}{a^2} \frac{da}{dt} = - \frac{d}{dt}\left(\frac{1}{a}\right) = \frac{4Q_1}{\mu}(\lambda_1 a)$$

to give

$$\frac{a_2}{a(t)} = 1 - \frac{4Q_1 a_2}{\mu} \lambda_{10} a_2 t + \frac{2Q_1 a_2}{\mu} H_1 t^2$$

For θ we have

$$\frac{d\theta}{dt} = \frac{5}{8} \frac{Q_1}{\mu} \lambda_{20} a$$

$$= \frac{\frac{5}{8} \frac{Q_1}{\mu} \lambda_{20} a_2}{1 - \frac{4Q_1}{\mu} \lambda_{10} a_2{}^2 t + \frac{2Qa_2}{\mu} H_1 t^2}$$

Thus

$$\theta - \theta_2 = \sqrt{\frac{5}{8}} \left[\tan^{-1}\left(\frac{-2\lambda_1 a}{\sqrt{\frac{5}{8}}\,\lambda_{20}}\right) - \tan^{-1}\left(\frac{-2\,\lambda_{10} a_2}{\sqrt{\frac{5}{8}}\,\lambda_{20}}\right)\right]$$

From basic trigonometry we get

$$\sin\sqrt{\frac{8}{5}}\,(\theta-\theta_2) = \sqrt{\frac{5}{8}}\,\lambda_{20} \frac{Q_1 a_2}{\mu} \sqrt{\frac{a(t)}{a_2}}\, t$$

The initial conditions on $\underline{\lambda}$ can be determined from the terminal conditions on $\underline{x}$. These are

$$\lambda_{10} a_2 = \frac{1}{2} \frac{\mu}{Q_1 a_2 t_f} (1 - \gamma \cos \psi) \tag{D.14}$$

$$\lambda_{20} = \sqrt{\frac{8}{5}} \frac{\mu}{Q_1 a_2 t_f} (\gamma \sin \psi) \tag{D.15}$$

The other solution

$$\lambda_{10} a_2 = \frac{1}{2} \frac{\mu}{Q_1 a_2 t_f} (1 + \gamma \cos \psi)$$

was discarded as it does not maximize the payload m_π. Thus for the Hamiltonian we have

$$H_1 = \frac{1}{2} \frac{\mu h}{Q_1 a_2 t_f{}^2}$$

with

$$h = 1 - 2\gamma \cos \psi + \gamma^2$$

Similarly the boundary conditions on $\underline{\lambda}$ can be substituted into the time history of $\underline{x}$. The low thrust acceleration is given by

$$\underline{A}_\ell(t) = Q_1 \underline{B}^T \underline{\lambda}$$

$$\underline{A}_\ell{}^2 = Q_1{}^2 \underline{\lambda}^T \underline{B}\ \underline{B}^T \underline{\lambda} = 2 Q_1 H$$

The low thrust cost $L_1{}^2$ is an integral of $\underline{A}_\ell{}^2$ over the entire transfer. Over an integral number of orbits, H and H_1 have identical integrals. Thus

$$L^2(t) = \frac{\alpha}{2} \int_0^t K^2(t)\underline{A}_\ell^{\,2}(t)\; dt$$

$$= \alpha Q K_1^{\,2} H_1 t$$

$$L(t) = K_1 \beta_2 \sqrt{\frac{2h}{\tau}} \sqrt{\frac{t}{t_f}}$$

The only time varying portion of the threshold $\delta_1(t)$ is $L^2(t)$. For this problem $\delta_1(t)$ is a linearly increasing function of time. The rate of increase is related to the amount of low thrust used during the transfer. If there is no low thrust, $L(t) = 0$ and the threshold will be constant.

In the notation of this appendix, the primer at an apse is given by

$$\underline{p} = \underline{B}^T \underline{\lambda} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$$

with

$$p = T \sqrt{\frac{a}{\mu}} \left[2\lambda_1 a \frac{G(\theta)}{F(\theta)} + \lambda_2 \right]$$

$$= \left(\frac{T}{Q_1 \alpha c}\right) \left(\frac{2\beta_2}{\tau}\right) \sqrt{\frac{a(t)}{2_2}} \left[\left(1 - \gamma \cos\psi - h \frac{t}{t_f}\right) \frac{G(\theta)}{F(\theta)} + \sqrt{\frac{8}{5}} \sin\psi\right]$$

At the start of the low thrust phase

$$p_2 = \frac{T_0}{Q_1 \alpha c} \frac{2\beta_2}{\tau} \left[(1 - \gamma \cos\psi) \frac{G_2}{F_2} + \sqrt{\frac{8}{5}}\, \gamma \sin\psi \right]$$

At the end of the low thrust phase

$$p_3 = \frac{T_f}{Q_1 \alpha c} \frac{2\beta_2}{\tau} \left[(\cos\psi - \gamma) \frac{G_3}{F_3} + \sqrt{\frac{8}{5}} \sin\psi \right]$$

Thus the low thrust phase is completely defined by

$$\sin \sqrt{\frac{8}{5}}\ (\theta - \theta_2) = \gamma \sin \psi \sqrt{\frac{a(t)}{a_2}}\ \frac{t}{t_f}$$

$$\frac{a_2}{a(t)} = 1 - 2(1-\gamma \cos \psi) \frac{t}{t_f} + h\left(\frac{t}{t_f}\right)^2$$

$$L^2(t) = L_1{}^2 \frac{t}{t_f}$$

$$L_1 = K_1\beta_2 \sqrt{\frac{2h}{\tau}}$$

This section of the appendix has derived expressions for the change in orbital elements during the low thrust phase of a transfer. Expressions for these maxima of the primer (at the apses) during the transfer have been derived. Further a simple expression has resulted for the time varying portion of the threshold. The fuel cost of the low thrust phase ($L_1{}^2$) is also given. Thus the total dynamics have been explicitly solved and parameters of interest identified. It is now possible to match this one low thrust phase with initial and terminal high thrust phases.

D.3 The high thrust phases

As shown in the previous section, the primer has a maximum magnitude at an apse and at that maximum the primer has only a tangential component. Thus any high thrust impulses will be tangential at an apse. Based upon that knowledge, an expression will be derived for the cost of such impulses. Then the conditions on the primer at an impulse will be given in terms of the expression for the primer at the boundaries of the low thrust phase.

An optimal impulse at an apse imparts a change in the velocity in the tangential direction. Since the orbital velocity an an apse is also tangential, this represents a change in the scalar magnitude of velocity. It is assumed that the direction of rotation on the orbit is never reversed by an impulse. The velocity at an apse is given by

$$V = \sqrt{\frac{\mu}{R}}\ \sqrt{1-T\ e}$$

where

$$R = a(1+T\ e)$$

As shown in Chapter 2, the payload for a high thrust impulse is determined by

$$K_1 = \exp\left[-\frac{1}{2c}\sum|\Delta V|\right]$$

The radius at the first impulse is

$$R_1 = 2a_0G_0{}^2$$

and the change in velocity is

$$\Delta V_1 = \sqrt{\frac{2\mu}{R_1}}\,(F_1-F_0)$$

$$\frac{\Delta V_1}{2c} = \frac{\beta_0}{T_0G_0}\,(F_1-F_0)$$

The radius at the second impulse is

$$R_2 = 2a_2F_2^{\,2}$$

and the change in velocity is

$$\Delta V_2 = T_0 \sqrt{\frac{2\mu}{R_2}} (G_2 - G_1)$$

$$\frac{\Delta V_2}{2c} = T_0 \frac{\beta_2}{F_2} (G_2 - G_1)$$

The radius at the third (and final) impulse is

$$R_3 = 2a_3F_3^{\,2}$$

and the change in velocity is

$$\Delta V_3 = T_f \sqrt{\frac{2\mu}{R_3}} (G_f - G_3)$$

$$\frac{\Delta V_3}{2c} = T_f \frac{\beta_3}{F_3} (G_f - G_3)$$

Finally define three more signums S_1, S_2, S_3 such that

$$S_1 = -T_0 \text{signum } (\Delta V_1)$$

$$S_2 = T_0 \text{signum } (\Delta V_2) \tag{D.16}$$

$$S_3 = T_f \text{signum } (\Delta V_3)$$

To complete the solution

$$K_1 = \exp\left[- \text{abs}\left(\frac{\Delta V_1}{2c}\right) - \text{abs}\left(\frac{\Delta V_2}{2c}\right)\right]$$

$$= \exp\left[- S_1 \frac{\beta_0}{G_0} (F_1 - F_0) - S_2 \frac{\beta_2}{F_2} (G_2 - G_1)\right]$$

and

$$J_1 = \exp\left[- \text{abs}\left(\frac{\Delta V_3}{2c}\right)\right]$$

$$= \exp\left[- S_3 \frac{\beta_3}{F_3} (G_f - G_3)\right]$$

The definition of the S functions also implies that

$$S_1(\theta_1 - \theta_0) \geq 0$$

$$S_2(\theta_2 - \theta_1) \geq 0$$

$$S_3(\theta_f - \theta_3) \geq 0$$

The high thrust phase is thus completely defined by the payload functions K_1 and J_1 and the variables θ which describe the transfer. In order to determine the relative amounts of high and low thrust (choice of θ's) the appropriate necessary conditions are required. As shown in Chapter 2, the primer must be matched with the thresholds δ_1 or δ_3 at the time of each impulse. Since two initial impulses are allowed, the level of the primer at the first apse is required in terms of the primer at the time of the second impulse.

Both the primer and its derivative must be continuous across an impulse. From the previous section, at the beginning of the low thrust phase, the primer is given by

$$p_2 = T_0 \left(C_2 \frac{G_2}{F_2} + D_2 \right)$$

where

$$C_2 = \frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} (1 - \gamma \cos \psi)$$

$$D_2 = - \frac{1}{Q_1 \alpha c} \frac{2\beta_2}{\tau} \left(\sqrt{\frac{8}{5}} \gamma \sin \psi \right)$$

and the derivative of the primer is related to

$$p_2' = C_2 \frac{1}{F_2} + D_2 G_2$$

These values for the primer are fixed by the low thrust phase. In terms of these definitions of C and D we also have

$$p_2 = T_0 \left(C_1 \frac{G_1}{F_1} + D_1 \right)$$

$$p_2' = C_1 \frac{1}{F_1} + D_1 G_1$$

C_1 and D_1 must be chosen to satisfy these relations. Having done so we get

$$p_1 = T_0 \left(C_1 \frac{F_1}{G_1} - D_1 \right)$$

$$= -(1 + \frac{1}{F_1^2}) p_2 + \frac{1}{F_1^2 T_0 G_1} p_2'$$

and

$$p_1' = -\frac{2}{F_1} p_2 + \frac{1+G_1^{\,2}}{F_1 T_0 G_1} p_2'$$

Similarly we get

$$p_0 = -\left(1 + \frac{1}{G_0^{\,2}}\right) p_1 + \frac{1}{G_0^{\,2} F_0} p_1'$$

Also it follows easily that

$$p_2^{+} = T_0 \left(C_2 \frac{F_2}{G_2} - D_2 \right)$$

At the final time

$$p_3 = T_f \left(C_3 \frac{G_3}{F_3} + D_3 \right)$$

$$p_3' = C_3 \frac{1}{F_3} + D_3 G_3$$

where

$$C_3 = \frac{1}{Q_1 \alpha c} \quad \frac{2\beta_2}{\tau} \left(\cos \psi - \gamma\right)$$

$$D_3 = -\frac{1}{Q_1 \alpha c} \quad \frac{2\beta_2}{\tau} \left(\sqrt{\frac{8}{5}} \sin \psi \right)$$

Similar to the preceding we get

$$p_3^{-} = T_f \left(C_3 \frac{F_3}{G_3} - D_3 \right)$$

$$p_f = -\left(1 + \frac{1}{F_f^{\,2}}\right) p_3 + \frac{1}{F_f^{\,2} T_f G_f} p_3'$$

These conditions during the high thrust phase can now be matched with the parameters from the low thrust phase to completely define the necessary conditions for the mixed mode transfer.

D.4 The mixed thrust necessary conditions

The previous sections have derived expressions for the cost functions and the nature of the primer for each phase of an optimal trajectory. Upon definition of the thresholds, the mixed thrust necessary conditions can be summarized. This section uses the nomenclature and definitions of the previous sections.

The initial threshold

$$\delta_1(0) = \frac{1}{c}\begin{cases} J_1^{\,2}(K_1 - L_1)^2 & m_e \leq m_p \\ \dfrac{m_\pi}{1 + \dfrac{1}{m_e} L_1^{\,2}} & m_e \geq m_p \end{cases}$$

can be used to define the time history of the threshold

$$\delta_1(t) = \delta_1(0)\begin{cases} 1 + \dfrac{1}{m_p} L_1^{\,2} \dfrac{t}{t_f} & m_e \leq m_p \\ 1 + \dfrac{1}{m_e} L_1^{\,2} \dfrac{t}{t_f} & m_e \geq m_p \end{cases}$$

and

$$\delta_3 = \frac{1}{c} m_\pi$$

There will be an impulse at the first apse if the primer at that time is equal to the initial threshold. Thus either

$$|p_1| = \delta_1(0)$$

or there is no first impulse, which is indicated by

$$\theta_1 = \theta_0$$

For this to be optimal we must also have

$$|p_1| < \delta_1(0)$$

Similarly at the second apse either

$$|p_2| = \delta_1(0) \qquad \text{or} \qquad \begin{cases} \theta_2 = \theta_1 \\ |p_2| < \delta_1(0) \end{cases}$$

At the final time either

$$|p_3| = \delta_3 \qquad \text{or} \qquad \begin{cases} \theta_3 = \theta_f \\ |p_3| < \delta_3 \end{cases}$$

These three conditions completely define the necessary conditions for transfers which maximize the payload under the assumptions of this chapter. To verify that no other impulses should be used, the primer at all other times must be less than the threshold appropriate to that time. Thus if

$$|p_0| < \delta_1(0) \qquad |p_2^{+}| < \delta_1(0)$$

$$|p_f| < \delta_3 \qquad |p_3^{-}| < \delta_3$$

and

$$|p(t)| < \delta_1(t)$$

the assumptions on the timing of the impulses are valid.

D.5 The first derivative of the payload

The equations in the previous section are useful for the analysis of the problem, but in their present form, they are not convenient for a numerical iteration. The first derivative vector of the payload with respect to the θ's will be determined here and related to the necessary conditions of the previous section. These equations will then be used to obtain the second derivative matrix for later use in the numerical iterations.

Expressing the payload as

$$m_\pi = \begin{cases} J_1^2 K_1^2 (1-x)^2 + \dfrac{m_e}{K_1^2} & m_e \leq m_p \\[2ex] \dfrac{J_1^2 K_1^2}{1 + \dfrac{K_1^2}{m_e} x^2} & m_e \geq m_p \end{cases}$$

where

$$x = \frac{L_1}{K_1}$$

the derivative of the payload with respect to $\underline{\theta}$ can be expressed by

$$\underline{g} = \frac{\partial m_\pi}{\partial \underline{\theta}} = B[\delta_A \frac{1}{K_1} \frac{\partial K_1}{\partial \underline{\theta}} + \delta_B \frac{1}{J_1} \frac{\partial J_1}{\partial \underline{\theta}} - \frac{\partial x}{\partial \underline{\theta}}] \tag{D.17}$$

where

$$\underline{\theta} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \qquad \underline{g} = \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \end{Bmatrix} \tag{D.18}$$

$$B = \frac{x}{\alpha Q_1} = 2J_1{}^2K_1{}^2 \begin{cases} (1-x) & m_e \leq m_p \\ \dfrac{K_1{}^2 x}{m_e\left(1 + \dfrac{K_1{}^2}{m_e} x^2\right)^2} & m_e \geq m_p \end{cases}$$

$$\delta_A = \begin{cases} (1-x) & m_e \leq m_p \\ \dfrac{m_e}{K_1{}^2 x} & m_e \geq m_p \end{cases} \tag{D.19}$$

$$\delta_B = \delta_A + \frac{m_e}{K_1{}^2 \delta_A} \tag{D.20}$$

In a numerical optimization of payload all first derivatives will be driven to zero, or the appropriate θ specified, and no longer free to be chosen.

It is easy to relate g_2 to the necessary condition

$$0 = \delta_1(0) - |p_2|$$

By algebra we have the relationships

$$\delta_1 = \frac{x}{\alpha c Q_1} \delta_A$$

$$p_2 = -\frac{T_0}{\beta_2} \frac{x}{Q_1 \alpha c} \frac{\partial x}{\partial \theta_2}$$

and

$$\frac{1}{K_1}\frac{\partial K_1}{\partial \theta_2} = - S_2\beta_2$$

and finally

$$g_2 = \frac{\partial m_\pi}{\partial \theta_2} = - S_2\beta_2 c\left\{\delta_1 - S_2 T_0 p_2\right\}$$

Optimally the impulse is applied in the direction of the primer and since $S_2\ T_0$ is the signum of the second ΔV, it is also the signum of p_2. Thus the first derivative here is simply related to the earlier necessary condition by a non-zero coefficient. The derivative with respect to θ_3 is similarly related to its necessary condition. However, more algebra is required to relate the derivative with respect to θ_1 to its necessary condition. Those steps will not be carried out here.

For completeness, all of the derivatives required for all first derivatives of the payload are expressed below.

$$g_1 = B[\delta_A(K_1)_1 - x_1]$$

$$g_2 = B[\delta_A(K_1)_2 - x_2]$$

$$g_3 = B[\delta_B(J_1)_3 - x_3]$$

where the other derivatives are given below.

$$(K_1)_1 = \frac{1}{K_1}\frac{\partial K_1}{\partial \theta_1} = \frac{S_2\beta_2}{F_1G_1F_2}\left[(1 + (1-S_1S_2)F_1{}^2)G_1 - G_2\right]$$

$$(K_1)_2 = \frac{1}{K_1} \frac{\partial K_1}{\partial \theta_2} = - S_2\beta_2$$

$$(J_1)_3 = \frac{1}{J_1} \frac{\partial J_1}{\partial \theta_3} = S_3\beta_3 = S_3\beta_2\gamma$$

$$\frac{\partial K_1}{\partial \theta_3} = \frac{\partial J_1}{\partial \theta_1} = \frac{\partial J_1}{\partial \theta_2} = 0$$

$$x_1 = \frac{\partial x}{\partial \theta_1} = \frac{\beta_2}{F_1 G_1} \sqrt{\frac{2}{\tau h}} (1 - \gamma \cos \psi)$$

$$x_2 = \frac{\partial x}{\partial \theta_2} = -\beta_2 \sqrt{\frac{2}{\tau h}} \left[(1 - \gamma \cos \psi) \frac{G_2}{F_2} - \gamma \sqrt{\frac{8}{5}} \sin \psi\right]$$

$$x_3 = \frac{\partial x}{\partial \theta_3} = -\beta_2\gamma \sqrt{\frac{2}{\tau h}} \left[(\cos \psi - \gamma) \frac{G_3}{F_3} - \sqrt{\frac{8}{5}} \sin \psi\right]$$

D.6 The second derivative matrix

The matrix of second derivatives of the payload with respect to the θ's is given in this section. The terms were obtained by a straightforward differentiation of the first derivatives, and are presented here for completeness. These terms are used in the Newton-Raphson step in the numerical iteration which is described in the next appendix. For a solution which satisfies the necessary conditions, a negative definite second derivative matrix is a sufficient condition for the solution to be at least locally maximum.

The second derivative matrix is

$$\underline{G} = \frac{\partial^2 m_\pi}{\partial \underline{\theta}^2} = \begin{bmatrix} \frac{\partial^2 m_\pi}{\partial \theta_1{}^2} & \frac{\partial^2 m_\pi}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 m_\pi}{\partial \theta_1 \partial \theta_3} \\ \frac{\partial^2 m_\pi}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 m_\pi}{\partial \theta_2{}^2} & \frac{\partial^2 m_\pi}{\partial \theta_2 \partial \theta_3} \\ \frac{\partial^2 m_\pi}{\partial \theta_1 \partial \theta_3} & \frac{\partial^2 m_\pi}{\partial \theta_2 \partial \theta_3} & \frac{\partial^2 m_\pi}{\partial \theta_3{}^2} \end{bmatrix} \quad (D.21)$$

with the individual terms given by

$$\frac{\partial^2 m_\pi}{\partial \theta_1{}^2} = \frac{\partial g_1}{\partial \theta_1}$$

$$= B[B_1 g_1 + \frac{\partial \delta_A}{\partial \theta_1} (K_1)_1 + \delta_A \frac{\partial}{\partial \theta_1} (K_1)_1 - x_{11}]$$

$$\frac{\partial^2 m_\pi}{\partial \theta_1 \partial \theta_2} = \frac{\partial g_2}{\partial \theta_1} = \frac{\partial g_1}{\partial \theta_2}$$

$$= B[B_1 g_2 + \frac{\partial \delta_A}{\partial \theta_1} (K_1)_2 + \delta_A \frac{\partial}{\partial \theta_1} (K_1)_2 - x_{12}]$$

$$\frac{\partial^2 m_\pi}{\partial \theta_2{}^2} = \frac{\partial g_2}{\partial \theta_2}$$

$$= B[B_2 g_2 + \frac{\partial \delta_A}{\partial \theta_2} (K_1)_2 + \delta_A \frac{\partial}{\partial \theta_2} (K_1)_2 - x_{22}]$$

$$\frac{\partial^2 m_\pi}{\partial\theta_1 \partial\theta_3} = \frac{\partial g_1}{\partial\theta_3} = \frac{\partial g_3}{\partial\theta_1}$$

$$= B[B_3 g_1 + \frac{\partial\delta_A}{\partial\theta_3} (K_1)_1 - x_{13}]$$

$$\frac{\partial^2 m_\pi}{\partial\theta_2 \partial\theta_3} = \frac{\partial g_2}{\partial\theta_3} = \frac{\partial g_3}{\partial\theta_2}$$

$$= B[B_3 g_2 + \frac{\partial\delta_A}{\partial\theta_3} (K_1)_2 - x_{23}]$$

$$\frac{\partial^2 m_\pi}{\partial\theta_3{}^2} = \frac{\partial g_3}{\partial\theta_3}$$

$$= B[B_3 g_3 + \frac{\partial\delta_B}{\partial\theta_3} (J_1)_3 + \delta_B \frac{\partial}{\partial\theta_3} (J_1)_3 - x_{33}]$$

Some of these terms have two expressions depending upon the relative sizes of m_e and m_p. For $m_e \leq m_p$ we have

$$\frac{\partial\delta_A}{\partial\theta_1} = - x_1$$

$$\frac{\partial\delta_A}{\partial\theta_2} = - x_2$$

$$\frac{\partial\delta_A}{\partial\theta_3} = -x_3$$

$$\frac{\partial \delta_B}{\partial \theta_3} = - x_3 \left(1 - \frac{m_e}{K_1^{\,2} \delta_A^{\,2}} \right)$$

$$B_1 = \frac{1}{B^2} \frac{\partial B}{\partial \theta_1} = \frac{1}{B} \left[2(K_1)_1 - \frac{x_1}{\delta_A} \right]$$

$$B_2 = \frac{1}{B^2} \frac{\partial B}{\partial \theta_2} = \frac{1}{B} \left[2(K_1)_2 - \frac{x_2}{\delta_A} \right]$$

$$B_3 = \frac{1}{B^2} \frac{\partial B}{\partial \theta_3} = \frac{1}{B} \left[2(J_1)_3 - \frac{x_3}{\delta_A} \right]$$

For $m_e \geq m_p$ we have

$$\frac{\partial \delta_A}{\partial \theta_1} = - \delta_A \left[2(K_1)_1 + \frac{x_1}{x} \right]$$

$$\frac{\partial \delta_A}{\partial \theta_2} = - \delta_A \left[2(K_1)_2 + \frac{x_2}{x} \right]$$

$$\frac{\partial \delta_A}{\partial \theta_3} = - \delta_A \frac{x_3}{x}$$

$$\frac{\partial \delta_B}{\partial \theta_3} = x_3 \left[1 - \frac{\delta_A}{x} \right]$$

$$B_1 = \frac{1}{B^2} \frac{\partial B}{\partial \theta_1} = \frac{1}{B} \frac{\delta_A}{x+\delta_A} \left[4(K_1)_1 + \left(1 - 3 \frac{x}{\delta_A} \right) \frac{x_1}{x} \right]$$

$$B_2 = \frac{1}{B^2}\frac{\partial B}{\partial \theta_2} = \frac{1}{B}\frac{\delta_A}{x+\delta_A}\left[4(K_1)_2 + \left(1 - 3\frac{x}{\delta_A}\right)\frac{x_2}{x}\right]$$

$$B_3 = \frac{1}{B^2}\frac{\partial B}{\partial \theta_3} = \frac{1}{B}\left[2(J_1)_3 + \frac{\delta_A - 3x}{x+\delta_A}\frac{x_3}{x}\right]$$

The other derivatives are given by

$$\frac{\partial}{\partial \theta_1}(K_1)_1 = \frac{\partial}{\partial \theta_1}\left(\frac{1}{K_1}\frac{\partial K_1}{\partial \theta_1}\right)$$

$$= \frac{S_2\beta_2}{F_2 G_1}\left[2\frac{G_1}{F_1^{\,2}}(G_1 - G_2) + 1 + (1 - S_1 S_2)F_1^{\,2}\right]$$

$$\frac{\partial}{\partial \theta_1}(K_1)_2 = \frac{\partial}{\partial \theta_1}\left(\frac{1}{K_1}\frac{\partial K_1}{\partial \theta_2}\right)$$

$$= \frac{1}{F_1 G_1}(K_1)_2$$

$$\frac{\partial}{\partial \theta_2}(K_1)_2 = \frac{\partial}{\partial \theta_2}\left(\frac{1}{K_1}\frac{\partial K_1}{\partial \theta_2}\right)$$

$$= -(K_1)_2\frac{G_2}{F_2}$$

$$\frac{\partial}{\partial\theta_3}(J_1)_4 = \frac{\partial}{\partial\theta_3}\left(\frac{1}{J_1}\quad\frac{\partial J_1}{\partial\theta_3}\right)$$

$$= -(K_1)_3\,\frac{G_3}{F_3}$$

$$x_{11} = \frac{\partial x_1}{\partial\theta_1}$$

$$= \frac{2\beta_2^{\,2}}{\tau x F_1 G_1^{\,2}}\left[2F_1(1-\gamma\cos\psi) + \frac{1}{F_1}\right] - \frac{x_1^{\,2}}{x}$$

$$x_{12} = \frac{\partial x_2}{\partial\theta_1}$$

$$= \frac{2\beta_2^{\,2}}{\tau x F_1 G_1}\left\{\gamma\left(\sqrt{\frac{8}{5}}\sin\psi + \frac{G_2}{F_2}\cos\psi\right) - 2\,\frac{G_2}{F_2}\right] - \frac{x_1 x_2}{x}$$

$$x_{22} = \frac{\partial x_2}{\partial\theta_2}$$

$$= \frac{2\beta_2^{\,2}}{\tau x}\left[\frac{13}{5}\gamma\cos\psi - 1 + \frac{G_2}{F_2}\left(\frac{G_2}{F_2} - 2\gamma\sqrt{\frac{8}{5}}\sin\psi\right)\right] - \frac{x_2^{\,2}}{x}$$

$$x_{13} = \frac{\partial x_1}{\partial\theta_3}$$

$$= \frac{2\beta_2^{\,2}\gamma}{\tau x F_1 G_1}\left[\frac{G_3}{F_3}\cos\psi - \sqrt{\frac{8}{5}}\sin\psi\right] - \frac{x_1 x_3}{x}$$

$$x_{23} = \frac{\partial x_2}{\partial \theta_3}$$

$$= \frac{2\beta_2^{\,2}\gamma}{\tau x}\left[\sqrt{\frac{8}{5}}\,\sin\psi\left(\frac{G_2}{F_2} - \frac{G_3}{F_3}\right) - \cos\psi\left(\frac{8}{5} + \frac{G_2 G_3}{F_2 F_3}\right)\right] - \frac{x_2 x_3}{x}$$

$$x_{33} = \frac{\partial x_3}{\partial \theta_3}$$

$$= \frac{2\beta_2^{\,2}\gamma}{\tau x}\left[\frac{13}{5}\cos\psi - \gamma + \frac{G_3}{F_3}\left(\gamma\,\frac{G_3}{F_3} + 2\sqrt{\frac{8}{5}}\,\sin\psi\right)\right] - \frac{x_3^{\,2}}{x}$$

D.7 A simpler transfer

In the process of investigating transfers between coplanar coaxial elliptic orbits, with two initial impulses, it was observed that the optimal transfer often approached a simpler transfer. This other transfer was investigated and found to be near optimal for many cases. It also provides a good initial condition for the numerical optimization of the complete problem. It is presented here, along with the necessary and sufficient conditions for its optimality

This near optimal transfer uses two impulses to go from the initial orbit to an intermediate orbit with an eccentricity equal to the final eccentricity. Then the low thrust phase spirals to the final orbit at constant eccentricity. No final impulses are allowed. The only free parameter is the semimajor axis of the intermediate orbit. In terms of the θ's which define the complete problem, we are assuming

$$\theta_2 = \theta_3 = \theta_f$$

Thus $\psi = 0$. Instead of θ_1, $\sqrt{a_2/a_0}$ is more convenient to use as the free parameter.

For these special assumptions, let

$$y = \sqrt{\frac{a_2}{a_0}}$$

be the independent variable in the algorithm. The low thrust cost is

$$x = \frac{L_1}{K_1} = S_y \beta_0 \sqrt{\frac{2}{\tau}} \left(\frac{1}{y} - \rho \right)$$

with

$$S_y = \text{signum } (R_f - R_0)$$

and all other variables defined as before. For the high thrust, the cost is

$$J_1 = 1$$

and from before

$$K_1 = \exp \left\{ - \frac{S_1 T \beta_0}{1-T\ e_0} \left[\sqrt{1+T\ e_1} - \sqrt{1+T\ e_0} + \sqrt{\frac{1-T\ e_1}{1+T\ e_1}} \left(\sqrt{1-T\ e_f} - \sqrt{1-T\ e_1} \right) \right] \right\}$$

having substituted $e_2 = e_f$ and explicitly expressed the F and G functions. With

$$T\ e_1 = \frac{y^2 - \eta}{y^2 + \eta}$$

and η defined by

$$\eta = \frac{1 - T e_0}{1 + T e_f}$$

we get

$$K_1 = \exp\left\{ - \frac{S_y \beta_0}{\sqrt{1-T e_0}} \left[y\sqrt{\frac{2}{y^2+\eta}} - \sqrt{1+T e_0} \right.\right.$$

$$\left.\left. + \frac{1}{y}\left(\sqrt{\eta}\sqrt{1-T e_f} - \eta\sqrt{\frac{2}{\eta+y^2}}\right)\right]\right\}$$

Note that

$$S_y = S_1 T$$

Similar to the previous two sections, the derivatives which are necessary for determining y are

$$g = \frac{\partial m_\pi}{\partial y} = B[\delta_A \ (K_1)_y - \frac{\partial x}{\partial y}]$$

and

$$\frac{\partial^2 m_\pi}{\partial y^2} = B\left[\frac{\partial \delta_A}{\partial y} (K_1)_y + \delta_A \frac{\partial}{\partial y} (K_1)_y - \frac{\partial^2 x}{\partial y^2} + \frac{1}{B^2}\frac{\partial B}{\partial y} g\right]$$

where B and δ_A are as before and the derivatives are

$$\frac{\partial x}{\partial y} = - \frac{S_y \beta_0 \sqrt{\frac{2}{\tau}}}{y^2}$$

$$\frac{\partial^2 x}{\partial y^2} = - \frac{2}{y} \frac{\partial x}{\partial y}$$

$$(K_1)_y = \frac{1}{K_1} \frac{\partial K_1}{\partial y} = S_y \left\{ \left(\eta \sqrt{\frac{2}{y^2+\eta}} - \sqrt{\eta} \sqrt{1-T\ e_f} \right) \frac{1}{y^2} + \left(\frac{2}{\eta+y^2} \right)^{\frac{3}{2}} \left(\frac{\eta+1}{2} \right) \right\}$$

$$\frac{\partial}{\partial y} (K_1)_y = -S_y \left\{ \frac{2}{y^3} \left(\eta \sqrt{\frac{2}{y^2+\eta}} - \sqrt{\eta} \sqrt{1-T\ e_f} \right) \right.$$

$$\left. + \left(\frac{2}{\eta+y^2} \right)^{\frac{3}{2}} \frac{\eta+1}{4y(\eta+y^2)} \left[(5+2\eta^2) y^2 + \eta^2 \right] \right\}$$

Appendix E

THE NUMERICAL PROCEDURES USED

As described in Chapter 5, a sequence of changes in $\underline{\theta}$, called $\delta\underline{\theta}$, is desired which will increase the payload and lead to a final value of $\underline{\theta}$ such that m_π is (locally) maximized. The four steps which are used for the maximization of m_π are the gradient, Newton-Raphson, parabolic, and acceleration steps. The description of how and when to use these respective steps is in Chapter 5.

E.1 The gradient and Newton-Raphson steps.

Expressions are derived in Appendix D for the payload and its first two derivatives as functions of the free parameters $\underline{\theta}$. These derivatives can be used effectively in a numerical iteration toward the choice of $\underline{\theta}$ which maximizes m_π. The gradient step uses only first derivative information, while the Newton-Raphson (N-R) step uses both first and second derivative information. These two steps evolve naturally from the Taylor series expansion for m_π about the present value for $\underline{\theta}$.

$$m_\pi(\underline{\theta}+\delta\underline{\theta}) = m_\pi(\underline{\theta}) + \underline{g}^T\delta\underline{\theta} + \frac{1}{2}\,\delta\underline{\theta}^T\underline{G}\,\delta\underline{\theta} + \ldots$$

where

$m_\pi(\underline{\theta})$ = the value of the payload for the current value of $\underline{\theta}$.

$\underline{g} = \frac{\partial m_\pi}{\partial \underline{\theta}}$ = the vector of first derivatives of the payload with respect to the $\underline{\theta}$'s evaluated at $\underline{\theta}$. ($\underline{g}$ is often called the gradient vector)

$$\underline{G} = \underline{G}^T = \frac{\partial \underline{g}^T}{\partial \underline{\theta}} = \text{the matrix of second derivatives of the payload with respect to the } \underline{\theta}\text{'s evaluated at } \underline{\theta}.$$

Analytic expressions for m_π, $\underline{g}$, and $\underline{G}$ are given in Appendix D. This series converges for small enough $\delta\underline{\theta}$ if all derivatives are bounded. It is assumed that those conditions apply.

The gradient step

The payload can always be increased by a (small enough) step in the gradient direction

$$\delta\underline{\theta} = k\ \underline{g}$$

where k is chosen so that the step is small enough. From experience in the numerical iteration to the solution for this problem, it was observed that

$$k = .1$$

works successfully most of the time. When a smaller step size is required, the "parabolic step" described in the next section is used to decrease the step size. The gradient step works well when "far" from the solution. It provides an improvement in the payload, but has poor convergence properties when close to the solution.

The Newton-Raphson step:

If $\delta\underline{\theta}$ is small enough so that all derivatives higher than second order can be ignored, the choice of $\delta\underline{\theta}$ which maximizes the payload is found when the first derivative vector with respect to $\delta\underline{\theta}$ is zero

$$\frac{\partial m_\pi}{\partial(\delta\underline{\theta})} = \underline{0} = \underline{g} + \underline{G}\ \delta\underline{\theta}$$

and the second derivative matrix

$$\frac{\partial^2 m_\pi}{\partial \underline{\theta}^2} = \underline{G} < 0$$

is negative definite. This optimum step is classically called the Newton-Raphson step and is given by

$$\delta\underline{\theta} = - \underline{G}^{-1} \underline{g}$$

If $\underline{G}$ is negative definite, $\underline{G}^{-1}$ exists. As the optimum solution is approached, $\underline{g}$ (and $\delta\underline{\theta}$) will become smaller. The validity of the assumption on the size of $\delta\underline{\theta}$ is thus best near the maximum. If far enough from the maximum, this $\delta\underline{\theta}$ may be too large to ignore the higher order derivatives. A parabolic step size control is used if the result of the step is not acceptable. When

$$\underline{g} = \underline{0}$$

and $\underline{G}$ is negative definite, no (small) step, $\delta\underline{\theta}$, can increase the payload. For the existence of a local maximum, it is necessary that $\underline{g} = 0$ and $\underline{G}$ be negative semi-definite ($\underline{G} \leq 0$), and it is sufficient if $\underline{G}$ is negative definite ($\underline{G} < 0$).

E.2 Parabolic step size control

Whenever a gradient or N-R step is not acceptable (see Chapter 5), step size control is exercised. This method uses the value of m_π and its gradient at the reference point and the value of m_π which resulted from the unacceptable step. The magnitude of another step, from the same reference point and in the same vector direction as the previous step is determined. If the step

$$\delta\underline{\theta} = \underline{d}$$

was unacceptable (produced a smaller payload), a better size step in the same direction is desired. Expressing the Taylor series expansion for the payload with the choice of the step

$$\delta\underline{\theta} = k\ \underline{d}$$

we get

$$m_\pi(k) = m_\pi(\theta) + k\ \underline{g}^T\underline{d} + \frac{1}{2}\ k^2\underline{d}^T\underline{G}\ \underline{d} + \ldots$$

As before, this expression is maximized by

$$k = -\frac{\underline{g}^T\underline{d}}{\underline{d}^T\underline{G}\ \underline{d}}$$

if

$$\underline{d}^T\underline{G}\ \underline{d} < 0$$

We know $m_\pi(1) = m(\underline{\theta} + \underline{d})$ since that value was the payload which resulted from the unacceptable step $\delta\underline{\theta} = \underline{d}$. Using this we get

$$\underline{d}^T\underline{G}\ \underline{d} = -2[\underline{g}^T\underline{d} - (m_\pi(\underline{\theta}+\underline{d}) - m_\pi(\underline{\theta}))]$$

Thus we have

$$k = \frac{1}{2}\left[\frac{\underline{g}^T\underline{d}}{\underline{g}^T\underline{d} - (m_\pi(\underline{\theta}+\underline{d}) - m_\pi(\theta))}\right]$$

If a gradient step was used, this step provides information about the second derivative in the direction of the gradient only. If a N-R step was used, this step includes the information about the higher order derivatives which were ignored. If a parabolic step is still too large, the procedure is repeated and a better estimate of the desired quantities is obtained.

No step size control is used to increase the size of a step. Thus bounds must be placed upon k to prevent cutting the step size too much. On the other hand, we wish to insure that the step size is indeed reduced. Thus k is constrained by

$$.05 \leq k \leq .5$$

If the optimum k is outside these bounds, it is assumed that the original step was beyond the bounds of a reasonable quadratic approximation, and the arbitrary intermediate choice

$$k = .25$$

is used.

E.3 The acceleration step

Numerical difficulties are encountered for some of the cases considered. The gradient steps do not make any progress and the second derivative matrix is not negative definite so that a N-R step can not be used. For these cases, the "acceleration step" described below can be used effectively. The problem arises when the payload has a small increase in one vector direction and a sharp drop in other directions. Such characteristics are often described as a ravine in in a minimization problem, or a ridge in a maximization problem such as this. The magnitude of the gradient on the top of the ridge is usually quite small. Two problems are involved in the treatment of this difficulty. First, there must be a means of identifying such a region, and second, a vector in the direction of the ridge must be defined.

In the vicinity of a ridge, the gradient vector points to the top of the ridge instead of along it. Typical behavior is a zig-zag pattern as successive gradient steps go from one side of the ridge to the other. Little progress is made in the direction of the ridge. A single parabolic step size control will usually successfully define the top of the ridge. Thus the vector between the result of two successful parabolic steps can be used to define the direction of the ridge. Numerically, $\underline{\theta}$ is defined as being on top of a ridge if

i) it was the result of a single parabolic step size control on a gradient step

ii) $\underline{g}^2 < .03$

iii) $\underline{G}$ is not negative definite

Although not true for all problems with ridges, $\underline{G}$ was not negative definite near a ridge and the test on $\underline{G}$ is used to prevent taking an unnecessary acceleration step. $\Delta\underline{\theta}$ is the vector between two points on the top of the ridge which are separated by at least one intermediate iteration step. Once a $\Delta\underline{\theta}$ is defined, an acceleration step

$$\delta\underline{\theta} = 1.4\ \Delta\underline{\theta}$$

is used. If such a step improves the payload, the next step is in the same direction only larger

$$\delta\underline{\theta}_{i+1} = 1.4\ \delta\underline{\theta}_i$$

The step size is thus increased and steps taken along the ridge as long as there is an improvement in the payload. For this problem, the ridge in m_π as a function of $\underline{\theta}$ is a straight line (what luck!) and large changes in $\underline{\theta}$ often result from this procedure once the ridge is identified. As many as 12 successful steps, each with a larger step size, have been observed in iterations.

The numerical definition of $\Delta\underline{\theta}$ and the identification of a ridge are best understood through use of an illustrative iteration. Figure E.1 shows a sequence of steps which lead to the definition of the ridge. The top of the ridge is indicated by a dashed line. For this discussion, the subscript on $\underline{\theta}$ indicates the result of a successful iteration step. $\underline{\theta}_0$ is the initial value. The first trial gradient step was not successful. Parabolic step sizing resulted in $\underline{\theta}_1$. The next gradient step is successful. $\underline{\theta}_3$ results from step size control on an unsuccessful gradient step. The vector $\Delta\underline{\theta} = \underline{\theta}_3 - \underline{\theta}_1$, which joins the result of these two successful parabolic steps, defines the approximate direction of the ridge. The steps to $\underline{\theta}_4$ and $\underline{\theta}_5$ are successful since they increase the payload even though they are off of the ridge. Note the gradient at $\underline{\theta}_3$, $\underline{\theta}_4$, and $\underline{\theta}_5$. The acceleration step usually increases the payload even though $\Delta\underline{\theta}$ is not aligned with $\underline{g}$. From this it is easy to see that successive gradient steps, even with parabolic step size control, will not work well in the vicinity of a ridge.

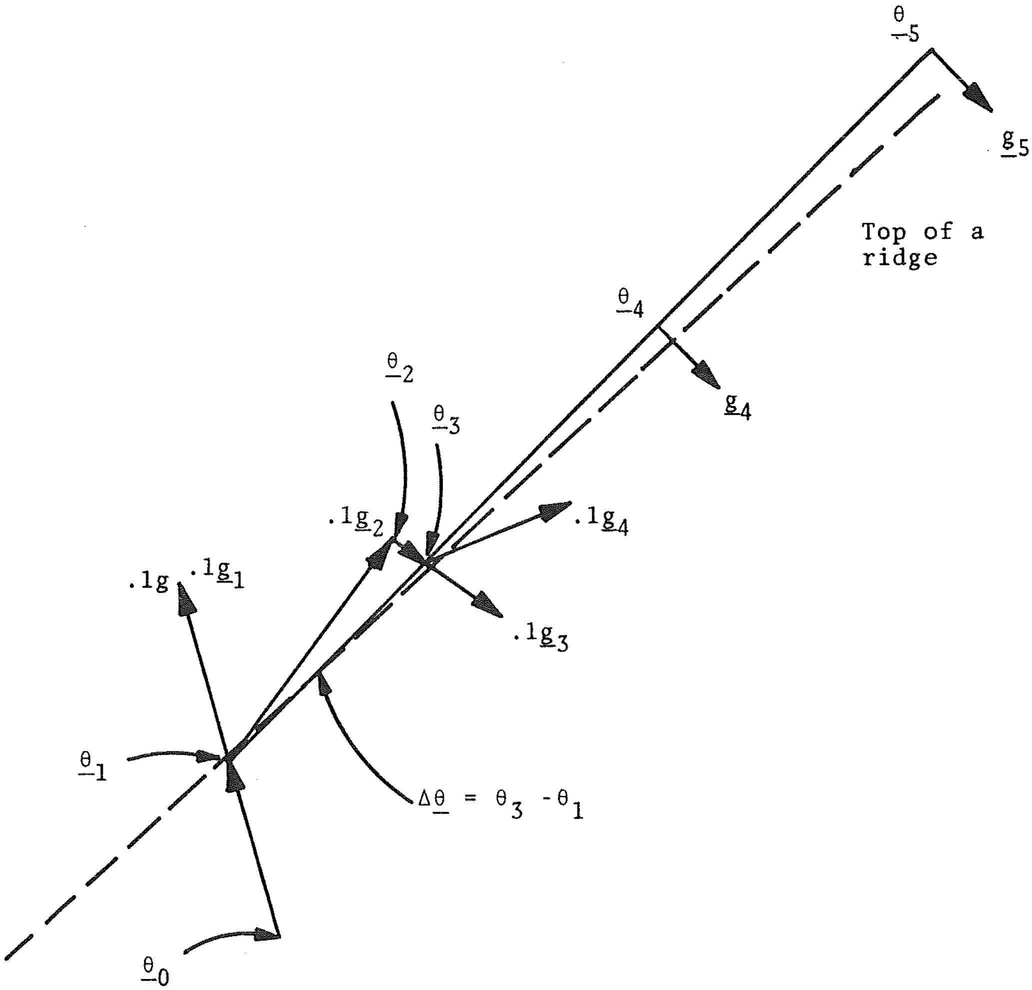


Figure E.1 Representation of the acceleration step

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BIOGRAPHICAL NOTE

C. Julian Vahlberg, Jr. was born on August 22, 1943 in Oklahoma City, Oklahoma. He was raised in Oklahoma City, where he graduated from Northwest Classen High School in 1961. He entered M.I.T. the following September and received the degree of Bachelor of Science in the Department of Aeronautics and Astronautics in June 1965.

Mr. Vahlberg continued his studies at M.I.T. and received the degree of Master of Science in September 1966. He held a research Assistantship in the M.I.T. Instrumentation Laboratory while earning this degree. His S.M. thesis was entitled "Computed Compensation of Gyroscope Compliance Drift". In September of 1966, Mr. Vahlberg was awarded a three-year NASA Traineeship to pursue his graduate study. To complete his doctoral thesis research, he became a Research Assistant in the Measurement Systems Laboratory in September of 1969.

During the course of his education, Mr. Vahlberg has held technical summer positions since 1963. The most recent of these were in the Apollo group at the M.I.T. Instrumentation Laboratory for the summers of 1966 and 1967. He has also held Teaching Assistantships in three courses in the Department of Aeronautics and Astronautics at M.I.T.

Upon receipt of the degree of Doctor of Science, Mr. Vahlberg will join the faculty of the School of Mechanical and Aerospace Engineering at Oklahoma State University, Stillwater, Okla.