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A MATHEMATICAL ANALYSIS OF THE STABILITY OF A LUNAR FLYING VEHICLE WITH MANUAL CONTROLS

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MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

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A MATHEMATICAL ANALYSIS OF THE STABILITY OF A LUNAR FLYING VEHICLE WITH MANUAL CONTROLS

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A MATHEMATICAL ANALYSIS OF THE STABILITY OF A LUNAR FLYING VEHICLE WITH MANUAL CONTROLS

By David Hall

SUMMARY

Equations of motion for a four-degree-of-freedom mathematical model of the lunar flying vehicle are developed and used to study the stability of the autonomous vehicle. The human pilot describing function is used to represent the control system and derive the stability criteria for the closed-loop attitude control system.

INTRODUCTION

The system under investigation is a free-flight manually controlled vehicle operating in the lunar environment. A mathematical model of the vehicle dynamics and the pilot control function is used to study the system stability. The system is represented by the block diagram in figure 1.

The vehicle consists of a platform which supports the pilot, propellant tanks, engine, landing gear, and equipment. Representative configurations of thrust vector-controlled and kinesthetic-controlled vehicles will be discussed. The coordinates which describe the motion of the bodies are shown in figure 2.

The vehicle dynamics are based on a four-degree-of-freedom mathematical model of plane motion, which considers two rigid bodies connected at a point which will be referred to as the gimbal point. Two coordinates X and Z locate the center of mass of one body in the plane. One coordinate θ determines the angular position of the body with respect to vertical, and the fourth coordinate β determines the angular position of the second body with respect to vertical.

For the purposes of this study, several assumptions were made which simplify the analysis but retain the fidelity of the model.

1. Flight time is small (approximately 4 minutes).

2. Mass, center of mass, and inertia of each body are constant.

3. Pilot describing function is time invariant and can be represented by the Laplace transform.

4. Pilot control force is applied normal to the thrust vector which is along the coordinate axis of body 2.

5. Pilot control force reaction is applied at the center of mass of body 1, parallel to the x_{b_1} body axis.

The equations of motion are developed using Lagrange's equation and solved for the angular accelerations. The Jacobian of the solution allows the equations to be written in state variable matrix notation. Routh's criteria are used to determine the stability requirements for the vehicle characteristic equation.

Vehicle control is represented by the experimentally determined human pilot describing function. The closed-loop system characteristic equation is derived and Hurwitz criteria used to study the effect on stability of varying the vehicle and control function parameters.

The mathematical model has been programed for the digital computer and particular solutions have been generated to verify the analytical results. Although this model is not an analog of the manually controlled vehicle with variable mass characteristics, it will hopefully provide some insight into the stability and control of vehicles in this class. The philosophy is that an approximate model will do in the beginning and that we will enhance our theoretical understanding by attempting engineering applications and learning from them.

Work related to this analysis is presented in the Bibliography.

SYMBOLS

a_{ij} matrix element

b_i coefficient of s^n in characteristic equation

c damping coefficient

D dissipation function

e system error

e base of natural logarithms

F_c control force

G transfer function

g acceleration of gravity

H Hurwitz matrix

H_n principal minor of Hurwitz matrix

I moment of inertia

J Jacobian matrix

K_p pilot gain

k spring coefficient

L_1 directed length from body 1 center of mass to gimbal

L_2 directed length from gimbal to body 2 center of mass

L_c directed length from gimbal to application of control force

M mass

Q generalized force

q generalized coordinate

s Laplace transform operator

T engine thrust

T kinetic energy function

T_I general lag time constant

T_L general leadtime constant

T_N lag time constant approximation of the neuromuscular system

t time

V potential energy function

X coordinate of body 1 center of mass

x_b body axis

Y_c controlled element transfer function

Y_p pilot describing function

Z coordinate of body 1 center of mass

z_b body axis

β coordinate of body 2 angular position

Δ determinant of matrix

δ gimbal angle

ζ_N damping ratio of the second-order component of the neuromuscular system

θ coordinate of body 1 angular position

τ pilot reaction time lag

ω_N undamped natural frequency of the second-order component of the neuromuscular system

Subscripts:

1 body 1

2 body 2

b	body
c	control
i	input
N	neuromuscular system
n	integer
p	pilot

Operators:

$\frac{d(\ )}{dq}$	derivative of ( ) with respect to q
$\frac{\partial(\ )}{\partial q}$	partial derivative of ( ) with respect to q
$(\dot{\ })$	derivative of ( ) with respect to time
$(\ddot{\ }$	second derivative of ( ) with respect to time
$\approx$	approximately equal to

ANALYSIS

Vehicle Equations of Motion

The analysis of vehicle dynamics is limited to plane motion to allow the effect of vehicle design on stability to be emphasized without unnecessary complexity. This results in a system with four degrees of freedom and consisting of two rigid bodies connected at a point.

The vehicle equations of motion can be derived using Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial D}{\partial \dot{q}_n} + \frac{\partial V}{\partial q_n} = Q_n \tag{1}$$

The functions are

$$T = \frac{1}{2} M_1\left(\dot{X}^2 + \dot{Z}^2\right) + \frac{1}{2} I_1\dot{\theta}^2 + \frac{1}{2} M_2\Big[\dot{X}^2 + \dot{Z}^2 + L_1{}^2\dot{\theta}^2$$

$$+ L_2{}^2\dot{\beta}^2 + 2L_1\dot{\theta}(\dot{X} \cos \theta - \dot{Z} \sin 0)$$

$$+ 2L_2\dot{\beta}(\dot{X} \cos \beta - \dot{Z} \sin \dot{\beta}) + 2L_1 L_2\dot{\theta}\dot{\beta} \cos(\theta - \beta)\Big]$$

$$+ \frac{1}{2} I_2\dot{\beta}^2$$

$$D = \frac{1}{2} c\left(\dot{\theta} - \dot{\beta}\right)^2$$

$$V = M_1gZ + M_2g\left(Z + L_1 \cos \theta + L_2 \cos \beta\right) + \frac{1}{2} k\left(\theta - \beta\right)^2$$

Let $q_1 = \theta$

$q_2 = \beta$

$q_3 = X$

$q_4 = Z$

Then $Q_1 = -TL_1 \sin(\theta - \beta)$

$Q_2 = -F_cL_c$

$Q_3 = T \sin \beta + F_c \cos \theta$

$Q_4 = T \cos \beta - F_c \sin \theta$

Differentiating, substituting into the Lagrange equation, and rearranging the equations into matrix form give the equations of motion

$$\begin{bmatrix} I_1 + M_2L_1^2 & M_2L_1L_2 \cos(\theta - \beta) & M_2L_1 \cos\theta & -M_2L_1 \sin\theta \\ M_2L_1L_2 \cos(\theta - \beta) & I_2 + M_2L_2^2 & M_2L_2 \cos\beta & -M_2L_2 \sin\beta \\ M_2L_1 \cos\theta & M_2L_2 \cos\beta & M_1 + M_2 & 0 \\ -M_2L_1 \sin\theta & -M_2L_2 \sin\beta & 0 & M_1 + M_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\beta} \\ \ddot{X} \\ \ddot{Z} \end{bmatrix}$$

$$= \begin{bmatrix} -k(\theta - \beta) - c(\dot{\theta} - \dot{\beta}) - M_2L_1L_2\dot{\beta}^2\sin(\theta - \beta) + M_2gL_1 \sin\theta - TL_1 \sin(\theta - \beta) \\ k(\theta - \beta) + c(\dot{\theta} - \dot{\beta}) + M_2L_1L_2\dot{\theta}^2\sin(\theta - \beta) + M_2gL_2 \sin\beta - F_cL_c \\ M_2L_1\dot{\theta}^2\sin\theta + M_2L_2\dot{\beta}^2\sin\beta + T \sin\beta + F_c \cos\theta \\ M_2L_1\dot{\theta}^2\cos\theta + M_2L_2\dot{\beta}^2\cos\beta - \left(M_1 + M_2\right)g + T \cos\beta - F_c \sin\theta \end{bmatrix} \qquad (2)$$

The solution of equation (2) will be of the form

$$\ddot{\theta} = f_1\left(\theta, \dot{\theta}, \beta, \dot{\beta}, F_c\right)$$

$$\ddot{\beta} = f_2\left(\theta, \dot{\theta}, \beta, \dot{\beta}, F_c\right)$$

$$\ddot{X} = f_3\left(\theta, \dot{\theta}, \beta, \dot{\beta}, F_c\right)$$

$$\ddot{Z} = f_4\left(\theta, \dot{\theta}, \beta, \dot{\beta}, F_c\right)$$

Since the accelerations are independent of the linear position and velocity, the stability of the rotational motion may be investigated separately. Consequently, equation (2) was solved for the angular accelerations by premultiplying by the inverse of the coefficient matrix. Letting Δ represent the determinant of the coefficient matrix, the angular accelerations are

$$\begin{aligned}
\Delta\ddot{\theta} = & -k\left(M_1 + M_2\right)\left\{\left(M_1 + M_2\right)I_2 + M_1M_2L_2\left[L_2 + L_1 \cos(\theta - \beta)\right]\right\}(\theta - \beta) \\
& - c\left(M_1 + M_2\right)\left\{\left(M_1 + M_2\right)I_2 + M_1M_2L_2\left[L_2 + L_1 \cos(\theta - \beta)\right]\right\}(\dot{\theta} - \dot{\beta}) \\
& - M_1^2M_2^2L_1^2L_2^2\cos(\theta - \beta)\sin(\theta - \beta)\dot{\theta}^2 \\
& - M_1M_2L_1L_2\left(M_1I_2 + M_1M_2L_2^2 + M_2I_2\right)\sin(\theta - \beta)\dot{\beta}^2 \\
& - M_1TL_1\left(M_1I_2 + M_1M_2L_2^2 + M_2I_2\right)\sin(\theta - \beta) \\
& - M_2F_cL_1\left\{\left(M_1 + M_2\right)\left[I_2 - M_1L_2L_c \cos(\theta - \beta)\right]\cdot\right. \\
& \left. + M_1M_2L_2^2\sin^2(\theta - \beta)\right\}
\end{aligned} \tag{3}$$

$$\Delta\ddot{\beta} = k\left(M_1 + M_2\right)\left\{\left(M_1 + M_2\right)I_1 + M_1M_2L_1\left[L_1 + L_2\cos(\theta - \beta)\right]\right\}(\theta - \beta)$$
$$+ c\left(M_1 + M_2\right)\left\{\left(M_1 + M_2\right)I_1 + M_1M_2L_1\left[L_1 + L_2\cos(\theta - \beta)\right]\right\}(\dot{\theta} - \dot{\beta})$$
$$+ M_1M_2L_1L_2\left(M_1I_1 + M_1M_2L_1{}^2 + M_2I_1\right)\sin(\theta - \beta)\dot{\theta}^2$$
$$+ M_1{}^2M_2{}^2L_1{}^2L_2{}^2\cos(\theta - \beta)\sin(\theta - \beta)\dot{\beta}^2$$
$$+ M_1{}^2M_2TL_1{}^2L_2\cos(\theta - \beta)\sin(\theta - \beta)$$
$$- \left(M_1 + M_2\right)F_c\left\{M_2I_1\left[L_c + L_2\cos(\theta - \beta)\right] + M_1L_c\left(I_1 + M_2L_1{}^2\right)\right\} \quad (4)$$

where

$$\Delta = \left(M_1 + M_2\right)\left[I_2M_2\left(I_1 + M_1L_1{}^2\right) + I_1M_1\left(I_2 + M_2L_2{}^2\right)\right]$$
$$+ M_1{}^2M_2{}^2L_1{}^2L_2{}^2\sin^2(\theta - \beta) \quad (5)$$

Vehicle Stability

Equilibrium is defined as that position of the variables for which the derivatives of the variables are zero. From equations (3), (4), and (5), the equilibrium position for the rotational motion of the autonomous vehicle is $(\theta - \beta) = 0$. To study the vehicle stability in the neighborhood of equilibrium, the Jacobian is used to obtain a linear approximation of the acceleration equations.

The Jacobian is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \frac{\partial f_1}{\partial q_3} & \frac{\partial f_1}{\partial q_4} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_2}{\partial q_3} & \frac{\partial f_2}{\partial q_4} \\ \frac{\partial f_3}{\partial q_1} & \frac{\partial f_3}{\partial q_2} & \frac{\partial f_3}{\partial q_3} & \frac{\partial f_3}{\partial q_4} \\ \frac{\partial f_4}{\partial q_1} & \frac{\partial f_4}{\partial q_2} & \frac{\partial f_4}{\partial q_3} & \frac{\partial f_4}{\partial q_4} \end{bmatrix}$$

where $f_1 = \dot{\theta}$

$f_2 = \ddot{\theta}$

$f_3 = \dot{\beta}$

$f_4 = \ddot{\beta}$

The equations for rotational motion may then be written in state variable matrix form

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\beta} \\ \ddot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \beta \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} 0 \\ a_{25} \\ 0 \\ a_{45} \end{bmatrix} F_c \qquad (6)$$

where

$$
\left.
\begin{aligned}
a_{21} &= -a_{23} = -\frac{1}{\Delta}\Big\{k\left(M_1 + M_2\right)\left[\left(M_1 + M_2\right)I_2 + M_1M_2L_2\left(L_2 + L_1\right)\right] \\
&\quad + M_1TL_1\left(M_1I_2 + M_1M_2L_2^{\,2} + M_2I_2\right)\Big\} \\
a_{22} &= -a_{24} = -\frac{1}{\Delta}\,c\left(M_1 + M_2\right)\left[\left(M_1 + M_2\right)I_2 + M_1M_2L_2\left(L_2 + L_1\right)\right] \\
a_{41} &= -a_{43} = \frac{1}{\Delta}\Big\{k\left(M_1 + M_2\right)\left[\left(M_1 + M_2\right)I_1 + M_1M_2L_1\left(L_1 + L_2\right)\right] \\
&\quad + M_1^{\,2}M_2TL_1^{\,2}L_2\Big\} \\
a_{42} &= -a_{44} = \frac{1}{\Delta}\,c\left(M_1 + M_2\right)\left[\left(M_1 + M_2\right)I_1 + M_1M_2L_1\left(L_1 + L_2\right)\right] \\
\Delta &= \left(M_1 + M_2\right)\left[I_2M_2\left(I_1 + M_1L_1^{\,2}\right) + I_1M_1\left(I_2 + M_2L_2^{\,2}\right)\right]
\end{aligned}
\right\} \quad (7)
$$

The adequacy of the linear approximation of the equations of motion is illustrated by figures 3 and 4 which were obtained by numerical integration of equations (2) and (6), respectively.

The third column of [A] is the negative of the first column; consequently, the matrix is singular and the system has infinitely many equilibrium states. This agrees with the statement that the vehicle is in equilibrium for $\theta = \beta$.

The vehicle characteristic equation is obtained by

$$\det\Big[s[I] - [A]\Big] = 0$$

where [I] is the identity matrix and [A] is the coefficient matrix.

The characteristic equation is

$$s^4 + \left(-a_{22} - a_{44}\right)s^3 + \left(a_{22}a_{44} - a_{21} - a_{24}a_{42} - a_{43}\right)s^2$$
$$+ \left(a_{21}a_{44} - a_{24}a_{41} + a_{43}a_{22} - a_{23}a_{42}\right)s$$
$$+ \left(a_{43}a_{21} - a_{23}a_{41}\right) = 0$$

As a result of the relationships between the a_{ij} given above, the last two coefficients are zero, and part of the coefficient of s^2 cancels.

The characteristic equation reduces to

$$s^4 + \left(-a_{22} - a_{44}\right)s^3 + \left(-a_{21} - a_{43}\right)s^2 = 0$$

The stability of the vehicle may be investigated by Routh's criteria. The Routhian array for a fourth-order system is

$$\begin{array}{c|lll}
s^4 & b_o & b_2 & b_4 \\
s^3 & b_1 & b_3 & \\
s^2 & b_2 - \dfrac{b_o b_3}{b_1} & b_4 & \\
s^1 & b_3 - \dfrac{b_1 b_4}{b_2 - \dfrac{b_o b_3}{b_1}} & & \\
s^0 & b_4 & &
\end{array}$$

where the b_i refer to the coefficient of s^{4-i} in the characteristic equation. Routh's criteria state that the number of roots of the characteristic equation with positive real parts is equal to the number of changes of sign of the terms in the first column of the Routhian array.

For the vehicle characteristic equation

$$b_o = 1$$

$$b_3 = b_4 = 0$$

and the Routhian array is

$$\begin{array}{c|cc} s^4 & 1 & b_2 \\ s^3 & b_1 & \\ s^2 & b_2 & \\ s^1 & 0 & \\ s^0 & 0 & \end{array}$$

The zero in the s^1 row indicates that there are roots that are the negatives of each other. From the auxiliary equation

$$b_2 s^2 = 0$$

both roots are zero. There will be no changes of sign in the first column and, therefore, no roots with positive real parts if

$$b_1 > 0$$

and

$$b_2 > 0$$

In terms of the vehicle parameters

$$b_1 = -a_{22} - a_{44} = \frac{c\left[\left(M_1 + M_2\right)\left(I_1 + I_2\right) + M_1M_2\left(L_1 + L_2\right)^2\right]}{I_2M_2\left(I_1 + M_1L_1^{\,2}\right) + I_1M_1\left(I_2 + M_2L_2^{\,2}\right)} \tag{8}$$

$$b_2 = -a_{21} - a_{43} = \frac{k\left(M_1 + M_2\right)\left[\left(M_1 + M_2\right)\left(I_1 + I_2\right) + M_1M_2\left(L_1 + L_2\right)^2\right] + M_1TL_1\left[\left(M_1 + M_2\right)I_2 + M_1M_2L_2\left(L_1 + L_2\right)\right]}{\left(M_1 + M_2\right)\left[I_2M_2\left(I_1 + M_1L_1^{\,2}\right) + I_1M_1\left(I_2 + M_2L_2^{\,2}\right)\right]} \tag{9}$$

Since all the terms in equation (8) are positive, b_1 is always greater than zero. The denominator of equation (9) is positive; therefore, the sign of b_2 will depend on the sign and magnitude of the terms in the numerator.

For example, with the gimbal point below the center of mass, L_1 is negative (fig. 2 and definition of symbols). Assuming L_2 is positive, the following equation must be satisfied for nondivergent oscillations.

$$k\left(M_1 + M_2\right)\left[\left(M_1 + M_2\right)\left(I_1 + I_2\right) + M_1M_2\left(L_2 - L_1\right)^2\right] \geq M_1TL_1\left[\left(M_1 + M_2\right)I_2 + M_1M_2L_2\left(L_2 - L_1\right)\right] \tag{10}$$

In the kinesthetic-controlled vehicle concept, it is assumed that the pilot acts as a rigid body pivoted at his ankles. Therefore, the mass, inertia, and characteristic length of body 1 are fixed, and the design of the platform is constrained to be compatible with the stability criteria.

The existence of a double root at $s = 0$ indicates that the origin is unstable. This can be understood by examining the block diagram of the vehicle equations of motion (fig. 5). The initial conditions

$$\dot{\theta}(0) = \dot{\beta}(0) \neq 0$$

will produce a linear and equal increase in θ and β.

For zero input, the solution of equation (6) is

$$\left[q(s)\right] = \left[s[I] - [A]\right]^{-1} \left[q(0)\right] \tag{11}$$

where

$$\left[s[I] - [A]\right]^{-1}$$

is the Laplace transform of the fundamental matrix.

For the vehicle system, the matrix is

$$
\left[s[I] - [A]\right]^{-1} = \frac{\begin{bmatrix} s^3 + \left(-a_{22} - a_{44}\right)s^2 - a_{43}s + \left(a_{22}a_{43} - a_{23}a_{42}\right) & s^2 - a_{44}s - a_{43} & a_{23}s + \left(a_{24}a_{43} - a_{23}a_{44}\right) & a_{24}s + a_{23} \\ a_{21}s^2 + \left(a_{24}a_{41} - a_{21}a_{44}\right)s & s^3 - a_{44}s^2 - a_{43}s & a_{23}s^2 + \left(a_{24}a_{43} - a_{23}a_{44}\right)s & a_{24}s^2 + a_{23}s \\ a_{41}s + \left(a_{21}a_{42} - a_{41}a_{22}\right) & a_{42}s + a_{41} & s^3 + \left(-a_{22} - a_{44}\right)s^2 - a_{21}s + \left(a_{21}a_{44} - a_{24}a_{41}\right) & s^2 - a_{22}s - a_{21} \\ a_{41}s^2 + \left(a_{21}a_{42} - a_{22}a_{41}\right)s & a_{42}s^2 + a_{41}s & a_{43}s^2 + \left(a_{23}a_{42} - a_{22}a_{43}\right)s & s^3 - a_{22}s^2 - a_{21}s \end{bmatrix}}{s^2\left[s^2 + \left(-a_{22} - a_{44}\right)s + \left(-a_{21} - a_{43}\right)\right]} \tag{12}
$$

The matrix shows that the solution will diverge for arbitrary initial conditions. However, if the damping coefficient c is zero

$$a_{22} = a_{24} = a_{42} = a_{44} = 0$$

and the solution is stable for arbitrary displacements and zero velocities. Therefore, misalinement between the thrust vector and the vehicle center of mass at lift-off is not critical since the oscillations do not diverge. However, any disturbance in flight will result in divergent oscillations unless corrected by the pilot. An example is shown in figures 6 and 7.

The form of the response is determined by the denominator of equation (12), which is the characteristic equation. Therefore, the natural frequency of oscillation is dependent on the location of the roots of the quadratic factor. Since the location of the gimbal point will effect the value of $(-a_{21} - a_{43})$, as shown in equations (9) and (10), it will effect the location of the roots. Obviously, the magnitude and frequency of the response is a function of gimbal point location.

Pilot Describing Function

The control system can be characterized as a manual process for minimizing visually perceived errors by exercising continuous control so as to match visually presented input and output signals. The pilot closes the control loop by determining the error between a reference attitude and the present attitude; this process is called compensatory tracking.

The output of a nonlinear element (e.g., a human pilot) always contains higher harmonics, the effect of which depends on the characteristics of the other components. A quasi-linear equivalent to the nonlinear element, for a specific input, can be represented by a describing function (the equivalent linear element) and a remnant (the difference between the output of the nonlinear system and the equivalent linear element). Since components with relatively large inertia and long time constants filter higher harmonics considerably, a reasonable description for the pilot response characteristics is a linearized model of an equivalent transfer function which is linearly correlated with the system input.

To develop some criteria for stability of the controlled vehicle, it is necessary to have an expression for the control system. The pilot describing function has been used to express the functional relationship between the input and output of a single-axis manual control system (McRuer, et al.).

For this analysis, the input is the pitch attitude error

$$e = \theta_i - \theta \tag{13}$$

and the output is the pilot limb position (i.e., the angle between body 1 and body 2; the gimbal angle)

$$\delta = \theta - \beta \tag{14}$$

Thus, angular control of the vehicle in the pitch plane is represented by

$$Y_p = \frac{\delta}{e} = \frac{K_p\left(T_L s + 1\right)e^{-\tau s}}{\left(T_I s + 1\right)\left(T_{N_1} s + 1\right)\left[\left(\frac{s}{\omega_N}\right)^2 + \frac{2\zeta_N s}{\omega_N} + 1\right]} \tag{15}$$

Since the parameters of the describing function are adjustable by the pilot, the function is only valid in the frequency domain and only exists under essentially stationary conditions.

Although the describing function represents a functional relationship, there appears to be some structural correlation. For example, the time delay $e^{-\tau s}$ is due to excitation of the retina, nerve conduction, and data processing in the central nervous system. The neuromuscular system characteristic

$$\left(T_{N_1} s + 1\right)\left[\left(\frac{s}{\omega_N}\right)^2 + \frac{2\zeta_N s}{\omega_N} + 1\right]$$

is compatible with physiological data for the muscle spindle and muscle tension, and the behavioral characteristics of the muscle/manipulator combination. The manipulator or control stick dynamics are not explicitly present in the term since virtually all experiments have been conducted with manipulators of negligible inertia, and the information which exists is insufficient to account for the effect analytically. However, the manipulator dynamics can be represented in terms of the gimbal angle and control force, using equations (6) and (14), as

$$\ddot{\delta} + \left(-a_{22} - a_{44}\right)\dot{\delta} + \left(-a_{21} - a_{43}\right)\delta = \left(a_{25} - a_{45}\right)F_c$$

and in Laplace transform notation as

$$\frac{\delta}{F_c} = \frac{\left(a_{25} - a_{45}\right)}{s^2 + \left(-a_{22} - a_{44}\right)s + \left(-a_{21} - a_{43}\right)} \tag{16}$$

The form of the manipulator dynamics is the same as the second-order component of the neuromuscular system. Since existing data support a first-order approximation to the neuromuscular system, the system is usually represented by $T_N s + 1$ where

$$T_N \approx T_{N_1} + \frac{2\zeta_N}{\omega_N}$$

The factor

$$\frac{T_L s + 1}{T_I s + 1}$$

in a compensation or equalizing characteristic which, together with the gain K, is adjustable by the pilot to obtain system stability.

To simplify the subsequent analysis, the time delay in equation (15) can be replaced by a Páde approximation

$$e^{-\tau s} \approx \left(\frac{\frac{-\tau}{2} s + 1}{\frac{\tau}{2} s + 1} \right)$$

The pilot describing function then becomes

$$Y_p = \frac{\delta}{e} \approx \frac{K_p \left(T_L s + 1\right)\left(- \frac{\tau}{2} s + 1\right)}{\left(T_I s + 1\right)\left(T_N s + 1\right)\left(\frac{\tau}{2} s + 1\right)} \tag{17}$$

Attitude Control System Stability

From equations (6) and (14), the linear equation for vehicle pitch attitude θ can be written in terms of the gimbal angle δ as

$$\ddot{\theta} = a_{21}\delta + a_{22}\dot{\delta} \tag{18}$$

Taking the Laplace transform and rearranging, the transfer function for the vehicle dynamics is

$$Y_c = \frac{\theta}{\delta} = \frac{a_{22}s + a_{21}}{s^2} \tag{19}$$

The product of the transfer function for the control system (eq. (17)) and the transfer function for the vehicle dynamics yields the system open-loop transfer function

$$G(s) = Y_p Y_c = \frac{\theta}{e} = \frac{K_p \left(T_L s + 1\right)\left(- \frac{\tau}{2} s + 1\right)\left(a_{22}s + a_{21}\right)}{s^2 \left(T_I s + 1\right)\left(T_N s + 1\right)\left(\frac{\tau}{2} s + 1\right)} \tag{20}$$

and the closed-loop transfer function is

$$\frac{G(s)}{1 + G(s)} = \frac{\theta}{\theta_i} = \frac{K_p\left(T_L s + 1\right)\left(-\frac{\tau}{2} s + 1\right)\left(a_{22}s + a_{21}\right)}{s^2\left(T_I s + 1\right)\left(T_N s + 1\right)\left(\frac{\tau}{2} s + 1\right) + K_p\left(T_L s + 1\right)\left(-\frac{\tau}{2} s + 1\right)\left(a_{22}s + a_{21}\right)} \tag{21}$$

The system characteristic equation is

$$\frac{\tau}{2} T_I T_N s^5 + \left[\frac{\tau}{2}\left(T_I + T_N\right) + T_I T_N\right] s^4 + \left[\frac{\tau}{2}\left(1 - K_p a_{22} T_L\right) + T_I + T_N\right] s^3 + \left\{1 + K_p\left[a_{22}\left(T_L - \frac{\tau}{2}\right) - \frac{\tau}{2} a_{21} T_L\right]\right\} s^2 + K_p\left[a_{22} + a_{21}\left(T_L - \frac{\tau}{2}\right)\right] s + K_p a_{21} \tag{22}$$

The Hurwitz algorithm determines the criteria which the system characteristic equation must satisfy for stability. The Hurwitz matrix for a fifth-order system is

$$H = \begin{bmatrix} b_1 & b_3 & b_5 & 0 & 0 \\ 1 & b_2 & b_4 & 0 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 \\ 0 & 1 & b_2 & b_4 & 0 \\ 0 & 0 & b_1 & b_3 & b_5 \end{bmatrix}$$

where

$$
\begin{aligned}
b_o &= \frac{\tau}{2} T_I T_N \\
b_1 &= \frac{\left[\frac{\tau}{2}\left(T_I + T_N\right) + T_I T_N\right]}{b_o} \\
b_2 &= \frac{\left[\frac{\tau}{2}\left(1 - K_p a_{22} T_L\right) + T_I + T_N\right]}{b_o} \\
b_3 &= \frac{\left\{1 + K_p\left[a_{22}\left(T_L - \frac{\tau}{2}\right) - \frac{\tau}{2} a_{21} T_L\right]\right\}}{b_o} \\
b_4 &= \frac{K_p\left[a_{22} + a_{21}\left(T_L - \frac{\tau}{2}\right)\right]}{b_o} \\
b_5 &= \frac{K_p a_{21}}{b_o}
\end{aligned}
\tag{23}
$$

A polynomial whose zeros all have negative real parts is called a Hurwitz polynomial. The necessary and sufficient condition for equation (22) to be a Hurwitz polynomial is that the principal minors of H all be positive.

The principal minors of H are

$$H_1 = b_1$$

$$H_2 = b_1 b_2 - b_3$$

$$H_3 = \left(b_1 b_2 - b_3\right) b_3 - \left(b_1 b_4 - b_5\right) b_1$$

$$\begin{aligned} H_4 = &\left[\left(b_1 b_2 - b_3\right) b_3 - \left(b_1 b_4 - b_5\right) b_1\right] b_4 \\ &- \left[\left(b_1 b_2 - b_3\right) b_2 - \left(b_1 b_4 - b_5\right)\right] b_5 \end{aligned}$$

$$\begin{aligned} H_5 = &\left\{\left[\left(b_1 b_2 - b_3\right) b_3 - \left(b_1 b_4 - b_5\right) b_1\right] b_4 \right. \\ &\left. - \left[\left(b_1 b_2 - b_3\right) b_2 - \left(b_1 b_4 - b_5\right)\right] b_5 \right\} b_5 \end{aligned}$$

From equation (23) it can be seen that

$$b_o > 0$$

$$b_1 > 0$$

since all the parameters are positive.

Therefore

$$H_1 = b_1 > 0$$

The remaining principal minors, which must be positive, are

$$H_2 = \frac{\frac{\tau^2}{4}\left(T_I + T_N\right) + \frac{\tau}{2}\left(T_I^2 + T_N^2\right) + T_IT_N\left(T_I + T_N + \tau\right) + K_p\left\{a_{22}\left[T_IT_N\left(\frac{\tau^2}{4} - \tau T_L\right) - \frac{\tau^2}{4}\left(T_I + T_N\right)T_L\right] + a_{21}\frac{\tau^2}{4}T_IT_NT_L\right\}}{\left(\frac{\tau}{2}T_IT_N\right)^2} \tag{24}$$

$$H_3 = \frac{H_2\left\{1 + K_p\left[a_{22}\left(T_L - \frac{\tau}{2}\right) - a_{21}\frac{\tau}{2}T_L\right]\right\}}{\frac{\tau}{2}T_IT_N} + \frac{\left[\frac{\tau}{2}\left(T_I + T_N\right) + T_IT_N\right]K_p}{\left(\frac{\tau}{2}T_IT_N\right)^3}\left\{a_{21}\left[T_IT_N\left(\tau - T_L\right) + \frac{\tau}{2}\left(T_I + T_N\right)\left(\frac{\tau}{2} - T_L\right)\right]\right.$$
$$\left. - a_{22}\left[\frac{\tau}{2}\left(T_I + T_N\right) + T_IT_N\right]\right\} \tag{25}$$

$$H_4 = \frac{K_p}{\left(\frac{\tau}{2}T_IT_N\right)^2}\left\{H_3\frac{\tau}{2}T_IT_N\left[a_{22} + a_{21}\left(T_L - \frac{\tau}{2}\right)\right] - H_2a_{21}\left[\frac{\tau}{2}\left(1 - K_pa_{22}T_L\right) + T_I + T_N\right]\right\} + \frac{K_p^2a_{21}}{\left(\frac{\tau}{2}T_IT_N\right)^3}\left\{a_{22}\left[\frac{\tau}{2}\left(T_I + T_N\right) + T_IT_N\right]\right.$$
$$\left. + a_{21}\left[\left(T_I + T_N\right)\left(T_L - \frac{\tau}{2}\right)\frac{\tau}{2} + T_IT_N\left(T_L - \tau\right)\right]\right\} \tag{26}$$

$$H_5 = \frac{H_4K_pa_{21}}{\frac{\tau}{2}T_IT_N} \tag{27}$$

The equations for the principal minors indicate that H_2, H_3, H_4, and H_5 could be positive or negative depending on the sign and magnitude of a_{21} and a_{22}. The parameters T_L, T_I, and K_p must be adjusted by the pilot to insure that the minors are positive, thus rendering the system stable. For a particular set of vehicle parameters and values of T_N and τ, the inequalities for H_n may be solved to obtain the permissible range of values of T_L, T_I, and K_p.

The impulse response of the attitude control system closed-loop transfer function is shown in figure 8 for a set of parameters which satisfy Hurwitz criteria and in figure 9 for a set which do not.

CONCLUSIONS

The autonomous vehicle is unstable except for the special case of zero damping and initial conditions on displacement only. The magnitude and frequency of the response differ for the configurations with the gimbal point above or below the center of mass since the vehicle parameters which define the configuration determine the location of the roots of the characteristic equation.

Kinesthetic control allows less freedom in design because the physical parameters of the pilot (i.e., mass, inertia, length) cannot be adjusted to meet the stability criteria.

The stability criteria for the vehicle and for the closed-loop system should be useful in defining the permissible range of vehicle parameters for stable operation. Extension of this procedure to multiaxis control will be complicated by vehicle axis coupling through products of inertia and by the requirement to generate different equilizations in different axes simultaneously.

BIBLIOGRAPHY

Baverschmidt, Donald K.; and Besco, Robert O.: Human Engineering Criteria for Manned Space Flight: Minimum Manual Systems. Behavioral Sciences Laboratory, Technical Documentary Rept. no. AMRL-TDR-62-87, Aug. 1962.

Besco, R. O.; Baverschmidt, D. K.; and McElwain, Carolyn S.: Studies of Manual Attitude Control-the Effect of Space Vehicle Configuration. IRE Transactions on Human Factors in Electronics, Sept. 1962, pp. 57-61.

Brown, B. Porter; and Johnson, Harold I.: Moving-Cockpit Simulator Investigation of the Minimum Tolerable Longitudinal Maneuvering Stability. NASA TN D-26, Sept. 1959.

Brown, B. Porter; Johnson, Harold I.; and Mungall, Robert G.: Simulator Motion Effects on a Pilot's Ability to Perform a Precise Longitudinal Flying Task. NASA TN D-367, May 1960.

Chen, C. F.; and Yates, R. E.: A New Matrix Formula for the Inverse Laplace Transformation. ASME Paper no. 66-WA/Aut-3, Dec. 1966.

Chen, Yu: Vibrations: Theoretical Methods. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1966.

D'Azzo, John J.; and Houpis, Constantine H.: Feedback Control System Analysis and Synthesis, McGraw-Hill Book Co., Inc., New York, 1966.

Elkind, Jerome I.: Characteristics of Simple Manual Control Systems. Massachusetts Institute of Technology, Lexington, Mass., Technical Rept. no. 111, Apr. 1956.

Elkind, Jerome I.; and Miller, Duncan C.: Adaptive Characteristics of the Human Controller of Time-Varying Systems. Air Force Flight Dynamics Laboratory, Technical Rept. AFFDL-TR-66-60, Dec. 1967.

Elkind, Jerome I.; Starr, Edward A.; Green, David M.; and Darley, D. Lucille: Evaluation of a Technique for Determining Time-Invariant and Time-Variant Dynamic Characteristics of Human Pilots. NASA TN D-1897, May 1963.

Flexman, Ralph E.; Seale, Leonard M.; and Henderson, Campbell: Development and Test of the Bell Zero-G Belt. Behavioral Sciences Laboratory, Technical Documentary Rept. no. AMRL-TDR-63-23, Mar. 1963.

Frost, George C.: An Application of a Dynamic Pilot Model to System Design. Behavioral Sciences Laboratory, ASD Technical Note 61-57, Apr. 1961.

Gibson, John E.: Nonlinear Automatic Control. McGraw-Hill Book Co., Inc., New York, 1963.

Graham, Dunstan; and McRuer, Duane: Analysis of Nonlinear Control Systems. John Wiley & Sons, Inc., New York, 1961.

Greenwood, Donald T.: Principles of Dynamics. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

Halfman, Robert L.: Dynamics. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

Hookway, R. O.; and Moken, A. J.: A Survey of Human Controller Models Applicable to the Design of Rocket Control Systems Which Include the Astronaut as an Onboard Control Element. NASA CR-71739, Feb. 1966.

Jex, H. R.; McDonnell, J. D.; and Phatak, A. V.: A Critical Tracking Task for Manual Control Research. IEEE Transactions on Human Factors in Electronics, vol. HFE-7, no. 4, 1966, pp. 138-145.

Levison, William H.; and Elkind, Jerome I.: Studies of Two-Variable Manual Control Systems. Joint Automatic Control Conference, June 1967.

Levison, William H.; and Elkind, Jerome I.: Two-Dimensional Manual Control Systems with Separated Displays. IEEE Transactions on Human Factors in Electronics, vol. HFE-8, no. 3, 1967, pp. 202-209.

Lindquist, O. Herbert: Development of the Manual Thrust Vector Mode (MTVC) of the Apollo Command and Service Module Stabilization and Control System (SCS). AIAA Paper no. 67-243, Feb. 1967.

McDonnell, J. D.; and Jex, H. R.: A Critical Tracking Task for Man-Machine Research Related to the Operator's Effective Delay Time, Part II. Experimental Effects of System Input Spectra, Control Stick Stiffness, and Controlled Element Order. NASA CR-674, Jan. 1967.

McKee, John W.: A Three-Axis Fixed-Simulator Investigation of the Effects on Control Precision of Various Ways of Utilizing Rate Signals. NASA TN D-525, Jan. 1961.

McRuer, Duane; Graham, Dunstan; Krendel, Ezra; and Reisener, William, Jr.: Human Pilot Dynamics in Compensatory Systems. Air Force Flight Dynamics Laboratory, Technical Rept. no. AFFDL-TR-65-15, July 1965.

McRuer, D. T.; Hofmann, L. G.; Jex, H. R.; Moore, G. P.; Phatak, A. V.; Weir, D. H.; and Wolkovitch, J.: New Approaches to Human-Pilot/Vehicle Dynamic Analysis. Air Force Flight Dynamics Laboratory, Technical Rept. no. AFFDL-TR-67-150, Feb. 1968.

McRuer, Duane T.; and Krendel, Ezra S.: Dynamic Response of Human Operators. WADC Technical Report 56-524, Oct. 1957.

McRuer, Duane T.; and Krendel, Ezra S.: The Human Operator as a Servo System Element. J. Franklin Inst., 1959, Part I, pp. 381-403, Part II, pp. 511-536.

Miller, Duncan C.; and Elkind, Jerome I.: The Adaptive Response of the Human Controller to Sudden Changes in Controlled Process Dynamics. IEEE Transactions on Human Factors in Electronics, vol. HFE-8, no. 3, 1967, pp. 218-223.

Sadoff, Melvin: A Study of a Pilot's Ability to Control During Simulated Stability Augmentation System Failures. NASA TN D-155, Nov. 1962.

Sadoff, Melvin: Effects of High Sustained Acceleration on Pilot's Performance and Dynamic Response. NASA TN D-2067, July 1964.

Sadoff, Melvin; McFadden, Norma M.; and Heinle, Donovan R.: A Study of Longitudinal Control Problems at Low and Negative Damping and Stability with Emphasis on Effects of Motion Cues. NASA TN D-348, Jan. 1961.

Schwartz, Ralph J.; and Friedland, Bernard: Linear Systems. McGraw-Hill Book Co., Inc., New York, 1965.

Seckel, E.; Breul, H.; Keller, T.; Suh, S.; and Weston, R.: A Study of the Mechanics of Human Balancing for Potential Application to the Control of Vehicles, Part II. Toward a Mathematical Model of Vertical Balancing in Earth Gravity. Grumman Research Department Memorandum RM-369, July 1967.

Seckel, Edward; Hall, Ian A. M.; McRuer, Duane T.; and Weir, David H.: Human Pilot Dynamic Response in Flight and Simulator. Wright Air Development Center, WADC Technical Rept. 57-520, Aug. 1958.

Senders, John W.: Human Tracking Behavior. Minneapolis-Honeywell Regulator Company. MH Aero Document U-ED 6141, Nov. 1959.

Skolnick, Alfred: Stability and Performance of Manned Control Systems. IEEE Transactions on Human Factors in Electronics, vol. HFE-7, no. 3, 1966, pp. 115-124.

Stapleford, Robert L.; McRuer, Duane T.; and Magdaleno, Raymond E.: Pilot Describing Function Measurements in a Multiloop Task. IEEE Transactions on Human Factors in Electronics, vol. HFE-8, no. 2, 1967, pp. 113-125.

Synge, John L.; and Griffith, Byron A.: Principles of Mechanics. McGraw-Hill Book Co., Inc., New York, 1959.

Thomson, William T.: Vibration Theory and Applications. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

Todosiev, Ernest P.: Human Performance in a Cross-Coupled Tracking System. IEEE Transactions on Human Factors in Electronics, vol. HFE-8, no. 3, 1967, pp. 210-217.

Todosiev, Ernest P.; Rose, Richard E.; and Summers, Leland G.: Human Performance in Single and Two-Axis Tracking Systems, IEEE Transactions on Human Factors in Electronics, vol. HFE-8, no. 2, 1967, pp. 125-129.

Tomovic, Rajko: Introduction to Non-linear Automatic Control Systems. John Wiley & Sons, Inc., 1966.

Truxal, John G.: Automatic Feedback Control System Synthesis. McGraw-Hill Book Co., Inc., New York, 1955.

Tustin, A.: The Nature of the Operator's Response in Manual Control, and Its Implications for Controller Design. J. IEE, vol. 94, 1947, pp. 190-202.

Wierwille, Walter W.; and Gagne, Gilbert A.: Nonlinear and Time-Varying Dynamical Models of Human Operators in Manual Control Systems. Human Factors, vol. 8, 1966, pp. 97-120.

Wierwille, Walter W.; Gagne, Gilbert A.; and Knight, James R.: An Experimental Study of Human Operator Models and Closed-Loop Analysis Methods for High-Speed Automobile Driving. IEEE Transactions on Human Factors in Electronics, vol. HFE-8, no. 3, 1967, pp. 187-201.

Young, Laurence R.; and Meiry, Jacob L.: Manual Control of an Unstable System with Visual and Motion Cues. IEEE International Conversion Record, vol. 13, Part 6, 1965, pp. 123-127.

An Engineering Study and Preliminary Design of a One Man Propulsion Device for Lunar and Free-Space Environments. Hamilton Standard Division of United Aircraft Corp., Windsor Locks, Conn., May 1964.

Third Annual NASA-University Conference on Manual Control. University of Southern California, Los Angeles, Calif., NASA SP-144, Mar. 1-3, 1967.

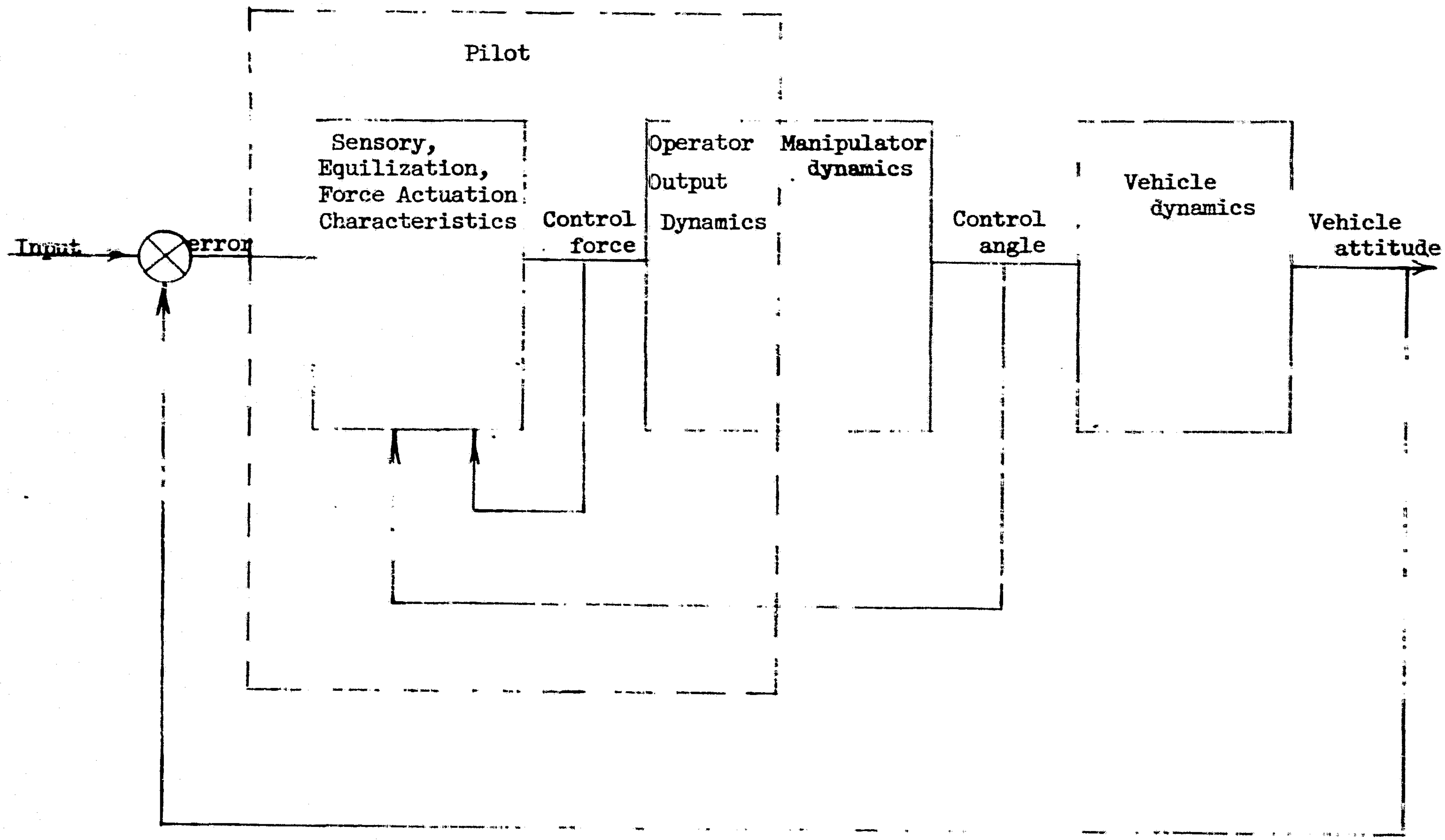

Figure 1.- Single-axis attitude control system block diagram.

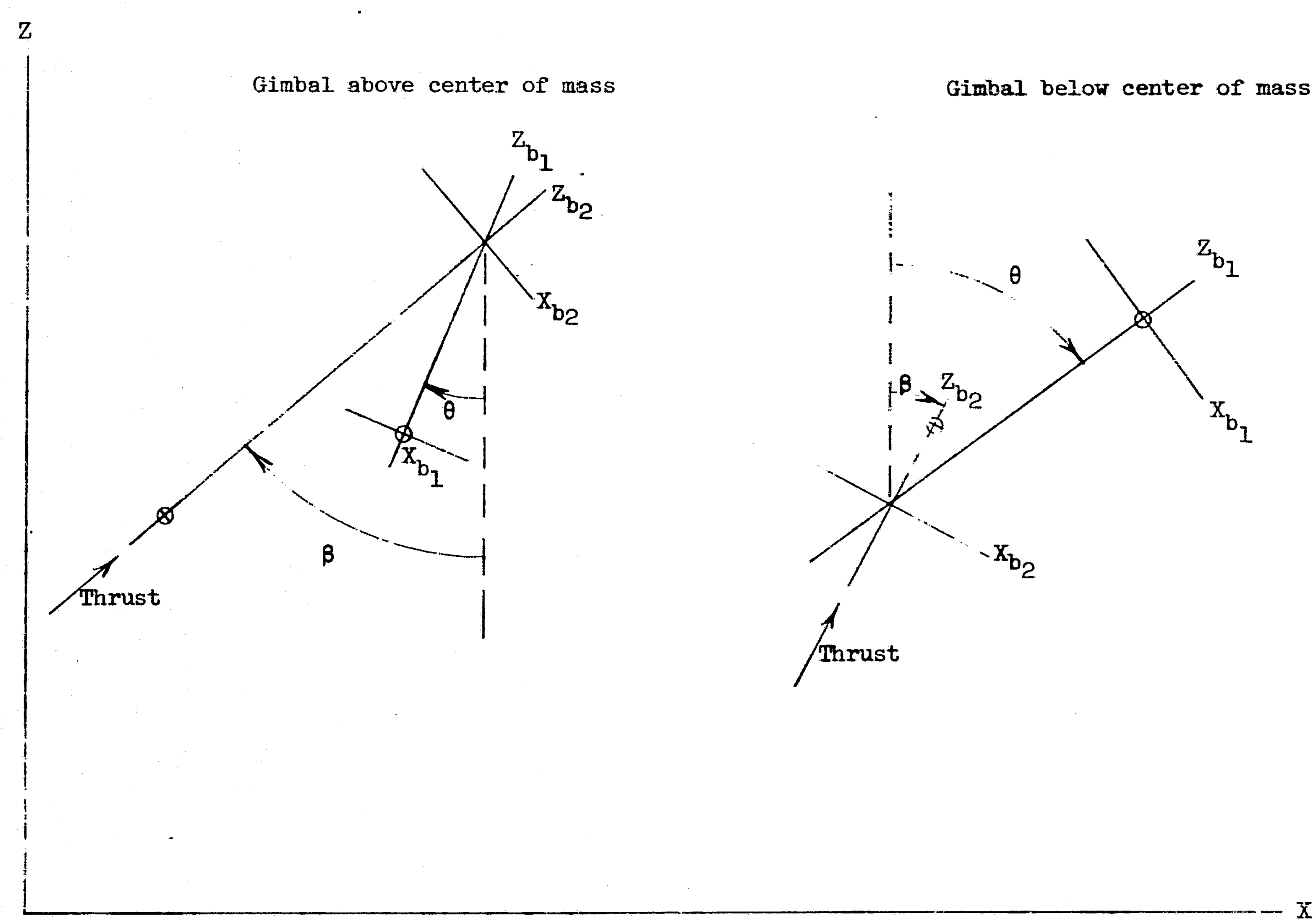

Figure 2.- Coordinate system.

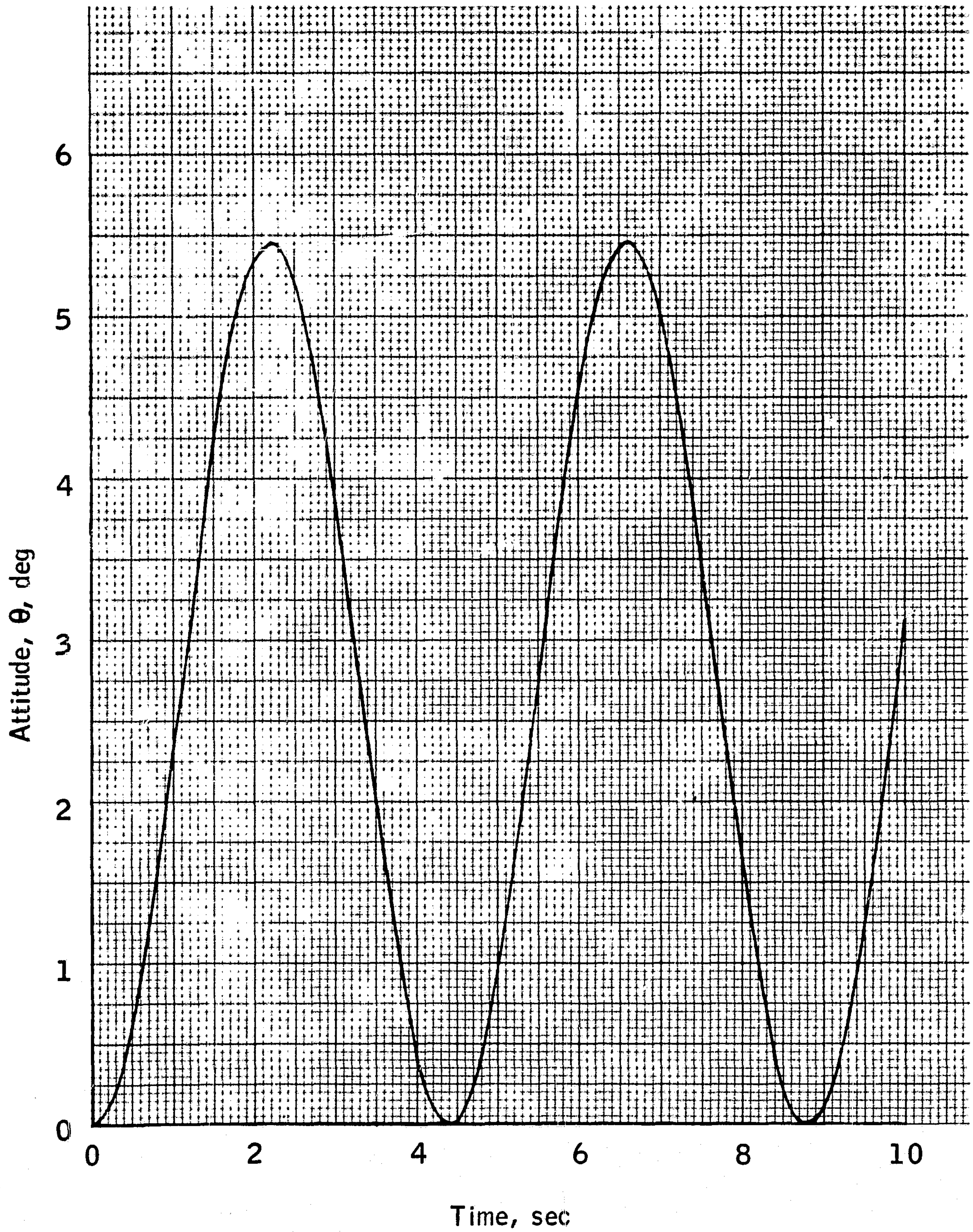

Figure 3.- Numerical solution of nonlinear equations of motion.

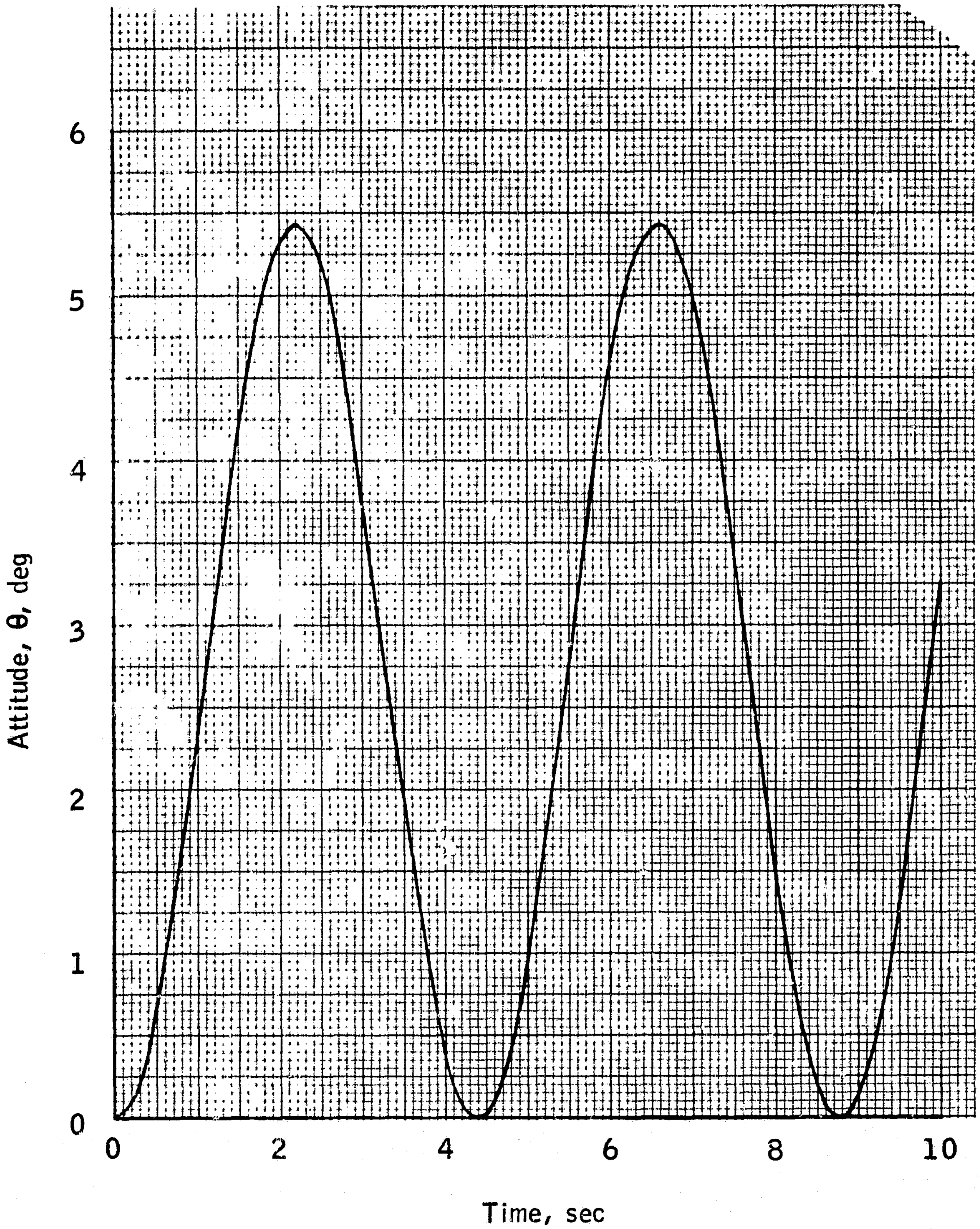

Figure 4.- Numerical solution of linear equations of motion.

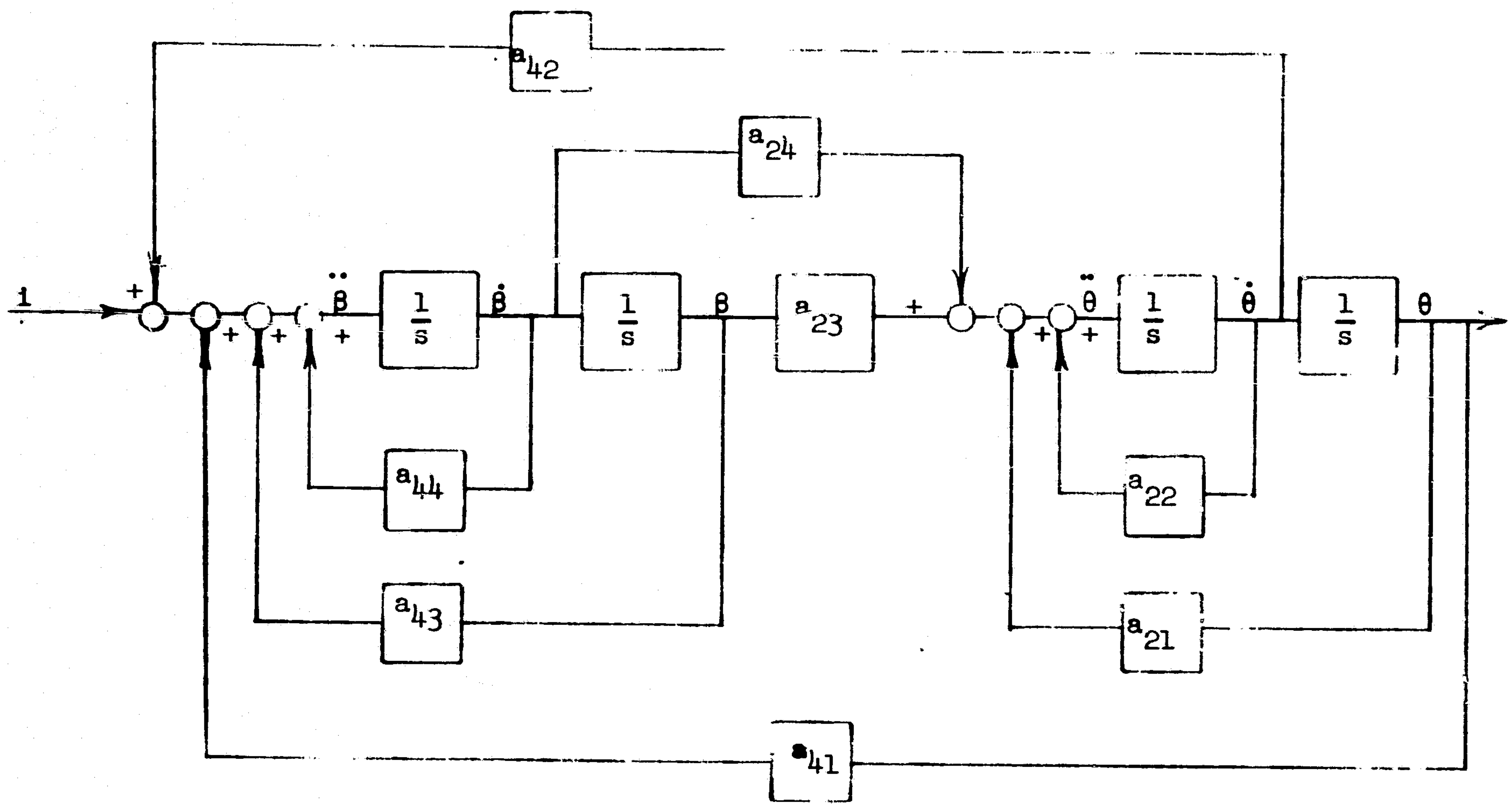

Figure 5.- Block diagram of vehicle dynamics.

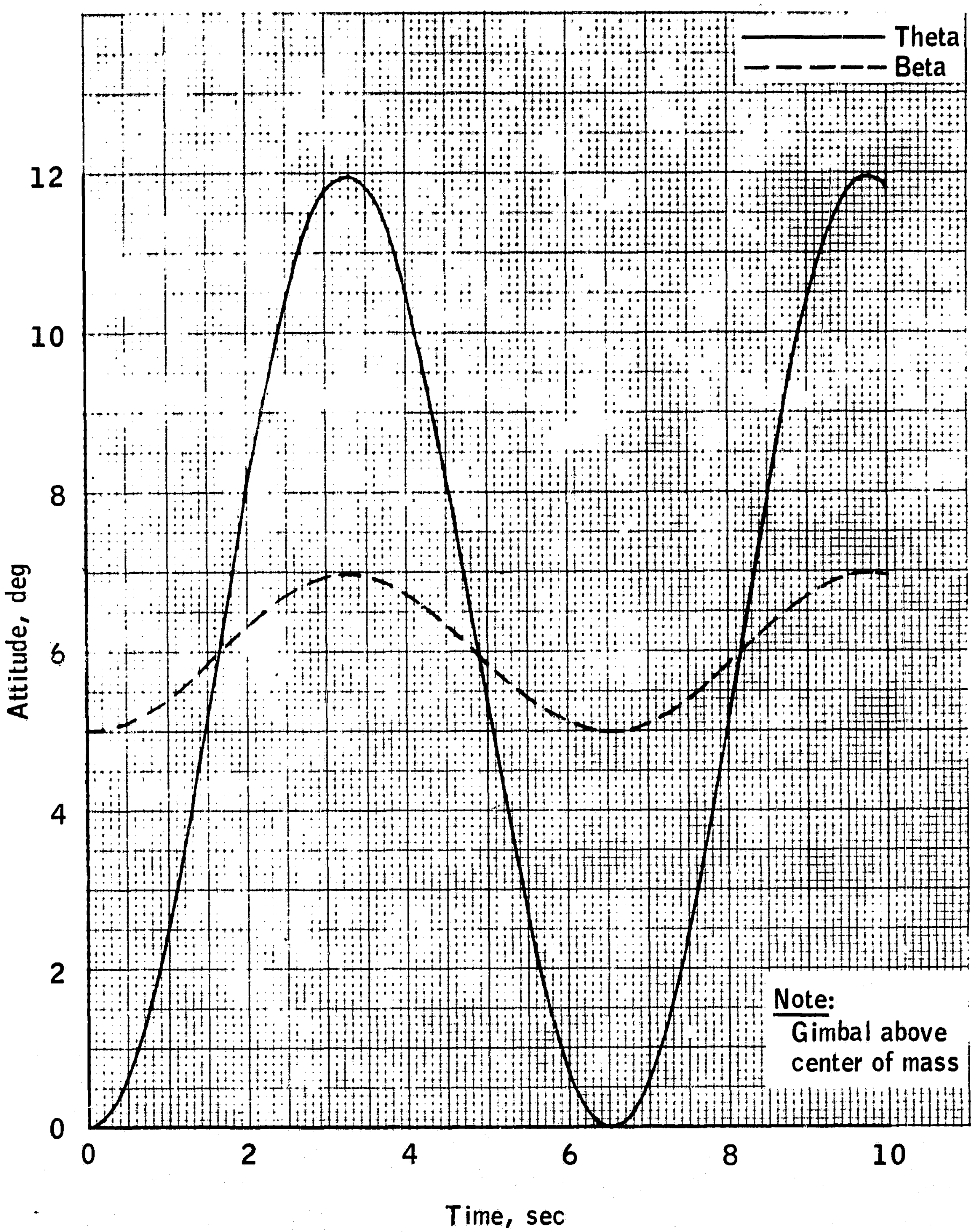

Figure 6.- Response of thrust vector controlled vehicle (autonomous, gimbal above center of mass).

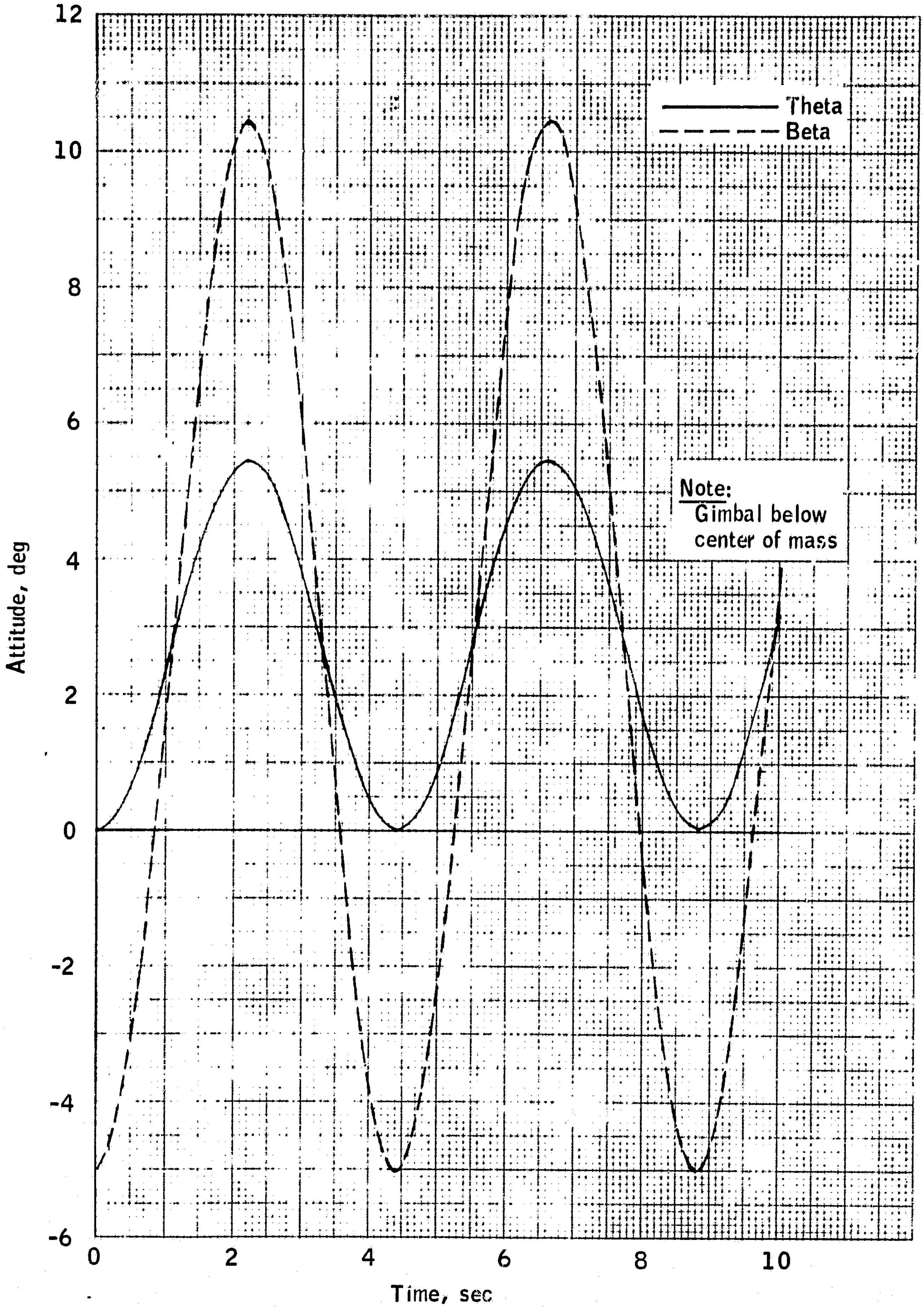

Figure 7.- Response of thrust vector controlled vehicle (autonomous, gimbal below center of mass).

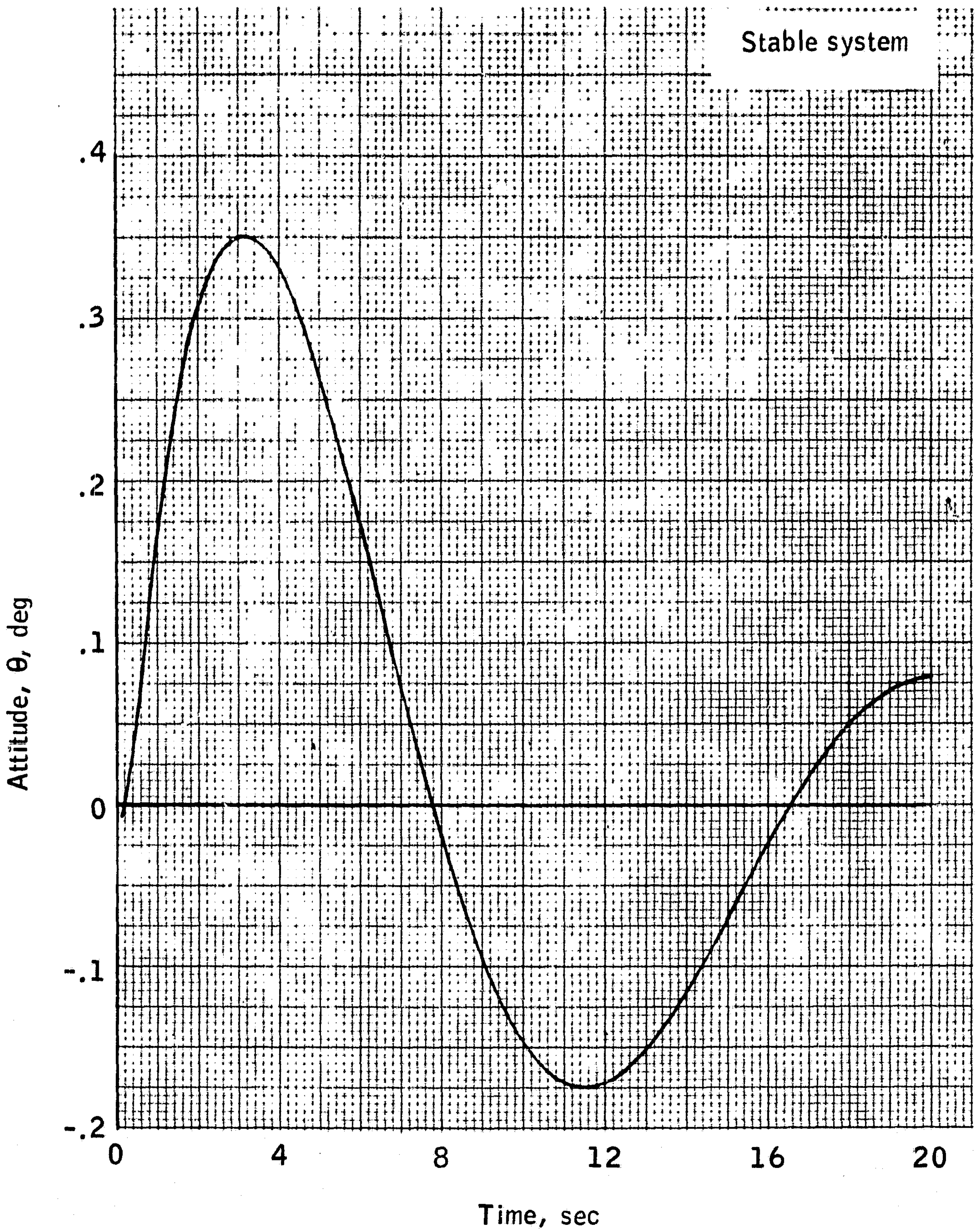

Figure 8.- Impulse response for attitude control system (stable).

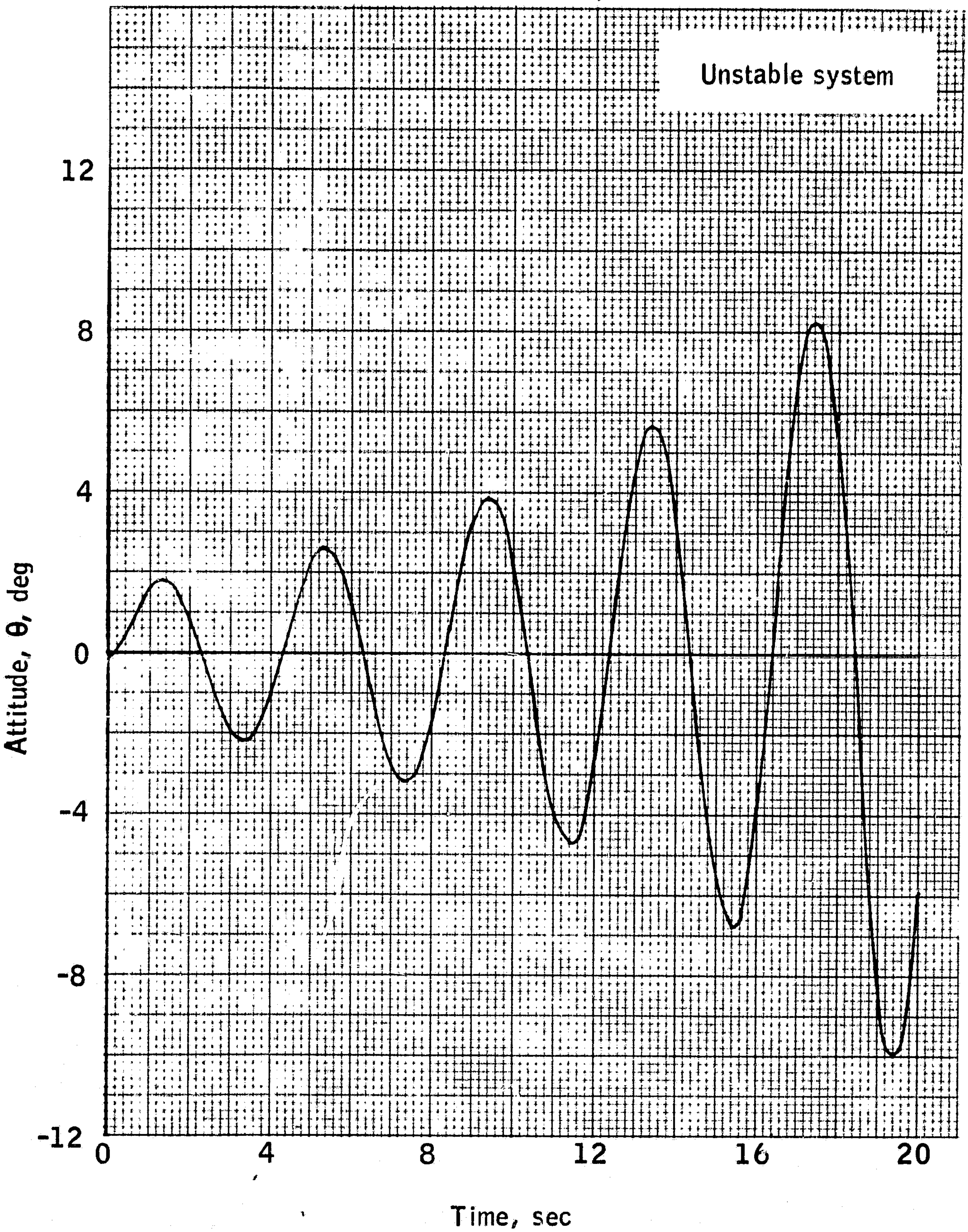

Figure 9.- Impulse response for attitude control system (unstable).