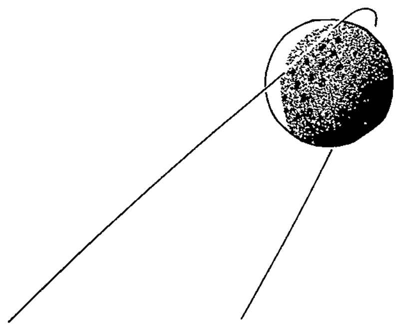

THE DERIVATION OF A GENERAL PERTURBATION SOLUTION AND ITS APPLICATION TO ORBIT DETERMINATION

BY

GEORGE H. BORN

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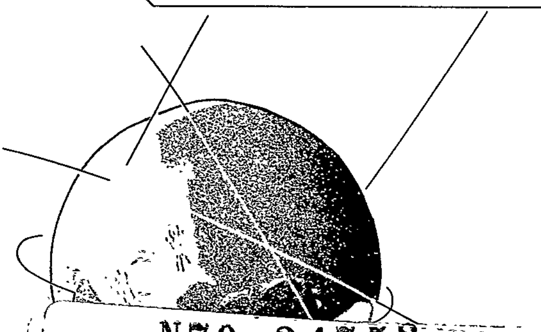

APPLIED MECHANICS RESEARCH LABORATORY

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PREFACE

There are a number of challenging problems facing the aerospace engineer in the area of spacecraft tracking and guidance. One requirement for the solution of these problems is the development of mathematical techniques which are computationally efficient and accurate. This study is concerned with the development of an analytical solution to a modified set of Lagrange's planetary equations. These solutions describe the variations with time of a spacecraft's orbit which is perturbed by an arbitrarily shaped primary body and a point mass third body. Analytical solutions are of value for their computational rapidity and for the insight which they provide into the behavior of the dynamic system. The analytical solutions also are incorporated into an orbit determination program which is of value as a research tool. Its value is demonstrated by using it for a study of several problems associated with orbit determination.

Special gratitude is expressed to Dr. B. D. Tapley for serving as advisor and for his encouragement and guidance during the author's course of study at The University of Texas. Thanks, also, is given to Dr. M. J. Thompson, Dr. E. J. Prouse, Dr. L. G. Clark, and Dr. W. T. Fowler for serving on the supervising committee.

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G. H. B.

August 1968
Austin, Texas

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

GENERAL

The advent of spaceflight has presented the engineer with many new and exciting challenges. A number of these lie in the field of spacecraft tracking and guidance. One fundamental requirement for spacecraft guidance is the accurate and rapid prediction of the spacecraft's trajectory. This necessitates the development of mathematical techniques which are computationally efficient while maintaining a high degree of accuracy. The development of analytical solutions to the spacecraft's equations of motion is useful in that the state vector may be generated at any desired time without intermediate numerical extrapolation. The study described here develops such a solution and applies it to the problem of lunar satellite orbit prediction and near-earth satellite orbit determination.

PROBLEM TO BE STUDIED

General Perturbation theory is concerned with the development of analytical solutions to the equations of motion of a satellite moving under the influence of an arbitrary gravitational force. In this study a first order general perturbation solution is developed for the motion of a satellite under the influence of an arbitrarily shaped primary body and a point mass third body. A set of nonsingular orbit elements is used to describe the satellite's motion and, therefore, the results are valid for all circular and elliptical motion. The solutions utilize the development of the disturbing function in terms of the Keplerian elements as given by Kaula. They are obtained by extending the previous work of Ingram to include the third body effects.

Next, the general perturbation solutions are incorporated into an orbit determination scheme. The result is an accurate and rapid orbit determination program which is valuable for doing preliminary design work. Its value as a research tool is demonstrated by using it for a study of several problems associated with orbit determination.

OUTLINE OF STUDY

Chapter II develops a modified first order general perturbation solution for the motion of a point mass satellite under the influence of an arbitrarily shaped primary body and a point mass third body. Results obtained with these solutions are compared with numerically integrated trajectories for various orbital configurations. It is shown that the analytical solutions are both accurate and amenable to computer evaluation.

In Chapter III linear estimation theory is reviewed, and a technique for recursively estimating the observation error covariance matrix is developed. The analytical solutions are used as the basis for an orbit determination scheme which compares favorably with an orbit determination program utilizing numerical integration. A brief study of the accuracy of transition matrices formed by numeric partial differentiation is made in order to justify its use in the analytical program. A numerical comparison of the effects of the coordinates used in the estimation process is made by using both Keplerian elements and Cartesian coordinates to determine the orbit of a near-earth satellite. In addition the recursive scheme for estimating the radar covariance matrix is tested by using the same sample problem.

The numerical studies were made by considering the problem of estimating the orbit of a near-earth satellite under the influence of the gravitational harmonics J_{20}, J_{30}, and J_{40}. Observation data of range, azimuth and elevation from four tracking stations was simulated and corrupted with normally distributed random noise.

A summary of results and a list of possible extensions to this work are presented in Chapter IV.

CHAPTER II

DERIVATION OF THE GENERAL PERTURBATION SOLUTION

INTRODUCTION

From the time of the first satellite launching in 1957 there have been numerous papers dealing with analytical solutions to the equations of motion of a satellite under the influence of a noncentral gravitational force field. Generally, these papers have considered the satellite's motion as perturbed by the zonal harmonics of the earth's gravity field.

The objective of this chapter is to develop an analytical solution to the equations of motion of a satellite moving under the influence of an aspherical primary and a perturbing third body. The third body disturbing function developed by Kaula[1] will provide the basis for including the third body effects. These results will be combined with Ingram's[2,3] solution for the primary body effects. It will be shown that the resulting analytical solution is both accurate and amenable to computer evaluation.

Kozai[4], Brouwer[5] and Garfinkel[6] published the classical studies of the motion of a satellite influenced by the zonal harmonics. They used the methods formulated by Hamilton, Jacobi, Delaunay and Von-Zeipel. In a later paper, Kozai[7] presented a solution which included second order periodic perturbations and third order secular perturbations for satellite orbits influenced by only the zonal harmonics. The classical method of Hansen has been modified by Musen[8] to give a theory which can be used to compute semi-analytically the perturbations up to any order. Vinti's theory[9] uses spheroidal coordinates in place of conventional spherical coordinates to obtain a closed form solution for the effects of oblateness. Lorell[10] et al., used the Method of Averages to develop the second order long period and secular effects of oblateness. More recently, the effects of the sectorial and tesseral harmonics have been studied by Garfinkel[11] and Vagners[12].

A paper by Kaula[13] which developes the primary body disturbing function in terms of the classical Keplerian orbit elements has provided the basis for several papers on general perturbation theory. Kaula also presents a first order theory for the variations of the classical orbit elements under the influence of an aspherical primary. In addition Kaula gives second order effects for the interaction of the oblateness with the other terms in the disturbing function. Ingram[2,3] et al., use Kaula's development of the primary body disturbing function as a basis for the first order solution of a set of orbit elements which have been modified to eliminate the singularities of the classical Keplerian elements. Ingram presents a first order solution for the elements which includes the influence of any harmonic in the primary body potential function as well as the first term in a binomial series expansion of the third body disturbing function. Results are presented which show the deviation of this solution from a numerically integrated trajectory.

A paper by Tapley and Born[14] shows the results of numerically integrating Lagrange's planetary equations for various Apollo-type lunar orbits in addition to comparing these results with those obtained from a first-order averaged solution.

Arsenault[15] et al., have surveyed published general perturbation solutions for satellite motion about an oblate planet. They have programmed several of the techniques and present data comparing the accuracy of the schemes for computing earth satellite motion as influenced by the dominant zonal harmonics. Results are presented for various inclinations and eccentricities.

Previously there have been relatively few papers dealing with the third body effects on artificial satellite motion. This is because lunar and solar effects are of second order for near earth satellites, and it is only recently that the study of lunar satellite motion has become significant. Although the related problem of planetary motion has been studied by astronomers for a number of years, the classical theories[16], such as those of Hansen, Delaunay, Hill and Brown, are concerned only with long term planetary ephemeris predictions. Therefore, they are not well suited for the prediction of short term lunar satellite orbits of current interest.

Kozai[17] gives the long term and secular effects for the node and argument of pericenter for the first term in the third body distrubing function. These results are obtained by integrating the disturbing function over the satellite's period, thus eliminating the short term effects. Later Kozai[18] extended this work to include short period lunar-solar perturbations. Anderson[19] and Lewis[20] present the long term and secular effects for all six classical orbit elements under the influence of the first term in the third body disturbing function. Lidov[21] uses averaging techniques to study the third body effects.

Kaula[1] has developed an infinite series form for the third body disturbing function in terms of the classical Keplerian elements. The results are quite similar to the corresponding expansion of the primary body disturbing function. As will be shown in the following discussion, Kaula's development of the primary and third body disturbing functions may be combined to obtain an approximate analytical solution for Lagrange's planetary equations.

In view of the advantages of analytical over numerical solutions, it is desirable to have an analytical solution to the equations of motion when dealing with the problem of orbit prediction and determination. If a satisfactory analytical solution cannot be found, the equations of motion must be integrated numerically. In addition to being time consuming, numerical integration also is subject to truncation and roundoff errors which tend to invalidate the results whenever a large number of integration steps are required. Analytical solutions are not subject to errors of this type; furthermore, they allow the state vector to be computed at any desired time, thus eliminating the stepwise extrapolation required by numerical integration. In addition, examination of the analytical solution yields general characteristics of a satellite's motion under the influence of a given perturbing force.

LAGRANGE'S PLANETARY EQUATIONS

Lagrange's planetary equations[16] describe the variations with time of a satellite's orbit elements and will provide the basis for the analytical solution developed here. This set of six first order nonlinear differential equations is given by Eq. (2-1) through Eq. (2-6)

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M} \tag{2-1}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\left[\sqrt{1-e^2}\,\frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega}\right] \tag{2-2}$$

$$\frac{dI}{dt} = \frac{\csc I}{na^2\sqrt{1-e^2}}\left[\cos I\,\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right] \tag{2-3}$$

$$\frac{d\Omega}{dt} = \frac{\csc I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} \tag{2-4}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2}\left[\frac{-\cot I}{1-e^2}\frac{\partial R}{\partial I} + \frac{1}{e}\frac{\partial R}{\partial e}\right] \tag{2-5}$$

$$\frac{dM}{dt} = n - \frac{1}{na}\left[\frac{1-e^2}{ae}\frac{\partial R}{\partial e} + 2\frac{\partial R}{\partial a}\right] \tag{2-6}$$

The elements a, e, and M are the dynamical elements; hence, they are invariant under coordinate transformation. The elements Ω, ω and I describe the orbit's orientation with respect to a specified Cartesian coordinate system as shown in Figure C-1 of Appendix C. The quantity R is the disturbing function and represents the portion of the potential function which causes deviation from two-body motion. The mean motion is denoted by n.

An examination of Lagrange's planetary equations reveals that the equations for five of the six elements will be singular if the eccentricity and inclination are zero. It can be seen from Fig. (C-1) that as the inclination approaches zero the node becomes completely arbitrary. Similarly the argument of pericenter and mean anomaly become aribtrary when the eccentricity approaches zero. Since the eccentricity and inclination are well defined even when they are zero, it is clear that the singularities in $\dot{e}$ and $\dot{I}$ are mathematical. Furthermore, the singularities in $\dot{\Omega}$, $\dot{\omega}$, and $\dot{M}$ are geometrical since they may be removed by a coorindate transformation. The mathematical singularities in $\dot{e}$ and $\dot{I}$ can be removed by simply rearranging the differential equations as follows

$$e\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na^2}\left[\sqrt{1-e^2}\frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega}\right] \tag{2-7}$$

$$\sin I\frac{dI}{dt} = \frac{1}{na^2\sqrt{1-e^2}}\left[\cos I\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right] \tag{2-8}$$

Even though the node, argument of pericenter and mean anomaly become arbitrary under certain conditions, the position of the vehicle with respect to the X-axis is still well defined, and the geometrical problems associated with the definition of Ω , ω , and M may be resolved by the redefinition of these orientation angles. It is clear that for elliptical motion the angle $\omega + M + \Omega$ always will be well defined even though the individual quantities may be poorly defined. The differential equation for $\omega + M + \Omega$ becomes

$$\frac{d}{dt}(\omega+M+\Omega) = n + \frac{1}{na^2\sqrt{1-e^2}}\frac{\sin I}{(1+\cos I)}\frac{\partial R}{\partial I} - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{e\sqrt{1-e^2}}{na^2(1+\sqrt{1-e^2})}\frac{\partial R}{\partial e} \qquad (2\text{-}9)$$

Although it appears that the above equation is singular when $I = \pi$, for the cases of interest $\frac{\partial R}{\partial I} = 0$ when $I = 0$ or $I = \pi$. However, any numerical difficulties can be eliminated by using $\omega + M - \Omega$ when I approaches π [2].

Hence, the differential equation may be written as

$$\frac{d}{dt}(\omega+M+\alpha\Omega) = n + \frac{1}{na^2}\left[F(\alpha)\frac{\partial R}{\partial I} - 2a\frac{\partial R}{\partial a} + \frac{eK}{1+K}\frac{\partial R}{\partial e}\right] \qquad (2\text{-}10)$$

where $\alpha = 1$ for $0 \le I \le 175°$ and $\alpha = -1$ for $175° < I \le 180°$ and

$$K = \sqrt{1-e^2}$$

$$F(\alpha) = \frac{\alpha \sin I}{1+\alpha \cos I}$$

The nonsingular elements are defined as follows[2]:

$$h = \sin I \sin \Omega$$

$$k = \sin I \cos \Omega$$

$$A = e \sin (\omega + \alpha\Omega)$$

$$B = e \cos (\omega + \alpha\Omega)$$

$$\delta = \omega + M + \alpha\Omega \, .$$

The sixth element is the semimajor axis, a . The associated differential equations are

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M} \tag{2-11}$$

$$\frac{dh}{dt} = k \frac{d\Omega}{dt} + \cos I \sin \Omega \frac{dI}{dt} \tag{2-12}$$

$$\frac{dk}{dt} = -h \frac{d\Omega}{dt} + \cos I \cos \Omega \frac{dI}{dt} \tag{2-13}$$

$$\frac{dA}{dt} = B\left[\frac{d\omega}{dt} + \alpha \frac{d\Omega}{dt}\right] + \sin (\omega+\alpha\Omega) \frac{de}{dt} \tag{2-14}$$

$$\frac{dB}{dt} = -A \left[\frac{d\omega}{dt} + \alpha \frac{d\Omega}{dt}\right] + \cos (\omega+\alpha\Omega) \frac{de}{dt} \tag{2-15}$$

$$\frac{d\delta}{dt} = \frac{d\omega}{dt} + \frac{dM}{dt} + \alpha \frac{d\Omega}{dt} \tag{2-16}$$

$$\frac{d(\cos I)}{dt} = \frac{1}{na^2\sqrt{1-e^2}} \left[\frac{\partial R}{\partial \Omega} - \cos I \frac{\partial R}{\partial \omega}\right] \tag{2-17}$$

The equation for $\cos I$ is used to determine I since $0 \leq I \leq \pi$ and $\cos I$ uniquely determines the quadrant.

In order to obtain an analytical solution to the above set of differential equations, it is necessary to express the disturbing function in terms of the orbit elements. The required results for the primary body[22] and the third body disturbing function[1] have been developed by Kaula. The transformations from polar spherical coordinates to the orbit elements for the primary body and the transformation from rectangular cartesian to the orbit elements for the third body disturbing function are quite involved. These references are rather brief; however, a detailed description of both these transformations is given by Born and Hildebrand[23]. A derivation of Hansen's coefficients which are required for the transformations also is given.

THE PRIMARY BODY POTENTIAL FUNCTION

The potential function for the primary body is[22, 23]

$$V = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{\mu a_e^{\ell}}{a^{\ell+1}} P(I, e) S_{\ell mpq} (\omega, M, \Omega, \theta) \tag{2-18}$$

The potential function reduces to that of a point mass for $\ell=m=0$. If the origin is located at the center of mass as is assumed for this investigation, $V = 0$ for $\ell = 1$. The portion of V for which $\ell \geq 2$ is known as the primary body disturbing function.

In Eq. (2-18) a_e is the mean radius of the primary body and

$$P(I, e) = \sum_{p=0}^{\ell} \sum_{q=-\infty}^{\infty} F_{\ell mp}(I) \; G_{\ell pq}(e) \tag{2-19}$$

where

$$F_{\ell mp}(I) = \sum_t \frac{(2\ell-2t)!}{t!(\ell-t)!(\ell-m-2t)!2^{(2\ell-2t)}} \sin^{\ell-m-2t} I$$

$$\sum_{s=0}^{m} \binom{m}{s} \cos^s I \sum_c \binom{\ell-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \qquad (2\text{-}20)$$

k is the integer part of $^{(\ell-m)}/2$, t is summed to the lesser of p or k , and c is summed over all values which make the binomial coefficient nonzero.

The function $G_{\ell pq}(e)$ is defined as follows: if $\ell - 2p + q = 0$

$$G_{\ell pq}(e) = \frac{1}{(1-e^2)^{\ell-\frac{1}{2}}} \sum_{d=0}^{p'-1} \binom{\ell-1}{2d+\ell-2p'} \binom{2d+\ell-2p'}{d} \left(\frac{e}{2}\right)^{2d+\ell-2p'} \qquad (2\text{-}21)$$

in which

$$p' = p \quad \text{for} \quad p \le \ell/2$$
$$p' = \ell-p \quad \text{for} \quad p \ge \ell/2 . \qquad (2\text{-}22)$$

For $\ell - 2p + q \neq 0$

$$G_{\ell pq}(e) = (-1)^{|q|} (1 + \beta^2)^{\ell} \beta^{|q|} \sum_{k=0}^{\infty} P_{\ell pqk} Q_{\ell pqk} \beta^{2k} \qquad (2\text{-}23)$$

where

$$\beta = \frac{e}{1+\sqrt{1-e^2}} \qquad (2\text{-}24)$$

$$P_{\ell pqk} = \sum_{r=0}^{h} \binom{2p'-2\ell}{h-r} \frac{(-1)^r}{r!} \left[\frac{(\ell-2p'+q')e}{2\beta}\right]^r \qquad (2\text{-}25)$$

$$h = k + q', \quad q' > 0; \quad h = k, \quad q' < 0;$$

and

$$Q_{\ell pqk} = \sum_{r=0}^{h} \binom{-2p'}{h-r} \frac{1}{r!} \left[\frac{(\ell-2p'+q')e}{2\beta}\right]^r \tag{2-26}$$

$$h = k, \quad q' > 0; \quad h = k - q', \quad q' < 0$$

$$p' = p, \quad q' = q \quad \text{for} \quad p \leq \ell/2;$$
$$p' = \ell-p, \quad q' = -q \quad \text{for} \quad p > \ell/2$$

also, in Eq. (2-18)

$$S_{\ell mpq}(\omega, M, \Omega, \theta) = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} \cos\left[(\ell-2p)\omega\right.$$
$$\left. + (\ell-2p+q)M + m(\Omega-\theta)\right] + \begin{bmatrix} S_{\ell m} \\ C_{\ell m} \end{bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}}$$
$$\sin\left[(\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\theta)\right] \tag{2-27}$$

where θ is the angle measured in the equatorial plane eastward from the inertial X-axis to the prime meridian. The quantity $S_{\ell mpq}$ also may be written as follows: if $\ell - m$ is even,

$$S_{\ell mpq}(\omega, M, \Omega, \theta) = J_{\ell m} \cos \Phi_{\ell mpq} \tag{2-28}$$

and if $\ell - m$ is odd,

$$S_{\ell mpq}(\omega, M, \Omega, \theta) = J_{\ell m} \sin \Phi_{\ell mpq} \tag{2-29}$$

where

$$\Phi_{\ell mpq} = (\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\theta-\lambda_{\ell m}) \tag{2-30}$$

The sets $[C_{\ell m}, S_{\ell m}]$ and $[J_{\ell m}, \lambda_{\ell m}]$ are constants determined by the shape and mass distribution of the body and are related by

$$C_{\ell m} = J_{\ell m} \cos m \lambda_{\ell m}$$
$$S_{\ell m} = J_{\ell m} \sin m \lambda_{\ell m} .$$

The $G_{\ell pq}(e)$ functions are the coefficients of the Fourier series expansion of the functions

$$\left(\frac{r}{a}\right)^n \sin mf \quad \text{and} \quad \left(\frac{r}{a}\right)^n \cos mf \tag{2-31}$$

in terms of the mean anomaly[23]. A recursive relationship for the $G_{\ell pq}(e)$ functions is developed in Appendix B.

THE THIRD BODY DISTURBING FUNCTION

The notation for the third body disturbing function is identical to that used for the primary except that the quantities associated with the third body's orbit are designated by asterisks. The third body disturbing function is[1,23]

$$R = \sum_{\ell=2}^{\infty} \frac{\mu^* a^\ell}{a^{*\ell+1}} \sum_{m=0}^{\ell} K_m \frac{(\ell-m)!}{(\ell+m)!} C(I, I^*, e, e^*) \tag{2-32}$$
$$T_{\ell mpsqj}(\omega, M, \Omega, \omega^*, M^*, \Omega^*)$$

where

$$C(I, I^*, e, e^*) = \sum_{p=0}^{\ell} \sum_{q=-\infty}^{\infty} \sum_{s=0}^{\ell} \sum_{j=-\infty}^{\infty} F_{\ell mp}(I)\, H_{\ell pq}(e)\, F_{\ell ms}(I^*)\, G_{\ell sj}(e^*) \tag{2-33}$$

The functions $F_{\ell mp}(I)$, $F_{\ell ms}(I^*)$ and $G_{\ell sj}(e^*)$ are defined exactly as they were for the primary body. Also $K_0 = 1$ and $K_m = 2$ for $m \neq 0$.

The function $H_{\ell pq}(e)$ is defined as follows: if $\ell - 2p + q = 0$

$$H_{\ell pq}(e) = \frac{(-\beta)^{\ell-2p'}}{(1+\beta^2)^{\ell+1}} \sum_{k=0}^{\infty} \binom{2p'+1}{k} \binom{2\ell-2p'+1}{\ell-2p'+k} \beta^{2k} \tag{2-34}$$

where

$$p' = p \text{ for } p \leq \ell/2 \text{ and } p' = \ell - p \text{ for } p \geq \ell/2 .$$

When $\ell - 2p + q \neq 0$, $H_{\ell pq}(e)$ is defined as

$$H_{\ell pq}(e) = (-1)^{|q|} (1+\beta^2)^{-s-1} \beta^{|q|} \sum_{k=0}^{\infty} P'_{\ell pqk}\, Q'_{\ell pqk}\, \beta^{2k} \tag{2-35}$$

where

$$P'_{\ell pqk} = \sum_{r=0}^{h} \binom{2p'+1}{h-r} \frac{(-1)^r}{r!} \left[\frac{(\ell-2p'+q')e}{2\beta}\right]^r \tag{2-36}$$

$$Q'_{\ell pqk} = \sum_{r=0}^{h} \binom{2\ell-2p'+1}{h-r} \frac{1}{r!} \left[\frac{(\ell-2p'+q')e}{2\beta}\right]^r \tag{2-37}$$

The definitions for the indices are the same as those given for $G_{\ell pq}(e)$.

Also,

$$T_{\ell mpsqj}(\omega, M, \Omega, \omega^*, M^*, \Omega^*) = \cos [(\ell-2p)\omega + (\ell-2p+q)M - (\ell-2s)\omega^* - (\ell-2s+j)M^* + m(\Omega-\Omega^*)] \tag{2-38}$$

The lead terms in both the $G_{\ell pq}(e)$ and $H_{\ell pq}(e)$ functions are of order $e^{|q|}$, and since they are power series in e, only a few terms need be considered for near circular orbits. Kaula[22] gives tables of the $F_{\ell mp}(I)$ and $G_{\ell pq}(e)$ functions for several values of the indices $\ell\ m\ p\ q$. Appendix A presents the table of $H_{\ell pq}(e)$ functions for the same values of the indices presented in Ref. (22). These results were generated by the IBM 7094 computer using the symbolic manipulation language FORMAC.

When the disturbing potential is written in this form it is easy to isolate the short period, long period and secular portions by selecting the proper indices. For example, the short period effects, those periodic in M, may be eliminated by letting $\ell - 2p + q = 0$ in the primary and third body disturbing function.

The disturbing function, R, which appears in Lagrange's planetary equations is the sum of the primary and third body disturbing functions, i.e.,

$$R = \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} V_{\ell m} + \sum_{\ell=2}^{\infty} R_{\ell} \qquad (2\text{-}39)$$

The derivatives of R with respect to the orbital elements required by Lagrange's planetary equations are obtained easily by differentiating Eq. (2-39).

THEORY OF THE PERTURBATION SOLUTION

Lagrange's planetary equations are of the form

$$\frac{dx_i}{dt} = \varepsilon f_i(x_j, t) \qquad i, j = 1 \ldots 6 \qquad (2\text{-}40)$$

and subject to the initial conditions

$$x_i(t_o) = \alpha_i \qquad i = 1 \ldots 6 \tag{2-41}$$

The quantity ε is a small parameter proportional to the magnitude of the perturbing force. A standard procedure originating with Euler is to express the solution of (2-40) as a power series in ε :

$$x_i(t) = \alpha_i + \varepsilon\, x_i^{(1)}(t) + \varepsilon^2\, x_i^{(2)}(t) + \ldots \tag{2-42}$$

where the $x_i^{(j)}(t)$ are to be determined.

Following the procedure of Chapter III in Ref. (24), Eq. (2-40) is expanded in a Taylor's series in $x_j - \alpha_j$ as follows,

$$\begin{aligned}\frac{dx_i}{dt} &= f_i(\alpha_j, t) + \sum_{j=1}^{6} \frac{\partial f_i}{\partial x_j} (x_j - \alpha_j) \\ &+ \frac{1}{2} \sum_{j=1}^{6} \frac{\partial^2 f_i}{\partial x_j^2} (x_j - \alpha_j)^2 \\ &+ \sum_{\substack{j=1 \\ j \neq k}}^{6} \sum_{k=1}^{6} \frac{\partial^2 f_i}{\partial x_j \partial x_k} (x_j - \alpha_j)(x_k - \alpha_k) + \ldots\end{aligned} \tag{2-43}$$

If Eq. (2-42) is substituted into Eq. (2-43) and terms with corresponding powers of ε are equated, an infinite series of systems of differential equations is obtained:

$$\begin{aligned}\frac{dx_i^{(1)}}{dt} &= f_i(\alpha_j, t) \\ \frac{dx_i^{(2)}}{dt} &= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, x_j^{(1)}\end{aligned} \tag{2-44}$$

$$\frac{dx_i^{(3)}}{dt} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} x_j^{(2)} + \frac{1}{2!}\left\{\sum_{j=1}^{6} \frac{\partial^2 f_i}{\partial x_j^2} (x_j^{(1)})^2 + \sum_{j=1}^{6} \sum_{\substack{k=1 \\ k\neq j}}^{6} \frac{\partial^2 f_i}{\partial x_j \partial x_k} x_j^{(1)} x_k^{(1)}\right\}$$

$$\frac{dx_i^{(k)}}{dt} = P_i^{(k)} (x_j^{(1)}, \ldots : x_j^{(k-1)}, t) \qquad (2\text{-}44)$$

where the $P_i^{(k)}$ are polynomials in the $x_j^{(1)} \ldots x_j^{(k-1)}$.

Since the $f_i(\alpha_j, t)$ are continuous functions of time, the set of differential equations, $x_i^{(1)}$, may be solved directly by quadratures to yield

$$x_i^{(1)} = \int f_i(\alpha_j, t)dt = F_i^{(1)} (\alpha_j, t) + C_i^{(1)} \qquad (2\text{-}45)$$

where the $C_i^{(1)}$ are arbitrary constants.

Now Eq. (2-45) may be substituted into the second of Eqs. (2-44) whose right-hand member will now be a known continuous function of time. Consequently, the $x_i^{(2)}$ and similarly all the $x_i^{(k)}$ may be obtained sequentially by quadratures. A general term of the solution is

$$x_i^{(k)} = \int P_i^{(k)} (x_j^{(1)}, \ldots, x_j^{(k-1)})dt = F_i^{(k)} (\alpha_j, t) + C_i^{(k)} \qquad (2\text{-}46)$$

where the $C_i^{(k)}$ are arbitrary constants.

The sets of constants $C_i^{(k)}$ may be determined in terms of the α_i by substituting Eq. (2-46) into Eq. (2-42) and evaluating the results at $t = t_0$. The result is

$$\alpha_i = \alpha_i + \mu(F_i(\alpha_j, t_0) + C_i^{(1)}) + \ldots + \mu^k(F_i(\alpha_j, t_0)C_i^{(k)}) + \ldots \qquad (2\text{-}47)$$

Since ε has been assumed nonzero, its coefficients must vanish. Consequently the $C_i^{(k)}$ have the values $C_i^{(1)} = -F_i^{(1)} (\alpha_j, t_0) \ldots , C_i^{(k)} = -F_i^{(k)} (\alpha_j, t_0)$ $\ldots$. Therefore, the function $x_i^{(k)}$ are the uniquely determined definite integrals

$$x_i^{(k)} = \int_{t_0}^{t} P_i^{(k)} (x_j^{(1)}, \ldots\ldots, x_j^{(k-1)})d\tau \qquad (2\text{-}48)$$

where

$$i = 1, \ldots, 6 , \quad j = 1, \ldots 6 , \quad k = 1, 2 \ldots .$$

Moulton[24] discusses conditions under which the series (2-42) converges to a solution of Eq. (2-40) and presents a proof of convergence.

Since the perturbing forces associated with the orbit of either an earth or a lunar satellite are quite small in comparison to the central force, the equations of motion are well suited to solution by a perturbation technique. However, since the labor associated with a second order solution is prohibitive and the solution itself is so lengthy and cumbersome as to be impractical, only a first order solution will be considered here. The first order solution will be modified somewhat in order to include some terms which would ordinarily be considered second order.

The first order solution for Lagrange's planetary equations involves the solution of the first set of Eqs. (2-44) which has been modified to include the linear dependence of the mean anomaly on time:

$$\frac{dx_i^{(1)}}{dt} = f_i(\alpha_j, M_0 + nt, t) \qquad \begin{array}{l} j = 1 \ldots 5 \\ i = 1 \ldots 6 \end{array}$$

where n is the mean motion given by

$$n = \frac{\mu^{1/2}}{a^{3/2}}$$

and the α_j are the epoch values of the elements.

When this assumption for the elements is substituted into the right-hand side of Lagrange's planetary equations, the equations are both linear and independent. This allows them to be integrated individually to obtain the first order solution. However, it is advantageous to allow for the secular variations (rates) of Ω , ω and M in evaluating the first-order perturbations arising from a given term in the disturbing function[16]. By including these secular rates, terms which ordinarily would come from a second order solution are obtained from a first order solution[1]. Also, combining the secular rates due to the primary and third body couples these effects and increases the accuracy of the solution. This modification of the first order equations still leaves them linear and independent. In addition, it is still permissible to deal separately with each term or group of terms of the disturbing function. The solutions obtained from the primary and third body disturbing function still may be superimposed.

SECULAR RATES OF THE ELEMENTS

If the argument of the trigonometric function vanishes, the only elements of the satellite's orbit which the disturbing function is dependent on are a , e and I . This portion of the disturbing function causes the elements to vary in a secular manner; consequently, it is designated as the secular portion of the disturbing function. The only elements which will have secular variations are Ω , ω and M since the differential equations for the remaining elements do not contain the partial derivative of R with respect to a , e or I .

Now it will be convenient to separate the disturbing function into its secular and periodic portions. Let

$$R = R_c + R_t \tag{2-49}$$

where R_c is the secular portion of the disturbing function. Note that R_c contains all of the terms of the primary body disturbing function for which

$$\begin{aligned} &\ell-m \text{ is even}, && \ell-2p = 0 \\ &\ell-2p+q = 0 , && m = 0 \end{aligned} \tag{2-50}$$

plus that portion of the third body disturbing function for which

$$\begin{aligned} &\ell \text{ is even}, && \ell-2p = 0 \\ &\ell-2p+q = 0, && \ell-2s = 0 \\ &\ell-2s+j = 0, && m = 0 . \end{aligned} \tag{2-51}$$

The quantity R_t is the periodic portion of the disturbing function and contains all remaining terms.

The general expressions for the secular rates of the orbit elements are derived by substituting R_c into Lagrange's planetary equations. The results for the primary body distrubing function are:

$$\dot{a}_c = \dot{e}_c = \dot{I}_c = 0 \tag{2-52}$$

$$\dot{\Omega}_c = \sum_{\ell} \frac{\mu \; a_e^{\ell}}{na^{\ell+3}K} \csc I \; \frac{\partial P(I,\, e)}{\partial I} \; J_{\ell 0} \tag{2-53}$$

$$\dot{\omega}_c = \sum_{\ell} \frac{\mu\, a_e^{\ell} K}{n a^{\ell+3}} \left[\frac{\cot I}{K^2} \frac{\partial P(I, e)}{\partial I} + \frac{1}{e} \frac{\partial P(I, e)}{\partial e}\right] J_{\ell 0} \tag{2-54}$$

$$\dot{M}_c = n - \sum_{\ell} \frac{\mu\, a_e^{\ell}}{n a^{\ell+2}} \left[\frac{K^2}{ae} \frac{\partial P(I, e)}{\partial e} - \frac{2(\ell+1)}{a} P(I, e)\right] J_{\ell 0} \tag{2-55}$$

The secular rates due to the third body disturbing function are

$$\dot{a}_c = \dot{e}_c = \dot{I}_c = 0 \tag{2-56}$$

$$\dot{\Omega}_c = \sum_{\ell} \frac{\mu^*}{a^{*\ell+1}} \frac{a^{\ell-2}}{nK} \csc I \frac{\partial C(I,I^*,e,e^*)}{\partial I} \tag{2-57}$$

$$\dot{\omega}_c = \sum_{\ell} \frac{\mu^* a^{\ell-2}}{a^{*\ell+1} n} K \left[\frac{-\cot I}{K^2} \frac{\partial C(I,I^*,e,e^*)}{\partial I} + \frac{1}{e} \frac{\partial C(I,I^*,e,e^*)}{\partial e}\right] \tag{2-58}$$

$$\dot{M}_c = n - \sum_{\ell} \frac{\mu^*}{a^{*\ell+1}} \frac{a^{\ell-2}}{n} \left[\frac{K^2}{e} \frac{\partial C(I,I^*,e,e^*)}{\partial e} + 2\ell\, C(I,I^*,e,e^*)\right] \tag{2-59}$$

where the conditions given for R_c in Eqs. (2-50) and (2-51) must be satisfied by the summation indices.

The secular rates due to the oblateness and the first term in the third body disturbing function are given below for convenience. For the oblateness,

$$\dot{\Omega}_c = n \left[\frac{-3/2\, J_{20}\, a_e^2}{a^2 K^2} \cos I\right] \tag{2-60}$$

$$\dot{\omega}_c = n \left[\frac{3/4\, J_{20}\, a_e^2 (5 \cos^2 I - 1)}{a^2 K^2}\right] \tag{2-61}$$

$$\dot{M}_c = n + n\left[\frac{3/4\ J_{20}\ a_e^2}{a^2\ K^3}\ (3\cos^2 I-1)\right] \tag{2-62}$$

For the first term in the third body disturbing function

$$\dot{\Omega}_c = 3/4\ \frac{n^{*2}}{n}\ \frac{\cos I}{K}\ \frac{(1+3/2\ e^2)}{(1-e^{*2})^{3/2}}\ (3/2\ \sin^2 I^*-1) \tag{2-63}$$

$$\dot{\omega}_c = 3/4\ \frac{n^{*2}}{n}\ \frac{(3/2\ \sin^2 I^*-1)}{(1-e^{*2})^{3/2}\ K}\left[5/2\ \sin^2 I - \frac{e^2}{2} - 2\right] \tag{2-64}$$

$$\dot{M}_c = n - 3/4\ \frac{n^{*2}}{n}\ \frac{(7/3+e^2)}{(1-e^{*2})^{3/2}}\ (3/2\ \sin^2 I-1)$$

$$\cdot\ (3/2)\ \sin^2 I^*-1) \tag{2-65}$$

where

$$n^* = \sqrt{\frac{\mu^*}{a^{*3}}}$$

and n is the value of the mean motion based on a mean value of the semimajor axis.

SOLUTION OF LAGRANGE'S PLANETARY EQUATIONS

A first order solution of the nonsingular set of Lagrange's planetary equations may now be obtained by holding a , e , I , a^* , e^* , I^* , Ω^* and ω^* constant when they appear on the right-hand side of the equations and by substituting for the remaining elements the expressions,

$$\Omega = \Omega_o + \dot{\Omega}_c(t-t_o)$$

$$\begin{aligned} \omega &= \omega_o + \dot{\omega}_c(t-t_o) \\ M &= M_o + n(t-t_o) \\ M^* &= M^*_o + n^*(t-t_o) \end{aligned} \tag{2-66}$$

With these substitutions the nonsingular set of Lagrange's planetary equations are linearized and independent and may be integrated individually. A simplified example showing the general procedure for the integration is given: assume

$$\alpha = \alpha_0 + \dot{\alpha}t \ ,$$

then

$$K \frac{d}{dt} \sin \alpha = K \cos \alpha \ \dot{\alpha}$$

since it has been assumed that $\dot{\alpha}$ is constant

$$K \int \cos \alpha \ dt = \frac{K}{\dot{\alpha}} \sin \alpha$$

and

$$K \int \sin \alpha \ dt = - \frac{K}{\dot{\alpha}} \cos \alpha \ .$$

Here α represents the argument of the trigonometric term in the disturbing function and K is a function of a , e and I which are held constant during the integration.

To further illustrate the method of solution, consider the equation for the semimajor axis:

$$a = a_0 + \int_{t_0}^{t} \frac{2}{na} \frac{\partial R}{\partial M} d\tau \ . \tag{2-67}$$

Substituting Eqs. (2-18) and (2-32) for R yields

$$a = a_o + \int_{t_o}^{t} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{2\mu a_e^{\ell}}{\bar{n}\bar{a}^{\ell+2}} P(I, e) \frac{\partial S_{\ell mpq}}{\partial M} d\tau$$

$$+ \int_{t_o}^{t} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{2\mu^* \bar{a}^{\ell-1}}{\bar{n}\bar{a}^{*\ell+1}} K_m \frac{(\ell-m)!}{(\ell+m)!} C(I, I^*, e, e^*)$$

$$\frac{\partial T_{\ell mpsqj}}{\partial M} d\tau \qquad (2\text{-}68)$$

The arguments for $S_{\ell mpq}$ and $T_{\ell mpsqj}$ are

$$\Phi_{\ell mpq} = \Phi_{\ell mpq} (\Omega_0, \omega_0, M_0, \theta_0) + \dot{\Phi}_{\ell mpq} (t-t_0)$$

and

$$\gamma_{\ell mpsqj} = \gamma_{\ell mpsqj} (\Omega_0, \omega_0, M_0, \Omega^*, \omega^*, M_0^*) + \dot{\gamma}_{\ell mpsqj} (t-t_0)$$

respectively, where

$$\dot{\Phi}_{\ell mpq} = (\ell - 2p) \dot{\omega}_c + (\ell - 2p + q) n + m(\dot{\Omega}_c - \dot{\theta})$$

and

$$\dot{\gamma}_{\ell mpsqj} = (\ell - 2p) \dot{\omega}_c + (\ell - 2p + q) n + m\dot{\Omega}_c$$
$$- (\ell - 2s + j) n^* .$$

The integral for Eq. (2-68) now may be written directly:

$$a = a_o + \left\{ \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{2\mu a_e^{\ell}}{\bar{n}\,\bar{a}^{\ell+2}} P(\bar{I}, e) \right.$$

$$(\ell-2p+q) \frac{S_{\ell mpq}}{\dot{\Phi}_{\ell mpq}} + \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \frac{2\mu^* \bar{a}^{\ell-1}}{\bar{n}}$$

$$\left. K_m \frac{(\ell-m)!}{(\ell+m)!} C(I, I^*, e, e^*) (\ell-2p+q) \frac{T_{\ell mpsqj}}{\dot{\Upsilon}_{\ell mpsqj}} \right\} \Bigg|_{t_o}^{t}$$

Next, consider the differential equation

$$\dot{h} = k\,\dot{\Omega} + \cos I \sin \Omega\, \dot{I} .$$

If the expression for $\dot{\Omega}$ is written as

$$\dot{\Omega} = \dot{\Omega}_c + \dot{\Omega}_p$$

where $\dot{\Omega}_c$ is the secular portion and $\dot{\Omega}_p$ is the periodic portion, then

$$\dot{h} = k\,(\dot{\Omega}_c + \dot{\Omega}_p) + \cos I \sin \Omega\, \dot{I} .$$

The homogeneous portion of the differential equation is

$$\dot{h} = k\,\dot{\Omega}_c \qquad (2\text{-}69)$$

The solution of Eq. (2-69) for the initial conditions

$$I = I_0, \quad \Omega = \Omega_0 \quad \text{for} \quad t = t_0 ,$$

is

$$h = \sin I_0 \sin [\Omega_0 + \dot{\Omega}_c(t-t_0)] .$$

The total solution is

$$h = \sin I_0 \sin [\Omega_0 + \dot{\Omega}_c(t-t_0)] + \int_{t_0}^{t} (k\,\dot{\Omega}_p + \cos I \sin \Omega\, \dot{I})d\tau$$

Since the solution of the particular integral involves a large number of terms, it is presented in the list of solutions. In order to illustrate several important points, the solutions for the nonsingular elements in terms of the partial derivatives of the disturbing function are given:

$$a = a_o + \int_{t_o}^{t} \frac{2}{na} \frac{\partial R}{\partial M}\, d\tau \tag{2-70}$$

$$h = \sin I_o \sin (\Omega_o + \dot{\Omega}_c(t-t_o)) + \frac{1}{na^2K} \int_{t_o}^{t} [\cos \Omega \frac{\partial R}{\partial I} + \cot I \sin \Omega (\cos I \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega})]d\tau \tag{2-71}$$

$$k = \sin I_o \cos (\Omega_o + \dot{\Omega}_c(t-t_o)) + \frac{1}{na^2K} \int_{t_o}^{t} [- \sin \Omega \frac{\partial R}{\partial I} + \cot I \cos \Omega (\cos I \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega})]d\tau \tag{2-72}$$

$$A = e_o \cos [\omega_o + \alpha\Omega_o + (\dot{\omega}_c + \alpha\dot{\Omega}_c)(t-t_o)] + \frac{1}{na^2} \int_{t_o}^{t} [\frac{-F(\alpha)e \sin (\omega+\alpha\Omega)}{K} \frac{\partial R}{\partial I} - K \sin (\omega+\alpha\Omega) \frac{\partial R}{\partial e} + \frac{K}{e} \cos (\omega+\alpha\Omega)(K \frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega})]d\tau \tag{2-73}$$

$$B = e_o \sin [\omega_o + \alpha\Omega_o + (\dot{\omega}_c + \alpha\dot{\Omega}_c)(t-t_o)]$$

$$+ \frac{1}{na^2} \int_{t_o}^{t} [\frac{F(\alpha)\, e \cos(\omega+\alpha\Omega)}{K} \frac{\partial R}{\partial I} + K \cos(\omega+\alpha\Omega)\frac{\partial R}{\partial e}$$

$$+ \frac{K}{e} \sin(\omega+\alpha\Omega)\, (K \frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega})] d\tau \qquad (2\text{-}74)$$

$$\delta = \delta_o + \dot{\delta}_c(t-t_o) + \frac{1}{na^2} \int_{t_o}^{t} \left[\frac{F(\alpha)}{K} \frac{\partial R}{\partial I} - 2a \frac{\partial R}{\partial a}\right.$$

$$\left. + \frac{e\,K}{1+K} \frac{\partial R}{\partial e}\right] d\tau + \int_{t_o}^{t} \frac{\partial n}{\partial a} \Delta a \, d\tau \qquad (2\text{-}75)$$

$$\cos I = \cos I_o + \frac{1}{na^2 K} \int_{t_o}^{t} \left[\frac{\partial R}{\partial \Omega} - \cos I \frac{\partial R}{\partial \omega}\right] d\tau \qquad (2\text{-}76)$$

The final integral in the solution for δ is the second order effect of the variation of a on M as given by Kaula[13]. Since the semimajor axis is subject to short period variations only, this second order variation in M is considered when the short period terms are computed.

In the above equations terms such as $\frac{1}{e}$ and $\frac{1}{\sin I}$ still exist. However, these are really not singularities[2] since

$$(\sqrt{1-e^2} \frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega}) \propto e \qquad (2\text{-}77)$$

and

$$(\cos I \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}) \propto \sin I \qquad (2\text{-}78)$$

Consequently, any numerical difficulties are eliminated since mean values are used for e and I and they will never be zero.

List of Solutions

The solutions for the variations of the nonsingular orbit elements under the influence of an aspherical primary body and a perturbing third body are:

$$a = a_o + \left\{ \sum_{\ell,m} \frac{2\bar{n}a_e^{\ell}}{\bar{a}^{\ell-1}} (\ell-2p+q) P\, E_6 \right\}\Bigg|_{t_o}^{t} + \sum_{\ell} \sum_{m} \left\{ 2\alpha_{\ell}^{*} \frac{\gamma_m}{\bar{n}} \bar{a}^{s-1} (\ell-2p+q) C\, E_{12} \right\}\Bigg|_{t_o}^{t} \qquad (2\text{-}79)$$

$$h = \sin I_o \sin(\Omega_o + \dot{\Omega}_c(t-t_o)) + \sum_{\ell,m} \frac{\bar{n}a_e^{\ell}}{K\bar{a}^{\ell}} \left\{ \frac{\partial P}{\partial I} E_1 + P \cot \bar{I}\, [(\ell-2p) \cos \bar{I} - m] E_4 \right\}\Bigg|_{t_o}^{t} + \sum_{\ell} \sum_{m} \alpha_{\ell}^{*} \gamma_m \frac{\bar{a}^{\ell-2}}{\bar{n}K} \left\{ \frac{\partial C}{\partial I} E_7 + C \cot \bar{I}\, [(\ell-2p) \cos \bar{I} - m] \right\} E_{10}\Bigg|_{t_o}^{t} \qquad (2\text{-}80)$$

$$k = \sin I_o \cos(\Omega_o + \dot{\Omega}_c(t-t_o)) + \sum_{\ell,m} \frac{\bar{n}a_e^{\ell}}{K\bar{a}^{\ell}} \left\{ \frac{-\partial P}{\partial I} E_2 + P \cot \bar{I}\, [(\ell-2p) \cos \bar{I} - m]\, E_3 \right\}\Bigg|_{t_o}^{t} + \sum_{\ell} \sum_{m} \alpha_{\ell}^{*} \frac{\bar{a}^{\ell-2}}{\bar{n}K} \gamma_m \left\{ \frac{-\partial C}{\partial I} E_8 + C \cot \bar{I}\, [(\ell-2p) \cos \bar{I} - m]\, E_9 \right\}\Bigg|_{t_o}^{t} \qquad (2\text{-}81)$$

$$
\begin{aligned}
A \;=\; & e_o \sin\left[\omega_o + \alpha\Omega_o + (\dot{\omega}_c + \alpha\dot{\Omega}_c)(t-t_o)\right] \\
& + \sum_{\ell,m} \frac{\bar{n} a_e^{\ell}}{\bar{a}^{\ell}} \left\{ \left[\frac{\bar{e}\, F(\alpha)}{K} \frac{\partial P}{\partial I} + K \frac{\partial P}{\partial e} \right] E_1 \right. \\
& \left. + K P \left[\frac{-\bar{e}(\ell-2p)}{1+K} + \frac{qK}{\bar{e}} \right] E_4 \right\} \Bigg|_{t_o}^{t} \\
& + \sum_{\ell} \sum_{m} \frac{\alpha_\ell^* \bar{a}^{\ell-2}}{\bar{n}} \gamma_m \left\{ \left[\frac{\bar{e} F(\alpha)}{K} \frac{\partial C}{\partial I} + K \frac{\partial C}{\partial e} \right] E_7 \right. \\
& \left. + KC \left[\frac{-(\ell-2p)\bar{e}}{1+K} + \frac{qK}{\bar{e}} \right] E_{10} \right\} \Bigg|_{t_o}^{t}
\end{aligned}
\qquad (2\text{-}82)
$$

$$
\begin{aligned}
B \;=\; & e_o \cos\left[\omega_o + \alpha\Omega_o + (\dot{\omega}_c + \alpha\dot{\Omega}_c)(t - t_o)\right] \\
& + \sum_{\ell,m} \frac{\bar{n}\, a_e^{\ell}}{\bar{a}^{\ell}} \left\{ \left[\frac{-\bar{e}\, F(\alpha)}{K} \frac{\partial P}{\partial I} - K \frac{\partial P}{\partial e} \right] E_2 \right. \\
& \left. + KP \left[\frac{-(\ell-2p)\bar{e}}{1+K} + \frac{qK}{\bar{e}} \right] E_3 \right\} \Bigg|_{t_o}^{t} \\
& + \sum_{\ell} \sum_{m} \frac{\alpha_\ell^*\, \bar{a}^{\ell-2}}{\bar{n}} \gamma_m \left\{ \left[\frac{-\bar{e}\, F(\alpha)}{K} \frac{\partial C}{\partial I} - K \frac{\partial C}{\partial e} \right] E_8 \right. \\
& \left. + K\, C \left[\frac{-(\ell-2p)\bar{e}}{1+K} + \frac{qK}{\bar{e}} \right] E_9 \right\} \Bigg|_{t_o}^{t}
\end{aligned}
\qquad (2\text{-}83)
$$

$$
\begin{aligned}
\delta \;=\; & \delta_o + \dot{\delta}_c(t-t_o) + \sum_{\ell,m} \frac{\bar{n}\, a_e^{\ell}}{\bar{a}^{\ell}} \left\{ \left[\frac{F(\alpha)}{K} \frac{\partial P}{\partial I} + 2(\ell+1) P \right. \right. \\
& \left. \left. + \frac{\bar{e} K}{1+K} \frac{\partial P}{\partial e} - 3\bar{n}\, P \frac{(\ell-2p+q)}{\dot{\Phi}_{\ell mpq}} \right] E_5 \right\} \Bigg|_{t_o}^{t} \\
& + \sum_{\ell} \sum_{m} \alpha_\ell^* \frac{\bar{a}^{\ell-2}}{\bar{n}} \gamma_m \left\{ \left[\frac{F(\alpha)}{K} \frac{\partial C}{\partial I} - 2\ell C + \frac{\bar{e}\, K}{1+K} \frac{\partial C}{\partial e} \right. \right. \\
& \left. \left. - 3\bar{n} C \frac{(\ell-2p+q)}{\dot{\gamma}_{\ell mpsqj}} \right] E_{11} \right\} \Bigg|_{t_o}^{t}
\end{aligned}
\qquad (2\text{-}84)
$$

$$\cos I = \cos I_o - \sum_{\ell,m} \left\{ \frac{\bar{n}\, a_e^{\ell}}{K\, \bar{a}^{\ell}} P[(\ell-2p)\cos \bar{I} - m]\, E_6 \right\} \Bigg|_{t_o}^{t}$$

$$- \sum_{\ell} \sum_{m} \alpha_{\ell}^{*} \frac{\bar{a}^{\ell-2}}{\bar{n}K} \gamma_m\, C[(\ell-2p)\cdot\cos \bar{I} - m]\, \dot{E}_{12} \Bigg|_{t_o}^{t} \quad (2\text{-}85)$$

where

$$K = \sqrt{1 - \bar{e}^2}\,, \qquad \alpha_{\ell}^{*} = \frac{\mu^{*}}{a^{*\,\ell+1}}$$

$$F(\alpha) = \frac{\alpha \sin \bar{I}}{1+\alpha \cos \bar{I}}\,, \qquad \gamma_m = K_m \frac{(\ell-m)!}{(\ell+m)!} \quad (2\text{-}86)$$

and $(\bar{\ })$ denotes mean value. For $\ell - m$ even, the E_i's are,

$$E_1 = \frac{J_{\ell m}}{2} \left\{ \frac{\sin \Phi^{-}_{\ell mpq}}{\dot{\Phi}^{-}_{\ell mpq}} + \frac{\sin \Phi^{+}_{\ell mpq}}{\dot{\Phi}^{+}_{\ell mpq}} \right\} \quad (2\text{-}87)$$

$$E_2 = \frac{J_{\ell m}}{2} \left\{ \frac{\cos \Phi^{-}_{\ell mpq}}{\dot{\Phi}^{-}_{\ell mpq}} - \frac{\cos \Phi^{+}_{\ell mpq}}{\dot{\Phi}^{+}_{\ell mpq}} \right\} \quad (2\text{-}88)$$

$$E_3 = \frac{J_{\ell m}}{2} \left\{ \frac{\cos \Phi^{-}_{\ell mpq}}{\dot{\Phi}^{-}_{\ell mpq}} + \frac{\cos \Phi^{+}_{\ell mpq}}{\dot{\Phi}^{+}_{\ell mpq}} \right\} \quad (2\text{-}89)$$

$$E_4 = \frac{J_{\ell m}}{2} \left\{ \frac{\sin \Phi^{-}_{\ell mpq}}{\dot{\Phi}^{-}_{\ell mpq}} + \frac{\sin \Phi^{+}_{\ell mpq}}{\dot{\Phi}^{+}_{\ell mpq}} \right\} \quad (2\text{-}90)$$

$$E_5 = J_{\ell m} \frac{\sin \Phi_{\ell mpq}}{\dot{\Phi}_{\ell mpq}} \quad (2\text{-}91)$$

$$E_6 = J_{\ell m} \frac{\cos \Phi_{\ell mpq}}{\dot{\Phi}_{\ell mpq}} \quad (2\text{-}92)$$

If $\ell - m$ is odd, E_1 is replaced by $-E_3$, E_2 by $-E_4$, E_3 by E_1 , E_4 by E_2 , E_5 by $-E_6$ and E_6 by E_5 . The quantities E_7 through E_{12} are defined as follows:

$$E_7 = 1/2 \left\{ \frac{\sin \gamma^-_{\ell mpsqj}}{\dot{\gamma}^-_{\ell mpsqj}} + \frac{\sin \gamma^+_{\ell mpsqj}}{\dot{\gamma}^+_{\ell mpsqj}} \right\} \tag{2-93}$$

$$E_8 = 1/2 \left\{ \frac{\cos \gamma^-_{\ell mpsqj}}{\dot{\gamma}^-_{\ell mpsqj}} - \frac{\cos \gamma^+_{\ell mpsqj}}{\dot{\gamma}^+_{\ell mpsqj}} \right\} \tag{2-94}$$

$$E_9 = 1/2 \left\{ \frac{\cos \gamma^-_{\ell mpsqj}}{\dot{\gamma}^-_{\ell mpsqj}} + \frac{\cos \gamma^+_{\ell mpsqj}}{\dot{\gamma}^+_{\ell mpsqj}} \right\} \tag{2-95}$$

$$E_{10} = -1/2 \left\{ \frac{\sin \gamma^-_{\ell mpsqj}}{\dot{\gamma}^-_{\ell mpsqj}} - \frac{\sin \gamma^+_{\ell mpsqj}}{\dot{\gamma}^+_{\ell mpsqj}} \right\} \tag{2-96}$$

$$E_{11} = \frac{\sin \gamma_{\ell mpsqj}}{\dot{\gamma}_{\ell mpsqj}} \tag{2-97}$$

$$E_{12} = \frac{\cos \gamma_{\ell mpsqj}}{\dot{\gamma}_{\ell mpsqj}} \tag{2-98}$$

The Φ and $\dot{\Phi}$ functions are defined by

$$\Phi_{\ell mpq} = (\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\theta-\lambda_{\ell m}) \tag{2-99}$$

$$\dot{\Phi}_{\ell mpq} = (\ell-2p)\dot{\omega}_c + (\ell-2p+q)\overline{n} + m(\dot{\Omega}_c-\dot{\theta}) \tag{2-100}$$

$$\Phi^+_{\ell mpq} = (\ell-2p+b)\omega + (\ell-2p+q)M + (m+\alpha)\Omega - m(\theta+\lambda_{\ell m}) \tag{2-101}$$

$$\dot{\Phi}^{+}_{\ell mpq} = (\ell-2p+b)\dot{\omega}_c + (\ell-2p+q)\overline{n} + (m+\alpha)\dot{\Omega}_c - m\dot{\theta} \tag{2-102}$$

$$\Phi^{-}_{\ell mpq} = (\ell-2p-b)\omega + (\ell-2p+q)M + (m-\alpha)\Omega - m(\theta+\lambda_{\ell m}) \tag{2-103}$$

$$\dot{\Phi}^{-}_{\ell mpq} = (\ell-2p-b)\dot{\omega}_c + (\ell-2p+q)\overline{n} + (m-\alpha)\dot{\Omega}_c - m\dot{\theta} \tag{2-104}$$

The γ and $\dot{\gamma}$ functions are defined by

$$\gamma_{\ell mpsqj} = (\ell-2p)\omega + (\ell-2p+q)M - (\ell-2s)\omega^{*} - (\ell-2s+j)M^{*} + M(\Omega-\Omega^{*}) \tag{2-105}$$

$$\dot{\gamma}_{\ell mpsqj} = (\ell-2p)\dot{\omega}_c + (\ell-2p+q)\overline{n} - (\ell-2s+j)n^{*} + m\dot{\Omega}_c \tag{2-106}$$

$$\gamma^{-}_{\ell mpsqj} = (\ell-2p-b)\omega + (\ell-2p+q)M - (\ell-2s)\omega^{*} - (\ell-2s+j)M^{*} + (m-\alpha)\Omega - m\Omega^{*} \tag{2-107}$$

$$\dot{\gamma}^{-}_{\ell mpsqj} = (\ell-2p-b)\dot{\omega}_c + (\ell-2p+q)\overline{n} - (\ell-2s+j)n^{*} + (m-\alpha)\dot{\Omega}_c \tag{2-108}$$

$$\gamma^{+}_{\ell mpsqj} = (\ell-2p+b)\omega + (\ell-2p+q)M - (\ell-2s)\omega^{*} - (\ell-2s+j)M^{*} + (m+\alpha)\Omega - m\Omega^{*} \tag{2-109}$$

$$\dot{\gamma}^{+}_{\ell mpsqj} = (\ell-2p+b)\dot{\omega}_c + (\ell-2p+q)\overline{n} - (\ell-2s+j)n^{*} + (m+\alpha)\dot{\Omega}_c \tag{2-110}$$

In Eqs. (2-79) through (2-81) $\alpha = +1$ and $b = 0$. In Eqs. (2-82) through (2-85) $b = 1$ and

$$\begin{aligned} \alpha &= -1 , \quad \text{for} \quad 175^\circ < I \le 180^\circ \\ \alpha &= +1 , \quad \text{for} \quad 0^\circ \le I \le 175^\circ \end{aligned} \tag{2-111}$$

For the primary body solutions the symbol $\sum_{\ell,m}$ represents the summation over a specified set of harmonics. For the third body solutions, the indices m , p and s are summed from zero to ℓ , and ℓ is summed from two through the number of terms desired in the third body disturbing function. The indices q and j are summed between limits dependent on the satellite and third body orbit's eccentricity.

The assumption that ω^* and Ω^* are constant leads to numerical difficulty in the solution for the third body effects. First, when the indices assume the values

ℓ	m	p	s	q	j
2	0	1	0	0	-2
2	0	1	2	0	2

then

$$\gamma_{20100-2} = 2\omega^* \quad \text{and} \quad \gamma_{201202} = -2\omega^* \tag{2-112}$$

but

$$\dot{\gamma}_{20100-2} = \dot{\gamma}_{201202} = 0 \; . \tag{2-113}$$

Consequently, E_{11} and E_{12} have zero divisors; however, this is no problem, since these coefficients are always multiplied by $G_{\ell sj}(e^*)$ and $G_{20-2}(e^*) = G_{222}(e^*) = 0$.

Unfortunately, the other troublesome terms are not handled this easily. When the indices assume the values

ℓ	m	p	s	q	j
2	1	1	1	0	0

and $b = 0$, then

$$\bar{\gamma}_{211100} = -\Omega^* \quad \text{and} \quad \dot{\bar{\gamma}}_{211100} = 0 \,. \tag{2-114}$$

Now E_7 , E_8 , E_9 and E_{10} have zero divisors, and the solutions for h and k given by Eqs. (2-80) and (2-81) must be modified. Actually these elements are periodic in Ω^* for this set of indices and since Ω^* has been assumed constant the variations in h and k will be secular. Consequently, the solutions for these indices must be obtained separately. The solutions are

$$(h)_{211100} = \frac{\alpha_2^*}{6\bar{n}K} F_{211}(I^*) H_{210}(\bar{e}) G_{210}(e^*)$$
$$\left\{ \frac{\partial F_{211}(\bar{I})}{\partial I} \left[\frac{\sin(2\Omega-\Omega^*)}{2\dot{\Omega}_c} + (\cos\Omega^*)\tau \right] \Bigg|_{t_o}^{t} \right. \tag{2-115}$$
$$\left. + F_{211}(\bar{I}) \cot \bar{I} \left[\frac{-\sin(2\Omega-\Omega^*)}{2\dot{\Omega}_c} + (\cos\Omega^*)\tau \right] \Bigg|_{t_o}^{t} \right\}$$

$$(k)_{211100} = \frac{-\alpha_2^*}{6\bar{n}K} F_{211}(I^*) H_{210}(\bar{e}) G_{210}(e^*)$$
$$\left\{ \frac{\partial F_{211}(\bar{I})}{\partial I} \left[\frac{-\cos(2\Omega-\Omega^*)}{2\dot{\Omega}_c} + (\sin\Omega^*)\tau \right] \Bigg|_{t_o}^{t} \right. \tag{2-116}$$
$$\left. - F_{211}(\bar{I}) \cot \bar{I} \left[\frac{\cos(2\Omega-\Omega^*)}{2\dot{\Omega}_c} + (\sin\Omega^*)\tau \right] \Bigg|_{t_o}^{t} \right\}$$

where the terms in $(\cos\Omega^*)\tau$ and $(\sin\Omega^*)\tau$ are the secular contributions to h and k respectively. For the indices

ℓ	m	p	s	q	j
3	1	2	2	1	1

and $b = 1$

$$\begin{aligned}
\gamma^{+}_{312211} &= (1+\alpha)\Omega - \Omega^{*} + \omega^{*} \\
\dot{\gamma}^{+}_{312211} &= (1+\alpha)\dot{\Omega} \\
\gamma^{-}_{312211} &= (1-\alpha)\Omega - 2\omega - \Omega^{*} + \omega^{*} \\
\dot{\gamma}^{-}_{312211} &= (1-\alpha)\,\dot{\Omega} - 2\dot{\omega}
\end{aligned} \tag{2-117}$$

If $175° < I \leq 180°$, then $\alpha = -1$ and $\dot{\gamma}^{+} = 0$; therefore, at least four of the E_i coefficients in the solutions for A and B will have zero divisors. The solutions for this set of indices are:

$$\begin{aligned}
A_{312211} = \frac{\alpha_3^*}{12}\,\frac{\bar{a}}{\bar{n}}\,F_{312}(I^*)\,G_{321}(e^*) \Bigg\{ & \left[\frac{\bar{e}\,F(\alpha)}{K}\,\frac{\partial F_{312}(\bar{I})}{\partial I}\,H_{321}(\bar{e}) + KF_{312}(\bar{I})\,\frac{\partial H_{321}(\bar{e})}{\partial e}\right] \left[[\cos(\omega^*-\Omega^*)]\tau + \frac{\sin[2(\Omega-\omega)+\omega^*-\Omega^*]}{2(\dot{\Omega}_c-\dot{\omega}_c)}\right]\Bigg|_{t_o}^{t} \\
& + KF_{312}(\bar{I})\,H_{321}(\bar{e})\left[\frac{\bar{e}}{1+K} + \frac{K}{\bar{e}}\right] \\
& \times \left[(\cos(\omega^*-\Omega^*))\tau - \frac{\sin[2(\Omega-\omega)+\omega^*-\Omega^*]}{2(\dot{\Omega}_c-\dot{\omega}_c)}\right]\Bigg|_{t_o}^{t} \Bigg\}
\end{aligned} \tag{2-118}$$

$$\begin{aligned}
B_{312211} = \frac{\alpha_3^*}{12}\,\frac{\bar{a}}{\bar{n}}\,F_{312}(I^*)\,G_{321}(e^*) \Bigg\{ & \left[\frac{-\bar{e}\,F(\alpha)}{k}\,\frac{\partial F_{312}(\bar{I})}{\partial I}\,H_{321}(\bar{e}) - KF_{312}(\bar{I})\,\frac{\partial H_{321}(\bar{e})}{\partial e}\right] \Bigg[[\sin(\omega^*-\Omega^*)]\tau \\
& + \frac{\cos[2(\Omega-\omega)+\omega^*-\Omega^*]}{2(\dot{\Omega}_c-\dot{\omega}_c)}\Bigg]\Bigg|_{t_o}^{t} \\
& + KF_{312}(\bar{I})\,H_{321}(\bar{e})\left[\frac{\bar{e}}{1+K} + \frac{K}{\bar{e}}\right]
\end{aligned} \tag{2-119}$$

$$\times \left[-(\sin (\omega^* - \Omega^*))\tau + \frac{\cos [2(\Omega-\omega) + \omega^* - \Omega^*]}{2(\dot{\Omega}_c - \dot{\omega}_c)} \right] \Bigg|_{t_o}^{t} \Bigg\} \qquad (2\text{-}119)$$

Also, when the indices assume the values

ℓ	m	p	s	q	j
4	1	2	2	0	0

the solutions for h and k again have zero divisors, and the solutions for these indices must be written explicitly in order to avoid numerical problems. However, since the third body disturbing function through three terms is seldom required, the results are not given here.

It should be noted that the first order solutions presented here do not contain first order long period terms proportional to J_{20}. These terms would come from a second order perturbation solution; however, the algebra involved in generating the second order solutions for the nonsingular elements is prohibitive. These first order terms are presented by Brouwer(5), page 394, for the Keplerian elements. An examination of these terms reveals that they can be neglected for satellite orbits which are near circular and have low inclinations.

COMPUTATIONAL PROCEDURE

The computational procedure for applying the analytical solutions to the prediction of a lunar satellite orbit will be presented here.

A. Coordinate System

The inertial coordinate system used in this study is the selenographic system fixed at epoch. The selenographic system(25) is fixed with its x-axis directed toward the earth at the moon's apogee or perigee; its z-axis is directed along the moon's spin axis, and its y-axis completes a right-hand triad. This system is fixed at the initial time and is the frame in which perturbations due to both the primary and third body are computed.

B. Computational Algorithm

The computational algorithm for computing the elements of a lunar satellite under the influence of an aspherical moon and a point mass earth will be presented.

Given a set of initial conditions the procedure for computing post epoch values of the orbit elements is as follows:

1. determine the orbit elements of the third body's orbit,
2. determine the mean values of a , e , and I and the mean motion,
3. determine the secular rates for Ω and ω , and
4. using the analytical solutions given in Eqs. (2-79) through (2-85), compute the perturbing effects due to the primary and third body.

The earth's orbit elements in the selenographic system may be obtained from the JPL ephemeris tapes[26]. These tapes give the position and velocity of the earth-moon-sun system in the interval of time from June 1, 1950 through July 31, 1999. The coordinate system to which all data is referenced is geocentric and is defined by the mean equator and ecliptic at the nearest beginning of a Besselian year, which differs from the beginning of the calendar year of the same number by only a fraction of a day[27]. However, for the application desired here it is necessary to have the position and velocity of the earth with respect to the moon in the selenographic of epoch system. The precession-nutation-libration matrix for transforming from the reference geocentric system to the selenographic of epoch system is available also on the ephemeris tapes. Once the selenographic position and velocity of the earth are obtained, the orbit elements are readily available by well known transformations[28]. The transformations for obtaining Cartesian coordinates from the nonsingular orbit elements are presented in Appendix C.

The mean values of a , e and I are obtained from the analytical expressions for these elements. The analytical expressions in an abbreviated form are

$$a = a_o + \int_{t_o}^{t} A'(\alpha,\tau)d\tau \tag{2-120}$$

$$e = e_o + \int_{t_o}^{t} E'(\alpha,\tau)d\tau \tag{2-121}$$

$$\cos I = \cos I_o + \int_{t_o}^{t} I'(\alpha,\tau)d\tau \tag{2-122}$$

where $A'(\alpha,\tau)$ represents the integrand in the solution for a , and $E'(\alpha,\tau)$ and $I'(\alpha,\tau)$ have corresponding definitions for e and $\cos I$.

If the values of the integrals at the lower limits are combined with the epoch values of the elements, Eqs. (2-120) through (2-122) may be written as

$$a = K_a + [A(\alpha,t)] \tag{2-123}$$

$$e = K_e + [E(\alpha,t)] \tag{2-124}$$

$$\cos I = K_I + [I(\alpha,t)] \tag{2-125}$$

Since $A(\alpha,t)$, $F(\alpha,t)$ and $G(\alpha,t)$ are almost periodic, they will oscillate about the values of K_a , K_e and K_I respectively. Hence K_a , K_e

and $\cos^{-1} K_I$ will be good approximations to the mean values of a , e and I . This is an iterative procedure, and for the first iteration the epoch values of the elements are used on the right-hand side of Eqs. (2-123) through (2-125). It was found that one iteration was sufficient to converge to the mean elements.

The calculation of the proper mean motion, n , is very important because any error in n will appear as a secular error in the mean anomaly. If the satellite's motion is perturbed only by the zonal harmonics, the total energy will be conserved since the potential function is independent of time.

For a satellite moving in a conservative force field, the mean value of n is given by[2]

$$\bar{n} = \sqrt{\mu/\bar{a}^3} \tag{2-126}$$

where $\bar{a} = a_0/[1 + (2Ra_0/\mu)]$. If the satellite's orbit is perturbed by sectorial or tesseral harmonics or by a third body, the potential function will be an explicit function of time and the energy is no longer conserved. In this case, it is necessary to use the secular portion of the differential equations for $\dot{M}$ to compute the mean motion.

In general the value of the mean motion, $\bar{n}$, is obtained by evaluating the equations for $\dot{M}_c$, i.e.

$$\begin{aligned} \dot{M}_c = \bar{n} = n - \sum_{\ell} \frac{\mu\, a_e^{\ell}}{\bar{n}\, \bar{a}^{\ell+2}} \left[\frac{K^2}{\bar{a}\,\bar{e}} \frac{\partial P}{\partial e} - \frac{2(\ell+1)}{\bar{a}} P \right] J_{\ell 0} \\ - \sum_{\ell} \frac{\mu^*}{a^{*\ell+1}} \frac{\bar{a}^{\ell-2}}{\bar{n}} \left[\frac{K^2}{\bar{e}} \frac{\partial C}{\partial e} + 2\ell C \right] . \end{aligned} \tag{2-127}$$

The quantity n is based on the mean value of a, as given by Eq. (2-123), i.e. $\bar{n} = \frac{\mu^{1/2}}{\bar{a}^{3/2}}$, and the calculation of $\bar{n}$ is included in the iteration loop for the mean elements.

The secular rates for $\dot{\Omega}$ and $\dot{\omega}$ may now be obtained by substituting $\bar{n}$ along with the mean values of the elements into Eqs. (2-53), (2-54), (2-57) and (2-58). The final values of the rates are obtained by summing the primary and third body contributions. The inclination functions, $F(I)$ and $F(I^*)$, and the eccentricity functions, $G(e)$ and $G(e^*)$, may be generated and stored at the same time the secular rates and mean values are computed. The symmetry properties for the eccentricity functions developed in Appendix B should be used to reduce the number of computations.

Given the mean values and secular rates of the elements, then at time t the quantities Φ, $\dot{\Phi}$, γ and $\dot{\gamma}$ may be computed according to Eqs. (2-99) through (2-110). The values for the elements may now be obtained by evaluating Eqs. (2-79) through (2-85). The solutions are well suited for computer programming since they may be rearranged so that all quantities which are functions of the summation indices are grouped inside the initial summation.

Since the perturbations in the elements are essentially a product of combinatorial functions, $F(I)$ and $G(e)$ and their derivatives, the perturbations apart from the divisors behave as $G(e)$ which is order $e^{|q|}$. Hence, for small values of eccentricity only a few terms need be carried in the q summation. For the primary body the limit on q is chosen so that

$$e^{|q|} \quad J_{\ell m} \leq J_{20}^2 \tag{2-128}$$

since terms in J_{20}^2 have not been included in the solution and J_{20} is the largest of the harmonics. The same reasoning applies to the q summations in the third body solutions; i.e.

$$e^{|q|} \frac{\mu^*}{a^{*s+1}} \leq \left[\frac{\mu^*}{a^{*2}}\right]^2 \tag{2-129}$$

For near-lunar satellite orbits or for preliminary design work it may be feasible to eliminate the short period effects due to the third body perturbations. This is accomplished easily by letting $q = 2p-\ell$ thus eliminating the summation on q from the third body solutions. The number of summations required to evaluate the system of equation for the third body effect is

$$(\ell - 1)(\ell + 1)\ (\ell + 1)\ (\ell + 1)\ (2 \cdot q_{max} + 1)\ (2 \cdot j_{max} + 1)\ .$$

Consequently, if the summation on q is eliminated there will be a significant reduction in the number of computations required.

The number of terms carried in the third body disturbing function will depend on the satellite's altitude as well as the accuracy desired. In general for Apollo-Type orbits the first term will be sufficient; however, if the ephemeris is desired for times greater than three or four days, it may be desirable to retain two terms in order to minimize long period errors in the position and velocity. These long period errors are discussed in the section on results. If more than one term is carried, the limits on q and j should be lowered for the second term since the perturbations for the higher values of these indices are insignificant.

Normally the equation for cos I is used to determine I since cos I is unique. However, for values of I within about .03° of 0° or

180°, single precision arithmetic is inadequate to determine $\cos^{-1} I$ since

$$\cos I = 1 - \frac{I^2}{2} + \dots$$

will be ±.999... to a large number of places and the computer will interpret this as ±1 and I as 0° or 180°. However, sin I given by

$$\sin I = I - I^3/3! + \dots$$

will be well defined for single precision arithmetic. Consequently, for this range of I the equations for h and k should be used to determine the inclination and the sign of cos I used to determine its quadrant.

If the solution for the orbit elements is desired over a period of time greater than one day, accuracy may be improved slightly if the third body's orbit elements are updated periodically. This requires the use of the JPL ephemeris tapes to obtain an updated set of selenographic orbit elements for the earth. Then these elements are used to determine new constants of integration for the analytical solutions. The updating procedure is equivalent to a piecewise integration of Lagrange's planetary equations. Consider as an example the solution for the semimajor axis at time t which has been updated at time t_1 :

$$a(t) = a_o + \int_{t_o}^{t_1} F'(\alpha,\tau)d\tau + \int_{t_1}^{t} G'(\alpha,\tau)d\tau \qquad (2\text{-}130)$$

where $F'(\alpha,\tau)$ is dependent on the original third body elements and $G'(\alpha,\tau)$ is a function of the updated elements. Equations (2-130) may be written as

$$a(t) = a(t_1) + \int_{t_1}^{t} G'(\alpha,\tau)d\tau \qquad t \geq t_1 \ . \tag{2-131}$$

a similar procedure may be used for updating the other orbit elements.

It may be desirable also to update e and I of the satellite's orbit if the long term effects alter these elements significantly. This could be done simultaneously with updating the third body's elements.

Although the computational algorithm for the nonsingular elements has been set up with the earth as the only secondary body, the first order effects of the sun probably should be included whenever the satellite's semimajor axis is large enough to require two terms of the earth's disturbing function. The sun's effects may be included by performing an additional summation over the third body solutions evaluated for the sun's orbit elements.

A potential computational problem which should be considered is the resonance phenomena. Recall that the form of the analytical solution is

$$\alpha_i = \alpha_0 + \dot{\alpha}_c t + K \frac{\begin{matrix}\cos\\ \sin\end{matrix}\ \Phi}{\dot{\Phi}}$$

where α represents one of the Keplerian elements and $\dot{\Phi}$ is of the form

$$\dot{\Phi}_{\ell mpq} = (\ell - 2p)\ \dot{\omega}_c + (\ell - 2p + q)\ \dot{M}_c + m(\dot{\Omega} - \dot{\theta}) \ .$$

For some orbital configurations it is possible that the secular rates of the arguments for some of the surface harmonics may approach zero. Consequently, the periodic variations of the arguments will be more significant than their secular variations. In other words, a resonance condition exists in which there will be libration rather than secular motion. The resonance terms in the analytical solutions are those for which $\dot{\Phi}_{\ell mpq} \simeq 0$.

The classical example of resonance is that associated with the critical inclination. Whenever the inclination approaches $I = \cos^{-1}\sqrt{1/5}$ or 63.435° the value of $\dot{\omega}_c$ due to the oblateness approaches zero (see Eq. (2-61)). The analytical solutions will have zero divisors when the indices ℓ , m , p , q assume values so that

$$\dot{\Phi}_{\ell mpq} = i\,\dot{\omega}_c \qquad (2\text{-}132)$$

where i is an integer. For Eq. (2-132) to be true the form of $\dot{\Phi}_{\ell mpq}$ imposes the following restrictions on ℓ , m , p , q :

$$\begin{aligned} m &= 0 \\ \ell-2p+q &= 0 \end{aligned} \qquad (2\text{-}133)$$

Since $m = 0$, only terms in the analytical solutions associated with the zonal harmonics are effected by the critical inclination. It is easily demonstrated that no resonance conditions exist for terms associated with the oblateness in the analytical solutions presented here. For $\ell = 2$ and $m = 0$ Eq. (2-133) requires the limits on q to be ± 2 and the only allowable values of ℓ , p and q are 2 , 0 , -2 and 2 , -2 , 2 . However, $G_{\ell pq}(e) = 0$ for these values and any potential resonance terms in J_{20} are eliminated.

Terms in the analytical solution most effected by the critical inclination are those in J_{30} , the pear shape effect. These terms may be expanded using a technique suggested by Ingram[29] so that the zero divisor is eliminated. However, the critical inclination in reality is no problem whenever a realistic model of the disturbing function is used since there will be contributions to $\dot{\omega}_c$ from the third body effect as well as from higher harmonics in the primary body's disturbing function. If only the secular effects of the oblateness are considered, an expansion technique must be used in the vicinity of the critical inclination.

Another case of resonance is that related to a satellite whose period is an integer multiple of the period of the primary body such as the communications satellites. Since $\dot{\omega}_c$ and $\dot{\Omega}_c$ are small compared to $\dot{M}$ and $\dot{\theta}$ they may be neglected. Then

$$\dot{\Phi}_{\ell mpq} = (\ell - 2p + q)\,\dot{M}_c - m\dot{\theta} \ .$$

Since ℓ , m , p and q are integers, $\dot{\Phi}$ will be zero for certain combinations of these indices whenever $\dot{M}_c$ is an integer multiple of $\dot{\theta}$. If only zero order terms in e are considered (i.e., $q = 0$), the condition for resonance is

$$\ell - 2p = m$$

or

$$\ell - m = 2p \ .$$

Hence $\ell - m$ must be even and the satellite's period must be equal to that of the primary body. Furthermore, the general (ℓ, m) term in the disturbing function depends on the radial distance as $r^{-\ell-1}$; consequently, the resonance term $V_{\ell m}$ with the lowest value of ℓ will be dominant. The tesseral harmonic which will dominate the resonance effect is J_{22} , the equatorial ellipticity.

This type of resonance will be insignificant for lunar satellites since the minimum distance for a synchronous orbit is the earth-moon distance. There is a resonance condition associated with a satellite located at the earth-moon libration points; however, this situation will not be considered in this investigation. For an earth satellite the minimum distance for a synchronous orbit is about 6.6 earth radii. For a more detailed discussion of resonance see Ref. (30) or (31).

The resonance phenomena will not be considered further since the problem of critical inclination is eliminated by other terms in the disturbing function while the altitude required for synchronous orbits is greater than that of interest in the present study.

RESULTS

The accuracy of the analytical solutions was evaluated by comparing results obtained with these solutions with those obtained by numerically integrating the equations of motion for various orbital configurations. The numerical integration was done with the ESPOD Program developed by TRW Systems[2]. The analytical and numerical results were compared for various initial conditions for time periods up to eight days. Comparisons were made for orbits perturbed by the primary body only, the third body only and the combined effects of the primary and third body disturbing functions. The nonsingular elements were converted to Keplerian orbit elements in order to give more physical insight into the error propagation and also to allow comparison with the Keplerian elements given by the numerical integrator. The results presented here are for a typical Lunar Orbiter and Apollo orbit. The Apollo orbit has approximately a 100 n.mi. altitude, it is near circular and has a near equatorial inclination, while the Lunar Orbiter, as is shown later, has a higher altitude and higher eccentricity. The results for the Apollo orbit were compared over an eight day period and the results for the Lunar Orbiter over a four day period. To demonstrate the characteristic and magnitude of the errors, they are plotted at fifteen minute increments for the Apollo orbit over a period of two days or about twenty-four satellite revolutions. Similarly, the errors for the Lunar Orbiter's elements were plotted at thirty minute increments for a two day time span. The disturbing function included J_{20} and J_{22} for the primary body and either one or two terms in the third body disturbing function.

The epoch date and initial conditions for the Apollo orbit are

EPOCH DATE	INITIAL CONDITIONS (Selenographic)
Year - 1969	$a_0 = .61 \times 10^7$ ft
Month - 7	$e_0 = .01$
Day - 7	$I_0 = 177°$
Hour - 2	$\Omega_0 = 45°$
Min - 0	$\omega_0 = 10°$
Sec - 0	$M_0 = 0°$
	$\theta_0 = 0°$

The limits on the q summation for both the primary and the third body solutions were chosen as ± 2 , while the limit on the j summation for the third body solution also was chosen as ± 2 .

The results for the Apollo-Type orbit are presented in Figs. (1) through (4). It is known that the variation in a is short period only; consequently, the errors in a also exhibit a short period variation as shown in Fig. (1). The errors in a have a bias of about five feet because a set of mean elements for the earth's orbit over the eight day period was used. Ideally the values of the earth's orbit elements at epoch should have been used initially and periodic updates made of these elements. However, since the earth's orbit elements are practically constant, a mean set was used to simplify programming and conserve computer time. The set of mean elements will yield an error bound which will include that of true elements. When the true set of earth's orbit elements at epoch was used this bias disappeared.

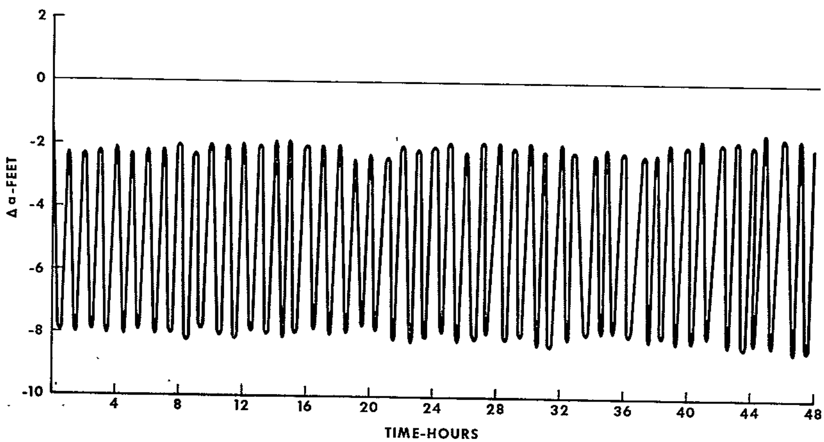
The errors in I and Ω are presented in Fig. (2). They can be explained by examining the long period effects on I and Ω due to J_{22}. These are[14]

$$\frac{dI}{du} = \frac{-a_e^2}{\bar{p}^2} J_{22} \sin \bar{I} \sin 2\theta \qquad (2\text{-}134)$$

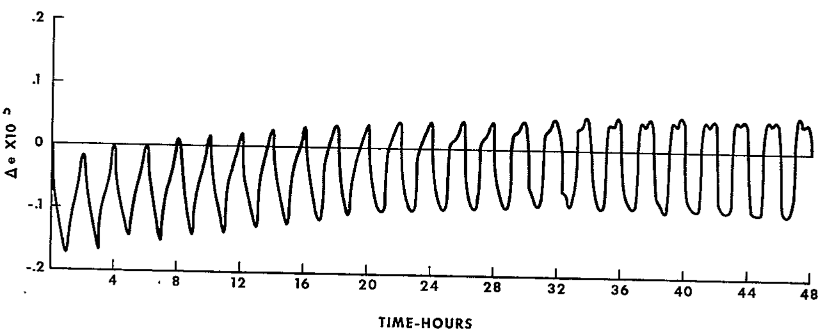
$$\frac{d\Omega}{du} = \frac{a_e^2}{\bar{p}^2} J_{22} \cos \bar{I} \cos 2\bar{\theta} \ . \qquad (2\text{-}135)$$

where $\bar{p} = \bar{a}(1-\bar{e}^2)$ and u is the argument of latitude.

The notation of Ref. (14) has been altered slightly to agree with that used here. From Eq. (2-134) and (2-135) it is seen that both I and Ω will have long period variations on the order of one-half the period of the primary body. For a lunar satellite this will be slightly less than two weeks. Even though the long period terms in J_{22} are included in the solution, the errors in Ω also exhibit this two week periodicity as the following table demonstrates:

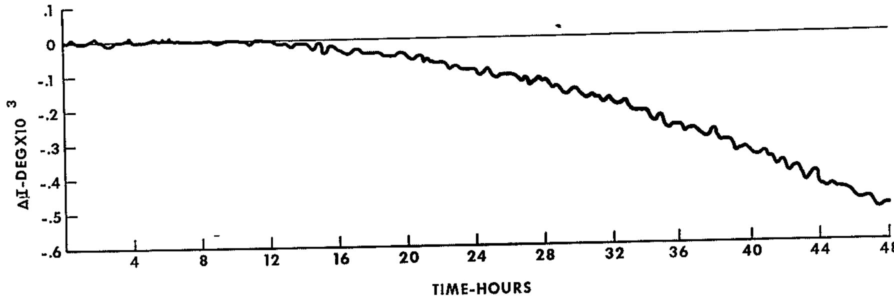


ERROR IN SEMIMAJOR AXIS AS A FUNCTION OF TIME FOR APOLLO ORBIT

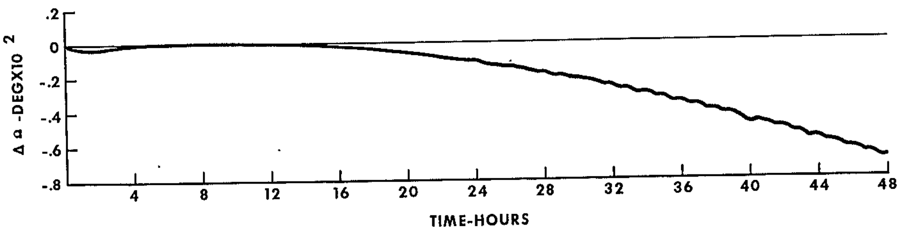


ERROR IN ECCENTRICITY AS A FUNCTION OF TIME FOR APOLLO ORBIT

FIGURE 1



ERROR IN INCLINATION AS A FUNCTION OF TIME
FOR APOLLO ORBIT



ERROR IN NODE AS A FUNCTION OF TIME
FOR APOLLO ORBIT
FIGURE 2

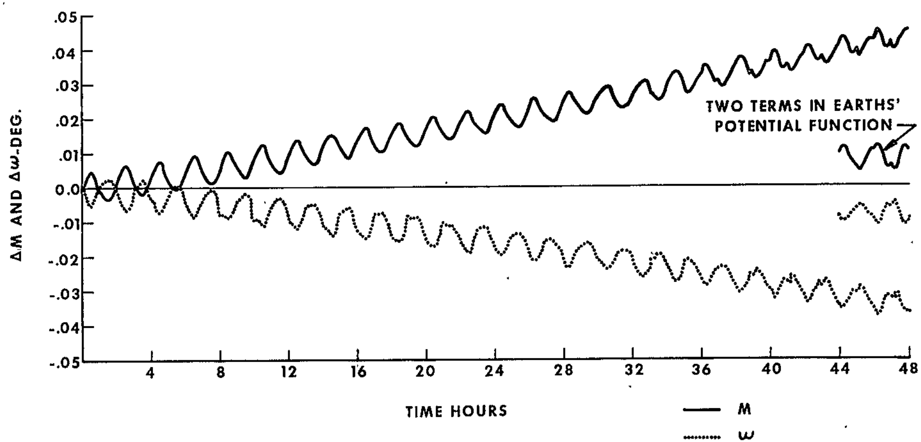
Time (Days)	$\Delta\Omega$ (Degrees)
1	-.00104
2	-.00670
3	-.0151
4	-.0230
5	-.0270
6	-.0244
7	-.0138
8	+.00541

TABLE I - Errors in Node

The errors in $\dot{I}$ do not exhibit this two week period and are dominated by other effects.

The neglected first order long period effects in the oblateness, which exist for all the elements except the semimajor axis[5], will be small for near-circular, low-inclination orbits and should have a negligible effect on the elements of the Apollo orbit.

Figure (3) presents the errors in ω and M. Since the eccentricity is small, both ω and M have large short period variations which are reflected in the large short period errors shown in Fig. (3). The short period variations in $\omega + M$ are small for all values of eccentricity, and it is seen that adding the errors in ω and M will result in very small errors in this quantity. Consequently, even for low eccentricity orbits, the position of the vehicle is still well defined by the analytical solutions even though the individual quantities ω and M may be poorly defined. It was found that the long term errors in ω and M are inversely proportional to the magnitude of the eccentricity and are primarily due to the omission of



ERROR IN MEAN ANOMALY AND ARGUMENT OF PERICENTER
AS A FUNCTION OF TIME FOR APOLLO ORBIT
FIGURE 3

the second term in the earth's disturbing function. The result of including the second term in the earth disturbing function is plotted in Fig. (3) for the last two satellite revolutions of the second day.

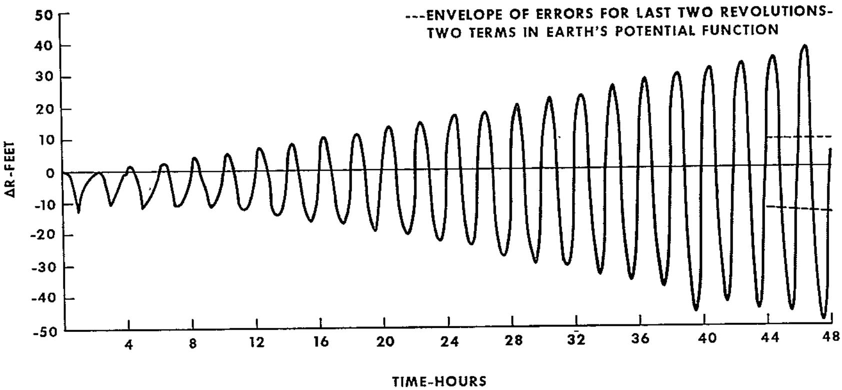
Carrying the second term in the earth disturbing functions also reduces the errors in a , e and I ; however, the improvement in these elements is small, therefore it is not shown on the plots.

An examination of the long period effects in the analytical solution reveals that there are terms with eccentricity divisors for both ω and M for the second term in the third body disturbing function. Consider the differential equations for ω and M given by Eq. (2-5) and (2-6). Both contain the term

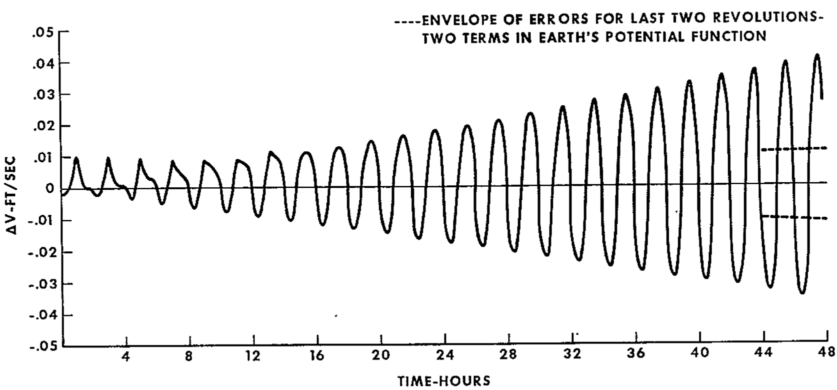
$$\frac{f(a,e,n)}{e}\left[\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}\right] . \tag{2-136}$$

The requirement for long period terms is that $\ell - 2p + q = 0$ or $q = 2p - \ell$. For $\ell = 3$, q must be odd, and when $q = 1$, the lead term in $\frac{\partial H(e)}{\partial e}$ will be a constant, therefore, the resulting terms in the analytical solutions will have e as a divisor. Consequently, for low eccentricity orbits omission of the second term in the third body disturbing function results in large long period errors in ω and M as demonstrated by Fig. (3). The solutions for a , e , I and Ω do not have terms of this nature.

The errors in position and velocity magnitudes versus time for one term in the third body disturbing function are shown in Fig. (4). Also shown is the envelope of the errors in range and velocity whenever two terms in the earth disturbing function are retained. Since the errors in the dynamical element, M , are significantly reduced by including two terms in



ERROR IN RANGE AS A FUNCTION OF TIME
FOR APOLLO ORBIT



ERROR IN VELOCITY AS A FUNCTION OF TIME
FOR APOLLO ORBIT
FIGURE 4

the earth disturbing function the errors in range and velocity also will be reduced because these quantities are functions of the dynamical elements.

Table 2 presents the errors at fifteen minute increments for the last satellite revolution of the eighth day for both one and two terms in the third body disturbing function. This table demonstrates the magnitude of the errors in the elements over longer periods of time. From this table it is seen that the analytic solutions are still in excellent agreement with the numerically integrated ephemeris. The result of carrying the second term in the earth disturbing function as demonstrated by Table 2 is that the error reduction in a , I and Ω is significant while errors for the other quantities are reduced by more than fifty percent.

A set of initial conditions for Lunar Orbiter I was obtained from the Mission Planning and Analysis Division of the Manned Spacecraft Center. The epoch date and initial conditions are:

EPOCH DATE	INITIAL CONDITIONS (Selenographic)
Year - 1966	a_0 = 9106683.9 ft
Month - 8	e_0 = .30543108
Day - 14	I_0 = 12.090135°
Hour - 16	Ω_0 = 325.79860°
Minute - 31	ω_0 = 180.47295°
Second - 52.013	M_0 = .99784197°
	θ_0 = 0.0

The initial conditions correspond to aposelene and periselene altitudes of about 1500 and 400 nautical miles respectively.

ONE TERM IN THIRD BODY POTENTIAL FUNCTION

Time, Min.	Δa (ft)	Δe	ΔI (deg)	ΔΩ (deg)	Δω (deg)	ΔM (deg)	ΔR (ft)	ΔV (ft/sec)
11400	-.4	$-.193_{10}{}^{-4}$	-.00583	+.00231	-.112	+.143	+187.6	-.165
11415	-3.7	$-.206_{10}{}^{-4}$	-.00582	+.00274	-.115	+.147	+165.3	-.146
11430	-9.6	$-.202_{10}{}^{-4}$	-.00583	+.00327	-.127	+.159	+40.1	-.0391
11445	-6.2	$-.200_{10}{}^{-4}$	-.00586	+.00330	-.128	+.160	-115.5	+.0940
11460	-.5	$-.205_{10}{}^{-4}$	-.00587	+.00317	-.130	+.162	-202.8	+.1759
11475	-3.4	$-.193_{10}{}^{-4}$	-.00584	+.00356	-.125	+.158	-177.4	+.153
11490	-9.1	$-.199_{10}{}^{-4}$	-.00585	+.00414	-.114	+.148	-51.8	+.0519
11505	-6.3	$-.202_{10}{}^{-4}$	-.00589	+.00420	-.111	+.146	+101.2	-.0919
11520	0	$-.197_{10}{}^{-4}$	-.00587	+.00510	-.110	+.143	+190	-.167

TWO TERMS IN THIRD BODY POTENTIAL FUNCTION

Time, Min.	Δa (ft)	Δe	ΔI (deg)	ΔΩ (deg)	Δω (deg)	ΔM (deg)	ΔR (ft)	ΔV (ft/sec)
11400	-.4	$-.978_{10}{}^{-5}$	-.0058	+.0027	-.0201	+.0517	+81.3	-.0715
11415	-3.1	$-.107_{10}{}^{-4}$	-.00581	+.00316	-.0230	+.0553	+63.3	-.0564
11430	-9.2	$-.104_{10}{}^{-4}$	-.00582	+.00370	-.0322	+.0641	+2	-.006
11445	-6.1	$-.100_{10}{}^{-4}$	-.00584	+.00372	-.0321	+.0640	-63	+.0525
11460	+.1	$-.113_{10}{}^{-4}$	-.0058	+.00359	-.0328	+.0656	-96	+.084
11475	-3.3	$-.103_{10}{}^{-4}$	-.00582	+.00399	-.0284	+.0617	-73	+.0623
11490	-9.2	$-.107_{10}{}^{-4}$	-.00584	+.00458	-.0194	+.0539	-11	+.062
11505	-5.8	$-.105_{10}{}^{-4}$	-.00587	+.00464	-.0186	+.0453	+106	-.0492
11520	0	$-.99_{10}{}^{-5}$	-.00586	+.0054	-.018	+.051	+82	-.0725

TABLE 2

ERRORS IN THE ORBIT ELEMENTS FOR ONE SATELLITE

REVOLUTION OF AN APOLLO ORBIT

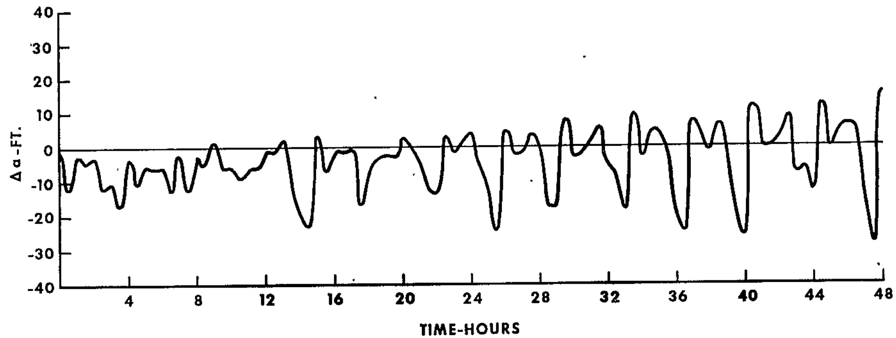
The limits on the q summation for the primary body solutions are ± 8 while those on the q summation for the third body solutions are ± 4 . The limits on the j summation associated with the eccentricity of the earth's orbit are ± 2 . Two terms were carried in the earth disturbing function, and for the second term the limits on q and j were reduced to ± 2 and ± 1 respectively.

Figures (5) through (8) present the deviations in the orbit elements and the position and velocity magnitudes of Lunar Orbiter I at thirty minute increments for a duration of two days.

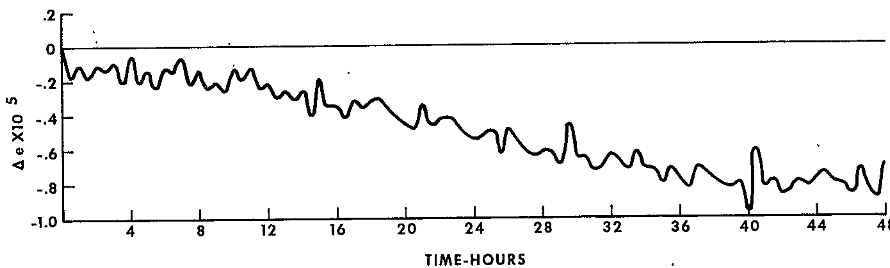
Because Lunar Orbiter I has a higher eccentricity orbit than the Apollo, the short period errors in Ω , ω and M are considerably smaller. The long period deviations in ω and M , shown in Fig. (7), are not of the same magnitude. Note that these deviations are considerably smaller then the corresponding ones for the Apollo orbit and as a result are exaggerated somewhat by the plots.

The neglected first-order long period variations due to J_{20} in the remaining elements will be more significant for Lunar Orbiter I than for the Apollo orbit. However, an examination of the results presented here reveals that for short-term ephemeris prediction the analytical solutions are in good agreement with numerical integration.

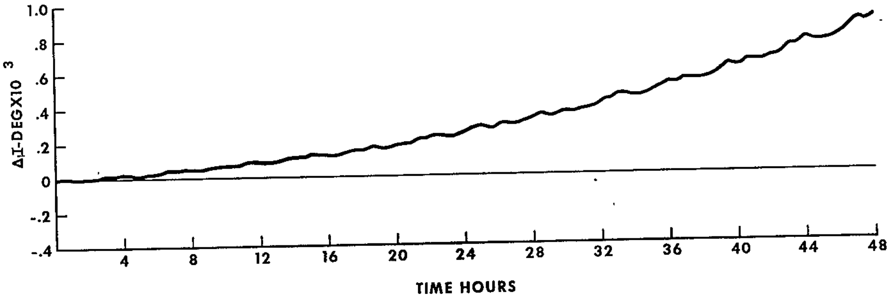
Additional possibilities for improving the accuracy of the analytic ephemeris generator were considered. The most obvious of these would be to include the first order long period effects of oblateness. Because of the algebra involved in generating these terms from a second order solution, the possibilities of using Brouwer's[5] results should be considered. For long-term ephemeris generation the second order secular effects of oblateness also



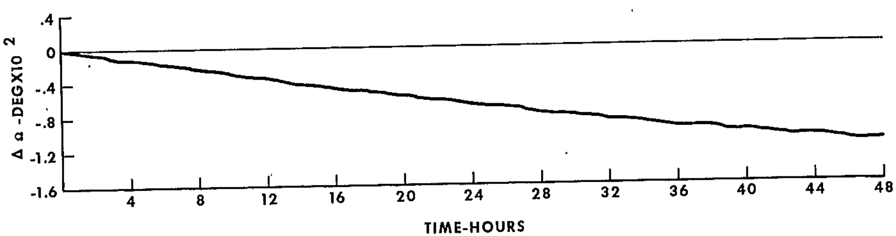
ERROR IN SEMIMAJOR AXIS AS A FUNCTION OF TIME
FOR LUNAR ORBITER



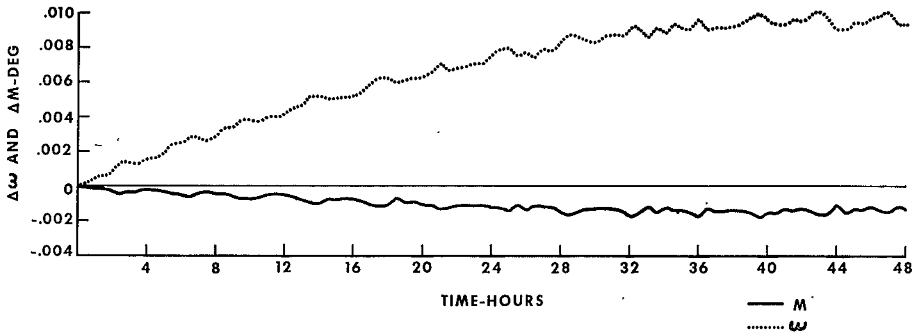
ERROR IN ECCENTRICITY AS A FUNCTION OF TIME
FOR LUNAR ORBITER
FIGURE 5



ERROR IN INCLINATION AS A FUNCTION OF TIME
FOR LUNAR ORBITER

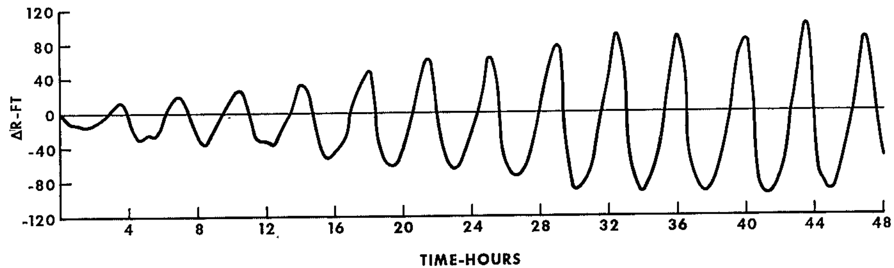


ERROR IN NODE AS A FUNCTION OF TIME
FOR LUNAR ORBITER
FIGURE 6

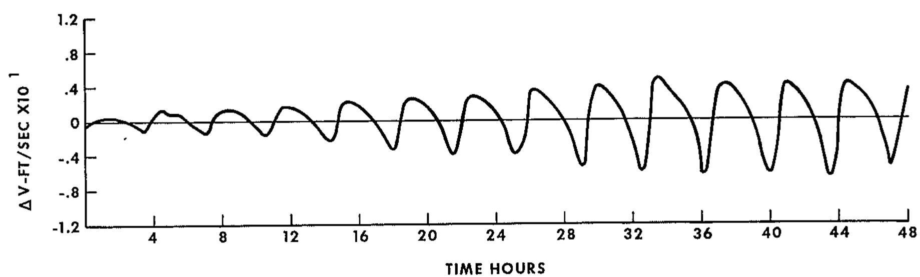


ERROR IN MEAN ANOMALY AND ARGUMENT OF PERICENTER AS A FUNCTION OF TIME FOR LUNAR ORBITER

FIGURE 7



ERROR IN RANGE AS A FUNCTION OF TIME
FOR LUNAR ORBITER



ERROR IN VELOCITY AS A FUNCTION OF TIME
FOR LUNAR ORBITER
FIGURE 8

should be included. The ratio of first order to second order secular effects of oblateness is proportional to $\frac{1}{J_{20}}$. Therefore, these effects would be negligible for the time periods considered here.

Next an attempt was made to improve accuracy by doing all calculations in double precision. However, it was found that the errors inherent with this perturbation solution are larger than the changes that double precision arithmetic makes in the results.

It was found that the accuracy of the solutions was about the same whether the perturbing force was assumed to be an aspherical primary, a third body or the combined effects of the primary and third body. In summary, it may be said that the analytical solutions presented here are an accurate and efficient means of predicting the ephemeris of a satellite moving under the influence of an aspherical primary and a third body. In the following chapter the analytical solutions will be incorporated into an orbit determination scheme for planetary satellites.

CHAPTER III

ON ORBIT DETERMINATION

INTRODUCTION

Having developed the general perturbation solution in Chapter II, now it is possible to demonstrate its application to the problem of orbit determination. A linear sequential estimator is derived in detail by using the maximum likelihood (abbreviated M. L.) method. The least squares and minimum variance methods for deriving linear estimators are discussed briefly, and their similarities are noted. In particular it is demonstrated that the general perturbation solution derived in Chapter II can be used as the basis for a preliminary design orbit determination program which is both accurate and computationally efficient. As a result of the reduction in computation time over programs which employ numerical integration, this program is useful for studying problems associated with orbit determination. Also, a realistic perturbing force model may be used rather than the two-body model used in most preliminary design programs.

By using the M. L. principle a theory of sequential estimation is derived for use when the radar covariance matrix is unkown. Application of this scheme to a specific example demonstrates that good results may be obtained if a reasonable guess for the covariance matrix is available.

In addition, the problem of estimating the classical Keplerian elements is considered, and a comparison of the efficiency of the process for estimating Cartesian coordinates and Keplerian elements is made. A brief study of the accuracy of a state transition matrix generated by numeric partial differentiation is made in order to justify use of this method in the orbit determination program.

Historical Background

Historically, one of the fundamental problems of astronomy has been orbit determination. As early as 1875 Gauss[32] proposed the method of least squares which is still the most widely used technique for fitting observational data. The classical method of least squares makes no use of statistical knowledge about the measurement errors in the data, but merely chooses as the best estimate of the parameter vector the one which minimizes the norm of the observation error vector.

R. A. Fisher was largely responsible for the introduction of statistical concepts into the field of estimation theory. In 1922 Fisher[33, 34] introduced the concept of maximum likelihood. This technique requires a priori knowledge of a likelihood function and chooses as the best estimate of the parameter vector the one which maximizes the likelihood function. The M. L. principle is intuitively appealing. Also, the procedure for deriving the estimator is straight forward, and the estimator has several desirable statistical properties. Swerling[35] used the M. L. principle to develop the first sequential differential correction procedure for satellite tracking.

Work in the field of communications theory has advanced the state of the art of estimation theory. Transmission of communications signals by electrical means is subject to random perturbations or noise from diverse sources such as thermal motion in resistors and galactic and ionospheric noise in propagation. Hence, the communications engineer is concerned with extracting the best estimate of the signal from the electrical transmission. Much of the fundamental work in this area was done independently by Wiener[36] and Kolmogorov[37]. The result of their work is the Wiener-Kolmogorov filter, usually referred to as the Wiener filter.

There are several difficulties associated with the Wiener filter[38]. The optimum filter is specified by an integral equation whose solution amounts to specifying the optimum filter by its impulse response. However, there generally is no simple method for synthesizing a filter with a prescribed impulse response. Furthermore, the numerical determination of the optimum impulse response is ill-suited to computer solution. In addition, the classical Wiener filter is valid only for stationary processes.

Kalman and Bucy[39] recognized the desirability of converting the integral equation of Wiener's into a nonlinear differential equation whose solution yields the covariance matrix of the estimation error. This matrix contains all the necessary information for the design of the optimum filter. The computation of the optimum filter is much simpler than that of Wiener, and the more general equations cover either stationary or time-varying situations. In addition, the theory accommodates both discrete-time and continuous-time linear systems. The Wiener-Kalman filter is used extensively in the field of orbit determination and celestial navigation and guidance.

To this point only linear estimators have been discussed, however, the dynamical equations describing a satellite's motion are nonlinear. Since the estimators used in orbit determination generally require linear state and observation relations, the equations of motion and the equations relating the observations to the state are linearized by expanding about a reference trajectory. It would be desirable to have an estimator capable of handling nonlinear systems, and some work has been done in this area. Morrison[40] considers an iterative scheme for estimating the state of a nonlinear system. Opah and Stubberud[41] discuss a technique for using the Kalman filter in conjunction with quasilinearization to do nonlinear estimation.

It is known that when only noise corrupted observation data are available, all of the information about the state of the system is contained in the probability density function of the system state conditioned on all the past measurements. Fisher and Stear[42] derive a dynamical equation for the conditional density function when the system disturbances and the measurement noise are both jointly Gaussian and white. Their work is applicable to nonlinear systems and both generalizes and unifies the results of previous work in this area.

Generally, it is assumed that the statistical model associated with the random noise in the observation data is known. The noise usually is assumed to be described by a normal distribution with a known mean and variance. Little work has been done on the problem of orbit determination when the statistical model of the noise is unknown. The usual procedure in this situation is to use a classical least squares estimator. Smith[43] has used a sequential form of Bayesian estimation theory given by Raiffa[44] et al., to develop a method for relaxing the assumption that the distributions of the observation errors are known. His approach is to regard the distributions as normal, but with unknown variances. The unknown variances are represented as random variables having inverted-gamma distributions. Applying Bayesian estimation theory in a multistage process then yields recursive equations for simultaneously estimating the system state and the variances.

Problem Definition

Fundamentally, the problem of orbit determination may be stated as follows: given observation data related to parameters of the vehicle's trajectory, determine the best estimate of the vehicle's state. The observation data are quantities such as range, range rate, and azimuth and elevation angles. The observation data invariably are corrupted by random and systematic errors. Further errors are introduced by inaccuracies in the mathematical model.

Random errors are accidental in nature and are caused by innumerable factors beyond the control of the analyst. Usually they are treated as a normally distributed stochastic process. Systematic errors are due to deficiencies in the tracking equipment. For example, an improperly calibrated radar may yield consistently biased data. In addition to errors of this type, inaccuracies in the mathematical model introduce errors into the system. Examples of model errors are improper station locations or inaccurate knowledge or premature truncation of the gravitational disturbing function. Other examples are improper correction for refraction, precession, nutation, or light transit time.

Due to errors of this nature the estimate of a trajectory never will agree identically with the true trajectory. Consequently, the process of observation and estimation must be repeated continually in order to minimize deficiencies in the estimation technique.

The estimation procedure lends itself naturally to the estimation of parameters such as station locations and values of gravitational constants which may not be known accurately. Biases in the observation data also may be estimated. However, when additional parameters are estimated, the state vector and associated matrices must be augmented, thereby increasing the complexity of the problem.

Mathematical Preliminaries

In general, the problem of orbit determination is formulated by linearizing the governing equations. This is accomplished by a Taylor's series expansion about a known reference or nominal trajectory. Presupposing knowledge of a nearby reference trajectory is a valid assumption, since present technology makes it possible to determine injection conditions to within a few tenths of one percent. Generally, the design trajectory is a sufficient first guess to start the estimation procedure. When a design trajectory is unavailable or inadequate, one of the classical orbit determination techniques, such as those of Gauss or Laplace[46], may be used in conjunction with observation data to determine an estimate of the initial conditions.

The linearization procedure will be presented briefly. Let ξ be an $n \times 1$ vector of state variables. Then, the equations of motion for ξ may be expressed as

$$\dot{\xi} = F(\xi, t) \tag{3-1}$$

where F is an $n \times 1$ column vector. Now, if x represents the $n \times 1$ vector of deviations of the nominal from the true state, the true state may be expressed as

$$\xi(t) = \xi^*(t) + x(t) \tag{3-2}$$

The symbol $(\ )^*$ indicates that the quantity is evaluated on the nominal trajectory. Differentiating Eq. (3-2), and using (3-1) yields

$$\dot{\xi}^* + \dot{x} = F(\xi^* + x, t) \tag{3-3}$$

Expanding Eq. (3-3) in a Taylor's series about the nominal at each point in time leads to

$$\dot{\xi}^* + \dot{x} = F(\xi^*, t) + \left(\frac{\partial F}{\partial \xi}\right)^* x + \ldots . \tag{3-4}$$

If only linear terms are retained, the equation for the state deviation is

$$\dot{x} = \left(\frac{\partial F}{\partial \xi}\right)^* x . \tag{3-5}$$

Equation (3-5) is a system of linear differential equations with time dependent coefficients. The solution to Eq. (3-5) is[45]

$$x(t) = \Phi(t, t_o)x_o \tag{3-6}$$

where $\Phi(t, t_o)$ satisfies the differential equation

$$\dot{\Phi}(t, t_o) = (\frac{\partial F}{\partial \xi})^* \Phi(t, t_o) \tag{3-7}$$

subject to the initial conditions

$$\Phi(t_o, t_o) = I .$$

The properties of the state transition matrix are discussed more fully in the following section.

Generally, it is possible to observe quantities which only are related to the state i.e., which are nonlinear functions of the state. Since the observations also contain random noise, they are related to the state by

$$\eta = G(\xi) + v \tag{3-8}$$

where

- η is the $p \times 1$ observation vector
- $G(\xi)$ is a nonlinear function relating the observation and the state
- v is a $p \times 1$ vector of observation noise

Equation (3-8) is linearized by expanding about the reference trajectory as follows:

$$\eta = \eta^* + \left(\frac{\partial G}{\partial \xi}\right)^* (\xi - \xi^*) + \dots + v \tag{3-9}$$

Using conventional notation and denoting $(\eta - \eta^*)$ by y , $(\xi - \xi^*)$ by x , and $\left(\frac{\partial G}{\partial \xi}\right)^*$ by H , Eq. (3-9) may be written as

$$y = Hx + v \tag{3-10}$$

Henceforth, y will be referred to as the observation vector and x as the state vector even though they are actually deviations from the nominal values. Since the state transition process also contains random noise from sources such as modeling errors or random fluctuations of control systems, Eqs. (3-6) and (3-10) usually are written as

$$x = \Phi(t, t_o)\, x_o + w \tag{3-11}$$

$$y = Hx + v \tag{3-12}$$

where w is a $n \times 1$ vector of state noise.

Generation of the State Transition Matrix

The function of the state transition matrix is to map small deviations in the state forward or backward in time. The matrix differential equation which generates the transition matrix is derived in the same manner as Eq. (3-5) and is given by

$$\dot{\Phi} = A\Phi \tag{3-13}$$

where

$$\Phi = \frac{\partial \xi}{\partial \xi_o} \quad \text{and} \quad A = \left(\frac{\partial F}{\partial \xi}\right)^* \tag{3-14}$$

Equations (3-13) are a set of n x n linear differential equations with time dependent coefficients denoted by the matrix A . This matrix is evaluated on the nominal trajectory. Equations (3-13) may be solved by numerically integrating the set of n x n differential equations n times, subject to the initial conditions

$$\Phi(t_o, t_o) = I \tag{3-15}$$

There are alternate techniques for generating the state transition matrix. If an analytical solution is available, such as the one presented in Chapter II, in theory it is possible to form Φ analytically. The analytical solutions were programmed in FORMAC, and an attempt was made to generate Φ analytically. However, the complexity of the solution and the inability of FORMAC to make any significant simplification rendered the generation of Φ in this manner infeasible. The major difficulty lies in the complexity of the short period terms in the solution. The transition matrix for the long period and secular terms is relatively simple to obtain.

An alternate technique is the generation of the state transition matrix by numeric partial differentiation[46]. This is a straightforward procedure which may be used with either analytical solutions for the equations of motion or by numerical integration of the equations of motion. Let α and ξ be generic terms representing any of the orbit elements. Then, $\alpha(t) = \alpha(a_o, e_o \ldots \xi_o \ldots M_o, t)$, and the partial derivative may be approximated directly by its definition, i.e.

$$\frac{\partial\alpha(t)}{\partial\xi_o} = \frac{\alpha(a_o, e_o, \ldots \xi_o + \Delta\xi_o \ldots M_o, t) - \alpha(a_o, e_o, \ldots M_o, t)}{\Delta\xi_o} \quad (3\text{-}16)$$

The first term on the right is obtained by incrementing a typical element ξ and evaluating the solution at the desired time either analytically or by numerical integration. The corresponding value of ξ from the nominal trajectory is then subtracted and the result divided by $\Delta\xi_o$. The result is an accurate approximation to the derivative. This process must be repeated n times since each perturbation yields one column of the transition matrix.

The technique outlined above worked quite well for the application considered here. Use of the analytical solutions makes this a particularly attractive technique since it is necessary to evaluate the solutions only at the required times. A study which was made to determine the accuracy of numeric partials is discussed in the section presenting results.

Linear Estimators

Given the system of equations (3-11) and (3-12), the problem is to choose an estimation scheme which somehow minimizes the estimation error and yields a best estimate of the state vector. There are several linear estimators available to perform this task, and, if all random quantities are assumed to be normally distributed, they all yield the same results.

A. Least Squares

The best known of all estimators is the least squares estimator. The least squares criterion is to choose as the best estimate of the state vector the one which minimizes the norm of the observation error vector.

Consider the case where the state vector, x, is not a random quantity and the observation vector, y, is related to x by Eq. (3-12). The least squares criterion requires that $\hat{x}$, the best estimate of x, be chosen in order to minimize

$$Q = [v^T v] \tag{3-17}$$

Assume that k observations have been made. Then, Eq. (3-12) may be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} = \begin{bmatrix} h_1 & \Phi(t_1, t_k) \\ h_2 & \Phi(t_2, t_k) \\ & \cdot \\ & \cdot \\ & \cdot \\ & \cdot \\ & \cdot \\ h_k & \Phi(t_K, t_k \end{bmatrix} [x_k] + \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{bmatrix} \tag{3-18}$$

Write Eq. (3-18) as

$$Y_k = H_k x_k + v_k . \tag{3-19}$$

Substituting Eq. (3-19) into (3-17) yields

$$Q = (Y_k - H_k x_k)^T (Y_k - H_k x_k) . \tag{3-20}$$

For a minimum, the first variation of Q must vanish; hence,

$$\delta Q = 0 = (Y_k - H_k \hat{x}_k)^T H_k \delta x_k - (H_k \delta x_k)^T (Y_k - H_k \hat{x}_k) \tag{3-21}$$

where $\hat{x}_k$ is the value of x_k that extremizes Q. Since the two terms on the right are the scalar transpose of each other, they are equal. Now, since δx is arbitrary,

$$H_k^T (Y_k - H_k \hat{x}_k) = 0 \tag{3-22}$$

or

$$\hat{x}_k = (H_k^T H_k)^{-1} H_k^T Y_k . \tag{3-23}$$

In order to insure a minimum the second variation must be positive definite. Thus,

$$\delta^2 Q = 2\delta x_k^T (H_k^T H_k) \delta x_k > 0 . \tag{3-24}$$

The matrix $(H_k^T H_k)$ must be positive definite for a unique solution for $\hat{x}$ to exist. It will be assumed here that sufficient independent observations have been made to render $(H_k^T H_k)$ positive definite. Consequently, the second variation will be positive definite.

The shortcoming of the least squares method is that all observations are treated equally, and no attempt is made to weight the observations according to their relative accuracy. The obvious solution to this problem is to introduce a positive definite weighting matrix, W, into the estimation scheme. The best estimate for x in this case is[48]

$$\hat{x}_k = (H_k^T W_k^{-1} H_k)^{-1} H_k^T W_k^{-1} Y_k \tag{3-25}$$

The disadvantage of weighted least squares is that it gives no criteria for choosing the weighting matrix. However, the observation error is a random variable. Consequently, the next logical step is to consider estimation as a statistical problem and to invoke the laws of probability and statistics to determine an estimate of the state. When the problem is treated from a statistical viewpoint, the weighting matrix evolves naturally as the observation error covariance matrix.

The theory of probability and statistics is considered in the derivation of the minimum variance estimator which is discussed next.

B. The Minimum Variance Estimator

The principle of minimum variance is to choose $\hat{x}$ so that the variance of the estimation error,

$$Q = E[(\hat{x} - x)(\hat{x} - x)^T] \tag{3-26}$$

is minimized subject to the constraint that $\hat{x}$ be a linear unbiased estimator. Therefore, an estimator is desired of the form

$$\hat{x} = By \tag{3-27}$$

with the restriction that $\hat{x}$ be unbiased, i.e.

$$E(\hat{x}) = x = E(By) = E[B(Hx + v)] = BHx \tag{3-28}$$

It is assumed that the mean of the observation error is zero. Equation (3-28) requires that

$$BH = I \tag{3-29}$$

In order for this to be true both H and B must be of full rank. This is related to the fact that, in the case of least squares, the matrix $(H^T H)^{-1}$ must exist.

Substituting Eq. (3-27) into (3-26), expanding, and utilizing Eqs. (3-29) and (3-11) yields

$$\begin{aligned} Q(B) &= E[(By - x)\,(By - x)^T] \\ &= BE[v\,v^T]B^T \qquad (3\text{-}30) \\ &= BRB \end{aligned}$$

where R is the covariance matrix of the observation error. Now, it is necessary to minimize Q(B) subject to Eq. (3-29). Adjoining these two equations through the use of a matrix Lagrange multiplier, λ , results in

$$Q_1(B) = BRB^T - BH\lambda - \lambda^T H^T B^T \qquad (3\text{-}31)$$

Taking the first variation yields

$$\delta Q_1(B) = \delta B[RB^T - H\lambda] + [BR - \lambda^T H^T]\delta B^T = 0 \qquad (3\text{-}32)$$

Since the two terms are the transpose of one another, it is necessary to deal with only one of them. Furthermore, since δB is arbitrary, it is necessary that

$$BR - \lambda^T H^T = \phi$$

or

$$B = \lambda^T H^T R^{-1} \qquad (3\text{-}33)$$

Using Eq. (3-29) to determine λ results in

$$BH = I = \lambda^T H^T R^{-1} H \tag{3-34}$$

or

$$\lambda = (H^T R^{-1} H)^{-1} . \tag{3-35}$$

Collecting results

$$B = (H^T R^{-1} H)^{-1} H^T R \tag{3-36}$$

and

$$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y . \tag{3-37}$$

It may be shown that the second variation of $Q_1(B)$ is positive since R is positive definite[48]. Note that the minimum variance estimator requires the weighting matrix to be the observation error covariance matrix. In addition, no assumption was necessary for the form of the probability distribution of the observation noise; only knowledge of its mean and covariance is required.

The Gauss-Markoff theorem gives the best minimum variance, linear, unbiased, estimator in the form of a theorem. It is essentially a statement of Eq. (3-37). For a detailed discussion and proof of the Gauss-Markoff theorem, see Ref. (49).

C. The Maximum Likelihood Estimate of the State of a Linear Dynamic System

Consider the situation where x is a $n \times 1$ parameter vector related to the observation vector by Eq. (3-12). Assume that the measurement noise is described by the joint Gaussian distribution function

$$p(v_k) = \frac{1}{(2\pi)^{p/2} |R_k|^{\frac{1}{2}}} \exp\left[-\tfrac{1}{2} v_k^T R_k^{-1} v_k\right] , \tag{3-38}$$

where $|R_k|$ denotes the determinant of R_k and p is the dimension of v_k. In addition assume that every set of measurements is independent. Hence, the joint probability density of the sets of measurement errors $v_1 \ldots v_k$ is simply a product of k expressions similar to Eq. (3-38). The transformation to the joint probability density function of $y_1 \ldots y_k$ is made by substituting

$$v_k = y_k - H_k x \tag{3-39}$$

into Eq. (3-38) and noting that the Jacobian of the transformation is 1. This yields

$$p(Y_k;x) = \frac{1}{(2\pi)^{\frac{m}{2}} |R_K|^{\frac{1}{2}}} \exp \left[-\tfrac{1}{2}(y_k - H_k x)^T R_K^{-1} (y_K - H_k x)\right] \tag{3-40}$$

where R_K^{-1} is a block diagonal consisting of $R_1^{-1} \ldots R_k^{-1}$ and m is the dimension of Y_k. The matrix H_k has dimensions $m \times n$.

The M.L. principle is to select the x which maximizes $p(y_k; x)$ when evaluated at the random sample $[Y_k]$. The heuristic reason for so doing is that, of all possible samples, this is the one actually observed and, therefore, the most likely. Thus, x should be chosen to maximize that probability.

The value of x which maximizes $p(Y_k; x)$ is the one which minimizes the exponent in Eq. (3-40). As shown in the discussion on least squares, this value of x is

$$\hat{x} = (H_k^T R_K^{-1} H_k)^{-1} H_k^T R_K^{-1} y_k . \tag{3-41}$$

Hence, for Gaussian random variables the methods of maximum likelihood and minimum variance yield the same estimator. Both specify the weighting matrix to be the observation error covariance matrix. For a more detailed discussion of the maximum likelihood estimator see Ref. (34).

Now, consider the situation where x is a time dependent stochastic process and a sequential estimator for x is desired. The equation which describes the state propagation is

$$x_{k+1} = \Phi(t_{k+1}, t_k) x_k + w_k \tag{3-42}$$

and the observations are related to the state by

$$y_k = H_k x_k + v_k \tag{3-43}$$

The noise or error vectors are assumed to be independent Gaussian vectors with zero means and the following covariances

$$E[w_j w_k^T] = Q \delta_{jk} \tag{3-44}$$

$$E[v_j v_k^T] = R \delta_{jk} \tag{3-45}$$

$$E[w_j v_k^T] = \phi \tag{3-46}$$

where δ_{jk} is the Kronecker delta. The matrices R and Q are assumed to be positive definite. Both Q and R could have covariance terms (terms which lie off the main diagonal); however, as long as the errors are not time correlated, the form of the maximum likelihood estimator would not change.

It is assumed also that the initial state, x_o , is a Gaussian random vector with the a priori information

$$E[x_o] = \hat{x}_o \tag{3-47}$$

$$E[(x_o - \hat{x}_o)(x_o - \hat{x}_o)^T] = P_o \tag{3-48}$$

where P_o is the covariance matrix associated with the initial estimate of the state, x_o , and is assumed positive definite. The problem may be stated as follows: given the observation vectors y_o , $y_1, \cdots y_n$, find the "best" estimate of x_k . This will be regarded as a filtering problem if $k = n$, as a prediction problem if $k > n$, and as a smoothing problem if $k < n$.

A Bayesian approach to the estimation problem will be used. The Bayesian approach assumes that certain density functions are known a priori. Bayes' rule states that

$$p(x_k/Y_k) = \frac{p(x_k, Y_k)}{p(Y_k)} \tag{3-49}$$

where Y_k denotes the ensemble of the observation vectors, i.e.

$$Y_k = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix}$$

Using the assumptions on independence given by Eq. (3-46), Bayes' rule may be simplified in the following manner

$$p(x_k, Y_k) = p(y_k/x_k, Y_{k-1})\, p(x_k, Y_{k-1}) . \tag{3-50}$$

Note that

$$y_k = H_k x_k + v_k$$

implies that knowledge of Y_{k-1} is not necessary to determine y_k . Hence, Eq. (3-50) may be written as

$$p(x_k, Y_k) = p(y_k/x_k)\ p(x_k/Y_{k-1})\ p(Y_{k-1}) \tag{3-51}$$

Substituting Eq. (3-51) into Eq. (3-50) leads to

$$\begin{aligned} p(x_k/Y_k) &= \frac{p(y_k/x_k)\ p(x_k/Y_{k-1})\ p(Y_{k-1})}{p(Y_k)} \\ &= \frac{p(y_k/x_k)\ p(x_k/Y_{k-1})}{p(y_k/Y_{k-1})} \end{aligned} \tag{3-52}$$

The three density functions on the right-hand side of Eq. (3-52) are assumed known. The density function $p(x_k/Y_k)$ is known as the a posteriori density function of x_k . It provides knowledge about the state of the system in terms of the observations Y_k . By definition, it contains all of the information necessary for estimation of the state. The M.L. criterion will be used to obtain $\hat{x}_k$ from $p(x_k/Y_k)$.

It is demonstrated in Ref. (50) that under the Gaussian assumptions imposed on w_k and v_k , the conditional density function $p(x_k/Y_k)$ will be of the form

$$p(x_k/Y_k) = \frac{1}{(2\pi)^{n/2}\ |U_k|^{\frac{1}{2}}} \exp[-\tfrac{1}{2}(x_k - E(x_k/Y_k))^T\ U_k^{-1} (x_k - E(x_k/Y_k))] \tag{3-53}$$

where

$$\text{cov}[x_k/Y_k] = U_k$$

The M.L. principle requires that x_k be chosen so that $p(x_k/Y_k)$ is maximized. This is equivalent to choosing x_k in order to maximize the probability of obtaining the set of observations which were actually obtained.

To maximize $p(x_k/Y_k)$ it is obvious from Eq. (3-53) that $\hat{x}_k$ must be chosen as

$$\hat{x}_k = E(x_k/Y_k) \tag{3-54}$$

Hence, the principle of maximum likelihood states that for normal random variables the best estimate of x_k is the conditional mean, $E(x_k/Y_k)$.

Since the noise vectors v and w are assumed normal and x and y are linear combinations of v and w , all density functions in Eq. (3-52) will be normal. Thus, it will be necessary to determine the mean vectors and covariance matrices in order to specify unique normal distributions. However, it will be convenient to first define the statistics of the estimation error.

The estimation error is defined by

$$\tilde{x}_{k-1/k-1} = x_{k-1} - E[x_{k-1}/Y_{k-1}] \equiv x_{k-1} - \hat{x}_{k-1/k-1} \tag{3-55}$$

$$\tilde{x}_{k/k-1} = x_k - E[x_k/Y_{k-1}] \equiv x_k - \hat{x}_{k/k-1} \tag{3-56}$$

with unconditional mean

$$E(\tilde{x}_{k-1/k-1}) = E(x_{k-1} - \hat{x}_{k-1/k-1}) \ . \tag{3-57}$$

Substituting Eq. (3-55) into (3-57) yields,

$$E(\tilde{x}_{k-1/k-1}) = E(x_{k-1}) - E_y(E_x[x_{k-1}/Y_{k-1}]) = \phi \ . \tag{3-58}$$

The x and y subscripts of E denote the variable of integration. Likewise,

$$E(x_{k/k-1}) = E(x_k) - E_y[E_x(x_k/Y_{k-1})] = \phi \tag{3-59}$$

and

$$\mathrm{cov}(\tilde{x}_{k-1/k-1}) = E[(x_{k-1} - \hat{x}_{k-1/k-1})(x_{k-1} - \hat{x}_{k-1/k-1})^T] \equiv P_{k-1} \ . \tag{3-60}$$

In a similar manner

$$\mathrm{cov}(\tilde{x}_{k/k-1}) = E[(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T] \equiv P_{k/k-1} \equiv \overline{P}_k \ . \tag{3-61}$$

The mean of x_k conditioned on Y_{k-1} is

$$E[x_k/Y_{k-1}] = \hat{x}_{k/k-1} \equiv \overline{x}_k \ . \tag{3-62}$$

The covariance of x_k conditioned on Y_{k-1} is independent of Y_{k-1} (see page 64 of Ref. 44). Hence,

$$\begin{aligned} & E[(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T/Y_{k-1}] \\ & = E[(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T] \equiv \overline{P}_k \end{aligned} \tag{3-63}$$

A recursive scheme for developing $\overline{P}_k$ given P_{k-1} may be obtained by expanding Eq. (3-63) as follows

$$\overline{P}_k = E[(\Phi(t_k,t_{k-1})x_{k-1} + w_{k-1} - \Phi(t_k,t_{k-1})\hat{x}_{k-1/k-1})(\Phi(t_k,t_{k-1})x_{k-1} + w_{k-1} - \Phi(t_k,t_{k-1})\hat{x}_{k-1/k-1})^T]$$

$$= \Phi(t_k,t_{k-1})\, E[(x_{k-1}-\hat{x}_{k-1/k-1})(x_{k-1} - \hat{x}_{k-1/k-1})^T]\, \Phi(t_k,t_{k-1})$$

$$+ E[w_{k-1}w_{k-1}^T] + \Phi(t_k,t_{k-1})\, E[(x_{k-1} - \hat{x}_{k-1/k-1})w_{k-1}^T]$$

$$+ E[w_{k-1}(x_{k-1} - \hat{x}_{k-1/k-1})^T]\, \Phi^T(t_k,t_{k-1}) .$$

The last two terms are zero since x_{k-1} is independent of w_{k-1} and

$$\hat{x}_{k-1/k-1} = E_x[x_{k-1}/Y_{k-1}]$$

is a function of Y which is independent of w . Hence,

$$\overline{P}_k = \Phi(t_k,t_{k-1})P_{k-1}\Phi^T(t_k,t_{k-1})+Q . \tag{3-64}$$

The density function of y_k conditioned on x_k has associated with it a conditional mean

$$E_y[y_k/x_k] = H_k\, x_k , \tag{3-65}$$

and covariance

$$\begin{aligned} &E_y[(y_k - H_k\, x_k)(y_k - H_k\, x_k)^T/x_k] \\ &= E_y[v_k\, v_k^T/x_k] = R . \end{aligned} \tag{3-66}$$

The density function of y_k conditioned on Y_{k-1} has associated with it a conditional mean

$$E[y_k/Y_{k-1}] = H_k \overline{x}_k ,$$

and covariance

$$\begin{aligned} \text{cov}(y_k/Y_{k-1}) &= E[(y_k - H_k \overline{x}_k)(y_k - H_k \overline{x}_k)^T/Y_{k-1}] \\ &= E[(H_k(x_k - \overline{x}_k) + v_k)(H_k(x_k - \overline{x}_k) + v_k)^T] \end{aligned} \tag{3-67}$$

Since the noise, v_k , is not dependent on the state, all cross products will be zero and

$$\begin{aligned} \text{cov}(y_k/Y_{k-1}) &= E[(H_k(x_k - \overline{x}_k)(x_k - \overline{x}_k)^T H_k^T] + E[v_k v_k^T] \\ &= H_k \overline{P}_k H_k^T + R \end{aligned} \tag{3-68}$$

Substitution of Eqs. (3-62) through (3-68) into Eq. (3-52) yields

$$\begin{aligned} p(x_k/Y_k) = \frac{|H_k \overline{P}_k H_k^T + R|^{\frac{1}{2}}}{(2\pi)^{n/2} |R|^{\frac{1}{2}} |\overline{P}_k|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}[(y_k - H_k x_k)^T R^{-1} \\ (y_k - H_k x_k)] \\ + (x_k - \overline{x}_k)^T \overline{P}_k^{-1} (x_k - \overline{x}_k) \\ - (y_k - H_k \overline{x}_k)^T (H_k \overline{P}_k H_k^T + R)^{-1} (y_k - H_k \overline{x}_k)]\} . \end{aligned} \tag{3-69}$$

Define the following quantities

$$\Delta x \equiv x_k - \overline{x}_k \tag{3-70}$$

$$\Delta y \equiv y_k - H_k \overline{x}_k \tag{3-71}$$

$$P_k^{-1} \equiv \overline{P}_k^{-1} + H_k^T R^{-1} H_k . \tag{3-72}$$

Define J to be the exponential term of Eq. (3-69). Adding and subtracting $H_k \bar{x}_k$ from the first term in J and dropping subscripts leads to

$$
\begin{aligned}
J &= -\tfrac{1}{2}[(\Delta y - H\,\Delta x)^T \; R^{-1}(\Delta y - H\,\Delta x^T \; + \Delta x)^T \, \bar{P}^{-1} \, \Delta x \\
&\quad -\Delta y^T (H\,\bar{P}\,H^T + R)^{-1} \, \Delta y \\
&= -\tfrac{1}{2}[\Delta y[R^{-1} - (H\,\bar{P}\,H^T + R)^{-1} \, \Delta y^T - \Delta y^T \; R^{-1} \; H\,\Delta x \\
&\quad -\Delta x^T \; H^T \; R^{-1} \; \Delta y + \Delta x^T \; P^{-1} \; \Delta x] \; .
\end{aligned}
\tag{3-73}
$$

It can be shown that

$$
R^{-1} - (H\,\bar{P}\,H^T + R)^{-1} = R^{-1} \; H \cdot P \; H^T \; R^{-1}
\tag{3-74}
$$

Hence,

$$
\begin{aligned}
J &= -\tfrac{1}{2}[\Delta y \; R^{-1} \; HPH^T \; R^{-1} \; \Delta y^T - \Delta y^T \; R^{-1} \; H\Delta x \\
&\quad -\Delta x^T \; H^T \; R^{-1} \; \Delta y + \Delta x^T \; P^{-1} \; \Delta x] \; .
\end{aligned}
\tag{3-75}
$$

Then, on multiplying the first three terms of J by PP^{-1} , the following expression is obtained

$$
\begin{aligned}
J &= -\tfrac{1}{2}[\Delta y \; R^{-1} \; HPP^{-1} \; PH^T \; R^{-1} \; \Delta y^T - \Delta y^T \; R^{-1} \; HPP^{-1} \; \Delta x \\
&\quad -\Delta x^T \; P^{-1} \; PH^T \; R^{-1} \; \Delta y + \Delta x^T \; P^{-1} \; \Delta x\,] \\
&= -\tfrac{1}{2}[(\Delta x - PH^T \; R^{-1} \; \Delta y)^T \; P^{-1}(\Delta x - PH^T \; R^{-1} \; \Delta y)] \; .
\end{aligned}
\tag{3-76}
$$

Substituting Eq. (3-76) into (3-69) and substituting for Δx and Δy

$$P(x_k/Y_k) = \frac{|H_k \overline{P}_k H_k^T + R|^{1/2}}{(2\pi)^{n/2} |R|^{1/2} |\overline{P}_k|^{1/2}} \exp [-\tfrac{1}{2}\{(x_k - [\overline{x}_k + P_k H_k^T R^{-1} (y_k - H_k \overline{x}_k)])^T P_k^{-1}(x_k - [\overline{x}_k + P_k H_k^T R^{-1}(y_k - H_k \overline{x}_k)])\}] \tag{3-77}$$

On comparing Eq. (3-69) with Eq. (3-53) it is seen that

$$U_k = P_k \tag{3-78}$$

and

$$|P_k|^{-1/2} = \frac{|H_k \overline{P}_k H_k^T + R|^{1/2}}{|R|^{1/2} |\overline{P}_k|^{1/2}} \tag{3-79}$$

Also,

$$\hat{x}_k = E[x_k/Y_k] = \overline{x}_k + P_k H_k^T R^{-1}(y_k - H_k \overline{x}_k) . \tag{3-80}$$

An alternate expression for $\hat{x}_k$ written in terms of a recursive relationship for P may be obtained by using Eq. (3-72). Accordingly,

$$P_k H_k^T R^{-1} = (\overline{P}_k^{-1} + H_k^T \cdot R^{-1} H_k)^{-1} H_k^T R^{-1} . \tag{3-81}$$

Using the result of Theorem I-52 of Appendix A of Ref. (48), this may be written as

$$P_k H_k^T R^{-1} = \overline{P}_k H_k^T (H_k \overline{P}_k H_k^T + R)^{-1} \equiv K_k . \tag{3-82}$$

Substituting Eq. (3-82) into (3-80) yields

$$\hat{x}_k = \overline{x}_k + K_k(y_k - H_k \overline{x}_k) . \tag{3-83}$$

Applying the matrix identity (known as the inside out rule) given in Ref. (48) to Eq. (3-72), an alternate expression for P_k may be developed

$$P_k = \overline{P}_k - \overline{P}_k H_k^T (H_k \overline{P}_k H_k^T + R)^{-1} H_k \overline{P}_k . \tag{3-84}$$

The results required for computing $\hat{x}_k$ can be summarized as follows

$$\hat{x}_k = \overline{x}_k + K_k(y_k - H_k \overline{x}_k) \tag{3-85a}$$

$$K_k = \overline{P}_k H_k^T (R + H_k \overline{P}_k H_k^T)^{-1} \tag{3-85b}$$

$$\overline{P}_k = \Phi(t_k, t_{k-1}) P_{k-1} \Phi^T(t_k, t_{k-1}) + Q \tag{3-85c}$$

$$\overline{x}_k = \Phi(t_k, t_{k-1}) \hat{x}_{k-1} \tag{3-85d}$$

$$P_k = \overline{P}_k - K_k H_k \overline{P}_k . \tag{3-85e}$$

Values of x_o and P_o are assumed known a priori. This completes the solution to the filtering problem.

The prediction problem likewise has been solved since for any $n > k$

$$\hat{x}_{n/k} = \Phi(t_n, t_k) \hat{x}_k . \tag{3-86}$$

The Sequential Estimator

Prior to the introduction of the Kalman filter in 1960, the problem of filtering continuous time signals was characterized by the solution of the Wiener problem. The problem of filtering discrete-time observations was solved by using the technique of weighted least squares or maximum likelihood.

Swirling[35] used the M.L. principle to develop a sequential estimator. Under the proper assumptions, the Kalman filter and Swirling's estimator are identical. Kalman's theory yields a general set of equations which is valid for both continuous and discrete-time filtering problems. For the problem of orbit determination based on discrete observations of a Gaussian random process, the Kalman filter yields the same set of equations given by the M. L. principle in Eqs. (3-85).

In order to start the estimation process it is assumed that values for P_o and $\hat{x}_o$ are given. If no initial guess for P_o is available and the observations are uncorrelated in time, a first guess may be obtained by taking enough observations so that H is of full rank and then using

$$\hat{P}_o = (H^T R^{-1} H)^{-1} . \qquad (3\text{-}87)$$

If the observations are taken at different times the quantities in Eq. (3-87) must be mapped by the state transition matrix, i.e.

$$\hat{P}_o = \left\{ \begin{bmatrix} H_1 & \Phi(t_o,t_1) \\ H_2 & \Phi(t_o,t_2) \end{bmatrix}^T \begin{bmatrix} R_1 & \phi \\ \phi & R_2 \end{bmatrix}^{-1} \begin{bmatrix} H_1 & \Phi(t_o,t_1) \\ H_2 & \Phi(t_o,t_2) \end{bmatrix} \right\}^{-1} \qquad (3\text{-}88)$$

The initial estimate for P_o is important since it determines the relative weight that the estimator assigns to the observations and the previous estimate of the state. Generally, $\hat{x}_o$ is assumed to be zero.

The Batch Processor

The batch processor, as its name implies, processes the observation data in groups or batches to yield a best estimate of the initial state of the vehicle. The batch processor used for this study is a maximum likelihood filter whose governing equation for normally distributed random variables is

$$\hat{x}_o = (H_k^T R_K^{-1} H_k)^{-1} H_k^T R_K^{-1} Y_k , \tag{3-89}$$

where

$$H_k = \begin{bmatrix} h_1 & \Phi(t_o,t_1) \\ h_2 & \Phi(t_o,t_2) \\ . & . \\ . & . \\ . & . \\ . & . \\ h_k & \Phi(t_o,t_k) \end{bmatrix} \tag{3-90}$$

$$R_K = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_k \end{bmatrix} \tag{3-91}$$

$$Y_k = \begin{bmatrix} y_1 \\ . \\ . \\ . \\ . \\ y_k \end{bmatrix} \tag{3-92}$$

The matricies $(H_k^T R_K^{-1} H_k)^{-1}$ and $(H_k^T R_K^{-1} Y_k)$ may be augmented as each observation vector becomes available by noting that

$$H_k^T R_K^{-1} H_k = H_{k-1}^T R_{k-1}^{-1} H_{k-1} + h_k^T R_k^{-1} h_k \tag{3-93}$$

and

$$H_k^T R_K^{-1} Y_k = H_{k-1}^T R_{K-1}^{-1} Y_{k-1} + h_k^T R_k^{-1} h_k \tag{3-94}$$

Hence, Eq. (3-89) may be formed sequentially and always will be the product of an $n \times n$ matrix, $(H_k^T R_K^{-1} H_k)^{-1}$, and an $n \times 1$ matrix, $(H_k^T R_K^{-1} Y_k)$. The major disadvantage of the batch processor is that the inversion of a large order matrix may be necessary if the state vector contains a large number of parameters.

It is well known that the recursive formulation of the Kalman filter can be derived from the nonrecursive batch processor by employing a matrix inversion lemma known as the inside out rule (see for example Ho)[51].

Comparison of the Batch Processor and Sequential Estimator

As stated previously, the sequential estimator is a recursive estimator that provides an estimate of the spacecraft's current state by processing each observation as soon as it becomes available. On the other hand, the batch processor yields an estimate of the state at some epoch by processing an entire arc of observation data simultaneously. There are several other noteworthy factors regarding the sequential estimator and the batch processor:

a. The sequential estimator substitutes the inversion of a large order $n \times n$ matrix for the inversion of a smaller $p \times p$ matrix, where n is the number of state variables and p is the dimension of the observation vector.

b. Both estimators tend to saturate after a large number of observations have been made. Saturation occurs when the covariance matrices of the estimators become so small that subsequent observations are ignored, i.e., the estimator assumes that it has perfect knowledge of the state.

c. The sequential estimator can easily handle state noise as well as time correlated measurement noise. These quantities are much more difficult to include with the batch processor. In fact, state noise is often introduced into the sequential estimator as a means of preventing the estimator from saturating.

d. The sequential estimator allows the nominal trajectory to be updated periodically which reduces the size of the deviations between the nominal and the true trajectory and results in an improvement in the linearity assumption. Consequently, it is possible for the sequential estimator to yield a better estimate of the state for one pass through the data.

e. The sequential estimator lends itself to real time filtering problems since it provides an estimate of the current state. The batch processor provides an estimate of the state with a minimum of mathematical manipulation; hence, it is desirable because numerical errors are minimized. Consequently, the batch processor is often used for post flight analysis. The batch processor generally is iterated until there are no changes in the estimate of the state. The sequential estimator involves more numerical operations than the batch processor; therefore, it may tend to diverge after a large number of observations have been processed[52].

Recursive Estimation of the Observation Error Covariance Matrix

In the formulation of the sequential estimator it is assumed that the observation error covariance matrix is known. However, situations may arise where there is substantial uncertainty in the observation error variances. Under these conditions it is desirable to have a recursive scheme which estimates the observation error covariance matrix as well as the state vector.

Smith[43] develops such a scheme by assuming that the unknown variances can be represented as random variables having inverted-gamma distributions. The recursive scheme proposed here treats the variances as parameters and makes use of the M.L. estimate of the mean and covariance of a normally distributed observation error vector.

The density function of a normally distributed random variable v is

$$f(v) = \frac{1}{(2\pi)^{p/2} \, |R|^{\frac{1}{2}}} \exp -\tfrac{1}{2}[(v-\mu)^T R^{-1}(v-\mu)] \qquad (3\text{-}95)$$

Consider a sample of size N taken from the population of v. It may be shown that the M.L. estimate of the population mean and covariance matrix of v is given by[54]

$$\mu = \frac{1}{N} \sum_{i=1}^{N} v_i \qquad (3\text{-}96)$$

$$R = \frac{1}{N} \sum_{i=1}^{N} (v_i-\mu)\,(v_i-\mu)^T \qquad (3\text{-}97)$$

i.e., the M.L. estimate of the population mean and covariance is the sample mean and covariance.

These results now will be applied to the orbit determination problem. The linearized equations are given by

$$x_N = \Phi(t_N,t_{N-1})\, x_{N-1} \qquad (3\text{-}98)$$

$$y_N = H_N x_N + v_N \qquad (3\text{-}99)$$

Assume that

$$E[v_N] = \phi \tag{3-100}$$

$$E[v_N v_N^T] = R \tag{3-101}$$

Assume further that R is unknown and that recursive estimators for $\hat{x}$ and R are desired.

Since x_N is not known, v_N cannot be determined. The nearest thing which can be obtained from the sampling of y_N is

$$\tilde{v}_N = y_N - H_N \overline{x}_N \tag{3-102}$$

Since $\tilde{v}_N$ is the best approximation available for v_N, it will be used to develop the estimator for R. The mean of $\tilde{v}_N$ is

$$\begin{aligned} E[\tilde{v}_N] &= E[y_N] - H_N E[\overline{x}_N] \\ &= H_N E[x_N] - H_N E_y E_x[x_N/Y_{N-1}] \\ &= H_N[E(x_N) - E(x_N)] = \phi . \end{aligned} \tag{3-103}$$

Hence, the sample mean is equal to the population mean. According to Eq. (3-97), the M.L. estimate of the population variance will be the sample variance given by

$$R_N = \frac{1}{N} \sum_{i=1}^{N} [(y_i - H_i \overline{x}_i)(y_i - H_i \overline{x}_i)^T] \tag{3-104}$$

The expected value or mean of the sample variance is

$$E[R_N] = \frac{1}{N} \sum_{i=1}^{N} E[(y_i - H_i \overline{x}_i)(y_i - H_i \overline{x}_i)^T] \tag{3-105}$$

Now, add and subtract $H_i x_i$, and define $\Delta x_i = x_i - \overline{x}_i$. Then,

$$E[(y_i - H_i x_i + H_i \Delta x_i)(y_i - H_i x_i + H_i \Delta x_i)^T] \tag{3-106}$$

$$= E[v_i v_i^T] + E[v_i \Delta x_i^T]H_i^T + H_i E[\Delta x_i v_i^T] + H_i E[\Delta x_i \Delta x_i^T]H_i^T$$

Since v_i is assumed independent of the state,

$$E[v_i \Delta x_i^T] = E[v_i] E[\Delta x_i^T] = \phi \tag{3-107}$$

and

$$E[R_N] = R + \frac{1}{N} \sum_{i=1}^{N} H_i \overline{P}_i H_i^T . \tag{3-108}$$

Therefore, R_N is a biased estimator of R. An unbiased estimator may be obtained by subtracting the bias. Accordingly,

$$\tilde{R}_N = \frac{1}{N} \sum_{i=1}^{N} [(y_i - H_i\overline{x}_i)(y_i - H_i\overline{x}_i)^T - H_i \overline{P}_i H_i^T] \tag{3-109}$$

A recursive estimator for R is easily obtained by noting that

$$\begin{aligned} \tilde{R}_{N+1} = {} & \frac{1}{N+1} \frac{N}{N} \sum_{i=1}^{N} [(\tilde{v}_i \tilde{v}_i^T) - H_i \overline{P}_i H_i^T] \\ & + \frac{1}{N+1} [(\tilde{v}_{N+1} \tilde{v}_{N+1}^T) - H_{N+1} \overline{P}_{N+1} H_{N+1}^T] , \end{aligned} \tag{3-110}$$

which simplifies to

$$\tilde{R}_{N+1} = \frac{N}{N+1} \tilde{R}_N + \frac{1}{N+1} [\tilde{v}_{N+1} \tilde{v}_{N+1}^T - H_{N+1} \overline{P}_{N+1} H_{N+1}^T] \tag{3-111}$$

Equation (3-111) is a recursive, unbiased, M.L. estimator for R. The above results may be combined with the sequential estimator for $\hat{x}$ given by Eqs. (3-85) as

$$\hat{x}_{N+1} = \overline{x}_{N+1} + \overline{P}_{N+1} H_{N+1}^T (R_{N+1} + H_{N+1} \overline{P}_{N+1} H_{N+1}^T)^{-1} \tilde{v}_{N+1} \tag{3-112}$$

Substituting Eq. (3-111) into (3-112) yields the desired relationship

$$\begin{aligned} \hat{x}_{N+1} = \overline{x}_{N+1} + \overline{P}_{N+1} H_{N+1}^T \Big[\frac{N}{N+1} \tilde{R}_N + \frac{1}{N+1} (y_{N+1} - H_{N+1}\overline{x}_{N+1}) \\ (y_{N+1} - H_{N+1}\overline{x}_{N+1})^T + \frac{N}{N+1} H_{N+1} \overline{P}_{N+1} H_{N+1}^T\Big]^{-1} (y_{N+1}-H_{N+1}\overline{x}_{N+1}) \end{aligned} \tag{3-113}$$

It was discovered that results given by Eq. (3-113) could be improved by modifying the weighting function, $\frac{1}{N+1}$. The reason for this, as well as the proposed change, is discussed in the section on results.

Use of the General Perturbation Solution in the Orbit Determination Program

The speed and accuracy of the analytical solutions discussed in Chapter II makes them very attractive for orbit determination studies. Using the analytical solutions to generate the nominal trajectory and the state transition matrix eliminates time consuming numerical integration. This is particularly valuable during intervals of time when the spacecraft is not in view of a tracking station since the analytical solutions need not be evaluated at all during these times. The analytical solutions were used also to generate the simulation data for the numerical results described here.

Results of Numeric Partials Study

A numerical study was made to determine the accuracy with which the state transition matrix would map initial perturbations in the state to subsequent times in the orbit. The study was made for transition matrices generated by numeric partial differentiation based on both Cartesian coordinates and orbital elements. The trajectory of Lunar Orbiter I was chosen as a test case since both the primary and third body perturbations have a significant influence on it. The following table shows the initial perturbations in the state of Lunar Orbiter I which were considered.

case	Δx_o-ft	Δy_o	Δz_o	$\Delta \dot{x}_o$-ft/sec	$\Delta \dot{y}_o$	$\Delta \dot{z}_o$
1	6032	6410	1484	-6.80	2.91	1.85
2	-6026	-6421	-1489	6.80	-2.91	-1.84
3	24178	25581	5901	-27.18	11.71	7.39
4	-24058	-25743	-5988	27.24	-11.57	-7.36

Table 3
Initial Perturbations for Numeric Partials Study

Figure (9) presents the position and velocity error metric during the second and fourth revolution of Lunar Orbiter I. The position error metric is defined by

$$\Delta R = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{\frac{1}{2}} \tag{3-114}$$

where

$$\Delta x = x_T - x_\Phi \tag{3-115}$$

and

x_T = true value of x on perturbed trajectory

x_Φ = value of x computed from transition matrix .

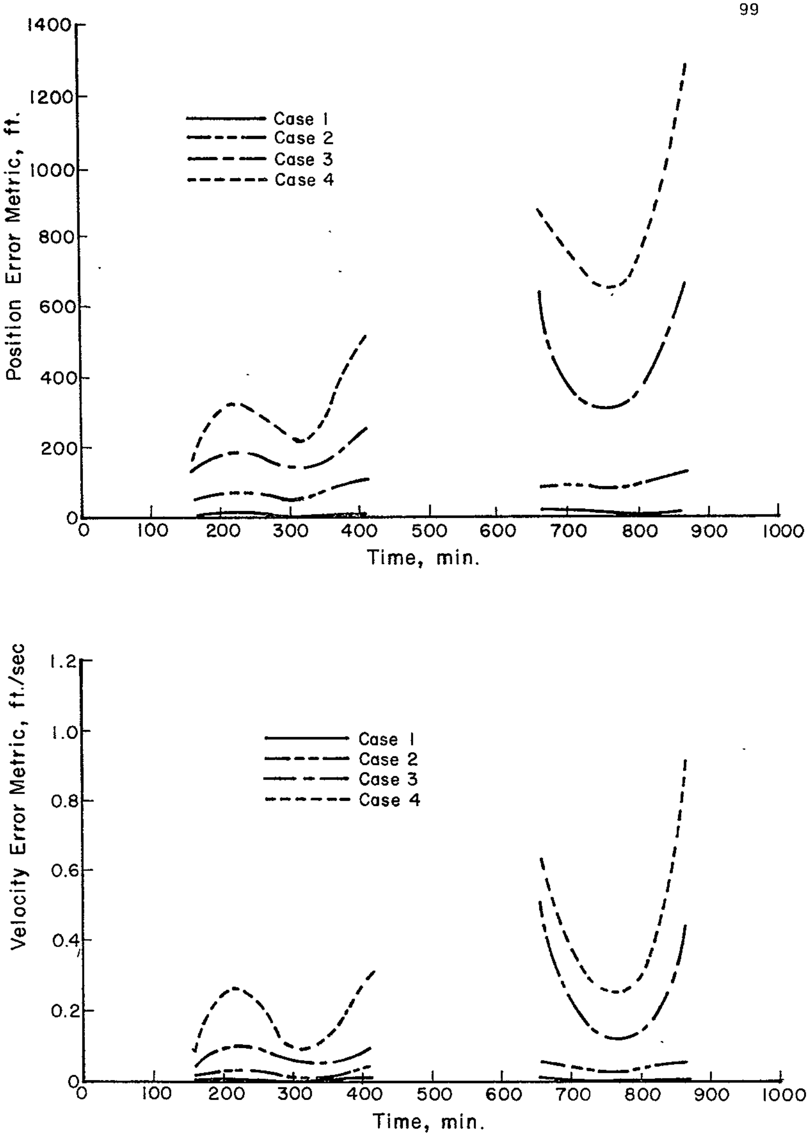


Figure 9 - Position and Velocity Error Metrics as a Function of Time for Lunar Orbiter I

A similar definition applies for the velocity metric. These results are based on a Keplerian element transition matrix generated by numeric partial differentiations. As can be seen from Fig. (9), the transition matrix accurately maps small perturbations in the state. However, as the initial perturbations become larger, nonlinear effects are more significant, and the error metric increases with time. Figure (9) shows the error metric to be nearly periodic with the maximum error occurring at pericenter and the minimum error at apocenter. This is to be expected since pericenter is the point of maximum velocity, and a perturbation of one ft./sec. in velocity has roughly the same perturbing influence as 10,000 ft. in the magnitude of the radius vector.

The errors generated by the Keplerian transition matrix were compared with those of a Cartesian transition matrix at the beginning of the fourth revolution (657 minutes). The position and velocity error metrics as well as the percentage of the error is tabulated below:

	R metric-ft		V metric-ft/sec		% error in R metric		% error in V metric	
case	Cart.	Kep.	Cart.	Kep.	Cart.	Kep.	Cart.	Kep.
1	9,306	79.43	8.5	.059	8.6	.074	11.1	.07685
2	10,460	17.41	8.707	.0119	9.74	.0162	11.35	.0155
3	43,803	872.8	41.03	.640	10.19	.203	13.38	.209

Table 4

Position and Velocity Error Metrics

The percent error gives an indication of the amount of the total perturbation that the transition matrix predicts and is given by

$$\%\ \text{error} = \frac{(\Delta x^2 + \Delta y^2 + \Delta z^2)^{\frac{1}{2}}}{(\delta x^2 + \delta y^2 + \delta z^2)^{\frac{1}{2}}}$$

where

$$\delta x = x_N - x_T$$

and

$$x_N = \text{value of } x \text{ from nominal trajectory}$$

$$x_T = \text{value of } x \text{ from perturbed trajectory} .$$

Reference (47) presents several examples of the error propagation resulting from a state transition matrix generated by the integration of the variational equations in Cartesian coordinates. Several of these cases were computed using numeric partials to form the transition matrix. The errors resulting from the Cartesian matrix were on the same order of magnitude as those shown in Ref. (47).

It should be pointed out that a forward differencing technique was used to generate the transition matrix. Accuracy would be increased by using a central differencing technique; however, this would require the generation of another perturbed trajectory, and it is believed that the forward differencing scheme gives sufficient accuracy to assure convergence in the estimation process.

Results of Orbit Determination Study

An existing orbit determination program[54] for processing range, azimuth and elevation observations of an earth satellite was modified to accommodate the analytical solutions developed in Chapter II. Additional subroutines were developed to permit the option of doing sequential estimation or batch processing as well as estimating Keplerian elements or Cartesian coordinates. The resulting program was totally dependent on analytical solutions, and no numerical integration was involved.

In order to allow comparison with the orbit determination program utilizing numerical integration, an earth satellite whose orbit corresponds to one of the Gemini series of flights was chosen. The initial conditions for this orbit are

$$t_0 = 47{,}349 \text{ sec.}$$
$$a_0 = 21622917.0 \text{ ft.}$$
$$e_0 = .008769$$
$$I_0 = 32.542°$$
$$\Omega_0 = 209.391°$$
$$\omega_0 = 69.648°$$
$$M_0 = 369.222°$$

The satellite was assumed to be tracked by the Bermuda, Carnarvon, Hawaii and White Sands tracking stations. The observation schedule is as follows

Time Interval, sec.	Observing Station
47349-47495	Bermuda
47979-48267	Carnarvon
50091-50457	Hawaii
52425-52647	White Sands

Observations of range, azimuth and elevation were taken at 6 second increments, and only those observations which had elevations above five degrees were processed.

The satellite's orbit was assumed to be perturbed by the harmonics J_{20} , J_{30} , and J_{40} . For an orbit of this altitude the moon's influence is negligible. While a real time orbit determination program would have to consider atmospheric drag, it was felt that for the preliminary design applications considered here, drag could be ignored.

The observation data was generated by using the analytical solutions developed in Chapter II. These observations were corrupted by normally distributed random noise. Hence, no perturbations to the data were considered specifically; it was assumed that the net result of all disturbances was a normally distributed random process. The observation error variances used for the simulation data are

$$\sigma_R^2 = 400 \text{ ft.}^2$$

$$\sigma_E^2 = \sigma_A^2 = .625 \times 10^{-3} \text{ deg}^2$$

Numerical Results

First, the analytical orbit determination program was compared to the program utilizing numerical integration by processing identical observations under identical initial conditions to see if the results were compatable. Figure (10) presents a comparison of a position and velocity variance as obtained by the numerical integrator and the analytical program. These results are for one pass over the Bermuda tracking station. The covariance matrix, P , is dependent only on the state transition matrix, Φ , the radar covariance matrix, R , and the initial guess for P . Since P_o and R are identical for both cases, any deviation in P must be due to the state transition matrix. Hence, the close agreement between the numerical and analytical results indicates close agreement for the state transition matrix generated by numeric partials and numerical integration. Figure (11) is a comparison of the errors in x and $\dot{x}$ for one pass over the Bermuda tracking station. These errors

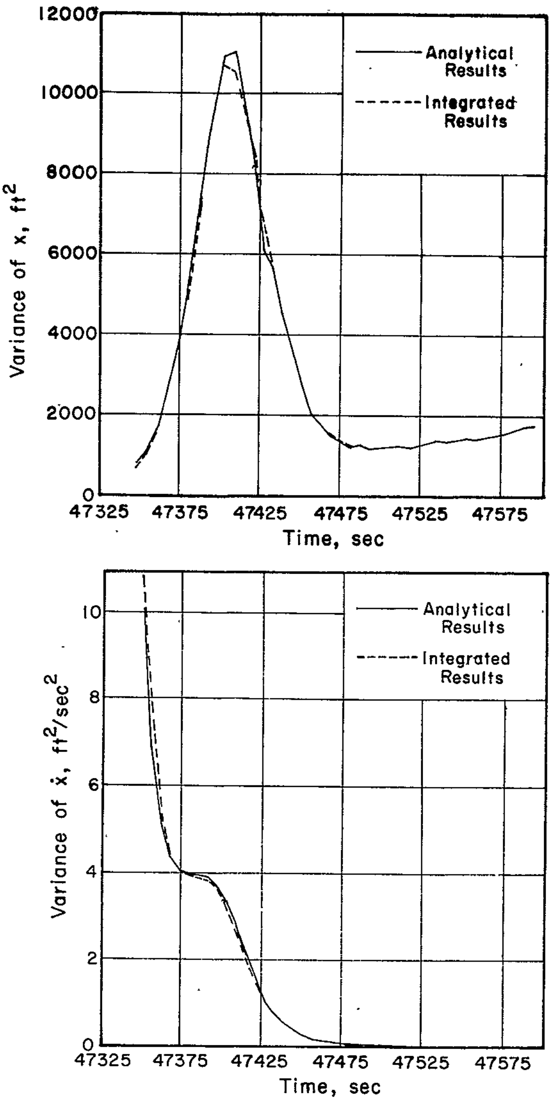


Figure 10 - Comparison of Position and Velocity Variances for Analytical and Numerical Integration Programs - Bermuda Tracking Station

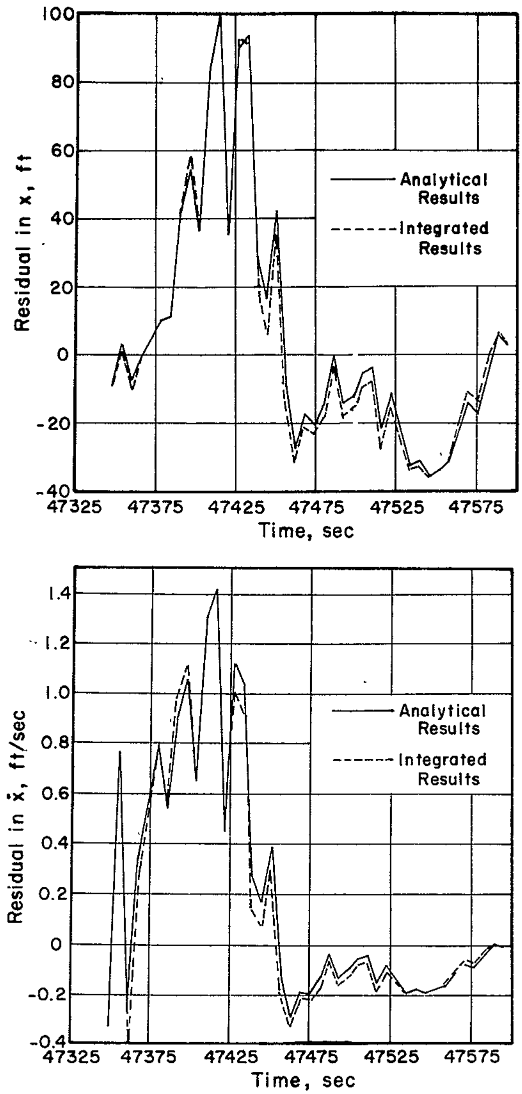


Figure 11 - Comparison of Typical Residuals for Analytical and Numerical Integration Programs - Bermuda Tracking Station

are the difference between the true and estimated values. Once again agreement between the analytical and numerical integration program is very good.

The analytical program was used next to do a study of the effect of estimating Keplerian elements in place of Cartesian coordinates. Intuitively, it seems possible to estimate the elements more accurately than the Cartesian coordinates since all elements except the mean anomaly are slowly varying functions of time. The mean anomaly essentially varies linearly with time. However, the Cartesian coordinates vary rapidly and nonlinearly with time and it is reasonable to assume that a set of constants could be estimated more accurately than a set of rapidly varying quantities.

Figures (12) through (15) are a comparison of the position and velocity error metrics obtained by estimating Keplerian elements and Cartesian coordinates. The form of the error metric is given by Eq. (3-114). No initial knowledge was assumed for P_o in either case, i.e., $P_o = \infty$. Initial errors of 100 feet in each position coordinate and 1 ft/sec in each velocity coordinate were assumed.

As shown by the figures there is very little difference between the position and velocity error metrics for the Keplerian elements and Cartesian coordinates for the first two stations. However, these two stations are relatively close together in time and nonlinearities have not influenced the system to a great extent. Figures (14) and (15), which compare the error metrics over Hawaii and White Sands, show that the errors in the Keplerian elements are considerably smaller than those in the Cartesian coordinates. In fact, the errors over the White Sands station are about five times smaller for the Keplerian elements. Note that the time interval between the Carnarvon and Hawaii stations is about 30 minutes or 120° of arc. As a result of

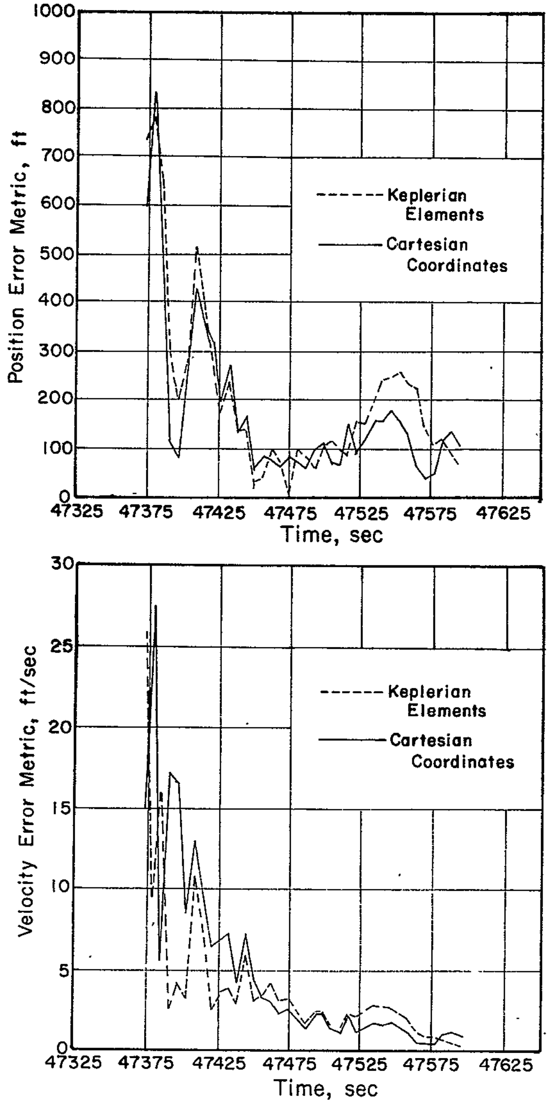


Figure 12 - Comparison of Position and Velocity Error Metrics for Keplerian and Cartesian Coordinates - Bermuda Tracking Station

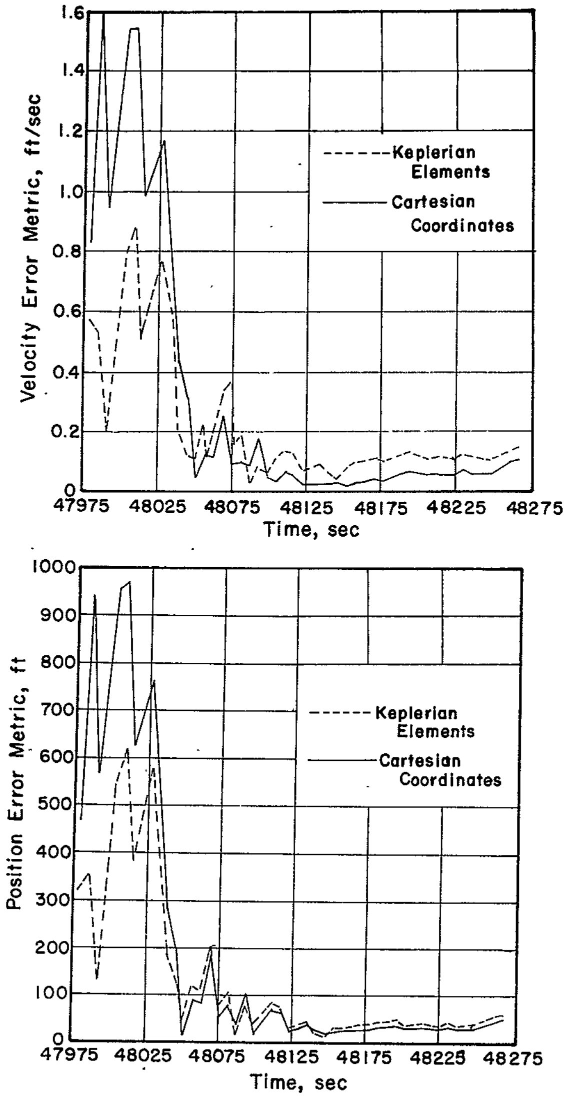
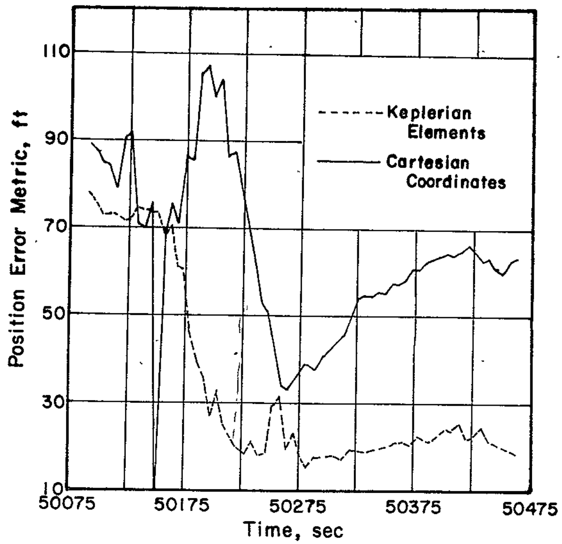


Figure 13 - Comparison of Position and Velocity Error Metrics for Keplerian and Cartesian Coordinates - Carnarvon Tracking Station



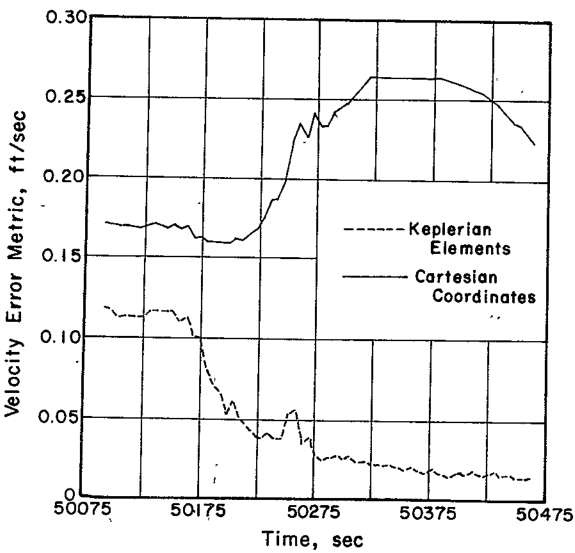


Figure 14 - Comparison of Position and Velocity Error Metrics for Keplerian and Cartesian Coordinates - Hawaii Tracking Station

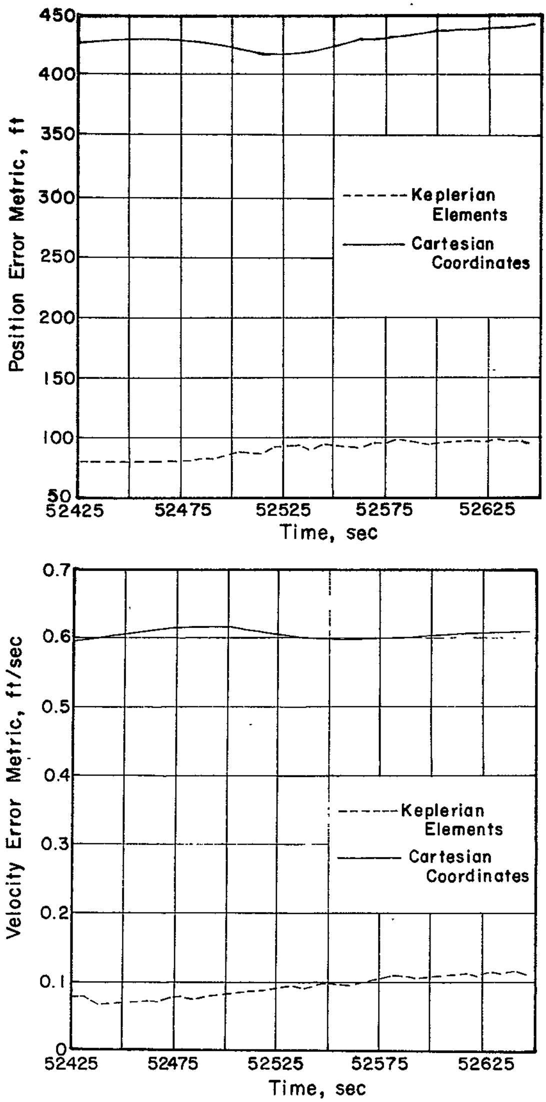


Figure 15 - Comparison of Position and Velocity Error Metrics for Keplerian and Cartesian Coordinates - White Sands Tracking Station

nonlinearities, the errors existing when the satellite left the Carnarvon station have multiplied into much larger errors when the satellite is picked up by the Hawaii station. However, even significant errors in the coordinates will be reflected as only small errors in the Keplerian elements and consequently they can be estimated more accurately. In order to improve the linearity assumption, the best estimate of the state was used to update the reference trajectory at the end of each tracking interval.

Equation (3-111), which gives the estimate for the observation error covariance matrix, R, was tested with the same sample problem considered for the comparison of the Keplerian elements and Cartesian coordinates. This estimate was incorporated into the estimation scheme through Eq.(3-113). It was found that the estimate of the coordinates was relatively insensitive to errors which increased the variance. However, a decrease in the variance reduces the value of P and causes premature saturation of the filter. This may be verified by examining Eq. (3-85). Hence, if the assumption is made that R is smaller than its true value, the estimator will be inaccurate since less emphasis will be placed on the data than should be. On the other hand, if R is selected to be larger than its true value, P will not saturate as soon, and the estimator will place more emphasis on the data and less on the previous best estimate of the state. Consequently, the maximum likelihood estimator of R was tested on a case where the initial value of R was chosen much larger than the true value, R_T. The true value is the value used to generate the observation data. Several cases were tested, and it was found that for this example R could be three or four times larger than R_T without altering the errors significantly. The results shown here are for R equal to 15 times R_T. The initial conditions were the same as those used for comparing the Keplerian elements and Cartesian coordinates except that the initial variances of the estimation error in position were chosen to be $.2 \times 10^5$ ft.^{2}, and the initial velocity variances were chosen as 10 ft.^{2}/sec.^{2}. This case was run also with the true value of R and with R assumed constant with a value 15 times larger than R_T.

Since each tracking station will have a different covariance matrix, this scheme was applied only to the Bermuda station, and the data was iterated to improve the estimate of R.

Figures (16) through (18) present the errors in the standard deviation of range, azimuth and elevation for five iterations of the data. As seen in Eq. (3-111) each correction to R is weighted by the factor $\frac{1}{N+1}$; therefore, as the number of observations becomes larger, the correction to R becomes smaller. This is reflected in the figures by the decreasing slope for each iteration.

Plots comparing the position and velocity error metrics for the first two iterations of the data are given in Figs. (19) and (20). The results are shown for R equal to R_T, for R fifteen times larger than R_T, and for R fifteen times larger than R_T initially, but sequentially estimated by Eqs. (3-111) and (3-113). These figures indicate that the sequential scheme for estimating R results in considerably smaller errors in position and velocity than obtained if R is held fixed with the initial error. A large value for R causes the estimator to place less emphasis on individual observations, hence the errors for $R = R_T$ are much more sensitive to the observations than for the other two cases. This is evident by the large variations in the errors from one observation to the next. For the second pass through the data the errors for $R = R_T$ do not vary nearly so much as they did during the first pass. After two passes through the data its information content has been exhausted, and subsequent iterations failed to reduce the errors.

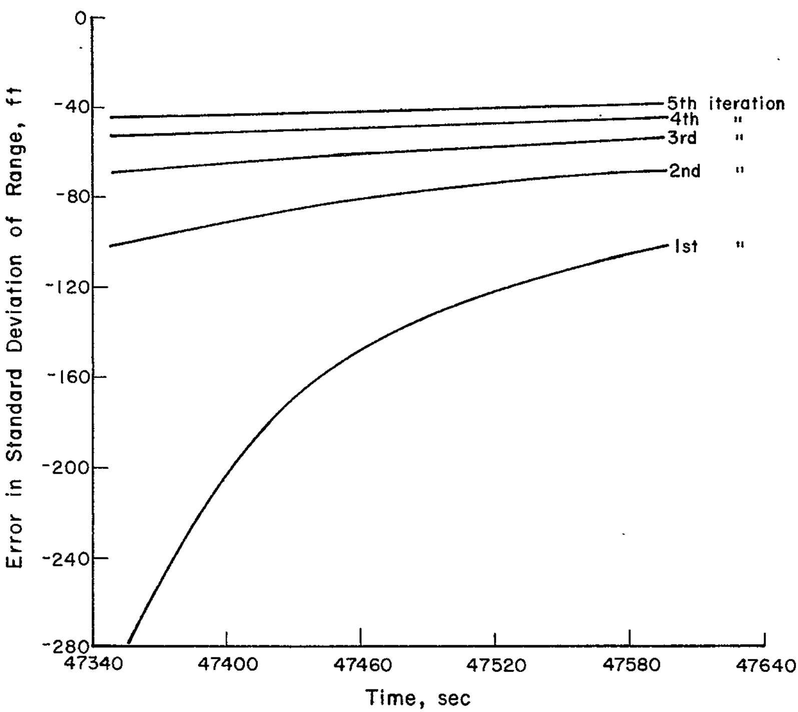


Figure 16 - Error in Standard Deviation of Range for Five Iterations of the Data

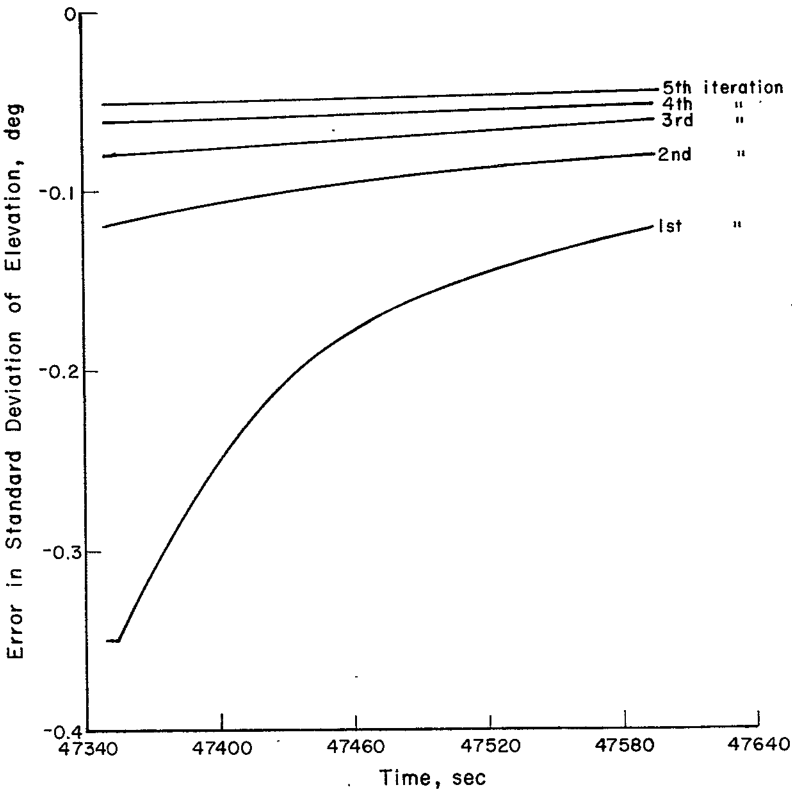


Figure 17 - Error in Standard Deviation of Elevation for Five Iterations of the Data

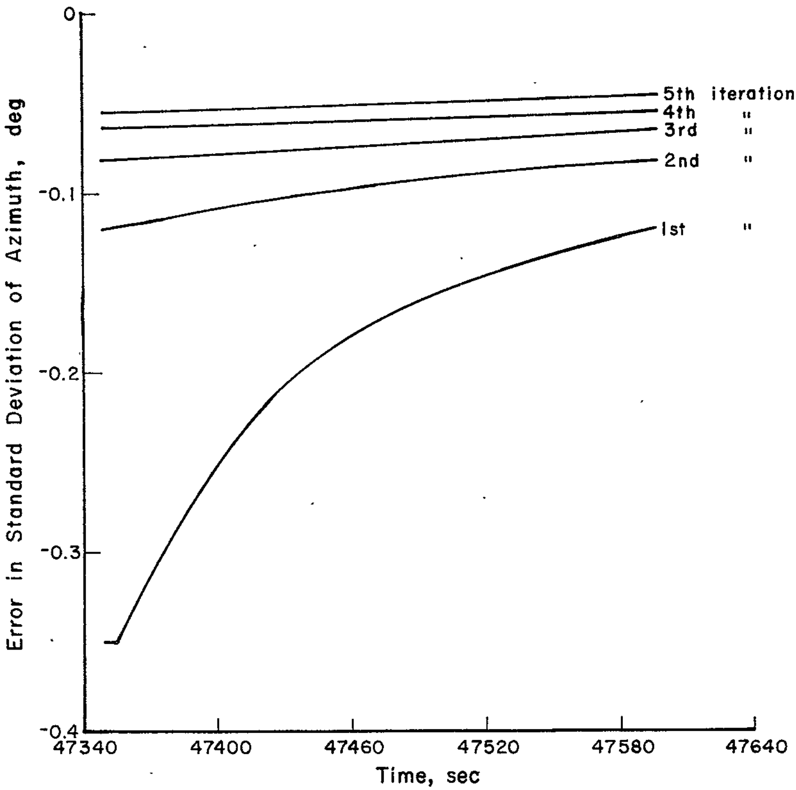


Figure 18 - Error in Standard Deviation of Azimuth for Five Iterations of the Data

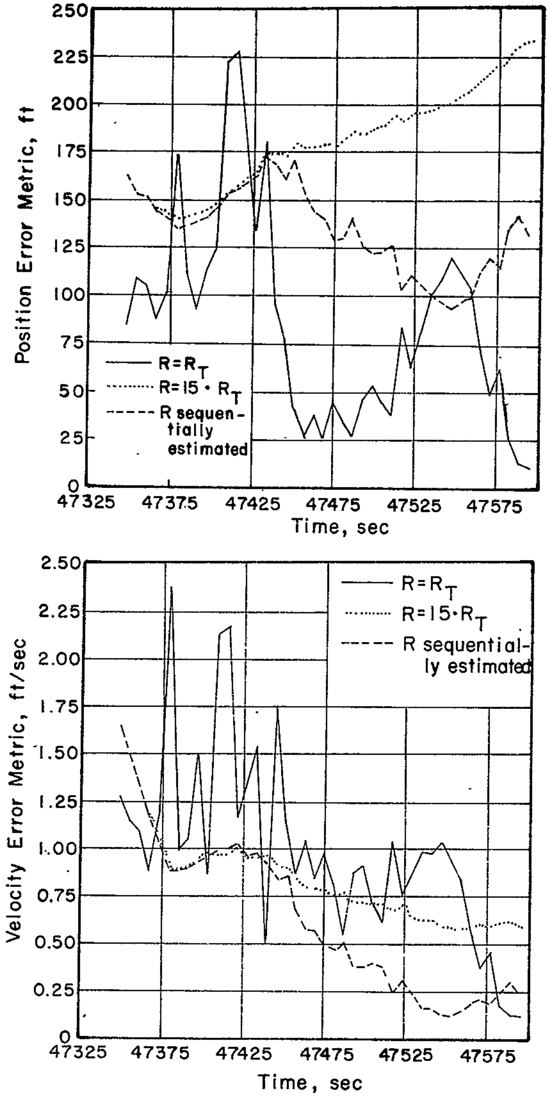
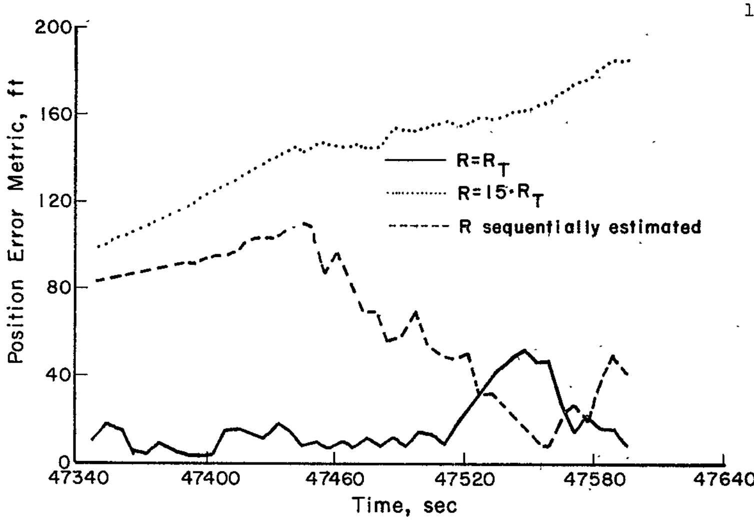


Figure 19 - Comparison of Position and Velocity Error Metrics for True, Incorrect and Sequentially Estimated Radar Covariance Matrix - First Iteration



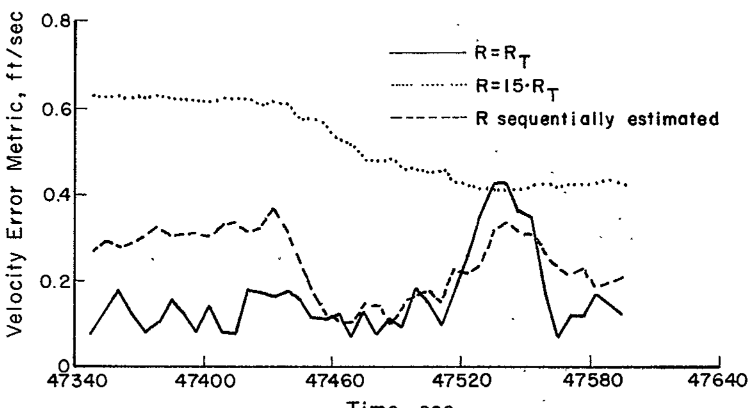


Figure 20 - Comparison of Position and Velocity Error Metrics for True, Incorrect and Sequentially Estimated Radar Covariance Matrix - Second Iteration

After each iteration of the data, the best estimate of the initial state is used to update the reference trajectory thereby improving the linearity assumption. The estimation error covariance matrix also is mapped back to the initial time by using Eq. (3-85c).

If the error in the initial estimate for R is known to be small, it may be desirable to place less emphasis on the corrections to R for the first few observations. This modification would give the sequential estimator an opportunity to stabilize its estimate of the state before attempting to use this estimate to correct the observation error covariance matrix. This could be accomplished by modifying the weighting function, $\frac{1}{N+1}$, by which the corrections are multiplied (see Eq. (3-111)). One suitable weighting function would be of the form

$$WT = \frac{(N-1)(N-2)\ \ldots\ (N-k)}{N^{k+1}}\ , \tag{3-116}$$

where k is an integer. This weighting function would cause the estimator to ignore the correction for the first k observations and would reduce the influence of the remaining corrections. For k small, this weighting function and $\frac{1}{N+1}$ approach the same value as N becomes large, i.e.,

$$\underset{N\to\infty}{L}\ \frac{1}{N+1} = \underset{N\to\infty}{L}\ \frac{(N-1)(N-2)\ \ldots\ (N-k)}{N^{k+1}} = \frac{1}{N} \tag{3-117}$$

Substituting Eq. (3-116) into (3-111) yields the estimator for R_{N+1}

$$\tilde{R}_{N+1} = (1-WT)\tilde{R}_N + WT[\tilde{v}_{N+1}\ \tilde{v}_{N+1}^T - H_{N+1}\ \overline{P}_{N+1}\ H_{N+1}^T] \tag{3-118}$$

The weighting function given by Eq. (3-116) places more or less emphasis on the initial conditions depending on the value chosen for k. For the results shown here k was chosen to be 2.

In a real time tracking situation it may not be feasible to iterate the data. In this instance the covariance estimation scheme would be applied each time the satellite crosses the station with the estimate of R from the previous pass over this station as the initial guess. The most useful application of this technique probably would be for post flight analysis where there is sufficient time to iterate the data.

The results presented thus far all were generated by using the sequential estimation scheme without state noise. A comparison of the results obtained from the sequential estimator and the batch processor was made. Under the same initial conditions both estimators gave the same results for a four station pass of the data. This is to be expected since all matrix operations were performed in double precision, and numerical difficulties should not cause divergence until many more observations have been processed.

As stated previously, estimation of the Keplerian elements offered a marked improvement over Cartesian coordinates for the example considered here. Because of the singularities associated with circular and low inclination orbits, the use of Keplerian elements is restricted. Therefore, it would be desirable if an estimation scheme utilizing the nonsingular elements were developed. This problem was considered. However, it appears as if there may be singularities associated with the mapping matrix, H, even for these nonsingular elements. It should be possible to eliminate this problem by rearranging and canceling terms in the elements of this matrix.

As an extension of the work associated with estimating the radar covariance matrix, it would be interesting to relax the assumption on the distribution of the variances as proposed by Smith and use an empirical Bayes estimator to estimate both the variances and their distribution functions. Empirical Bayes estimation(53) does not require knowledge of the distribution functions for the variances but utilizes the observation data to estimate their distribution functions. Of course, the Bayes estimator will yield better results if the variances are distributed as inverted gamma; however, there is no guarantee that this is true. If they are not distributed as inverted gamma, the empirical Bayes estimator may well give much better results than the Bayes estimator. It has been shown that even if the variances are distributed as assumed the empirical Bayes estimator asymptotically approaches the Bayes estimator(53).

Another area which needs additional study is the assumption of normally distributed observation errors. A test case was run in which the observation noise was assumed normally distributed in the estimator, but the noise in the observations was distributed uniformly with the variance assumed for the estimator. Based on this limited study, it appears that the estimator is relatively insensitive to the assumption of normally distributed observation noise.

CHAPTER IV
CONCLUSIONS AND RECOMMENDATIONS

This study has dealt with the derivation of a first order general perturbation solution and its application to the problem of orbit determination for a near-planetary satellite. A set of nonsingular orbit elements was used; therefore, the solution is valid for all elliptical motion. The perturbing force was assumed to be the gravitational force due to an arbitrarily shaped primary body and a point mass third body. Results from the general perturbation solution were compared with numerically integrated trajectories to determine their accuracy. Comparisons were made for both lunar and earth satellite orbits with a variety of initial conditions.

Based on an examination of the perturbation solutions and the numerical results, the following conclusions may be drawn:

1. Kaula's development of the disturbing function in terms of the Keplerian elements allows the primary and third body general perturbation solutions to be coupled through the secular rates in the angular quantities. Consequently, the solutions are very accurate for a period of a few days. However, first order long period effects in J_{20} as well as higher order effects which have not been considered cause accuracy to degenerate after longer time periods.

2. The similarity of the solutions for the primary and third body effects makes them amenable to computer evaluation.

3. An examination of the solutions for the third body effects reveals that long period terms with eccentricity divisors exist for the argument of

pericenter and mean anomaly for the second term in the third body disturbing function. Hence, inclusion of this term is important for near-circular orbits. This is verified by numerical results.

4. A systematic means of generating the mean elements for the perturbation solutions is proposed.

In Chapter III the general perturbation solutions developed in Chapter II were incorporated with an orbit determination program. Linear estimation theory was reviewed briefly, and a recursive scheme for estimating the observation error covariance matrix was derived and incorporated with the sequential estimator for the state. A comparison also was made of the accuracy of the resulting estimate when Keplerian elements and Cartesian coordinates are used to describe the motion. Based on the results presented in Chapter III, the following conclusions may be drawn:

1. Numeric partial differentiation, which is simpler to implement than integration of the differential equations for the state transition matrix, yields a state transition matrix which is in good agreement with the one obtained by numerical integration.

2. The orbit determination program utilizing the analytical solution of Chapter II yields results which compare very well to those generated by a program using numerical integration. The time required to execute the analytical program was approximately 1/5 of the time required by the program which used numerical integration to solve the same problem. Consequently, the analytical program is a valuable tool for doing preliminary design studies.

3. The estimate of the vehicle's state using the Keplerian elements is more accurate than the estimate obtained using the Cartesian coordinates. This was demonstrated by considering a specific example. The superiority of the elements becomes more pronounced with increasing time.

4. The recursive scheme for estimating the observation error covariance matrix was shown to give good results if a reasonable guess for the radar covariance is available to start the process.

5. The assumption of a normal distribution for the observation noise was tested by processing observation data which were uniformly distributed with the same variance which was assumed for the normal distribution. Based on this limited study it was concluded that the estimator is fairly insensitive to the assumption of normally distributed noise.

Some of the contributions of this study which the author believes to be original are:

1. Inclusion of the third body effect for a set of nonsingular orbit elements in the manner presented here is unique. Previous studies which included both the primary and third body disturbing functions have considered only long period and secular effects of the first term in the third body disturbing function. The first order solutions given here include short period, long period and secular effects for all terms in the third body expansion. Any one or any combination of these effects may be neglected by the selection of the proper values for the summation indices. Another advantage of this solution is that it provides for coupling of the primary and third body solutions through the secular rates of the angular variables.

2. The importance of the second term in the third body expansion for near-circular orbits had not been evaluated prior to this study.

3. Demonstration of the feasibility of using analytical solutions which contain a sophisticated perturbing force model in an orbit determination program is accomplished.

4. The recursive form of the maximum likelihood estimator for the observation error covariance matrix and its incorporation with the sequential estimator for the state is original.

RECOMMENDATIONS FOR FUTURE STUDY

It is believed that further study in the following areas would be profitable:

1. The general perturbation solutions should be extended to include first order long period effects of oblateness. Because of the algebra involved in generating these terms from a second order solution, the possibility of using Brouwer's[5] results should be considered. The second order solution for the third body effect also should be examined for first order long period terms proportional to the mass of the third body. The secular rate for the mean anomaly should be extended to include second order oblateness effects.

2. An error analysis should be made of the solutions to determine the importance of higher order effects.

3. The general perturbation solutions also could be extended to include effects of atmospheric drag and solar radiation pressure.

4. The feasibility of using the perturbation solutions as a basis for a modified Encke integration scheme should be examined. This would be of particular interest for a satellite lifetime program in which only long period and secular effects are included.

5. An estimation scheme utilizing the nonsingular orbit elements should be developed. Based on the results shown here for the classical elements, this would be of considerable value.

6. Extending the orbit determination program to allow processing of lunar satellite observation data would be worthwhile.

7. Relaxing the assumption of knowledge of the distribution of the observation error variance by using an empirical Bayes estimation scheme would be of value.

APPENDICES

APPENDIX A

The eccentricity function, $H_{\ell pq}(e)$, is presented here. The results are presented as a power series in the eccentricity for $\ell - 2p + q \neq 0$. For $\ell - 2p + q = 0$ the result is a closed form expression in β. The results were generated by using FORMAC. The quantity β is defined as $\beta = \frac{e}{1+\sqrt{1-e^2}}$.

ℓ	p	q	ℓ	p	q	$H_{\ell pq}(e)$
2	0	-2	2	2	2	$10\beta^2/(1+\beta^2)^2$
2	0	-1	2	2	1	$-3e + 1.625e^3 + .02604e^5 + \ldots$
2	0	0	2	2	0	$1 - 2.5e^2 + 1.4375e^4 - .22569e^6 + \ldots$
2	0	1	2	2	-1	$e - 2.375e^3 + 1.671875e^5 + \ldots$
2	0	2	2	2	-2	$e^2 - 2.5e^4 + 2.1042e^6 + \ldots$
2	1	-2	2	1	2	$-.25e^2 + .083333e^4 - .010412e^6 + \ldots$
2	1	-1	2	1	1	$-e + .125e^3 - .005208e^5 + \ldots$
			2	1	0	$\frac{1}{(1+\beta^2)^3}(1 + 9\beta^2 + 9\beta^4 + \beta^6)$
3	0	-2	3	3	2	$7.125e^2 - 4.0625e^4 - .01855e^6 + \ldots$
3	0	-1	3	3	1	$-4.5e + 8.25e^3 - 4.15625e^5 + \ldots$
3	0	0	3	3	0	$1 - 6e^2 + 9.23437e^4 - 5.453125e^6$
3	0	1	3	3	-1	$1.5e - 7.125e^3 + 11e^5 + \ldots$
3	0	2	3	3	-2	$1.875e^2 - 8.4375e^4 + 13.6475e^6 + \ldots$
3	1	-2	3	2	2	$1.375e^2 + 1.4583e^4 + .086263e^6 + \ldots$

ℓ	p	q	ℓ	p	q	$H_{\ell pq}^{(e)}$
3	1	-1	3	2	1	$\frac{-\beta}{(1+\beta^2)^4}\ (1 + 6\beta^2 + 6\beta^4 + \beta^6)$
3	1	0	3	2	0	$1 + 2e^2 - .640625e^4 - .064236e^3 + \ldots$
3	1	1	3	2	-1	$-.5e + e^3 - .36458e^5 + \ldots$
3	1	2	3	2	-2	$-.375e^2 + .6875e^4 - .34277e^6 + \ldots$
4	0	-2	4	4	2	$14e^2 - 22.83333e^4 + 10.6875e^6 + \ldots$
4	0	-1	4	4	1	$-6e + 23.25e^3 - 29.15625e^5 + \ldots$
4	0	0	4	4	0	$1 - 11e^2 + 31.625e^4 - 38.4444e^6 + \ldots$
4	0	1	4	4	-1	$2e - 15.75e^3 + 41.5521e^5 + \ldots$
4	0	2	4	4	-2	$3e^2 - 21e^4 + 54.375e^6 + \ldots$
4	1	-2	4	3	2	$\frac{21\beta^2}{(1+\beta^2)^5}\ [1 + 5\beta^2 + 5\beta^4 + \beta^6]$
4	1	-1	4	3	1	$-4e - 3e^3 + 1.645e^5 + \ldots$
4	1	0	4	3	0	$1 + e^2 - 2.6875e^4 + 97222e^6$
4	1	1	4	3	-1	$1.5e^3 - 2.25e^5 + \ldots$
4	1	2	4	3	-2	$-.25e^2 + 1.5417e^4 - 2.1719e^6 + \ldots$
4	2	-2	4	2	2	$.5e^2 - .583333e^4 + .125e^6 + \ldots$
4	2	-1	4	2	1	$-2e - 2.25e^3 + .19792e^5 + \ldots$
			4	2	0	$\frac{1}{(1+\beta^2)^5}\ [1+25\beta^2+100\beta^4+100\beta^6+25\beta^8+\beta^{10}]$

APPENDIX B

Recursive Relation for Hansen's Coefficients

Proof of a recursive relation for Hansen's coefficients which is not given in any of the references cited here is presented in this appendix.

The general expression for Hansen's coefficients is given by Eq. (A-13) of Ref. (18) as

$$X_i^{n,m} = \tfrac{1}{2}\pi \int_0^{2\pi} \left(\frac{r}{a}\right)^n x^m z^{-i}\, dM \qquad \text{(B-1)}$$

where

$$x = E^{jf}, \qquad z = e^{jM} \qquad \text{(B-2)}$$

and

E - base of natural logarithms

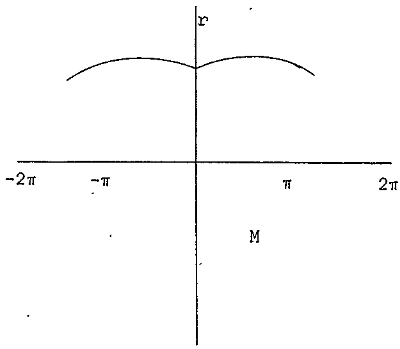
j $\sqrt{-1}$

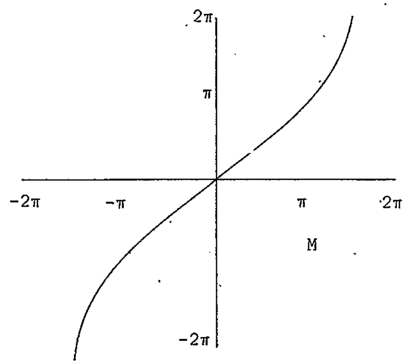
The notation is that of Ref. (18).

In order to perform the integration indicated in Eq. (B-1) it is necessary to express r and f as functions of M. For the purposes of this proof it is sufficient to indicate this functional dependence symbolically. In addition, it will be convenient to use the relations

$$\begin{aligned} r(M) &= r(-M) \\ f(-M) &= -f(M) , \end{aligned} \qquad \text{(B-3)}$$

i.e. $r(M)$ is an even function and $f(M)$ is an odd function of M. Eq. (B-3) may be proved intuitively by sketching r vs. M and f vs. M as follows





From these sketches it is seen that Eq. (B-3) is correct.

Substituting Eq. (B-2) into (B-1) yields

$$X_i^{n,m} = \frac{1}{2\pi}\int_0^{-2\pi} \left(\frac{r(M)}{a}\right)^n E^{jmf} E^{-jiM}\, dM \,. \tag{B-4}$$

Now, make the change of variables $M = -\tilde{M}$ in Eq. (B-4). Then

$$X_i^{n,m} = \frac{1}{2\pi}\int_0^{-2\pi} \left(\frac{r(-\tilde{M})}{a}\right)^n E^{j(mf(-\tilde{M})+i\tilde{M})}\, (-d\tilde{M}) \tag{B-5}$$

The limits of integration in Eq. (B-5) may be changed to $0 \to 2\pi$ by using the minus sign in $(-d\tilde{M})$ to reverse the limits on the integral. Since the integrand is periodic with period 2π, the integral $-2\pi \to 0$ is equal to the integral $0 \to 2\pi$. By using this information and Eq. (B-3), Eq. (B-5) may be written as

$$X_i^{n,m} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r(\tilde{M})}{a}\right)^n E^{j(-mf(\tilde{M})+i(\tilde{M}))} \, d\tilde{M} \; . \tag{B-6}$$

A comparison of Eq. (B-6) and Eq. (B-1), while keeping in mind that $\tilde{M}$ is a dummy variable, reveals that

$$X_i^{n,m} = X_{-i}^{n,-m} \tag{B-7}$$

The recursive relation (B-7) must be related to $G_{\ell pq}(e)$ for the primary body and $H_{\ell pq}(e)$ for the third body disturbing function. Equation (35) of Ref. (18) relates n , m , i and ℓ , p , q for the primary body,

$$i = \ell-2p+q \; , \quad m = \ell-2p \; , \quad n = -\ell-1 \tag{B-8}$$

Substituting these results into $X_i^{n,m}$ yields,

$$X_i^{n,m} = X_{\ell-2p+q}^{-\ell-1,\ell-2p} \equiv G_{\ell pq}(e) \tag{B-9}$$

Given a value of ℓ , p , and q , Eqs. (B-7) and B-8) are used to determine the set of indicies ℓ' , p' , q' which yield the same value for $G(e)$. The results are

$$\begin{aligned} \ell &= \ell' \\ \ell-2p &= -\ell'+2p' \\ \ell-2p+q &= -\ell'+2p'-q' \end{aligned} \tag{B-10}$$

Solving Eqs. (B-10) for ℓ' , p' , and q' yields

$$\ell' = \ell$$
$$p' = \ell - p \qquad \text{(B-11)}$$
$$q' = -q$$

Consider, for example, $G_{31-1}(e)$. According to Eq. (B-11)

$$\ell' = 3 \,, \quad p' = 2 \,, \quad q' = 1$$

Consequently,

$$G_{31-1}(e) = G_{321}(e)$$

Equation (B-11) also applies for $H_{\ell pq}(e)$

APPENDIX C

Transformation from Nonsingular Orbit Elements to Classical Orbit Elements and Cartesian Coordinates

Figure C-1 defines the classical orbital elements which orient the orbit with respect to the inertial Cartesian coordinates.

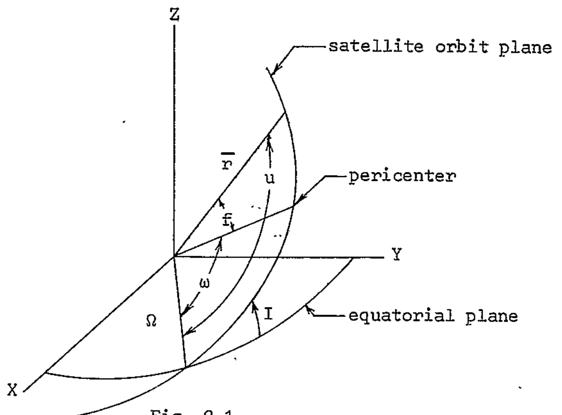


Fig. C-1
Orientation of the Orbit Plane

In Fig. C-1

- I - inclination
- Ω - longitude of ascending node
- ω - argument of pericenter
- f - true anomaly
- u - argument of latitude.

The transformation from classical elements to Cartesian coordinates is given by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r \begin{bmatrix} \sin u \cos \Omega - \sin u \sin \Omega \cos I \\ \cos u \sin \Omega + \sin u \cos \Omega \cos I \\ \sin u \sin I \end{bmatrix} \tag{C-1}$$

The transformation from nonsingular elements to classical elements is

$$e = \sqrt{A^2 + B^2} \tag{C-2}$$

$$I = \sin^{-1} \sqrt{h^2 + k^2} \tag{C-3}$$

$$\Omega = \tan^{-1} (\frac{h}{k}) \tag{C-4}$$

$$\omega = \tan^{-1} (\frac{A}{B}) - \tan^{-1} (\frac{h}{k}) \tag{C-5}$$

$$M = \delta - \tan^{-1} (\frac{A}{B}) \tag{C-6}$$

$$I = \cos^{-1} \gamma \tag{C-7}$$

where $\gamma = \cos I$.

By defining

$$u' = u + \Omega \ , \tag{C-8}$$

and using the trigonometric identities for the sine and cosine of the sum and difference of two angles, r may be written as

$$r = \frac{a \ (1 - \sqrt{A^2 + B^2}\ )}{1 + B \cos u' + A \sin u'} \tag{C-9}$$

Substituting Eqs. (C-2) through (C-9) into Eq. (C-1) yields the transformation from the nonsingular elements to Cartesian coordinates

$$\begin{bmatrix} X \\ \\ Y \\ \\ Z \end{bmatrix} = r \begin{bmatrix} \dfrac{k}{\sqrt{h^2 + k^2}} \cos u - \dfrac{h}{\sqrt{h^2 + k^2}} \sqrt{1 - h^2 + k^2} \sin u \\ \dfrac{h}{\sqrt{h^2 + k^2}} \cos u + \dfrac{k}{\sqrt{h^2 + k^2}} \sqrt{1 - h^2 + k^2} \sin u \\ \sqrt{h^2 + k^2} \sin u \end{bmatrix} \tag{C-10}$$

If desired, $\cos u$ and $\sin u$ may be written as

$$\begin{aligned} \cos u &= \frac{1}{\sqrt{h^2 + k^2}} (k \cos u' + h \sin u') \\ \sin u &= \frac{1}{\sqrt{h^2 + k^2}} (h \sin u' - k \cos u') \end{aligned} \tag{C-11}$$

In Eq. (C-10) the positive root of $\sqrt{h^2 + k^2}$ always is used, while the sign of $\sqrt{1 - h^2 + k^2}$ is chosen to be the same as that of γ .

APPENDIX D

Generation of the Mapping Matrix for Cartesian Coordinates

The mapping matrix for an observation vector consisting of range, R, elevation, E, and azimuth, A, will be outlined in this appendix.

The tracker coordinate system, (x_t, y_t, z_t) is a topocentric system with the x_t axis directed along the radius vector of the tracking station, its z_t axis directed north, and the y_t axis directed east. In this system R, E and A are given by

$$R = \sqrt{x_t^2 + y_t^2 + z_t^2} \tag{D-1}$$

$$E = \cos^{-1} \frac{\sqrt{y_t^2 + z_t^2}}{R} \tag{D-2}$$

$$A = \tan^{-1}\left(\frac{y_t}{z_t}\right) \tag{D-3}$$

Solving Eqs. (D-1) through (D-3) for x_t, y_t and z_t yields

$$x_t = R \sin E \tag{D-4}$$

$$y_t = R \cos E \sin A \tag{D-5}$$

$$z_t = R \cos E \cos A \tag{D-6}$$

Now, Eqs. (D-4) through (D-6) are expanded in a Taylor's series about the nominal trajectory resulting in

$$\begin{bmatrix} dx_t \\ dy_t \\ dz_t \end{bmatrix} = \begin{bmatrix} \sin E^* & \cos E^* & 0 \\ \cos E^* \sin A^* & -\sin E^* \sin A^* & \cos A^* \\ \cos E^* \cos A^* & -\sin E^* \cos A^* & -\sin A^* \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & R^* & 0 \\ 0 & 0 & R^* \cos E^* \end{bmatrix} \cdot \begin{bmatrix} dR \\ dE \\ dA \end{bmatrix} \tag{D-7}$$

The first matrix to the right of the equal sign is designated as M . Now, the product of the last two matrices will be defined as the observation vector. Then,

$$\begin{bmatrix} dR \\ RdE \\ R \cos E \, dA \end{bmatrix} = M^T \begin{bmatrix} dx_t \\ dy_t \\ dz_t \end{bmatrix} \tag{D-8}$$

The desired state vector is the Cartesian coordinates of the vehicle measured in an inertial geocemtric reference frame. Hence the vector

$$\begin{bmatrix} dx_t \\ dy_t \\ dz_t \end{bmatrix}$$

must be augmented to include the velocities and then transformed to yield the inertial coordinates. It is easily shown that[54]

$$\begin{bmatrix} dx_t \\ dy_t \\ dz_t \end{bmatrix} = \Delta\Phi G \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{D-9}$$

Here $\Delta\phi$ is the local vertical correction given by

$$\Delta\phi = \begin{bmatrix} \cos\Delta\phi & 0 & \sin\Delta\phi \\ 0 & 1 & 0 \\ -\sin\Delta\phi & 0 & \cos\Delta\phi \end{bmatrix} \quad \text{(D-10)}$$

and $\Delta\phi$ is the difference between the geodetic and geocentric latitude. The matrix, G , is the transformation matrix relating the geocentric inertial coordinates to the topocentric tracker system and is defined as

$$G = \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \end{bmatrix} \quad \text{(D-11)}$$

where

ϕ - latitude of the tracking station

λ - Greenwich hour angle plus station longitude.

The mapping matrix now is defined as

$$H = [M^T \Delta\phi G \,\vdots\, \phi] \quad \text{(D-12)}$$

The quantity ϕ is a 3×3 null matrix relating the observations to the velocity components.

APPENDIX E

Generation of the Mapping Matrix for the Classical Orbit Elements

The mapping matrix relating the observations to the state vector for the classical orbit elements is derived in this appendix.

Consider the classical orbit elements described in Fig. C-1 of Appendix C. They are related to the satellite's inertial Cartesian coordinates by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r \begin{bmatrix} \cos u \cos \Omega - \sin u \sin \Omega \cos I \\ \cos u \sin \Omega + \sin u \cos I \cos \Omega \\ \sin u \sin I \end{bmatrix} \qquad \text{(E-1)}$$

The H matrix is defined by

$$H = \begin{bmatrix} \frac{\partial R}{\partial a} & \frac{\partial R}{\partial e} & \cdots\cdots & \frac{\partial R}{\partial M} \\ \frac{\partial E}{\partial a} & \frac{\partial E}{\partial e} & \cdots\cdots & \frac{\partial E}{\partial M} \\ \frac{\partial A}{\partial a} & \frac{\partial A}{\partial e} & \cdots\cdots & \frac{\partial A}{\partial M} \end{bmatrix} \qquad \text{(E-2)}$$

Now,

$$R = (x_t^2 + y_t^2 + z_t^2)^{\frac{1}{2}} \qquad \text{(E-3)}$$

If ε represents one of the orbit elements, then

$$\begin{aligned} \frac{\partial R}{\partial \varepsilon} &= \frac{\partial R}{\partial x_t} \frac{\partial x_t}{\partial \varepsilon} + \frac{\partial R}{\partial y_t} \frac{\partial y_t}{\partial \varepsilon} + \frac{\partial R}{\partial z_t} \frac{\partial z_t}{\partial \varepsilon} \\ &= \frac{x_t}{R} \frac{\partial x_t}{\partial \varepsilon} + \frac{y_t}{R} \frac{\partial y_t}{\partial \varepsilon} + \frac{z_t}{R} \frac{\partial z_t}{\partial \varepsilon} \end{aligned} \qquad \text{(E-4)}$$

Similar equations may be written for A and E . Using Eq. (E-9), it is seen that

$$\frac{\partial}{\partial E}\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \Delta\Phi G \, \frac{\partial}{\partial E}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{E-5}$$

From Eq. (E-1) it may be shown that

$$\frac{\partial X}{\partial a} = \frac{X}{r}\frac{\partial r}{\partial a} \tag{E-6}$$

$$\frac{\partial x}{\partial e} = \frac{X}{r}\frac{\partial r}{\partial e} - (\sin u \cos \Omega + \cos u \sin \Omega \cos I) r \frac{\partial u}{\partial e} \tag{E-7}$$

$$\frac{\partial X}{\partial I} = Z \sin \Omega \tag{E-8}$$

$$\frac{\partial X}{\partial \Omega} = -Y \tag{E-9}$$

$$\frac{\partial X}{\partial \omega} = -r \{\cos \Omega \sin u + \cos u \sin \Omega \cos I\} \tag{E-10}$$

$$\frac{\partial X}{\partial M} = \frac{X}{r}\frac{\partial r}{\partial M} - r \{\cos \Omega \sin u + \cos u \sin \Omega \cos I\} \frac{\partial u}{\partial M} \tag{E-11}$$

$$\frac{\partial Y}{\partial a} = \frac{Y}{r}\frac{\partial r}{\partial a} \tag{E-12}$$

$$\frac{\partial Y}{\partial e} = \frac{Y}{r}\frac{\partial r}{\partial e} + r (-\sin u \sin \Omega + \cos u \cos I \cos \Omega) \frac{\partial u}{\partial e} \tag{E-13}$$

$$\frac{\partial Y}{\partial I} = -Z \cos \Omega \tag{E-14}$$

$$\frac{\partial Y}{\partial \Omega} = X \tag{E-15}$$

$$\frac{\partial Y}{\partial \omega} = r (-\sin u \sin \Omega + \cos u \cos I \cos \Omega) \tag{E-16}$$

$$\frac{\partial Y}{\partial M} = \frac{Y}{r}\frac{\partial r}{\partial M} + r (-\sin u \sin \Omega + \cos u \cos I \cos \Omega) \frac{\partial u}{\partial M} \tag{E-17}$$

$$\frac{\partial Z}{\partial a} = \frac{Z}{r}\frac{\partial r}{\partial a} \tag{E-18}$$

$$\frac{\partial Z}{\partial e} = \frac{Z}{r}\frac{\partial r}{\partial e} + r \cos u \sin I \frac{\partial u}{\partial e} \tag{E-19}$$

$$\frac{\partial Z}{\partial I} = r \sin u \cos I \tag{E-20}$$

$$\frac{\partial Z}{\partial \Omega} = 0 \tag{E-21}$$

$$\frac{\partial Z}{\partial \omega} = r \cos u \sin I \tag{E-22}$$

$$\frac{\partial Z}{\partial M} = \frac{Z}{r}\frac{\partial r}{\partial M} + r \cos u \sin I \frac{\partial u}{\partial M} \tag{E-23}$$

The partials of r and u with respect to the orbit elements are given by

$$\frac{\partial r}{\partial a} = r/a \tag{E-24}$$

$$\frac{\partial r}{\partial e} = -a \cos f \tag{E-25}$$

$$\frac{\partial r}{\partial M} = \frac{a\ e \sin f}{(1 - e^2)^{\frac{1}{2}}} \tag{E-26}$$

$$\frac{\partial u}{\partial M} = \frac{a(1 + e \cos f)}{r(1 - e^2)^{\frac{1}{2}}} \tag{E-27}$$

$$\frac{\partial u}{\partial e} = \frac{(2 + e \cos f)\sin f}{(1 - e^2)} \tag{E-28}$$

From the expressions for R, A and E given in Appendix D, it may be shown that

$$\begin{bmatrix} \frac{\partial R}{\partial x_t} & \frac{\partial R}{\partial y_t} & \frac{\partial R}{\partial z_t} \end{bmatrix} = \begin{bmatrix} \frac{x_t}{R} & \frac{y_t}{R} & \frac{z_t}{R} \end{bmatrix} \tag{E-29}$$

$$\begin{bmatrix} \frac{\partial E}{\partial x_t} & \frac{\partial E}{\partial y_t} & \frac{\partial E}{\partial z_t} \end{bmatrix} = \begin{bmatrix} \frac{\cos E}{R} & -\frac{\sin E \sin A}{R} & -\frac{\cos A \sin E}{R} \end{bmatrix} \tag{E-30}$$

$$\begin{bmatrix} \frac{\partial A}{\partial x_t} & \frac{\partial A}{\partial y_t} & \frac{\partial A}{\partial z_t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{z_t}{y_t^2 + z_t^2} & -\frac{y_t}{y_t^2 + z_t^2} \end{bmatrix} \tag{E-31}$$

Using Eqs. (E-2) and (E-4) the expression for H becomes

$$H = \begin{bmatrix} \dfrac{\partial R}{\partial x_t} & \dfrac{\partial R}{\partial y_t} & \dfrac{\partial R}{\partial z_t} \\ \\ \dfrac{\partial E}{\partial x_t} & \dfrac{\partial E}{\partial y_t} & \dfrac{\partial E}{\partial z_t} \\ \\ \dfrac{\partial A}{\partial x_t} & \dfrac{\partial A}{\partial y_t} & \dfrac{\partial A}{\partial z_t} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_t}{\partial a} & \dfrac{\partial x_t}{\partial e} & \cdots\cdots & \dfrac{\partial x_t}{\partial M} \\ \\ \dfrac{\partial y_t}{\partial a} & \dfrac{\partial y_t}{\partial e} & \cdots\cdots & \dfrac{\partial y_t}{\partial M} \\ \\ \dfrac{\partial z_t}{\partial a} & \dfrac{\partial z_t}{\partial e} & \cdots\cdots & \dfrac{\partial z_t}{\partial M} \end{bmatrix}$$

The first matrix is given by Eq. (E-19) through (E-31) and the second by Eqs. (E-6) through (E-28).

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