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MANNED SPACECRAFT CENTER
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PROJECT APOLLO

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# SENSITIVITY OF EARTH AND LUNAR ORBITS TO VELOCITY CHANGES AT PERIAPSIS AND APOAPSIS

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## SUMMARY

This note presents maneuver-sensitivity coefficients for various orbits around both the earth and the moon. The sensitivity coefficient represents the change in altitude ($\Delta h$) which occurs 180° of orbital travel after a horizontal delta velocity ($\Delta V$) maneuver of 1 fps performed at either periapsis or apoapsis. The sensitivity coefficient $\left(\frac{\Delta h}{\Delta V}\right)$ is a variable ratio and can be expressed as a function of the apoapsis and periapsis altitudes of a particular orbit; such functions are developed herein.

## INTRODUCTION

The maneuver-sensitivity coefficients in general use for mission planning of earth and lunar orbits are 0.56 n.mi./fps and 0.72 n.mi./fps, respectively. The coefficient being used for earth orbits is correct for a 125-n.mi. circular orbit and is accurate within 10 percent for circular earth orbits up to approximately 400-n.mi. Similarly, the lunar coefficient is correct for a 30-n.mi. circular orbit and is accurate within 10 percent for circular lunar orbits up to approximately 100-n.mi.

For high apoapsis orbits, these two sensitivity coefficients are no longer suitable. For example, the maneuver-sensitivity coefficient for a 2250-n.mi. circular earth orbit is actually 1.12 n.mi./fps or twice the coefficient for a 125-n.mi. circular orbit.

Since the sensitivity coefficients in this note consist of differential changes in altitude and velocity, they represent the effects of impulsive type $\Delta V$ maneuvers. However, they may be used to closely approximate effects of maneuvers with relatively short burn times. It should be noted that the sensitivity coefficients are derived from two-body equations and do not reflect oblateness or sun, moon (for earth coefficients) or earth (for moon coefficients) perturbations. Proper allowances should be made for such factors when using these sensitivity coefficients.

The enclosed figures give accurate maneuver-sensitivity coefficients for various earth and moon orbits.

## SYMBOLS

| | |
|---|---|
| $V$ | orbital velocity |
| $r$ | orbital radius from center of central body |
| $a$ | semimajor axis |
| $\mu$ | gravitational constant |
| $h$ | altitude |
| $\Delta V$ | delta velocity |
| $R$ | radius of central body |

### Subscripts

| | |
|---|---|
| a | apoapsis |
| p | periapsis |
| c | circular |
| e | earth |
| m | moon |

## ANALYSIS

Equations expressing the horizontal maneuver-sensitivity coefficients for the earth and the moon as functions of the apoapsis and periapsis altitudes can be developed starting with the vis-viva integral,

$$V^2 = \mu \left[\frac{2}{r} - \frac{1}{a}\right]$$

Assuming that there are differential changes in velocity and the semimajor axis and that $r$ is a constant, then

$$2V \, \Delta V = \frac{\mu}{a^2} \, \Delta a$$

or

$$\frac{\Delta a}{\Delta V} = \frac{2Va^2}{\mu} .$$

Now, knowing that

$$a = \frac{r_p + r_a}{2}$$

a differential change in $a$ is given by

$$\Delta a = \frac{\Delta r_a}{2}$$

if $r_p$ is a constant, and similarly,

$$\Delta a = \frac{\Delta r_p}{2}$$

if $r_a$ is a constant.

For a horizontal $\Delta V$ maneuver performed at periapsis, $r_p$ is a constant and

$$\frac{\Delta r_a}{2 \, \Delta V_p} = \frac{2V_p a^2}{\mu}$$

or

$$\frac{\Delta r_a}{\Delta V_p} = \frac{4V_p a^2}{\mu}$$

where $V_p$ is the velocity at periapsis before any maneuver.

This result is given in reference 1. Likewise, for a horizontal $\Delta V$ maneuver performed at apoapsis, $r_a$ is a constant and

$$\frac{\Delta r_p}{2\ \Delta V_a} = \frac{2V_a a^2}{\mu}$$

or

$$\frac{\Delta r_p}{\Delta V_a} = \frac{4V_a a^2}{\mu}$$

where $V_a$ is the velocity at apoapsis before any maneuver. Then since $\Delta r_a = \Delta h_a$ and $\Delta r_p = \Delta h_p$ the two preceding equations can finally be written as

$$\frac{\Delta h_a}{\Delta V_p} = \frac{4V_p a^2}{\mu} \qquad (1)$$

and

$$\frac{\Delta h_p}{\Delta V_a} = \frac{4V_a a^2}{\mu} \qquad (2)$$

where

$$\frac{\Delta h_a}{\Delta V_p}$$ = sensitivity coefficient for a horizontal $\Delta V$ maneuver performed at periapsis; the ratio between the change in apoapsis altitude and the preceding change in velocity at periapsis

and

$$\frac{\Delta h_p}{\Delta V_a}$$ = sensitivity coefficient for a horizontal $\Delta V$ maneuver performed at apoapsis; the ratio between the change in periapsis altitude and the preceding change in velocity at apoapsis.

Equations (1) and (2) were used to obtain the data presented in the accompanying figures. However, it is not always convenient to calculate the sensitivity coefficients from equations (1) and (2). For mission planning, it is useful to have the sensitivity equations reduced in terms of the orbital parameters $h_a$ and $h_p$.

Going back to the vis-viva integral, $V_p$ can be expressed as

$$V_p = \sqrt{\mu \left[\frac{2}{r_p} - \frac{1}{a}\right]}$$

Substituting in equation (1),

$$\frac{\Delta h_a}{\Delta V_p} = \frac{4a^2}{\mu} \sqrt{\mu \left[\frac{2}{r_p} - \frac{1}{a}\right]}$$

and recombining terms,

$$\frac{\Delta h_a}{\Delta V_p} = 4\sqrt{\frac{a^3}{\mu}\left[\frac{2a - r_p}{r_p}\right]}$$

Now, since $r_p = R + h_p$, $r_a = R + h_a$ and $a = \dfrac{2R + h_a + h_p}{2}$

the above equation becomes

$$\frac{\Delta h_a}{\Delta V_p} = 4\sqrt{\frac{1}{\mu}\left(\frac{2R + h_a + h_p}{2}\right)^3 \left(\frac{R + h_a}{R + h_p}\right)}$$

Then by rearranging, we obtain

$$\frac{\Delta h_a}{\Delta V_p} = \sqrt{\frac{2}{\mu}\cdot\left(2R + h_a + h_p\right)^3 \left(\frac{R + h_a}{R + h_p}\right)}$$

For the earth,

$$\mu_e = 62750.595 \text{ n.mi.}^3/\text{sec}^2$$

$$R_e = 3441.3306 \text{ n. mi.}$$

By substituting the values into the preceding equation, it becomes

$$\frac{\Delta h_a}{\Delta V_p} = 5.645548 \times 10^{-3} \sqrt{\left(6882 + h_a + h_p\right)^3 \left(\frac{3441 + h_a}{3441 + h_p}\right)}$$

where $\frac{\Delta h_a}{\Delta V_p}$ is the sensitivity coefficient for the earth in seconds.

Dividing the right side of the preceding equation by 6076.115486 ft/n.mi. to get the proper units, the sensitivity equation for any given earth orbit becomes:

$$\frac{\Delta h_a}{\Delta V_p} = 9.29138 \times 10^{-7} \sqrt{\left(\frac{3441 + h_a}{3441 + h_p}\right) \left(6882 + h_a + h_p\right)^3}$$

where $\frac{\Delta h_a}{\Delta V_p}$ = sensitivity coefficient for the earth in n.mi./fps; the ratio between the change in apogee altitude and the preceding change in velocity at perigee.

For the moon,

$$\mu_m = 771.8258 \text{ n.mi.}^3/\text{sec}^2$$

$$R_m = 938.4935 \text{ n.mi.}$$

and the sensitivity coefficient for the moon in the proper units becomes

$$\frac{\Delta h_a}{\Delta V_p} = 8.378 \times 10^{-6} \sqrt{\left(\frac{938.5 + h_a}{938.5 + h_p}\right) \left(1877 + h_a + h_p\right)^3}$$

where $\frac{\Delta h_a}{\Delta V_p}$ = sensitivity coefficient for the moon in n.mi./fps; the ratio between the change in apocynthion altitude and the preceding change in velocity at pericynthion.

A similar development, starting with equation (2) gives for any earth orbit

$$\frac{\Delta h_p}{\Delta V_a} = 9.29138 \times 10^{-7}\sqrt{\left(\frac{3441 + h_p}{3441 + h_a}\right)\left(6882 + h_a + h_p\right)^3}$$

where $\frac{\Delta h_p}{\Delta V_a}$ = sensitivity coefficient for the moon in n.mi/fps; the ratio between the change in pericynthion altitude and the preceding change in velocity at apocynthion.

and for any given moon orbit

$$\frac{\Delta h_p}{\Delta V_a} = 8.378 \times 10^{-6}\sqrt{\left(\frac{938.5 + h_p}{938.5 + h_a}\right)\left(1877 + h_a + h_p\right)^3}$$

For the special case of a circular orbit, the above equations reduce to

$$\frac{\Delta h}{\Delta V} = 4\sqrt{\frac{1}{\mu}\left(r_e + h_c\right)^3}$$

Therefore, given any circular earth orbit,

$$\frac{\Delta h}{\Delta V} = 2.628 \times 10^{-6}\sqrt{\left(3441.3 + h_c\right)^3}$$

where $\frac{\Delta h}{\Delta V}$ = sensitivity coefficient for the earth in n.mi./fps.

Finally, for a circular lunar orbit

$$\frac{\Delta h}{\Delta V} = 2.3696 \times 10^{-5}\sqrt{\left(938.5 + h_c\right)^3}$$

where $\frac{\Delta h}{\Delta V}$ = sensitivity coefficient for the moon in n.mi./fps.

The change in altitude for any of the six above equations occurs 180° of orbital travel after a horizontal $\Delta V$ maneuver. All six equations are summarized in table I.

## RESULTS

Figures 1 and 2 present horizontal maneuver-sensitivity coefficients for various earth orbits. Figure 1 shows the change in altitude which occurs 180° of orbital travel away from the point of application of a 1-fps horizontal $\Delta V$ maneuver as a function of the altitude above a 3441-n.mi. earth radius for circular earth orbits. Figure 2 presents two lines; one shows the change in perigee altitude for a horizontal $\Delta V$ of 1-fps applied at perigee. The sensitivity coefficients in figure 2 are given as functions of the height of apogee; the perigee altitude is a constant 100 n.mi.

Figure 3 is similar to figure 1 except that the sensitivity coefficients are for lunar orbits. Figure 4 is also similar to figure 2 except that the sensitivity coefficients are for lunar orbits and the pericynthion altitude is a constant 60 n.mi.

Tables I(a) and (b) list the horizontal sensitivity coefficients as functions of $h_a$, $h_p$, and $h_c$ for the earth and the moon, respectively.

### Use of Coefficients

As an example of the use of the maneuver-sensitivity coefficients, suppose a spacecraft is in a 100- by 7000-n.mi. earth orbit, and it is desired to lower the perigee 65 n.mi. by performing a horizontal $\Delta V$ maneuver at apogee. Referring to figure 2, it is seen that $\frac{\Delta h_p}{\Delta V_a}$ is approximately 0.90 n.mi./fps for this orbit. That is, for each 1-fps $\Delta V$ applied horizontally at apogee, the perigee altitude will change 0.9 n.mi. Dividing 65 nautical miles by 0.9 n.mi./fps, it is found that approximately 72 fps horizontal $\Delta V$ is needed to lower the perigee altitude by the required amount.

The sensitivity equations can also be used to predict the effects of $\Delta V$ on orbital parameters. Suppose a spacecraft is in a 110-n.mi.

circular lunar orbit, and a retrograde horizontal maneuver of 5-fps $\Delta V$ is applied at a certain point in the orbit. Referring to figure 3, it is seen that $\frac{\Delta h}{\Delta V}$ is approximately 0.80-n.mi./fps for a 110-n.mi. circular lunar orbit. Thus, by multiplying 5 fps by the sensitivity coefficient, 0.8-n.mi./fps, it is found that the perigee altitude will drop 4 n.mi. and the spacecraft will then be in a 106- by 110-n.mi. lunar orbit.

Finally, if the sensitivity coefficients in figures 1 and 3 (circular orbits) are divided by 4, they give precise values for radial (vertical) $\Delta V$ maneuvers; in this case the change in altitude occurs only 90° of orbital travel after the maneuver, as described in reference 2.

## CONCLUDING REMARKS

Use of the attached figures for the calculations of $\Delta V$ maneuvers or the effects of $\Delta V$ maneuvers will give much better results than the use of a constant-value sensitivity coefficient. If a sensitivity coefficient is needed for an orbit not represented in the attached figures, it can readily be computed using the equations in tables 1 and 2. Division of any horizontal maneuver sensitivity coefficient by 4 will give the correct coefficient for vertical $\Delta V$ maneuvers where the altitude change occurs 90° of orbital travel after the maneuver. For horizontal $\Delta V$ maneuvers, the altitude change occurs 180° of orbital travel after the maneuver. The sensitivity coefficients presented in this note have been calculated assuming that there are no third-body perturbations, that changes in velocity are imparted instantaneously, and that the central body is spherical.

TABLE I.- MANEUVER-SENSITIVITY COEFFICIENTS

(a) Earth orbital maneuvers

$$\frac{\Delta h_c}{\Delta V} = 2.628 \times 10^{-6} \sqrt{\left(3441.3 + h_c\right)^3}$$

$$\frac{\Delta h_a}{\Delta V_p} = 9.29138 \times 10^{-7} \sqrt{\left(\frac{3441.3 + h_a}{3441.3 + h_p}\right)\left(6882.6 + h_a + h_p\right)^3}$$

$$\frac{\Delta h_p}{\Delta V_a} = 9.29138 \times 10^{-7} \sqrt{\left(\frac{3441.3 + h_p}{3441.3 + h_a}\right)\left(6882.6 + h_a + h_p\right)^3}$$

(b) Lunar orbital maneuvers

$$\frac{\Delta h_c}{\Delta V} = 2.3696 \times 10^{-5} \sqrt{\left(938.5 + h_c\right)^3}$$

$$\frac{\Delta h_a}{\Delta V_p} = 8.378 \times 10^{-6} \sqrt{\left(\frac{938.5 + h_a}{938.5 + h_p}\right)\left(1877 + h_a + h_p\right)^3}$$

$$\frac{\Delta h_p}{\Delta V_c} = 8.378 \times 10^{-6} \sqrt{\left(\frac{938.5 + h_p}{938.5 + h_a}\right)\left(1877 + h_a + h_p\right)^3}$$

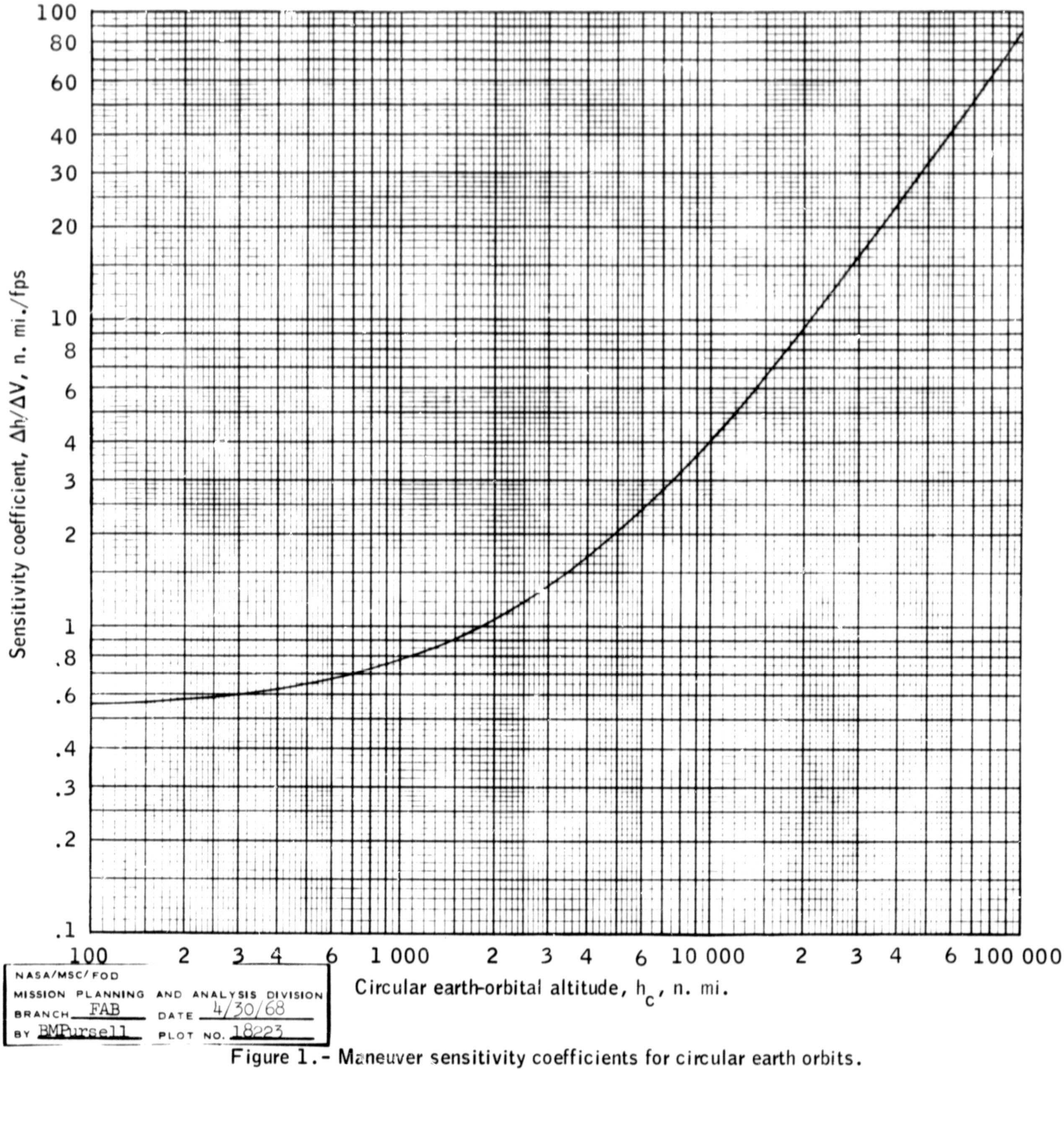


Figure 1.- Maneuver sensitivity coefficients for circular earth orbits.

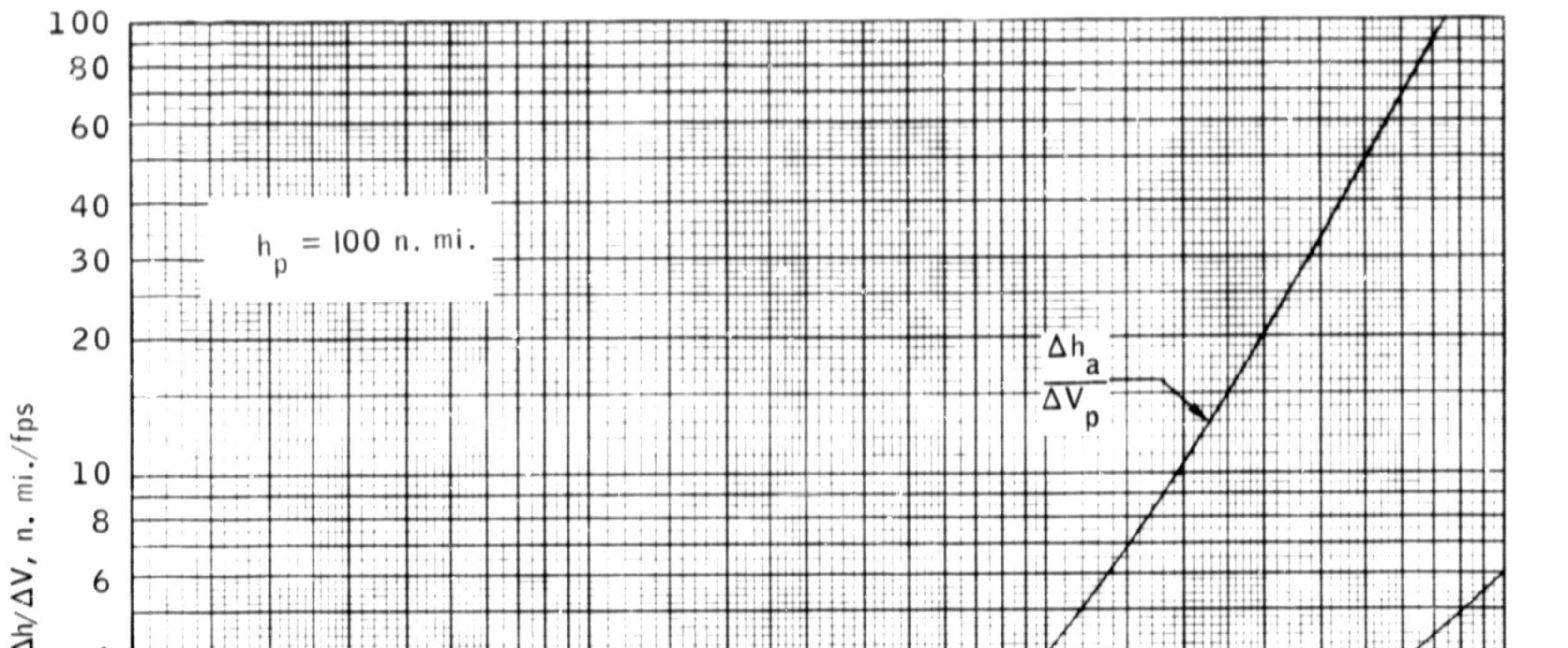


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Figure 2.- Maneuver sensitivity coefficients for elliptical earth orbits.

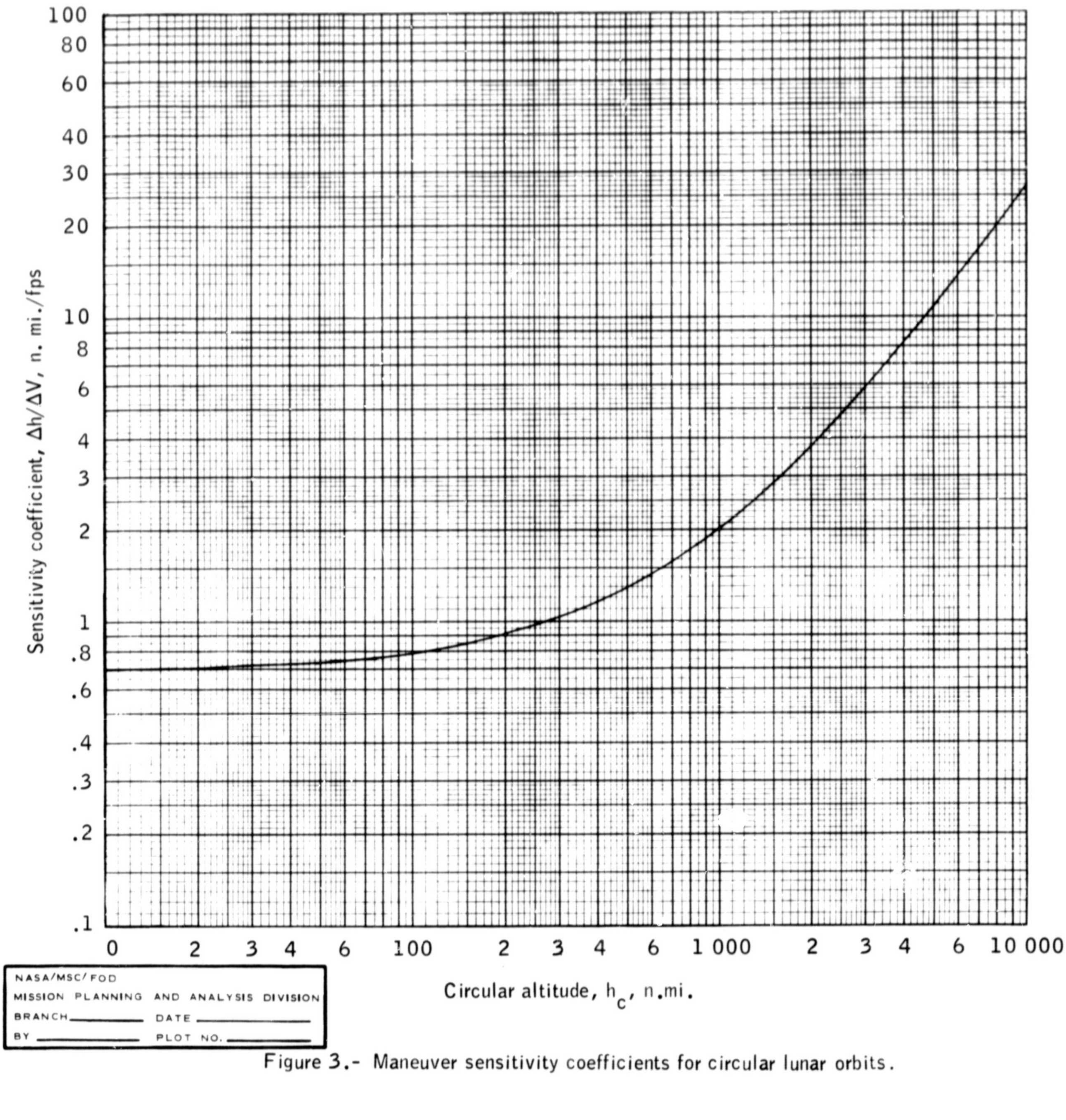


Figure 3.- Maneuver sensitivity coefficients for circular lunar orbits.

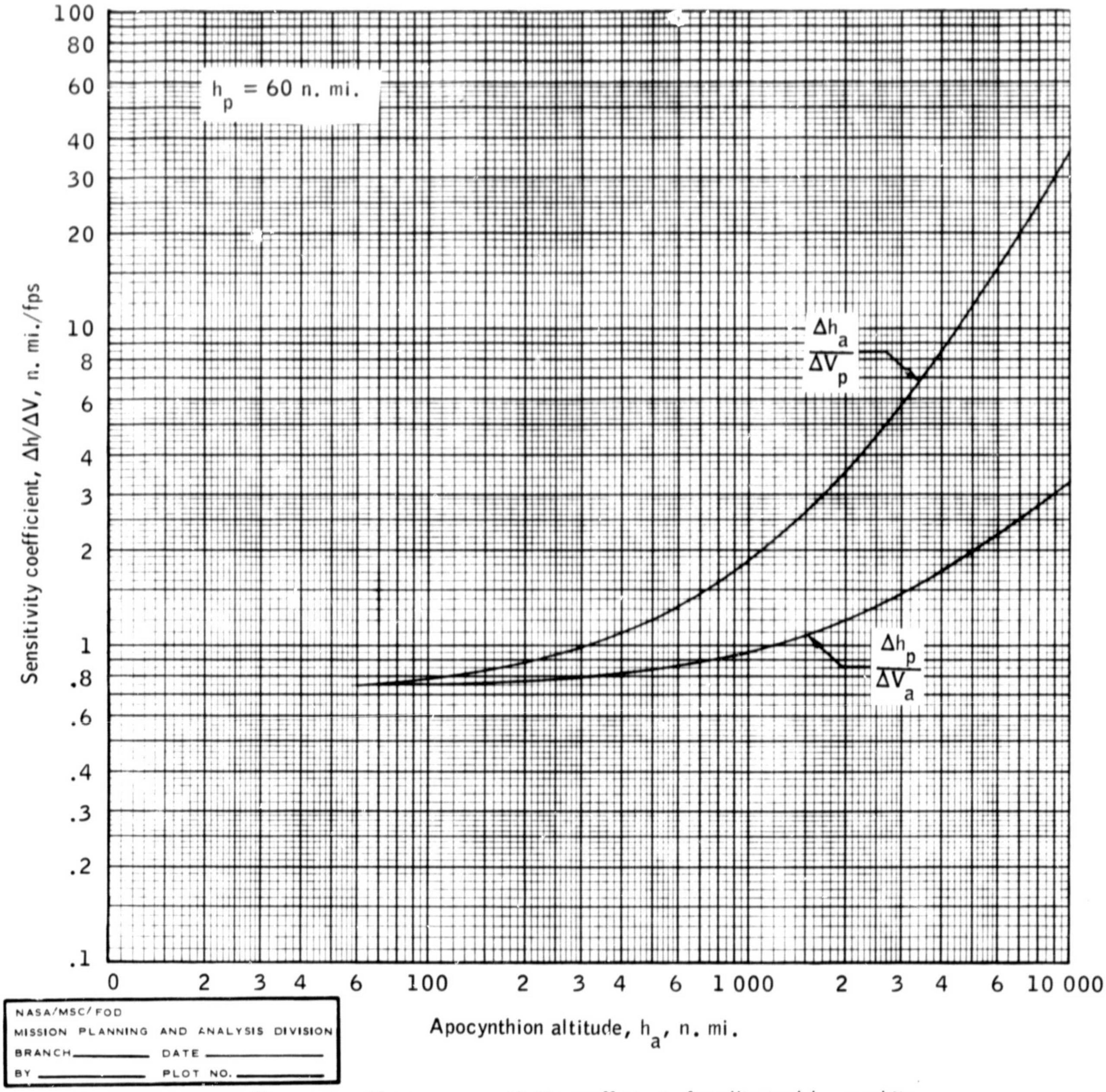


Figure 4.- Maneuver sensitivity coefficients for elliptical lunar orbits.

## REFERENCES


1. Douglas Aircraft Corporation: Stabilization and Control Design Considerations for Zero-g Space Stations. DAC report 3140, August 3, 1965.

2. Tindall, Howard J.: Maneuver Sensitivity Coefficients. MSC Memorandum 67-FM-T-74, August 29, 1967.