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# MSC INTERNAL NOTE

# NUMERICAL ALGORITHM FOR SOLVING OVERDETERMINED SYSTEMS OF NONLINEAR EQUATIONS

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## ABSTRACT

The purpose of this paper is to present a method for determining the simultaneous solution of an overdetermined system of nonlinear equations in $R^n$. A numerical algorithm is developed, and some examples of convergent solutions are contained in the last section of the paper. The algorithm can be used for problems such as obtaining the intersection mach cones which is used in the method of characteristics. In the development of the solution, it is assumed that each function can be expanded in an $n$ dimensional Taylor's Series in a neighborhood of the solution. The method utilizes an iterative procedure, involving the inversion of an $n \times n$ matrix, which does not necessarily converge. Thus, the method might be time consuming or inapplicable to some problems.


## INTRODUCTION

The purpose of this paper is to present a method for determining the simultaneous solution of an overdetermined system of nonlinear functions in $R^n$. Consider a set of functions $f_i : R^n \to R^n$ such that, for some $y \in R^n$, $f_i(y) = 0$ for $i = 1,2,\ldots,K$. Expressing the coordinates in $R^n$ as $(x_1, x_2, \ldots x_n)$, it will be assumed that each of the functions $f_i$ can be expanded in a convergent $n$-dimensional

Taylor's series about any point in some neighborhood of $y$ . Being similar to Newton's method for the solution of a single equation in one variable, the convergence of the solution will be subject to the initial guess for a solution and the characteristics of the functions.

## THE SPECIAL CASE OF $n = k$

For the case $n = k$ consider the usual approach to this problem. Choose a point $y_0$ as a guess to the solution. Then, provided all of the Taylor's series converge, find $\Delta y$ such that

$$0 = f_1(y_0 + \Delta y) = f_1(y_0) + \frac{\partial f_1(y_0)}{\partial x_1} \Delta x_1$$

$$+ \frac{\partial f_1}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial f_1}{\partial x_n} \Delta x_n + \ldots$$

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$$0 = f_n(y_0 + \Delta y) + \frac{\partial f_n}{\partial x_n} \Delta x_1 + \frac{\partial f_n}{\partial x_2} \Delta x_2$$

$$+ \ldots \frac{\partial f_n}{\partial x_n} \Delta x_n + \ldots$$

If the notation

$$A(y_0) = \{a_{ij}\} = \frac{\partial f_i(y_0)}{\partial x_j} \quad i = 1,n, \; j = 1,n, \quad \text{and}$$

$\Delta x^{(1)} = (\Delta x_1, \Delta x_2, \ldots \Delta x_n)$ is used then a first order approximation for $\Delta x^{(1)}$ is given by

$$\Delta x^{(1)} = A^{-1}_{(y_0)} \begin{pmatrix} -f_1(y_0) \\ -f_2(y_0) \\ \cdot \\ \cdot \\ \cdot \\ -f_n(y_0) \end{pmatrix}$$

In the usual fashion an iterative procedure can then be initiated by setting $y_\ell = y_{\ell-1} + \Delta x^{(\ell-1)}$ .

Because each of the series was truncated, the value of $\Delta x$ is an approximation and the method might not converge.

## THE OVERDETERMINED CASE

For the case $k > n$ , a solution for the equation

$$\begin{pmatrix} -f_1(y_0) \\ \cdot \\ \cdot \\ \cdot \\ -f_k(y_0) \end{pmatrix} = A(y_0)\, \Delta x^{(1)} \;\; \cdot\!=\!\cdot \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_k \end{pmatrix}$$

does not necessarily exist, since $A$ is a $k \times n$ matrix. Since $\Delta x^{(1)}$ is an approximation even for the case $n = k$, consider the solution $\Delta x^{(1)}$ such that $\sum_{i=1}^{k} (p_i - (-f_i(y_0))^2$ is minimized. If a unique solution exists, then it is given by

$$\Delta x^{(1)} = (A^T_{(y_0)} A_{(y_0)})^{-1} A^T_{(y_0)} \begin{pmatrix} -f_1(y_0) \\ \cdot \\ \cdot \\ \cdot \\ -f_k(y_0) \end{pmatrix}$$

An iterative solution can now be introduced just as before with

$$y_\ell = y_{\ell-1} + \Delta x^{(\ell-1)} .$$

Mr. Melvin J. Arldt of LEC has written a computer program using this method. Development and testing of this program has led to the following examples. All execution times are given for the UNIVAC 1108.

Example 1.

$$f_1 = x_2^2 + x_1^2 - 1$$

$$f_2 = x_2 - x_1 + 1$$

$$f_3 = x_2 + x_1 - 1$$

Initial guess $x_1 = 1/2$

$x_2 = 2$

Converged in 7 iterations

Computer time required .231 sec.

Example 2.

$$f_1 = x_2^2 + x_1^2 - 9$$

$$f_2 = x_1 - 3$$

$$f_3 = x_2 - x_1 + 3$$

$$f_4 = x_2^2 + (x_1 - 6)^2 - 9$$

Initial guess $x_1 = 1$

$x_2 = 0$

Converged in 6 iterations

Computer time required .224 sec.

Example 3.

$$f_1 = x_1^2 + x_2^2 + x_3^2 - 1$$

$$f_2 = x_1^2 + x_2^2 + (x_3 - 2)^2 - 1$$

$$f_3 = x_1 + x_2 + x_3 - 1$$

$$f_4 = x_1 + x_2 - x_3 + 1$$

$$f_5 = x_1^3 + 3x_2^3 + (5x_3 - x_1 + 1)^2 - 36$$

Initial guess $x_1 = 1$

$x_2 = 2$

$x_3 = 1$

Converged in 25 iterations

Computer time required 1.131 sec.

The remaining examples use the recursion formula

$$\begin{aligned} & f_1 = -(3 + \alpha x_1)\, x_1 + 2x_2 - \beta \\ & \vdots \\ i = 2,\ldots n - 1 ;\quad & f_i = x_{i-1} - (3 + \alpha x_i)\, x_i + 2x_{i+1} - \beta \\ & \vdots \\ & f_n = x_{n-1} - (3 + \alpha x_n)\, x_n - \beta \end{aligned}$$

with $\alpha = -.5$ $\beta = 1$

Example 4.

$$n = 5$$

Initial guess $x_i = -1 \quad i = 1,2,\ldots 5$

Converged in 5 iterations

Computer time required .508 sec.

Example 5.

$$n = 10$$

Initial guess $x_i = -1 \quad i = 1,2,\ldots,10$

Converged in 5 iterations

Computer time required 1.799 sec.

Example 6.

$$n = 20$$

Initial guess $x_i = -1 \quad i = 1,2,\ldots,20$

Converged in 6 iterations

Computer time required 10.709 sec.