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IMPULSIVE ORBIT TRANSFER OPTIMIZATION BY AN ACCELERATED GRADIENT METHOD

By Ivan L. Johnson, Jr.,
Mathematical Physics Branch

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IMPULSIVE ORBIT TRANSFER OPTIMIZATION BY AN ACCELERATED GRADIENT METHOD[a]

By Ivan L. Johnson, Jr., Manned Spacecraft Center, NASA

INTRODUCTION

An analysis of impulsive orbit transfer optimization using an accelerated gradient method is presented. Several years ago in the Trajectory Optimization Section of the Mathematical Physics Branch, research began on parameter optimization methods especially for trajectory optimization. Since that time, several different methods have been tried and abandoned. Among those methods were the gradient projection method (ref. 1 and 2), the method of steepest descent using a penalty function (ref. 1), various attempts at second order methods - all of which failed, and the conjugate gradient methods compared in reference 3.

A quadratically convergent gradient method developed by W. C. Davidon (ref. 4) was discovered. This method was refined slightly by Fletcher and Powell (ref. 5), and it was shown that it is probably the most powerful method known for the minimization of a general function of n variables.

For the constrained minimum problem, it was found that if one minimized a quadratic type of penalty function (ref. 1) for a given set of "large" penalty constants, an approximate solution can be obtained. If Davidon's method is used to minimize the penalty function, the inverse of the matrix of second partial derivatives of the penalty function at the minimum is obtained. If Newton's method is applied to the first-order necessary conditions for a constrained minimum, the exact solution for the constrained minimum problem can be obtained, provided the matrix of second partial derivatives of the augmented function can be calculated. Assuming that the approximate solution obtained by minimizing the penalty function using Davidon's method is "close" to the exact solution, the matrix of second partial derivatives needed for the solution by Newton's method can be approximated using the matrix from Davidon's method. This logic was used to develop a digital computer program known as the "Accelerated Gradient" program which is described in this paper.

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[a]Presented at the Astrodynamics Conference held December 12, 13 and 14, 1967, at the Manned Spacecraft Center, Houston, Texas.

Initially, the first type of orbit transfer models contained ellipses and circles only. Both the two-body and conic partials equations were good for ellipses and circles only. In order to solve problems of the more complex form involving hyperbolas as well as ellipses and circles, the universal variable formulation by S. Pines (ref. 6) and A. K. Nakashima (ref. 7) is now used. This formulation is also described briefly in this paper.

The formulation, programming, and accumulation of data for a three-impulse orbit transfer problem was accomplished mainly by W. C. Bean of the Mathematical Physics Branch, NASA, Houston, Texas, using the Accelerated Gradient program with the universal variable, two-body formulation. The optimal solution to this problem is presented in this paper as an example. In the problem, both an intermediate trajectory constraint and an inequality constraint are featured. In a report to be published, a further description and more data on this example problem will be presented by Mr. Bean.

STATEMENT OF IMPULSIVE ORBIT TRANSFER OPTIMIZATION PROBLEMS

With impulses approximating the rocket burns and two-body motion (conics) approximating the unpowered flight, orbit transfer models of various missions can be constructed for the purpose of parameter optimization.

The performance index for orbit transfer optimization problems is often the "characteristic velocity," the sum of the magnitudes of the impulsive-velocity vectors. The constraints, inequality and equality types, are usually specified on the terminal orbit, but constraints may be placed on intermediate trajectories as well.

The generalized eccentric anomalies, which are the independent variables governing the lengths of the coasting arcs, and the components of the impulsive-velocity vectors are the parameters of the optimization problem.

The orbit transfer optimization problem then is to find the parameters minimizing the performance index subject to the system of trajectory constraints.

A DIGITAL COMPUTER PROGRAM FOR PARAMETER OPTIMIZATION

The Accelerated Gradient program, a digital computer program, using the method of reference 8, is used for the impulsive orbit transfer optimization. The program numerically solves the following general optimization problem.

Find that set of parameters $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ which locally minimizes the general function

$$f \equiv f(\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{1}$$

subject to the system of $m < n$ nonlinear constraints

$$\left.\begin{array}{l} g_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \\ g_2(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \\ \quad\vdots \\ g_m(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \end{array}\right\} . \tag{2}$$

Included in this system are inequality constraints. If

$$h_j(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq 0 \tag{3}$$

is needed, then with the help of the function $S(h_j)$,

$$\begin{array}{ll} S(h_j) = 0 & \text{if } h_j < 0 \\ S(h_j) = 1 & \text{if } h_j \geq 0 \end{array} , \tag{4}$$

inequality constraints can be redefined as equality constraints

$$g_j \equiv h_j S(h_j) = 0 \text{ where } j \leq m. \tag{5}$$

The Accelerated Gradient program consists of two phases. The first phase treats the optimization problem in an approximate form. The unconstrained function (penalty function)

$$\bar{f} = f + \frac{1}{2}\sum_{j=1}^{m} k_j g_j^{\,2} \tag{6}$$

is formed and minimized for given "large" penalty constants $k_j > 0$. Should solutions of both the approximation problem and the original problem exist, the former approaches the latter as each $k_j \to \infty$ (ref. 1).

The method used for minimizing $\bar{f}$ was developed by William C. Davidon (ref. 4, 5, and 9). Davidon's method is a quadratically convergent gradient method for the minimization of a general function of n variables. Beginning with a starting point $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, the gradient vector $\bar{f}_\alpha$ and the function $\bar{f}$ are calculated. Using the formula

$$\Delta\alpha = -\gamma H \bar{f}_\alpha \tag{7}$$

a change in the vector α of parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ is made. The matrix H is symmetric and positive definite (it must be chosen as such initially), and the scalar γ is a positive step-size parameter. The one-dimensional minimum of $\bar{f}(\alpha + \Delta\alpha)$ versus γ is obtained (the method of reference 10 is used in the computer program), the gradient vector $\bar{f}_\alpha$ and the function $\bar{f}$ is recalculated at the new α, and H is updated according to the formula

$$H + \Delta H = H + \frac{\Delta\alpha \Delta\alpha^T}{\Delta\alpha^T \Delta\bar{f}_\alpha} - \frac{H\Delta\bar{f}_\alpha \Delta\bar{f}_\alpha^T H}{\Delta\bar{f}_\alpha^T H \Delta\bar{f}_\alpha} . \tag{8}$$

The procedure is then repeated using the new values of α, $\bar{f}_\alpha$, and H until the set of parameters $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is found which locally minimizes $\bar{f}$ for a given set of penalty constants $k_j > 0$. As a bonus, upon covergence to the minimum of $\bar{f}$, the H matrix approaches the inverse of the matrix of second partial derivatives of $\bar{f}$ with respect to the parameters α_j at the minimum. That is,

$$H \simeq \left(\frac{\partial^2 \bar{f}}{\partial\alpha_i \partial\alpha_j}\right)^{-1}_{\alpha=\alpha_{min}} \tag{9}$$

In order for the solution to the approximation problem to approach that of the original problem, the penalty constants k_j may be enlarged according to the relationship

$$(k_j)_{new} = (k_j)_{old}\frac{|g_j|}{\varepsilon_j} \quad \text{if } |g_j| > \varepsilon_j \tag{10}$$

and $\bar{f}$ is minimized again.

Before repeating the minimization of $\bar{f}$ with larger values of the k_j, the initial H matrix will be calculated using the converged one from the previous minimization and updating it according to the changes in the k_j. The initial H matrix is found by iterating m times using the formula

$$H \leftarrow H - Hg_{i\alpha}\left(\frac{\Delta k_i}{1 + \Delta k_i g_{i\alpha}^T H g_{i\alpha}}\right) g_{i\alpha}^T H, \tag{11}$$

where the symbol $\leftarrow$ denotes "is replaced by" and $g_{i\alpha}$ is the gradient vector of the i-th constraint at the previous minimum. The function $\bar{f}$ is then minimized again.

The second phase of the Accelerated Gradient program consists of taking the solution found by the first phase and refining it using an algorithm obtained by applying Newton's method.

The algorithm is obtained as follows. The first order necessary conditions for the constrained minimum problem is given by the system of equations

$$\left.\begin{aligned} f_\alpha + g_\alpha\lambda &= 0 \\ g &= 0 \end{aligned}\right\}, \tag{12}$$

where g_α is an $n \times m$ matrix whose columns are the gradient vectors of the constraints, λ is an m-dimensional vector of multipliers and g is an m-dimensional vector of the constraints. Application of Newton's method to the system of equations (12) yields

$$\begin{bmatrix} (f + \lambda^T g)_{\alpha\alpha} & g_\alpha \\ \\ g_\alpha{}^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} f_\alpha + g_\alpha \lambda \\ \\ g \end{bmatrix}. \tag{13}$$

Using the converged H matrix, the matrix of second partial derivatives $(f + \lambda^T g)_{\alpha\alpha}$ can be approximated by

$$H^{-1} - g_\alpha K g_\alpha^T, \tag{14}$$

where K is an $m \times m$ diagonal matrix, the diagonal elements being the penalty constants k_j. In terms of known quantities, the expression for $\Delta\alpha$ can be obtained from equation (13) as

$$\Delta\alpha = -\left[H - Hg_\alpha\ (g_\alpha^T H g_\alpha)^{-1} g_\alpha{}^T\ H\right] f_\alpha - Hg_\alpha (g_\alpha{}^T H g_\alpha)^{-1} g. \tag{15}$$

Repeated application of equation (15) starting with the "near" solution obtained by minimizing $\bar{f}$ using Davidon's method yields a solution to the original problem. Convergence is obtained when the $\Delta\alpha_i$'s are considered small enough.

TWO-BODY MOTION USING A UNIVERSAL VARIABLE FORMULATION

For the coast phases of the orbit transfer models, the two-body equations are presented in a form using a universal variable formulation. The advantage of this type of formulation is that only one set of two-body equations is needed to describe all of the different conics.

The generalized eccentric anomaly β is defined by

$$\gamma\beta^2 = \theta^2 \tag{16}$$

where

$$\gamma \equiv \frac{1}{a} = \frac{2}{r_o} - \frac{v_o^2}{\mu} \tag{17}$$

and r_o is the magnitude of the initial position vector, R_o, v_o is the magnitude of the initial velocity vector, V_o, and μ is the gravitational constant for the central body. For elliptical orbits

$$\theta = E - E_o, \tag{18}$$

where E is the eccentric anomaly. For hyperbolic trajectories θ is imaginary.

The equations of motion in a central force field are given by

$$\ddot{x}_i = \frac{-\mu x_i}{r^3}, \quad i = 1, 2, 3 \tag{19}$$

where r is the magnitude of the position vector R, x_i $(i = 1, 2, 3)$ are the components of R, and $\dot{x}_i$ $(i = 1, 2, 3)$ are the components of V. With the initial position vector, R_o, the initial velocity vector, V_o, and the initial time, t_o given, integrals of the equations of motion (19) with β as the independent variable are given by

$$\left.\begin{aligned} x_i &= f x_{oi} + g\dot{x}_{oi} \\ \dot{x}_i &= \dot{f} x_{oi} + \dot{g}\dot{x}_{oi} \end{aligned}\right\}, \quad i = 1, 2, 3 \tag{20}$$

The coefficients f, g, $\dot{f}$, and $\dot{g}$ are given by

$$f = 1 - \frac{G_2}{r_o}, \tag{21}$$

$$g = \frac{1}{\sqrt{\mu}}\left[r_o G_1 + \frac{d_o}{\sqrt{\mu}} G_2\right], \tag{22}$$

$$\dot{f} = -\frac{\sqrt{\mu}}{r r_o} G_1, \tag{23}$$

and

$$\dot{g} = 1 - \frac{G_2}{r}, \tag{24}$$

where

$$d_o = R_o \cdot V_o \tag{25}$$

and

$$r = r_o G_o + \frac{d_o}{\sqrt{\mu}} G_1 + G_2 \ . \tag{26}$$

The universal variables $G_p (p = 0, 1, 2, \ldots)$ are defined by

$$G_p = \beta^p \left[\frac{1}{p!} - \frac{\theta^2}{(p+2)!} + \frac{\theta^4}{(p+4)!} - \ldots \right] \ . \tag{27}$$

Calculating the G_p with the highest valued subscript, say G_p and G_{p-1}, by the series in equation (27), the remaining G_p's (G_{p-2}, G_{p-3}, ..., G_o) can be calculated using the recursive equation

$$G_{p-2} = \frac{\beta^{p-2}}{(p-2)!} - \gamma \, G_p. \tag{28}$$

The generalized Kepler's equation is given by

$$t = t_o + \frac{1}{\sqrt{\mu}} G_3 + r_o G_1 + \frac{d_o}{\sqrt{\mu}} G_2. \tag{29}$$

With the initial state, R_o, V_o, and t_o, given, the state of an orbit, R, V, and t, for any specified β is given by equations (20) and (29). Determining the state in this manner gets rid of iterations involving Kepler's equation.

A detailed account of the equations presented in this section is presented in references 6, 7, 11, 12, and 13.

TRAJECTORY DERIVATIVES

In using a gradient method for constrained minimization, the gradient vectors of both the function and the constraints are calculated. For orbit transfer optimization problems, the partial derivatives of both the function and constraints with respect to the components of the impulsive velocity vectors and the generalized eccentric anomalies must be calculated. For the gradient vectors of the constraints, the partial derivatives of the state at one time with respect to the state at an earlier time and the derivatives of the state with respect to the generalized eccentric anomaly are needed.

From equations (16) through (29), the state transition matrix Q of partial derivatives of the state at one time with respect to the state at an earlier time can be derived in closed form.

$$Q \equiv \left[\begin{array}{c|c|c} \left(\dfrac{\partial x_i}{\partial x_{oj}}\right) & \left(\dfrac{\partial x_i}{\partial \dot{x}_{oj}}\right) & \left(\dfrac{\partial x_i}{\partial t_o}\right) \\ \hline \left(\dfrac{\partial \dot{x}_i}{\partial x_{oj}}\right) & \left(\dfrac{\partial \dot{x}_i}{\partial \dot{x}_{oj}}\right) & \left(\dfrac{\partial \dot{x}_i}{\partial t_o}\right) \\ \hline \left(\dfrac{\partial t}{\partial x_{oj}}\right) & \left(\dfrac{\partial t}{\partial \dot{x}_{oj}}\right) & \left(\dfrac{\partial t}{\partial t_o}\right) \end{array}\right] , \tag{30}$$

The elements of the 3 × 3 submatrices $(\partial x_i/\partial x_{oj})$, $(\partial x_i/\partial \dot{x}_{oj})$, $(\partial \dot{x}_i/\partial x_{oj})$, and $(\partial \dot{x}_i/\partial \dot{x}_{oj})$, are given by

$$\frac{\partial x_i}{\partial x_{oj}} = f\delta_{ij} - \dot{x}_{oi}\left[\frac{p_1 x_{oj}}{r_o^3} + \frac{r}{\mu}(\dot{x}_j - \dot{x}_{oj})\right] + \frac{x_{oj}}{r_o^3}\left[\left(G_2 + \frac{2G_4 - \beta G_3}{r_o}\right)x_{oi} + \left(3G_5 - \beta G_4\right)\frac{\dot{x}_{oi}}{\sqrt{\mu}}\right] , \tag{31}$$

$$\frac{\partial x_i}{\partial \dot{x}_{oj}} = g\delta_{ij} - \frac{\dot{x}_{oi}}{\mu}(p_1\dot{x}_{oj} - G_2 x_{oj}) + \frac{\dot{x}_{oj}}{\mu}\left[(2G_4 - \beta G_3)\frac{x_{oi}}{r_o} + (3G_5 - \beta G_4)\frac{\dot{x}_{oi}}{\sqrt{\mu}}\right] , \tag{32}$$

$$\frac{\partial \dot{x}_i}{\partial x_{oj}} = \dot{f}\delta_{ij} + \frac{(\dot{x}_i - \dot{x}_{oi})}{r}\left[\frac{p_2 x_{oj}}{r_o^3} - \left(\frac{x_{oj}}{r_o} G_o + \frac{\dot{x}_{oj}}{\sqrt{\mu}} G_1\right)\right]$$

$$+ \frac{x_{oj}}{rr_o^3}\left[\left(G_1 + \frac{G_3 - \beta G_2}{r_o}\right)\sqrt{\mu}\, x_{oi} + (2G_4 - \beta G_3)\, \dot{x}_{oi}\right] , \tag{33}$$

$$\frac{\partial \dot{x}_i}{\partial \dot{x}_{oj}} = \dot{g}\delta_{ij} + \frac{(\dot{x}_i - \dot{x}_{oi})}{r\sqrt{\mu}}\left[\frac{p_2 \dot{x}_{oj}}{\sqrt{\mu}} - G_1 x_{oj}\right] + \frac{\dot{x}_{oj}}{\mu r}\left[(G_3 - \beta G_2)\frac{\sqrt{\mu}}{r_o} x_{oi}\right.$$

$$\left. + (2G_4 - \beta G_3)\, \dot{x}_{oi}\right] , \tag{34}$$

The elements of the 3 × 1 submatrices $(\partial x_i/\partial t_o)$ and $(\partial \dot{x}/\partial t_o)$ are given by

$$\frac{\partial x_i}{\partial t_o} = \frac{\partial \dot{x}_i}{\partial t_o} = 0, \tag{35}$$

The elements of the 1 × 3 submatrices $(\partial t/\partial x_{oj})$ and $(\partial t/\partial \dot{x}_{oj})$ are given by

$$\frac{\partial t}{\partial x_{oj}} = -\left[\frac{x_{oj}}{r_o^3} p_1 + \frac{r}{\mu}(\dot{x}_j - \dot{x}_{oj})\right] , \tag{36}$$

$$\frac{\partial t}{\partial \dot{x}_{oj}} = -\frac{1}{\mu}(p_1 \dot{x}_{oj} - G_2 x_{oj}), \tag{37}$$

and

$$\frac{\partial t}{\partial t_o} = 1. \tag{38}$$

In the preceding equations, δ_{ij} is the Kronecker delta,

$$\left.\begin{array}{ll} \delta_{ij} = 1, & \text{when } i = j \\ \delta_{ij} = 0, & \text{when } i \neq j \end{array}\right\} , \tag{39}$$

$$p_1 = \frac{1}{\sqrt{\mu}} \left[(3G_5 - \beta G_4) + r_\odot (G_3 - \beta G_2) + \frac{d_o}{\sqrt{\mu}} (2G_4 - \beta G_3) \right] , \quad (40)$$

and

$$p_2 = - \left[r_o \beta G_1 - \frac{d_o}{\sqrt{\mu}} (G_3 - \beta G_2) - (2G_4 - \beta G_3) \right] . \quad (41)$$

The derivatives of the state with respect to the generalized eccentric anomaly are given by

$$\frac{dx_i}{d\beta} = \frac{r}{\sqrt{\mu}} \dot{x}_i \quad , \quad (42)$$

$$\frac{d\dot{x}_i}{d\beta} = - \frac{\sqrt{\mu}}{r^2} x_i \quad , \quad (43)$$

and

$$\frac{dt}{d\beta} = \frac{r}{\sqrt{\mu}} . \quad (44)$$

AN OPTIMAL THREE-IMPULSE TRANSFER

The orbit transfer optimization method presented was applied to a three-impulse orbit transfer featuring an intermediate constraint.

The orbit transfer that was optimized may be described as follows. Starting with the initial state, R_o, V_o, and t_o, with respect to the earth as the central body, an initial impulsive velocity vector ΔV_o is applied, followed by a coast, whose length is governed by β_o, on an ellipse whose period P is governed by the constraint

$$g_1 = P - \bar{P} = 0. \quad (45)$$

Another impulse, ΔV_1, is applied followed by a coast whose length is governed by β_1. A third impulse, ΔV_2, is applied to transfer onto a hyperbola, partially specified by the constraints

$$\left.\begin{aligned} g_2 &= D - \bar{D} = 0 \\ g_3 &= v_\infty - \bar{v}_\infty = 0 \\ g_4 &= \bar{r}_p - r_p \le 0 \end{aligned}\right\} , \tag{46}$$

where D and v_∞ are the angle of declination and magnitude of the velocity vector at infinity, respectively. The quantity r_p denotes the magnitude of the radius vector at perigee, and $(\bar{\ })$ denotes given or specified quantities.

The parameters selected for optimization are

$$\alpha \equiv \begin{bmatrix} \Delta\dot{x}_2 \\ \Delta\dot{y}_2 \\ \Delta\dot{z}_2 \\ \beta_1 \\ \Delta\dot{x}_1 \\ \Delta\dot{y}_1 \\ \Delta\dot{z}_1 \\ \beta_o \\ \Delta\dot{x}_o \\ \Delta\dot{y}_o \\ \Delta\dot{z}_o \end{bmatrix} \tag{47}$$

The performance is measured by

$$f(\alpha) = |\Delta V_o| + |\Delta V_1| + |\Delta V_2| . \tag{48}$$

The function f is minimized subject to the constraining equations (45) and (46) where

$$\bar{v}_\infty = 37\ 002.647 \text{ fps}, \tag{49}$$

and

$$\bar{r}_p = 3543.934 \text{ n. mi. (100-n. mi. radial altitude).} \tag{50}$$

The initial state conditions are those of a circular orbit with a radial altitude of 262.0 n. mi. The vectors R_o and V_o are given by

$$R_o = \begin{bmatrix} 3705.934 \text{ n. mi.} \\ 0.0 \\ 0.0 \end{bmatrix} \tag{51}$$

and

$$V_o = \begin{bmatrix} 0.0 \\ 25\ 002.647 \text{ fps} \\ 0.0 \end{bmatrix}, \tag{52}$$

and t_o = 0.0 seconds.

Figure 1 is a pictoral display of the case where $\bar{D} = 0$. The optimum transfer is a double Hohmann transfer.

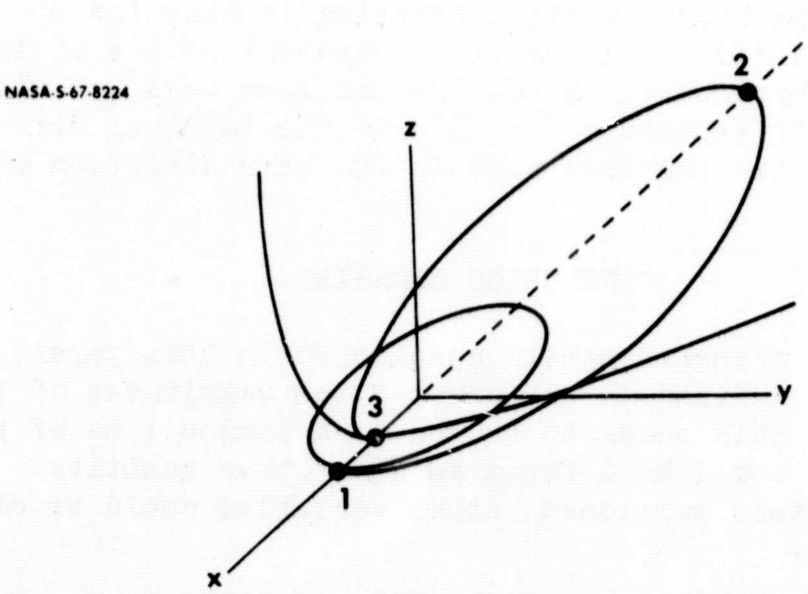

Figure 1.- Optimized three impulse orbit transfer.

Figure 2 is a graph of the performance f versus the angle of declination D for different values of the period P of the first transfer ellipse. Observing the graph, the performance gets better as the period of the second ellipse increases and gets worse as the angle of declination of the velocity vector at infinity for the hyperbola increases.

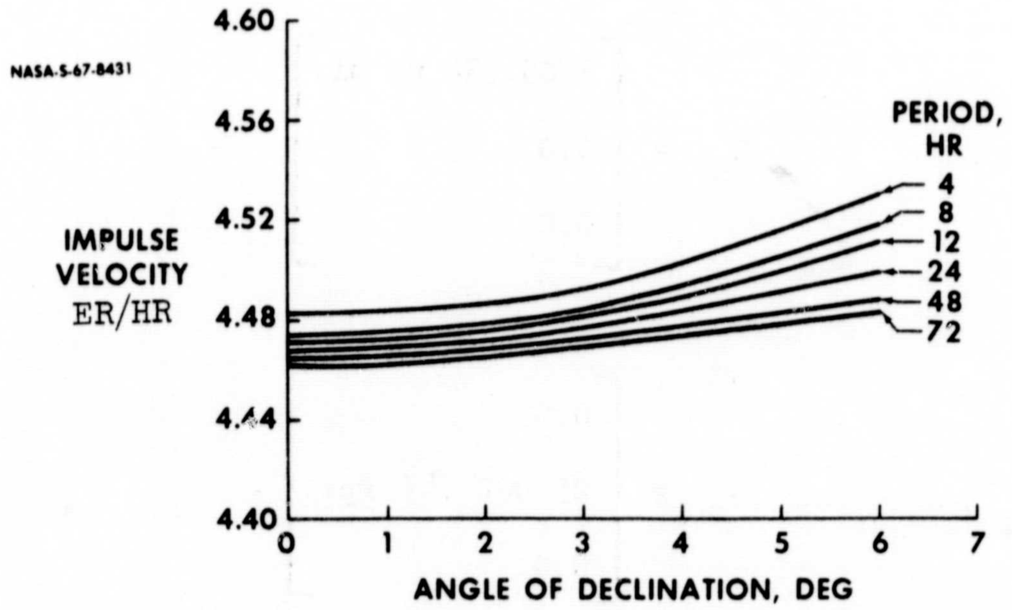

Figure 2.- Impulse velocity versus angle of declination for given periods.

One interesting result worth mentioning is that for the cases where $\bar{D} \neq 0$, the impulses did not occur at the apsidal points of the ellipses or perigee of the hyperbola. Also, the impulses were not along the same line as the velocity vectors. For $\bar{D} = 0$ the impulses did occur at the apsidal points and the impulses were in the same direction as the velocity vectors.

CONCLUDING REMARKS

For the orbit transfer models considered in this paper, the performance indices were defined as the sums of the magnitudes of the impulse velocity vectors. This seems to be the more common type of performance index; however, it could be defined as some other quantity. Also, besides the parameters mentioned, other variables could be chosen as parameters as well.

The reason for using a universal variable formulation for the solution to the two-body equations is that the single set of equations defining the solution is good for all the different types of trajectories (conics).

Specifying the state of a trajectory with seven dependent variables (the three components of both the position vector and velocity vector, and time) rather than the usual six gets rid of the usual iterations involving Kepler's equation. A good account of the numerical evaluation of the infinite series for G_p is given in reference 7.

The Accelerated Gradient program is not only used for orbit transfer optimization, but because it is written in a general form, it can be applied to optimization problems of a different nature. For example, the program is currently being used for the optimization of an electronics system and a propulsion system.

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