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PERTURBATION THEORY BASED ON LIE TRANSFORMS AND ITS APPLICATION TO THE STABILITY OF MOTION NEAR SUN-PERTURBED EARTH-MOON TRIANGULAR LIBRATION POINTS

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16. Abstract

General and simplified recurrence formulas based on Lie transforms and Lie series are obtained and discussed in relation to one another.

Some of these formulas and computerized symbolic manipulations are applied in a canonical perturbation treatment of sun-perturbed motion near the Earth-Moon triangular libration points.

Ignoring lunar eccentricity, third order analysis leads to two large stable one-month elliptic periodic orbits synchronized with the sun, semimajor axes are about 90,000 miles.

When lunar eccentricity was included, these stable orbits became quasi-periodic. The effect of this eccentricity on the size of these orbits was found to be small.

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Chapter I

INTRODUCTION

A. Previous Contributions

Three centuries ago, the mathematical basis of modern celestial mechanics was established by Newton who developed the calculus and expounded his laws of motion and gravitation. Kepler's laws of planetary motion, obtained from observations of the solar system, validated Newton's law of gravity. For this purpose, Newton utilized a model compatible with the instrument accuracy of that time. To describe the motion of a planet, he used this same model, in which the sun was taken as a fixed body attracting the planet by a force of attraction directly proportional to the product of the masses of the sun and the planet and inversely proportional to the square of the instantaneous distance between them.

Subsequently, more sophisticated models, which include perturbations of the other planets in the solar system, have been developed to verify the more precise measurements obtained by advanced instruments. Application of the Newtonian law of gravity to these models requires use of an "inertial" frame of reference (i.e., a frame fixed relative to the stars). In most cases, however, it is convenient to describe the motion relative to a "rotating" frame. For this purpose, the Coriolis law can be used to transform the obtained equations of motion to the desired rotating frame. This transformation gives rise to the so-called "Coriolis" and "centripetal" accelerations.

In 1772, Lagrange discovered five exact solutions to the problem of three bodies; an important specialization of which is the restricted problem of three bodies. In this particular case, one of the masses is so small that it does not affect the motion of the two larger masses.

In a rotating coordinate system, when the two primaries move in circles around their barycenter, the five Lagrange solutions become five fixed points, with the configuration as shown in Fig. 1. A particle with zero relative velocity placed at any one of these five fixed points will be in equilibrium because the gravitational and centripetal acceleration

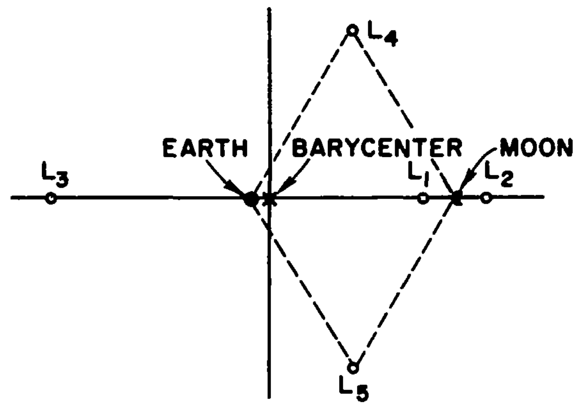

Fig. 1. LIBRATION POINTS IN THE RESTRICTED PROBLEM OF THREE BODIES.

acting on the particle will cancel out. The existence of periodic orbits about these points led researchers to refer to them as libration points.

Linearization of the equations of motion around each of the libration points suggests that the triangular points L_4 and L_5 are stable as long as the motion of the two primaries is circular and the mass ratio μ of the smaller primary to the sum of the masses of the two primaries is such that $\mu(1-\mu) < 1/27$. On the other hand, the collinear points are unstable for all mass ratios.

Studies of the stability of infinitesimal motions about the triangular points in the elliptic restricted problem of three bodies were made numerically by Danby [1], analytically by Bennett [2, 3], Alfriend and Rand [4], and recently by Nayfeh and Kamel [5]. In the last study, fourth-order analytical expressions for the transition curves that separate stable from unstable orbits in the μ-e plane are given in forms of power series in e (e is the eccentricity of the primaries orbit). The equation of one of the branches originating at $\mu = 0.02859^{\dagger}$ is

$$\mu^{(1)} = 0.02859 - 0.05641e + 0.01504e^2 + 0.02257e^3 - 0.01278e^4 + O(e^5) \tag{1.1}$$

†At this value, the zeroth-order equations admit a periodic solution of period 4π.

The equation of the second branch is

$$\mu^{(2)} = \mu^{(1)} (-e) \tag{1.2}$$

and the equation of the transition curve originating at $\mu = 0.03852$ [such that $\mu(1-\mu) < 1/27$] is

$$\mu^{(3)} = 0.03852 - 0.08025e^2 + 0(e^3) \tag{1.3}$$

The power series (1.1) was recast into a rational fraction that extended its validity to larger values of e. This fraction was a cubic divided by a linear term

$$\mu^{(1)} = 0.02859 \; \frac{1 - 1.428e - 0.55e^2 + 1.076e^3}{1 - 0.545e} \tag{1.4}$$

which is indistinguishable from the numerical curve up to $e = 0.8$, as seen in Fig. 2. In this figure, a comparison between numerical and the second-order, fourth-order, and fractional approximations is shown. A point whose coordinates are $\mu = 1/82.3 \simeq 0.012$ and $e \cong 0.055$ corresponds to the earth-moon system and belongs to the stable region.

In 1958, Klemperer and Benedikt [6] initiated the study of the earth-moon libration points. In 1961 and 1962, Kordylewski [7] reported observations of "clouds" near L_4 and L_5 which evoked interest and much investigation. An exhaustive list of references can be found in the recent treatise by Szebehely [8]. Placing satellites at earth-moon triangular libration points was recommended by a Summer Study Group [9] and more recently by Farquhar [10]. In both studies, some of the uses of such satellites and the advantages of their locations are outlined.

This research is a continuation of the analytic study of motion near the earth-moon equilateral points in the presence of the sun, using Hamiltonian techniques initiated by Breakwell and Pringle [11] and extended by Schechter [12].

Breakwell and Pringle used the von Zeipel perturbation method in a coplanar second-order analysis of the motion of a particle in the

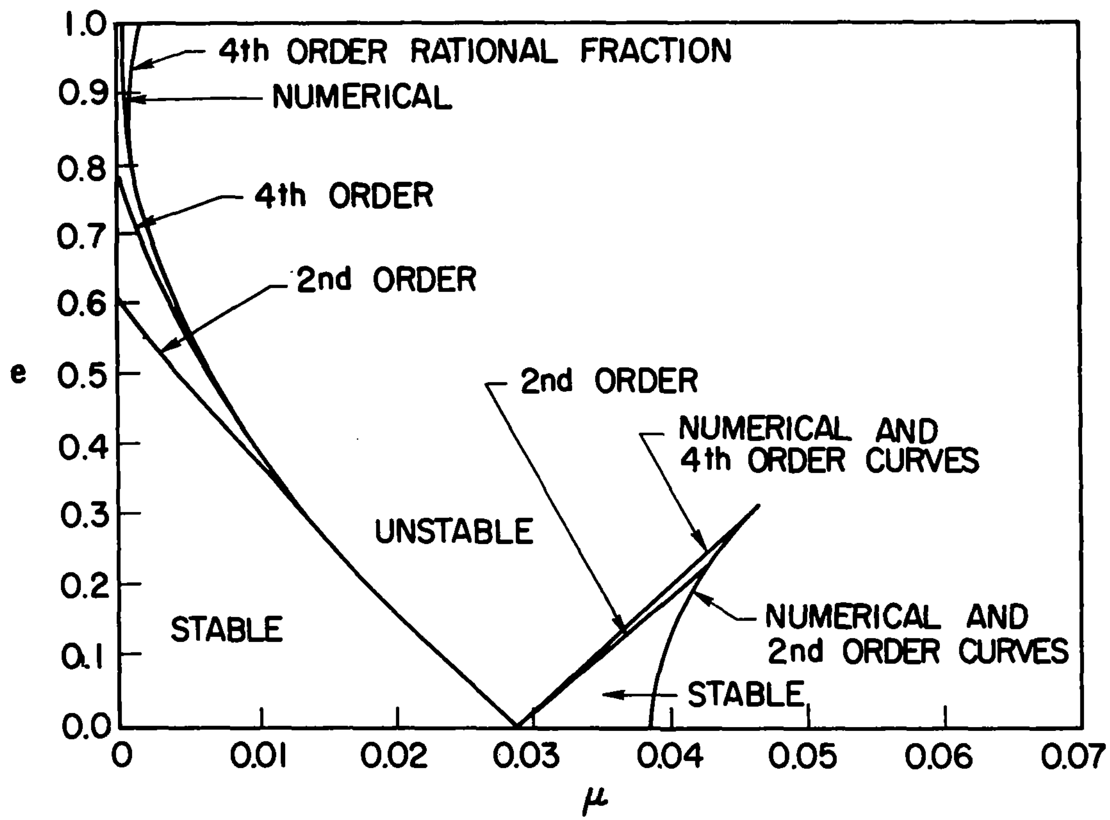

Fig. 2. COMPARISON OF THE ANALYTICAL AND NUMERICAL TRANSITION CURVES FOR INFINITESIMAL MOTIONS ABOUT THE TRIANGULAR POINTS IN THE ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES.

neighborhood of the earth-moon triangular libration points. They elaborated on the effect of near resonances caused by the faster natural frequency of the linearized analysis being approximately three times the slower one and by the solar twice-monthly perturbation frequency being almost twice the faster natural frequency.

After comparing with Deprit, Henrard, and Rom [13], Breakwell recognized (see Ref. 12) that the long-period part of the second-order Hamiltonian, derived in mixed variables utilizing the von Zeipel generating function, was misleading and that it was possible to obtain a different representation in terms of new variables only. Using this suggestion, Schechter [12] developed a more valid second-order expression and extended the analysis to the three-dimensional problem. The following major and interesting conclusions emerged from his study.

(1) As a result of nonlinear resonance, small coplanar motions near L_4 or L_5 will grow large because of parametric excitation by the sun; in fact, the growth of energy in the faster mode of the linearized theory was found to be governed by a Mathieu equation.

(2) The out-of-plane motion is not seriously excited by the sun and has a negligible effect on the coplanar motion.

(3) A one-lunar-month stable periodic coplanar orbit can exist in the presence of the sun. It consists of a retrograde elliptical motion around L_4, corresponds to the faster normal mode, and has a semimajor axis of approximately 60,000 mi. The external nonlinear excitation causes the mean angular motion of the particle to become synchronized with that of the sun; as a result, their angular positions coincide closely whenever the particle crosses one of the axes of the ellipse.

(4) A three-lunar-month unstable periodic orbit, somewhat larger than the stable one, exists.

Recently, Kolenkiewicz and Carpenter [14] obtained a seminumerical solution to confirm Schechter's third conclusion but with a semimajor axis of 90,000 mi rather than 60,000 mi, as found by Schechter. In addition, they discovered a second similar orbit having a phase difference of 180° and a semimajor axis of about 88,000 mi.

Because of the discrepancy between the results obtained by Schechter and Kolenkiewicz and Carpenter, it is concluded that a higher order analysis is desired.

B. Contribution of This Research

The purpose of this research is to extend Schechter's analysis to the fourth order; however, two major difficulties arise. The first is the need of a new perturbation theory to carry out systematically the higher order approximation. The second is the formidable algebra in the problem.

These difficulties were overcome by using two recent discoveries. The first is Deprit's perturbation theory based on Lie transforms [15] for which a simplified version is obtained in Chapter III (see also Ref. 16), and the second is the ability to carry out the enormous algebraic manipulations on the computer [17, 18]. It was possible to perform the algebraic analysis of this research on the Stanford IBM/360 computer, using the REDUCE language [17]. The results agreed closely with those of Kolenkiewicz and Carpenter [14].

An attempt was made to study the effect of lunar eccentricity on the obtained stable orbits. This eccentricity was carried out in the analysis up to third order rather than fourth order (because of limitations on computer storage), and its effect on the size of the orbits was found to be small; the orbits become quasi-periodic rather than periodic.

Chapter II

HAMILTONIAN AND NONLINEAR MECHANICS

This chapter will acquaint the unfamiliar reader with the basic materials needed to understand the main contribution of this research. These basic materials, and more, can be found in the literature [Refs. 19-24].

For a nonlinear oscillatory dynamical system in which all perturbing forces are derivable from a potential, the equations of motion take a very special form. This form is elegantly expressed by Hamilton's canonical equations generated by a scalar function called the Hamiltonian.

In this chapter, the Hamilton canonical equations are derived from the Lagrange equations. In Section B, these same equations again are derived directly from Hamilton's principle. The behavior of the Hamiltonian under the transformation of variables is also described. Outlined in the concluding section is the method of solution adopted for solving the examples of Chapter III and for solving the main problem presented in Chapters IV through VII.

A. The Hamilton Canonical Equations

Consider the nonlinear oscillatory dynamical system represented by the Lagrange equations†

$$\frac{d}{dt}\,\mathcal{L}_{\dot{q}} - \mathcal{L}_{q} = 0 \tag{2.1}$$

where q is the generalized coordinate vector, t is the independent variable, $\mathcal{L}(q,\dot{q},t) \equiv T - V$ is the Lagrangian, and T and V are kinetic and potential energies. The generalized momentum vector p can be defined as

---

† The subscripts $\dot{q}$ and q denote the arguments of partial differentiation. The differentiation of a scalar function with respect to a column (row) vector is taken as a column (row) vector; hence, a differentiation of n-elements column (row) vector with respect to m-elements row (column) vector is taken as an $n \times m$ $(m \times n)$ matrix.

$$p \equiv \mathfrak{L}_{\dot{q}} \tag{2.2}$$

and the Hamiltonian R as[†]

$$R \equiv p^T \dot{q} - \mathfrak{L} \tag{2.3}$$

Now, by using Eqs. (2.1) and (2.2), the variation δR of R can be written in the form

$$\delta R = -\dot{p}^T \delta q + \dot{q}^T \delta p \tag{2.4}$$

which yields the Hamilton canonical equations

$$\dot{q} = R_p \tag{2.5a}$$

$$\dot{p} = -R_q \tag{2.5b}$$

According to these equations,[‡]

$$\begin{aligned} \dot{R} &= R_{q^T}\dot{q} + R_{p^T}\dot{p} + R_t \\ &= -\dot{p}^T\dot{q} + \dot{q}^T\dot{p} + R_t \\ &= R_t \end{aligned} \tag{2.6}$$

therefore, if R does not depend explicitly on t, it is a constant of the motion.

In general, for a function f,

[†]The superscript T over the column vector p denotes its transpose (i.e., p^T is a row vector), and it should not be confused with the kinetic energy T.

[‡]R_{q^T} denotes a row vector of partial derivatives R_{q_i}.

$$\dot{f} = f_t + f_q{}^T\dot{q} + f_p{}^T\dot{p}$$

$$= f_t + (f;R) \tag{2.7}$$

where $(f;R)$ is the Poisson bracket defined by

$$(f;R) = f_q{}^T R_p - f_p{}^T R_q \tag{2.8}$$

As a result, if f does not depend explicitly on t, it is a constant of the motion if and only if $(f;R) = 0$.

B. The Hamilton Principle

Hamilton's principle for a conservative system asserts that the motion of the system from time t_0 to t_1 is such that the "variation" of the line integral $\int_{t_0}^{t_1} \mathfrak{L}\, dt$ for fixed t_0 and t_1 is zero; i.e.,

$$\delta \int_{t_0}^{t_1} \mathfrak{L}\, dt = 0 \tag{2.9}$$

In the modified form of this principle, p and q are regarded as independent. In this case, it is seen that it implies the canonical equations of motion. With reference to Eq. (2.3), the integrand of (2.9) can be written as

$$\mathfrak{L}(q,p,\dot{q},\dot{p},t) = p^T\dot{q} - R(q,p,t) \tag{2.10}$$

Then, if the end points in the phase space of (q,p) are fixed, Eq. (2.9) requires that $\mathfrak{L}$ satisfies the Euler-Lagrange equations in the form

$$\frac{d}{dt}\mathfrak{L}_{\dot{q}} = \mathfrak{L}_q \quad \text{and} \quad \frac{d}{dt}\mathfrak{L}_{\dot{p}} = \mathfrak{L}_p \tag{2.11}$$

but Eqs. (2.2) and (2.10) infer

$$\mathfrak{L}_{\dot{q}} = p \qquad \mathfrak{L}_q = -R_q \qquad \mathfrak{L}_{\dot{p}} = 0 \qquad \mathfrak{L}_p = \dot{q} - R_p \tag{2.12}$$

which, combined with (2.11), implies Eq. (2.5).

C. Transformation of Variables

Let the state of a system in the phase space be specified by a 2n vector,

$$x = \begin{pmatrix} q \\ p \end{pmatrix} \tag{2.13}$$

and define the matrix Φ_0 by

$$\Phi_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2.14}$$

where 0 denotes the $n \times n$ null-matrix, and I denotes the $n \times n$ identity matrix; also note the following identities:

$$|\Phi_0| = 1 \qquad \Phi_0^2 = -I \qquad \Phi_0^T = \Phi_0^{-1} = -\Phi_0 \tag{2.15}$$

therefore, Φ_0 is an orthogonal and skew-symmetric matrix.

Now, consider the "stationary" (time independent) transformation $x \rightarrow \bar{x}$ defined by

$$\bar{x} = \bar{x}(x) = \begin{pmatrix} \bar{q}(q,p) \\ \bar{p}(q,p) \end{pmatrix} \tag{2.16}$$

With reference to the definitions (2.13) and (2.14), the canonical equations of (2.5) assume the form

$$\dot{x} = \Phi_0 R_x \tag{2.17}$$

Under the stationary transformation of (2.16),

$$d\bar{x} = J dx \tag{2.18}$$

is obtained, where J is the Jacobian matrix of the transformation given by

$$J = \bar{x}_{x^T} = \begin{pmatrix} \bar{q}_{q^T} & \bar{q}_{p^T} \\ \bar{p}_{q^T} & \bar{p}_{p^T} \end{pmatrix} \tag{2.19}$$

Since

$$dR = R_{x^T} dx = R_{\bar{x}^T} d\bar{x} \tag{2.20}$$

then, with reference to Eq. (2.18),

$$R_{x^T} = R_{\bar{x}^T} J \tag{2.21}$$

Equations (2.17), (2.18), and (2.21) imply

$$\dot{\bar{x}} = \Phi R_{\bar{x}} \tag{2.22}$$

where Φ is the Poisson matrix defined by

$$\Phi = J \Phi_0 J^T \tag{2.23}$$

and the scalar invariant R is expressed as a function of $\bar{x}$ through Eq. (2.16).

D. Canonical Transformations

A transformation is said to be canonical if it preserves the canonical form of the equation of motion. Accordingly, and in view of Eqs. (2.17), (2.22), and (2.23), a stationary transformation (2.16) with a Jacobian J is canonical if and only if

$$J\Phi_0 J^T = \Phi_0 \tag{2.24}$$

Substitution of Eqs. (2.14) and (2.19) into (2.24) leads to

$$\begin{aligned}
\bar{q}_{q^T}\left(\bar{q}_{p^T}\right)^T - \bar{q}_{p^T}\left(\bar{q}_{q^T}\right)^T &= 0 \\
\bar{q}_{q^T}\left(\bar{p}_{p^T}\right)^T - \bar{q}_{p^T}\left(\bar{p}_{q^T}\right)^T &= I \\
\bar{p}_{q^T}\left(\bar{p}_{p^T}\right)^T - \bar{p}_{p^T}\left(\bar{p}_{q^T}\right)^T &= 0
\end{aligned} \tag{2.25}$$

which, in turn, can be reduced to the conditions,

$$\begin{aligned}
\left(\bar{q}_i;\bar{q}_j\right) &= 0 \\
\left(\bar{q}_i;\bar{p}_j\right) &= \delta_{ij} \\
\left(\bar{p}_i;\bar{p}_j\right) &= 0
\end{aligned} \tag{2.26}$$

where q_k and p_k are the k^{th} elements of the vectors q and p, respectively, and

$$\begin{aligned}
\delta_{ij} &= 1 \qquad \text{for } i = j \\
&= 0 \qquad \text{for } i \neq j
\end{aligned} \tag{2.27}$$

and

$$(f;g) \equiv f_{q^T} g_p - f_{p^T} g_q \tag{2.28}$$

An important technique for generating canonical transformations using "generating functions" is described as follows. Any generating function $S(q,\bar{p},t)$ with $|S_{q\bar{p}}| \neq 0$ can be employed to generate a canonical transformation (not stationary unless $S_t = 0$) from (q,p) to new variables $(\bar{q},\bar{p})$ associated with the new Hamiltonian,

$$\bar{R} = R + S_t \tag{2.29}$$

and defined by

$$\bar{q} = S_{\bar{p}} \qquad p = S_q \tag{2.30}$$

To prove the above statement, the Hamilton principle is used in its modified form,

$$\delta \int_{t_0}^{t_1} (p^T \dot{q} - R)\, dt \tag{2.31}$$

With the aid of Eq. (2.30), it then follows that

$$\begin{aligned} \delta \int_{t_0}^{t_1} (S_{q^T} \dot{q} - R)\, dt &= \delta \int_{t_0}^{t_1} \left[dS - \bar{q}^T d\bar{p} - (R + S_t)\, dt \right] \\ &= \delta (S - \bar{q}^T \bar{p}) \Big|_{t_0}^{t_1} \\ &\quad + \delta \int_{t_0}^{t_1} \left[\bar{p}^T \dot{\bar{q}} - (R + S_t) \right] dt \end{aligned} \tag{2.32}$$

and the first expression vanishes because the end points are assumed fixed in the phase space. It follows that

$$\dot{\bar{q}} = \bar{R}_{\bar{p}} \qquad \dot{\bar{p}} = -\bar{R}_{\bar{q}} \tag{2.33}$$

where $\bar{R} = R + S_t$.

If $|S_{q\bar{p}}| \neq 0$, the second equation of (2.30) can be solved for $\bar{p}$, and the result can be substituted into this equation so that the transformation is obtainable in the explicit form

$$\bar{q} = \bar{q}(q,p,t)$$

$$\bar{p} = \bar{p}(q,p,t) \tag{2.34}$$

Finally, $R(q,p,t) + S_t(q,\bar{p},t)$ is expressed as a function $\bar{R}(\bar{q},\bar{p},t)$.

A direct and interesting consequence of Eqs. (2.29), (2.30), and (2.33) is that if the generating function $S(q,\bar{p},t)$ of a canonical transformation is so chosen that the new Hamiltonian $\bar{R}$ is identically zero, then the new coordinates $\bar{q} = \beta$ and the new momenta $\bar{p} = \alpha$ will be constants of the motion. In addition, $S(q,\alpha,t)$ will satisfy the Hamilton-Jacobi partial differential equation

$$R(q,S_q,t) + S_t = 0 \tag{2.35}$$

E. Method of Solution

This section outlines the perturbation technique used to obtain the approximate solutions of the examples in Chapter III and the main problem presented in the following chapters.

Consider the system represented by the Hamiltonian R given as a power series in a small parameter ϵ in the form

$$R = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} R_n(q,p,t) \tag{2.36}$$

If the Hamilton-Jacobi equation (2.35) is assumed to be solvable for $\epsilon = 0$, the method of solution will involve the following three steps.

(1) Solve the Hamilton-Jacobi equation

$$R_0(q, S_q, t) + S_t = 0 \tag{2.37}$$

which yields $S(q,\alpha,t)$; then $S_\alpha = \beta$ furnishes the solution $q_0(\alpha,\beta,t)$ and $p_0(\alpha,\beta,t)$. The resulting new Hamiltonian $\bar{R}$ is

$$\bar{R} = R + S_t = R_0 + S_t + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} R_n$$

$$= \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} R_n \tag{2.38}$$

and the new variables (α,β) satisfy the standard form of

$$\dot{\alpha} = -\bar{R}_\beta(\alpha,\beta,t) \tag{2.39a}$$

$$\dot{\beta} = \bar{R}_\alpha(\alpha,\beta,t) \tag{2.39b}$$

Note that an equivalent representation for the standard form of these equations can be obtained by a canonical but stationary transformation $(q,p) \to (\tilde{\alpha}, \tilde{\beta})$ that reduces $\tilde{R} = R$ to

$$\tilde{R}(\tilde{\alpha},\tilde{\beta},t) = \tilde{R}_0(\tilde{\alpha}) + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \tilde{R}_n(\tilde{\alpha},\tilde{\beta},t) \tag{2.40}$$

In this case, the coordinate vector $\tilde{\beta}$ represents "fast variables" that satisfy

$$\dot{\tilde{\beta}} = \tilde{R}_{0\tilde{\alpha}} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \tilde{R}_{n\tilde{\alpha}} \tag{2.41a}$$

while the corresponding momenta still satisfy

$$\dot{\tilde{\alpha}} = -\sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \tilde{R}_{n\tilde{\beta}} \tag{2.41b}$$

(2) To reduce Eqs. (2.39) or (2.41) to a simpler form, a canonical transformation from (α,β) or $(\tilde{\alpha},\tilde{\beta})$ to $(\bar{\alpha},\bar{\beta})$ is (in general) desired. This can be elegantly performed by using the perturbation theory based on Lie transforms proposed by Deprit [15]. A simplified version of this method can be found in Ref. 16 and in the next chapter.

(3) Solve the transformed differential equations resulting from the previous step.

Chapter III

PERTURBATION THEORY BASED ON LIE TRANSFORMS

To reduce the standard forms of Eqs. (2.39) or (2.41) to a simpler form, it is desired to transform to new coordinates and momenta, as discussed in Chapter II. This transformation can be obtained by employing a von Zeipel generating function [21]. In such a case, the transformation is implicit because the generating function is in mixed variables (the old coordinates and the new momenta).

The shortcomings of the von Zeipel method were recognized by Breakwell and Pringle [11] and Deprit [25] when they used a von Zeipel generating function to remove the short-period terms from the Hamiltonian of a particle in the neighborhood of the triangular points in the restricted problem of three bodies. After comparing with Deprit et al [13], Breakwell [12] observed that the long-period terms of the second-order Hamiltonian, derived in mixed variables, was misleading and that it was possible to obtain a different representation in terms of new variables only. As a result of these observations, Schechter [12] obtained a more valid second-order expression. Deprit [15] attacked the problem using Lie transforms and extended the expansion to include higher orders.

This chapter obtains a simplified version of Deprit's method, which is outlined and clarified by two examples. Section E develops a general expansion of Hori's method, based on the Lie series, as a special case of the Lie transforms

A. General Expansions

A Lie transform can be defined by the differential equations

$$\frac{dx}{d\eta} = W_X(x, X, t; \eta) \tag{3.1a}$$

$$\frac{dX}{d\eta} = -W_x(x, X, t; \eta) \tag{3.1b}$$

$$\frac{dt}{d\eta} = 0 \tag{3.1c}$$

$$\frac{dR}{d\eta} = -W_t(x,X,t;\eta) \tag{3.1d}$$

whose initial conditions at $\eta = 0$ are $x = y(t;\epsilon)$, $X = Y(t;\epsilon)$, $t = t$, and $R = 0$, where

x,X = original generalized coordinate and momentum vectors

y,Y = transformed coordinate and momentum vectors

$R = K(y,Y,t;\epsilon) - H(x,X,t;\eta)$ = remainder function

K,H = transformed and original Hamiltonians

W = the generating function

t = independent variable

ϵ = a constant small parameter

η = a varying small parameter $(0 \leq \eta \leq \epsilon)$

The above equations can be shown to define a group of canonical transformations because

$$\frac{d}{d\eta}\,dx = dW_X = W_{Xx^T}\,dx + W_{XX^T}\,dX + W_{Xt}\,dt$$

$$\frac{d}{d\eta}\,\delta x = \delta W_X = W_{Xx^T}\,\delta x + W_{XX^T}\,\delta X$$

$$\frac{d}{d\eta}\,dX = -dW_x = -W_{xx^T}\,dx - W_{xX^T}\,dX - W_{xt}\,dt$$

$$\frac{d}{d\eta}\,\delta X = -\delta W_x = -W_{xx^T}\,\delta x - W_{xX^T}\,\delta X$$

$$\frac{d}{d\eta}\,\delta R = -\delta W_t = -W_{xt}\,\delta x^T - W_{Xt}\,\delta X^T$$

Thus, the differentials dx, dX, δx, δX, and δR that are produced by the initial changes dy, dY, δy, and δY satisfy

$$\frac{d}{d\eta}(dx^T \delta X - dX^T \delta x + dt\ \delta R) = 0 \tag{3.2}$$

From this equation it can be seen that $dx^T \delta X - dX^T \delta x - dt\ \delta R$ is an invariant of η and equals its value at $\eta = 0$, so that

$$\dot{x}^T \delta X - \dot{X}^T \delta x - \delta H = \dot{y}^T \delta Y - \dot{Y}^T \delta y - \delta K \tag{3.3}$$

Therefore, if x and X satisfy the canonical equations,

$$\dot{x} = H_X \qquad \dot{X} = -H_x \tag{3.4}$$

then y and Y also will satisfy the canonical form

$$\dot{y} = K_Y \qquad \dot{Y} = -K_y \tag{3.5}$$

It should be noted that Eq. (3.1) defines a group of canonical transformations whether W depends or does not depend explicitly on η. When W does not depend on η, Eq. (3.1) generates the so-called Lie series (see Ref. 26, for example); if W does depend on η, Eq. (3.1) generates the Lie transforms, so named by Deprit [15]. Thus, it may be stated that the Lie series is a special case of Lie transforms.

Now, take any indefinitely differentiable function $f(x,X,t;\epsilon)$ that can be expressed in terms of x, X, t, and ϵ as a power series in ϵ in the form

$$f(x,X,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} f_n(x,X,t) \tag{3.6}$$

where

$$f_n(x,X,t) = \left[\frac{\partial^n}{\partial \eta^n} f(x,X,t;\eta)\right]_{\eta=0}$$

In terms of y, Y, t, and ϵ, Eq. (3.6) then takes the form of

$$f(x,X,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} f^{(n)}(y,Y,t) \tag{3.7}$$

where

$$f^{(n)}(y,Y,t) = \left[\frac{d^n}{d\eta^n} f(x,X,t;\eta)\right]_{\eta=0}$$

and

$$\frac{df}{d\eta}(x,X,t;\eta) = \frac{\partial f}{\partial \eta} + f_x{}^T \frac{dx}{d\epsilon} + f_X{}^T \frac{dX}{d\epsilon} \tag{3.8}$$

Note that $f_0(x,X,t) = f(x,X,t;0)$ and $f^{(0)}(y,Y,t) = f(y,Y,t;0)$.

Given the sequence of functions $f_n(x,X,t)$ of (3.6), the corresponding sequence of functions $f^{(n)}(y,Y,t)$ of (3.7) will be constructed below. With reference to (3.1), Eq. (3.8) can be written as

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + L_W f \tag{3.9}$$

where L_W is a linear operator called the Lie derivative generated by W and is defined by the Poisson bracket

$$L_W f = (f;W) = f_x{}^T W_X - f_X{}^T W_x \tag{3.10}$$

Taking $f = x$, X, and R in Eq. (3.7) and using (3.1) obtains

$$x = y + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} y^{(n)}(y,Y,t) \tag{3.11a}$$

$$X = Y + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} Y^{(n)}(y,Y,t) \tag{3.11b}$$

$$H = K - \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} R^{(n)}(y,Y,t) \tag{3.11c}$$

where, for $n \geq 1$,

$$y^{(n)} = \left(\frac{d^{n-1}}{d\eta^{n-1}} W_X\right)_{\eta=0}$$

$$Y^{(n)} = -\left(\frac{d^{n-1}}{d\eta^{n-1}} W_x\right)_{\eta=0}$$

$$R^{(n)} = -\left(\frac{d^{n-1}}{d\eta^{n-1}} W_t\right)_{\eta=0}$$

In particular, for a generating function W of the form

$$W(x,X,t;\eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} W_{n+1}(x,X,t) \tag{3.12}$$

and $f(x,X,t;\epsilon)$ of the form given by (3.6), Eq. (3.9) yields

$$\frac{df}{d\eta}(x,X,t;\eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} f_n^{(1)}(x,X,t) \tag{3.13}$$

where, for $n \geq 0$,

$$f_n^{(1)}(x,X,t) = f_{n+1} + \sum_{m=0}^{n} C_m^n \, L_{m+1} \, f_{n-m}$$

$$C_m^n = \frac{n!}{m!\,(n-m)!}$$

$$L_i f = (f;W_i)$$

In general, for $k \geq 1$ and $n \geq 0$, one obtains

$$\frac{d^k}{d\eta^k} f = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} f_n^{(k)}(x,X,t) \tag{3.14}$$

where

$$f_n^{(k)}(x,X,t) = f_{n+1}^{(k-1)} + \sum_{m=0}^{n} C_m^n \, L_{m+1} \, f_{n-m}^{(k-1)}$$

Letting $\eta = 0$ in the above definition yields the following recursion equation which, for the remainder of this chapter, will be referred to as the Deprit equation:

$$f_n^{(k)}(y,Y,t) = f_{n+1}^{(k-1)} + \sum_{m=0}^{n} C_m^n \, L_{m+1} \, f_{n-m}^{(k-1)} \tag{3.15}$$

where, for $i \geq 1$,

$$L_i f = f_{y}{}^{T} \, W_{iY} - f_{Y}{}^{T} \, W_{iy}$$

In the Deprit equation,

$$f_n^{(0)}(y,Y,t) = f_n(y,Y,t) \qquad f_0^{(k)}(y,Y,t) = f^{(k)}(y,Y,t)$$

therefore, it can be used to obtain the sequence of functions $f^{(n)}(y,Y,t)$ of Eq. (3.7) in terms of the sequence of functions $f_n(y,Y,t) = [f_n(x,X,t)]_{\substack{x=y \\ X=Y}}$ given by (3.6). (This can be visualized from the triangle of Fig. 3.)

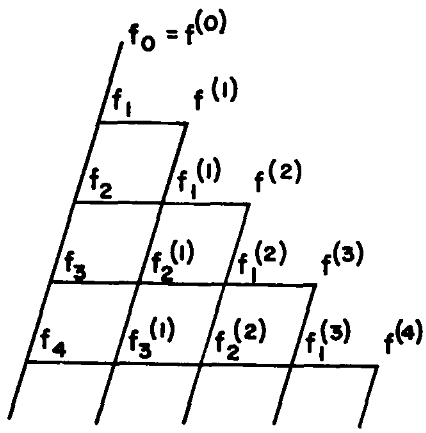

Fig. 3. RECURSIVE TRANSFORMATION OF AN ANALYTIC FUNCTION UNDER A LIE TRANSFORM.

For example,

$$f^{(1)} = f_1 + L_1 f_0 \tag{3.16a}$$

$$f_1^{(1)} = f_2 + L_1 f_1 + L_2 f_0 \tag{3.16b}$$

$$f^{(2)} = f_1^{(1)} + L_1 f^{(1)} \tag{3.16c}$$

$$f_2^{(1)} = f_3 + L_1 f_2 + 2L_2 f_1 + L_3 f_0 \tag{3.16d}$$

$$f_1^{(2)} = f_2^{(1)} + L_1 f_1^{(1)} + L_2 f^{(1)} \tag{3.16e}$$

$$f^{(3)} = f_1^{(2)} + L_1 f^{(2)} \tag{3.16f}$$

Similar triangles for $H^{(n)}$, $R^{(n)}$, $y^{(n)}$, and $Y^{(n)}$ are illustrated in Fig. 4.

Finally, the inverse transformation can be written as

$$y = x + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} x^{(n)}(x,X,t) \tag{3.17a}$$

$$Y = X + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} X^{(n)}(x,X,t) \tag{3.17b}$$

To find the relation between the $x^{(n)}$, $y^{(n)}$ and the $X^{(n)}$, $Y^{(n)}$, one can eliminate x-y, X-Y between Eqs. (3.11) and (3.17) and define the functions $q(x,X,t;\epsilon)$ and $Q(x,X,t,\epsilon)$ as

$$q(x,X,t;\epsilon) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} x^{(n)}(x,X,t)$$

$$= - \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} y^{(n)}(y,Y,t) \tag{3.18}$$

and

$$Q(x,X,t;\epsilon) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} X^{(n)}(x,X,t)$$

$$= - \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} Y^{(n)}(y,Y,t) \tag{3.19}$$

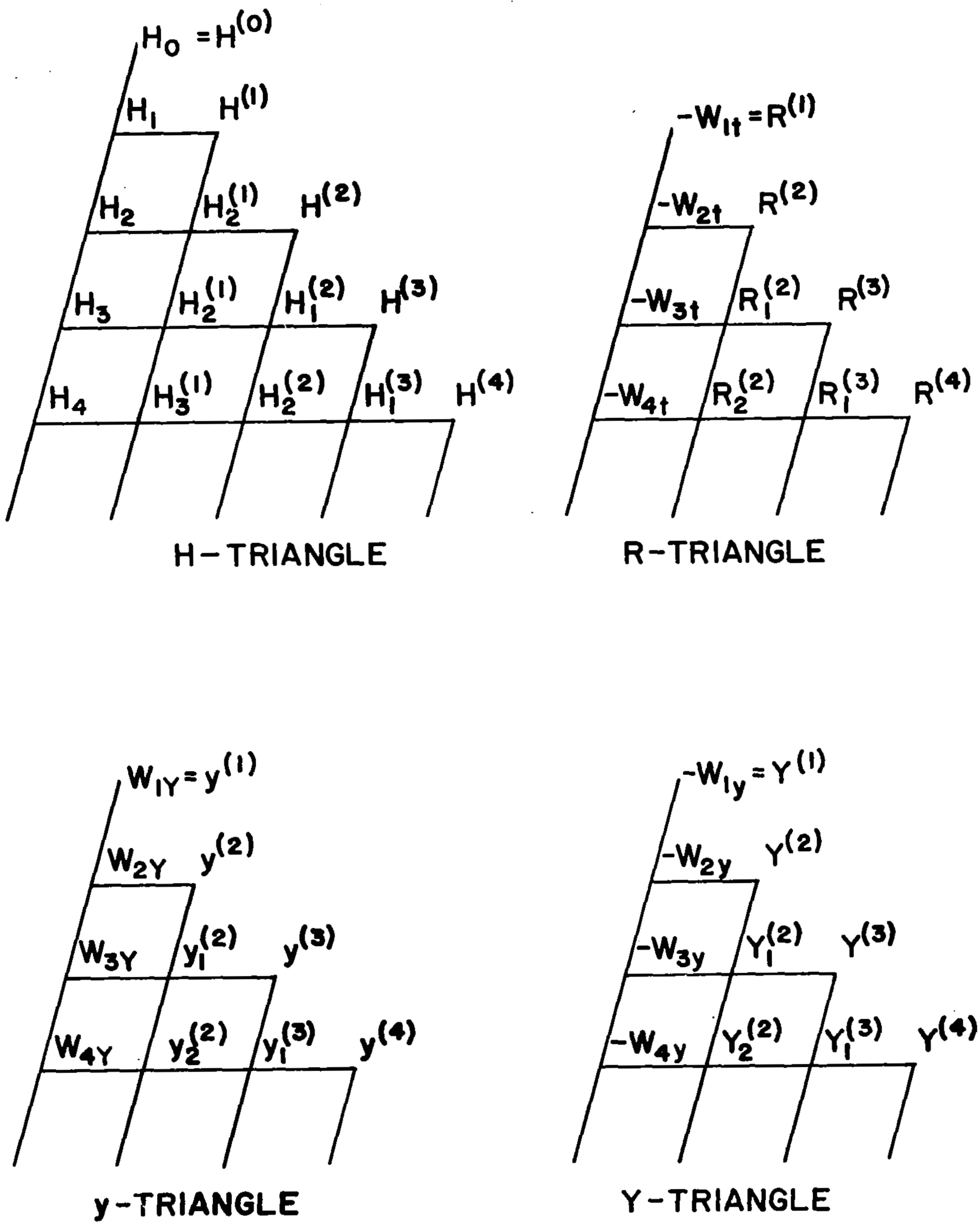

Fig. 4. TRIANGLES FOR THE HAMILTONIAN H, THE COORDINATES y, THE MOMENTA Y, AND THE REMAINDER R.

Comparison of Eqs. (3.18) and (3.19) with (3.6) and (3.7) leads to

$$q_0 = q^{(0)} = 0 \tag{3.20a}$$

$$Q_0 = Q^{(0)} = 0 \tag{3.20b}$$

and, for $n \geq 1$,

$$q_n = x^{(n)}(x,X,t) \qquad q^{(n)} = -y^{(n)}(y,Y,t) \tag{3.20c}$$

$$Q_n = X^{(n)}(x,X,t) \qquad Q^{(n)} = -Y^{(n)}(y,Y,t) \tag{3.20d}$$

B. Simplified General Expansions

Given the sequence of functions f_n, f_{n-1}, ..., and f_0, there are two approaches to constructing the required functions $f^{(n)}$ $(n \geq 0)$. The first is the Deprit approach which introduces certain auxiliary functions $f_n^{(k)}$ and moves recursively from the left to the right diagonal of Fig. 3.

The second approach, proposed here, is to construct $f^{(n)}$ $(n \geq 0)$ only in terms of f_n, $f^{(n-1)}$, ..., and $f^{(0)}$ by introducing a suitable linear operator. This approach is useful in constructing the inverse transformation and simplified general expansions.

To illustrate how this can be accomplished, let the Deprit equation be written as

$$f_n^{(k)} = f_{n-1}^{(k+1)} - \sum_{m=0}^{n-1} C_m^{n-1} L_{m+1} f_{n-m-1}^{(k)} \qquad \begin{matrix} n \geq 1 \\ k \geq 0 \end{matrix} \tag{3.21}$$

By successive elimination of the functions on the right-hand side of the above equation, $f_n^{(k)}$ eventually is obtained in terms of $f^{(k+n)}$, $f^{(k+n-1)}$, ..., and $f^{(k)}$; thus, for $f_n^{(k)}$, the form

$$f_n^{(k)} = f^{(k+n)} - \sum_{j=1}^{n} C_j^n \, G_j \, f^{(k+n-j)} \qquad \begin{matrix} n \geq 1 \\ k \geq 0 \end{matrix} \tag{3.22}$$

can be assumed, where G_j is a linear operator and is a function of L_j, L_{j-1}, ..., and L_1. Substitution of Eq. (3.22) into (3.21) yields the recursion relation

$$G_j = L_j - \sum_{m=1}^{j-1} C_{m-1}^{j-1} \, L_m \, G_{j-m} \qquad 1 \leq j \leq n \tag{3.23}$$

For example,

$$G_1 = L_1 \tag{3.24a}$$

$$G_2 = L_2 - L_1 L_1 \tag{3.24b}$$

$$G_3 = L_3 - L_1 (L_2 - L_1 L_1) - 2L_2 L_1 \tag{3.24c}$$

For $k = 0$ and $k = 1$, Eq. (3.22) yields

$$f^{(n)} = f_n + \sum_{j=1}^{n} C_j^n \, G_j \, f^{(n-j)} \tag{3.25a}$$

$$f_n^{(1)} = f^{(n+1)} - \sum_{j=1}^{n} C_j^n \, G_j \, f^{(n-j+1)} \tag{3.25b}$$

Also, if $G_j f^{(k)}$ is defined as $f_{j,i}$, Eqs. (3.25) can be written as

$$f^{(n)} = f_n + \sum_{j=1}^{n} C_j^n \, f_{j,n-j} \tag{3.26a}$$

$$f_n^{(1)} = f^{(n+1)} - \sum_{j=1}^{n} C_j^n \, f_{j,n-j+1} \tag{3.26b}$$

where

$$f_{j,i} = L_j f^{(i)} - \sum_{m=1}^{j-1} C_{m-1}^{j-1} \, L_m \, f_{j-m,i}$$

Using Eq. (3.26b), with $f = y,Y$ and with the help of the y and Y triangles, the general recursive relations for $y^{(n)}$ and $Y^{(n)}$ of Eqs. (3.11) can be obtained:

$$y^{(n)} = W_{nY} + \sum_{j=1}^{n-1} C_j^{n-1} \, y_{j,n-j} \tag{3.27a}$$

$$Y^{(n)} = -W_{ny} + \sum_{j=1}^{n-1} C_j^{n-1} \, Y_{j,n-j} \tag{3.27b}$$

where

$$y_{j,i} = L_j y^{(i)} - \sum_{m=1}^{j-1} C_{m-1}^{j-1} \, L_m \, y_{j-m,i} \tag{3.28a}$$

$$Y_{j,i} = L_j Y^{(i)} - \sum_{m=1}^{j-1} C_{m-1}^{j-1} \, L_m \, Y_{j-m,i} \tag{3.28b}$$

Using Eq. (3.26a) with $f = q,Q$ of Eqs. (3.18) and (3.19) yields

$$x^{(n)} = -y^{(n)} + \sum_{j=1}^{n-1} C_j^n \, y_{j,n-j} \tag{3.29a}$$

$$X^{(n)} = -Y^{(n)} + \sum_{j=1}^{n-1} C_j^n \, Y_{j,n-j} \tag{3.29b}$$

where $y_{j,n-j}$ and $Y_{j,n-j}$ are defined by Eqs. (3.28). Now $x^{(n)}(x,X,t)$ and $X^{(n)}(x,X,t)$ of (3.17) are given simply by

$$x^{(n)}(x,X,t) = \left[x^{(n)}\right]_{\substack{y=x \\ Y=X}} \tag{3.30a}$$

$$X^{(n)}(x,X,t) = \left[X^{(n)}\right]_{\substack{y=x \\ Y=X}} \tag{3.30b}$$

Given the Hamiltonian $H(x,X,t;\epsilon)$ in the form

$$H(x,X,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n(x,X,t) \tag{3.31}$$

the transformed Hamiltonian $K(y,Y,t;\epsilon)$ must be constructed in the form

$$K(y,Y,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} K_n(y,Y,t) \tag{3.32}$$

The relationship between K_n and H_n can be obtained as follows.

Referring to Eq. (3.7), H can be written as

$$H(x,X,t;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H^{(n)}(y,Y,t) \tag{3.33}$$

Combination of (3.11c) and the above equation yields, for $n \geq 1$,

$$K_0 = H_0 \tag{3.34a}$$

$$K_n = H^{(n)} + R^{(n)} \tag{3.34b}$$

Using (3.26b) with $f = H+R$ leads to

$$H_n^{(1)} + R_n^{(1)} = K_{n+1} - \sum_{j=1}^{n} C_j^n \, K_{j,n-j+1} \qquad n \geq 1 \tag{3.35}$$

but from the H and R triangles of Fig. 4,

$$H_n^{(1)} = H_{n+1} + \sum_{m=0}^{n} C_m^n \, L_{m+1} \, H_{n-m} \qquad n \geq 0 \tag{3.36a}$$

$$R_n^{(1)} = -\left[W_{n+1}\right]_t \qquad n \geq 0 \tag{3.36b}$$

Therefore, the simplified general recursive relation of the transformed Hamiltonian is, for $n \geq 1$,

$$K_0 = H_0 \tag{3.37a}$$

$$K_n = H_n + \sum_{j=1}^{n-1} \left(C_{j-1}^{n-1} \, L_j \, H_{n-j} + C_j^{n-1} \, K_{j,n-j} \right) - \frac{DW_n}{Dt} \tag{3.37b}$$

where

$$\frac{DW_n}{Dt} = \frac{\partial W_n}{\partial t} - L_n \, H_0 \qquad n \geq 1 \tag{3.38a}$$

$$K_{j,i} = L_j \, K_i - \sum_{m=1}^{j-1} C_{m-1}^{j-1} \, L_m \, K_{j-m,i} \tag{3.38b}$$

Equations (3.37) and (3.38) are applicable directly to nonlinear resonant problems in which H_0 is a function of only the action variables $X = \tilde{\alpha}$, and $H_n (n \geq 1)$ depends trigonometrically on the angle variables $x = \tilde{\beta}$ and possibly on time t. It is advantageous to transform to new variables so that the resulting Hamiltonian contain (together with the new action variables $Y = \bar{\alpha}$) only certain slowly varying "long period" combinations of the new angle variables $y = \bar{\beta}$ and time t. Equations (3.37) and (3.38) can be used to define W_n successively so as to remove all "short period" terms from the K_n; such a W_n is unique up to an arbitrary additive long-period term. It should be noted that, in this case, W_n are easily obtainable as solutions to partial differential equations of first order.

When resonances do not occur, the transformation

$$x = x(y, Y, t; \epsilon)$$

$$X = X(y, Y, t; \epsilon) \tag{3.39}$$

can be constructed, which reduces K to

$$K = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} K_n(Y) \tag{3.40}$$

The solution of the original system (2.41) then reduces to the solution of a system in the form of

$$\dot{y} = K_Y \qquad \dot{Y} = 0 \tag{3.41}$$

which is given simply by $Y = Y_0$ and $y = K_Y(Y_0)t + y_0$, where Y_0 and y_0 are arbitrary constants.

Note that Eq. (3.26a) does not require f_n to be the given functions; in fact, (3.26a) has the property of constructing f_n from the $f^{(n)}$, which can be valuable in reducing the computation requirements when the given Hamiltonian is limited in order. This can be achieved by letting

$K = K_0 + \epsilon K_1$ be the given Hamiltonian; then, in view of Eq. (3.38b), by finding $K_{j,i} = 0$ for $i \geq 2$ and for all possible values of j. This fact and Eqs. (3.37) lead to the desired reduced formula for constructing the transformed Hamiltonian

$$H(x,X,t;\epsilon) = H(y,Y,t;\epsilon)\Big|_{\substack{y=x \\ Y=X}} \tag{3.42}$$

C. <u>The Formal Technique</u>

Consider the system of vector differential equations in the standard form,

$$\dot{x} = H_X \qquad \dot{X} = -H_x \tag{3.43}$$

with Hamiltonian

$$H(x,X,t;\epsilon) = H_0(x,X,t) + \epsilon H_1(x,X,t) + \frac{\epsilon^2}{2!} H_2(x,X,t) + \frac{\epsilon^3}{3!} H_3(x,X,t) + \ldots \tag{3.44}$$

The essence of the technique proposed in this chapter consists of constructing a canonical transformation $(x,X) \rightarrow (y,Y)$ to achieve specific requirements (e.g., elimination of short-period terms) in the transformed Hamiltonian

$$K(y,Y,t;\epsilon) = K_0(y,Y,t) + \epsilon K_1(y,Y,t) + \frac{\epsilon^2}{2!} K_2(y,Y,t) + \frac{\epsilon^3}{3!} K_3(y,Y,t) + \ldots \tag{3.45}$$

This canonical transformation is expressed explicitly in the power series,

$$x = y + \epsilon\, y^{(1)}(y,Y,t) + \frac{\epsilon^2}{2!} y^{(2)}(y,Y,t) + \frac{\epsilon^3}{3!} y^{(3)}(y,Y,t) + \ldots \tag{3.46a}$$

$$X = Y + \epsilon Y^{(1)}(y,Y,t) + \frac{\epsilon^2}{2!} Y^{(2)}(y,Y,t) + \frac{\epsilon^3}{3!} Y^{(3)}(y,Y,t) + \ldots \tag{3.46b}$$

and in its inverse,

$$y = x + \epsilon x^{(1)}(x,X,t) + \frac{\epsilon^2}{2!} x^{(2)}(x,X,t) + \frac{\epsilon^3}{3!} x^{(3)}(x,X,t) + \ldots \tag{3.47a}$$

$$Y = X + \epsilon X^{(1)}(x,X,t) + \frac{\epsilon^2}{2!} X^{(2)}(x,X,t) + \frac{\epsilon^3}{3!} X^{(3)}(x,X,t) + \ldots \tag{3.47b}$$

Furthermore, any analytic function $f(x,X,t;\epsilon)$ given by

$$f(x,X,t;\epsilon) = f_0(x,X,t) + \epsilon f_1(x,X,t) + \frac{\epsilon^2}{2!} f_2(x,X,t) + \frac{\epsilon^3}{3!} f_3(x,X,t) + \ldots \tag{3.48}$$

can be expressed as

$$f(x,X,t;\epsilon) = f^{(0)}(y,Y,t) + \epsilon f^{(1)}(y,Y,t) + \frac{\epsilon^2}{2!} f^{(2)}(y,Y,t) + \frac{\epsilon^3}{3!} f^{(3)}(y,Y,t) + \ldots \tag{3.49}$$

The operations performed to carry out the canonical transformation are basically recursive and initiated by taking

$$K_0(y,Y,t) = H_0(y,Y,t) \tag{3.50a}$$

$$f^{(0)}(y,Y,t) = f_0(y,Y,t) \tag{3.50b}$$

The first-order operation begins by considering the linear partial differential relation

$$K_1(y,Y,t) = H_1(y,Y,t) - \frac{DW_1}{Dt} \tag{3.51}$$

Assuming that a choice has been made for K_1, Eq. (3.51) can be integrated to obtain $W_1(y,Y,t)$, and the following sequence is computed

$$\begin{aligned}
y^{(1)} &= W_{1Y} \\
Y^{(1)} &= -W_{1y} \\
x^{(1)} &= -y^{(1)} \\
X^{(1)} &= -Y^{(1)} \\
f_{1,0} &= L_1 f^{(0)} \\
f^{(1)} &= f_1 + f_{1,0}
\end{aligned} \tag{3.52}$$

to complete the process of first ordering.

To prepare for second-order expansion,

$$K_{1,1} = L_1 K_1 \tag{3.53}$$

is computed and, at second-order level, the partial differential relation

$$K_2 = H_2 + L_1 H_1 + K_{1,1} - \frac{DW_2}{Dt} \tag{3.54}$$

is set up. The unknown function K_2 is selected in compliance with the goals proposed for the transformation, and the resulting linear partial equation is integrated to yield $W_2(y,Y,t)$. The second-order expansion is completed by computing

$$
\begin{aligned}
y_{1,1} &= L_1 y^{(1)} \\
Y_{1,1} &= L_1 Y^{(1)} \\
y^{(2)} &= W_{2Y} + y_{1,1} \\
Y^{(2)} &= -W_{2y} + Y_{1,1} \\
x^{(2)} &= -y^{(2)} + 2y_{1,1} \\
X^{(2)} &= -Y^{(2)} + 2Y_{1,1} \\
f_{1,1} &= L_1 f^{(1)} \\
f_{2,0} &= L_2 f^{(0)} - L_1 f_{1,0} \\
f^{(2)} &= f_2 + 2f_{1,1} + f_{2,0}
\end{aligned}
\tag{3.55}
$$

To prepare for the third-order expansion,

$$K_{1,2} = L_1 K_2 \tag{3.56a}$$

$$K_{2,1} = L_2 K_1 - L_1 K_{1,1} \tag{3.56b}$$

is computed and, at third-order level, the partial differential equation

$$K_3 = H_3 + L_1 H_2 + 2L_2 H_1 + 2K_{1,2} + K_{2,1} - \frac{DW_3}{Dt} \tag{3.57}$$

is formed. The unknown function is chosen, and the resulting partial differential equation is solved to yield W_3. The following sequence of operations will complete the third order.

$$
\begin{aligned}
y_{1,2} &= L_1 y^{(2)} \\
Y_{1,2} &= L_1 Y^{(2)} \\
y_{2,1} &= L_2 y^{(1)} - L_1 y_{1,1} \\
Y_{2,1} &= L_2 Y^{(1)} - L_1 Y_{1,1} \\
y^{(3)} &= W_{3Y} + 2y_{1,2} + y_{2,1} \\
Y^{(3)} &= -W_{3y} + 2Y_{1,2} + Y_{2,1} \\
x^{(3)} &= -y^{(3)} + 3y_{1,2} + 3y_{2,1} \\
X^{(3)} &= -Y^{(3)} + 3Y_{1,2} + 3Y_{2,1} \\
f_{1,2} &= L_1 f^{(2)} \\
f_{2,1} &= L_2 f^{(1)} - L_1 f_{1,1} \\
f_{3,0} &= L_3 f^{(0)} - L_1 f_{2,0} - 2L_2 f_{1,0} \\
f^{(3)} &= f_3 + 3f_{1,2} + 3f_{2,1} + f_{3,0}
\end{aligned}
\tag{3.58}
$$

The entire procedure can be extended to any order by using Eqs. (3.6), (3.7), (3.11), (3.17), (3.26a) through (3.32), (3.37), and (3.38). Compared to the technique proposed by Deprit [15], one finds that the operations outlined above are simpler and require less computer time and storage.

D. Simple Examples

1. Example 1

Consider the nonlinear differential equation

$$\ddot{q} + q + \epsilon q^3 = 0 \tag{3.59}$$

Now, define

$$\dot{q} = p \tag{3.60a}$$

and, with reference to Eq. (3.59),

$$\dot{p} = -(q + \epsilon q^3) \tag{3.60b}$$

which can be put in the form,

$$\dot{q} = R_p \tag{3.61a}$$

$$\dot{p} = -R_q \tag{3.61b}$$

where R is the Hamiltonian given by

$$R = R_0 + \epsilon R_1 \tag{3.62a}$$

$$R_0 = \frac{1}{2} (p^2 + q^2) \tag{3.62b}$$

$$R_1 = \frac{1}{4} q^4 \tag{3.62c}$$

Following the steps outlined in Chapter II.E, the standard form of the above equations can be obtained through the stationary transformation $(q,p) \rightarrow (\tilde{\beta},\tilde{\alpha})$ defined by

$$q = \sqrt{2\tilde{\alpha}} \sin \tilde{\beta} \tag{3.63a}$$

$$p = \sqrt{2\tilde{\alpha}} \cos \tilde{\beta} \tag{3.63b}$$

which is canonical in view of Eq. (2.26). The new Hamiltonian $H = R$ now takes the form

$$H = H_0 + \epsilon H_1 \tag{3.64}$$

where

$$H_0 = \tilde{\alpha}$$

and

$$H_1 = \frac{\tilde{\alpha}^2}{8} (3 - 4 \cos 2\tilde{\beta} + \cos 4\tilde{\beta})$$

It is now desirable to transform from $(\tilde{\alpha}, \tilde{\beta})$ to $(\bar{\alpha}, \bar{\beta})$ so that the transformed Hamiltonian K contains only secular terms.

Following the steps outlined in the previous section (up to second order) with $x = \tilde{\beta}$, $X = \tilde{\alpha}$, $y = \bar{\beta}$, $Y = \bar{\alpha}$, and $f = q$ and eliminating the intermediate steps in the analysis obtains

$$W_1 = \frac{\bar{\alpha}^2}{32} (\sin 4\bar{\beta} - 8 \sin 2\bar{\beta}) \qquad (3.65a)$$

$$W_2 = - \frac{\bar{\alpha}^3}{192} (\sin 6\bar{\beta} + 9 \sin 4\bar{\beta} + 99 \sin 2\bar{\beta}) \qquad (3.65b)$$

$$K = \bar{\alpha} + \frac{3}{8} \bar{\alpha}^2 \epsilon - \frac{17}{64} \bar{\alpha}^3 \epsilon^2 \qquad (3.65c)$$

$$\tilde{\alpha} = \bar{\alpha} - \epsilon \frac{\bar{\alpha}^2}{8} (\cos 4\bar{\beta} - 4 \cos 2\bar{\beta})$$

$$+ \epsilon^2 \frac{\bar{\alpha}^3}{64} (2 \cos 6\bar{\beta} + 6 \cos 4\bar{\beta} - 42 \cos 2\bar{\beta} + 17) \qquad (3.65d)$$

$$\tilde{\beta} = \bar{\beta} + \epsilon \frac{\bar{\alpha}}{16} (\sin 4\bar{\beta} - 8 \sin 2\bar{\beta})$$

$$+ \epsilon^2 \frac{\bar{\alpha}^2}{512} (\sin 8\bar{\beta} - 16 \sin 6\bar{\beta} - 4 \sin 4\bar{\beta}$$

$$+ 400 \sin 2\bar{\beta}) \qquad (3.65e)$$

$$\bar{\alpha} = \tilde{\alpha} + \epsilon \frac{\tilde{\alpha}^2}{8} (\cos 4\tilde{\beta} - 4 \cos 2\tilde{\beta})$$

$$- \epsilon^2 \frac{\tilde{\alpha}^3}{64} (6 \cos 4\tilde{\beta} - 24 \cos 2\tilde{\beta} - 17) \qquad (3.65f)$$

$$\bar{\beta} = \tilde{\beta} - \epsilon \frac{\tilde{\alpha}}{16} (\sin 4\tilde{\beta} - 8 \sin 2\tilde{\beta})$$

$$+ \epsilon^2 \frac{\tilde{\alpha}^2}{512} (\sin 8\tilde{\beta} - 8 \sin 6\tilde{\beta} + 68 \sin 4\tilde{\beta}$$

$$- 392 \sin 2\tilde{\beta}) \tag{3.65g}$$

$$q = (2\bar{\alpha})^{1/2} \sin \bar{\beta} - \epsilon \frac{(2\bar{\alpha})^{3/2}}{32} (\sin 3\bar{\beta} + 6 \sin \bar{\beta})$$

$$+ \epsilon^2 \frac{(2\bar{\alpha})^{5/2}}{2048} (2 \sin 5\bar{\beta} + 78 \sin 3\bar{\beta} + 303 \sin \bar{\beta}) \tag{3.65h}$$

Referring to Eq. (3.65c) and according to the last step in the method of solution, the transformed system of equations

$$\dot{\bar{\alpha}} = -K_{\bar{\beta}} = 0 \tag{3.66a}$$

$$\dot{\bar{\beta}} = K_{\bar{\alpha}} = 1 + \frac{3}{4} \bar{\alpha}\epsilon - \frac{51}{64} \bar{\alpha}^2 \epsilon^2 \tag{3.66b}$$

can be solved. This solution is given simply by

$$\bar{\alpha} = \bar{\alpha}_0(\epsilon) \tag{3.67a}$$

$$\bar{\beta} = \left(1 + \frac{3}{4} \bar{\alpha}_0 \epsilon - \frac{51}{64} \bar{\alpha}_0^2 \epsilon^2\right)t + \bar{\beta}_0(\epsilon) \tag{3.67b}$$

where $\bar{\alpha}_0(\epsilon)$ and $\bar{\beta}_0(\epsilon)$ are constants determined by the initial conditions of (3.59). Equations (3.63) and the inverse transformations (3.65f) and (3.65g) are useful for this purpose.

It can be noted that there are two integrals of motion. The first is exact and determined by the fact that the Hamiltonian R is a constant (also, it is the total energy of the system). The second integral (3.67a) is approximate in terms of q and p and can be determined from the inverse transformation (3.65f) as

$$\frac{1}{2}(p^2 + q^2) - \frac{\epsilon}{32}(3p^2 + 6p^2q^2 - 5q^4) + \frac{\epsilon^2}{512}(q^2 + p^2)$$

$$(35p^4 + 70p^2q^2 - 13q^4) + O(\epsilon^3) = \bar{\alpha}_0(\epsilon) \qquad (3.68)$$

The algebra of the above example can be carried out in terms of the noncanonical variables for the amplitude and phase. This can be done by applying Eqs. (3.37) and (3.38) for a canonical transformation $(q,p) \rightarrow (\bar{q},\bar{p})$ and by defining $\bar{q}$ as $\bar{A} \sin \bar{\emptyset}$ and $\bar{p}$ as $\bar{A} \cos \bar{\emptyset}$. In this case, $DW_n/Dt = \partial W_n/\partial\bar{\emptyset}$, and W_n can be used to eliminate the short-period terms from K. For this elimination, $\bar{q}$ and $\bar{p}$ must be substituted in terms of $\bar{A}$ and $\bar{\emptyset}$. After the elimination, $\bar{A}$ and $\bar{\emptyset}$ are then substituted back to $\bar{q}$ and $\bar{p}$, and the necessary Poisson brackets for the next step are then computed in terms of these $\bar{q}$ and $\bar{p}$, and so on. For further details see Refs. 27 and 28.

2. Example 2

Consider the near-resonant Mathieu equation

$$\ddot{q} + (1 + \epsilon \cos 2\omega t)q = 0 \qquad (3.69)$$

where $\omega = (1 + \eta)$, and $\eta = O(\epsilon)$ is called the detuning. The effect of the trigonometric coefficient is to introduce a small fluctuation in the spring constant of the simple harmonic oscillator. Now, define

$$p = \dot{q} \qquad (3.70a)$$

and, with reference to Eq. (3.69),

$$\dot{p} = -(1 + \epsilon \cos 2\omega t)q \qquad (3.70b)$$

which can be put in the form,

$$\dot{q} = R_p \qquad (3.71a)$$

$$\dot{p} = -R_q \qquad (3.71b)$$

where R is the Hamiltonian given by

$$R = R_0 + \epsilon R_1 \tag{3.72a}$$

$$R_0 = \frac{1}{2}(p^2 + q^2) \tag{3.72b}$$

$$R_1 = \frac{1}{2} q^2 \cos 2\omega t \tag{3.72c}$$

As in the previous example, the canonical stationary transformation (3.63) is used. The new Hamiltonian H then takes the form of

$$H = H_0 + \epsilon H_1 \tag{3.73}$$

where

$$H_0 = \tilde{\alpha}$$

and

$$H_1 = \frac{\tilde{\alpha}}{2}\left[\cos 2\omega t - \frac{1}{2}\cos 2(\omega t + \tilde{\beta}) - \frac{1}{2}\cos 2(\omega t - \tilde{\beta})\right]$$

Because $\dot{\tilde{\beta}} = H_{\tilde{\alpha}} = 1 + O(\epsilon)$, one expects that $\omega - \dot{\tilde{\beta}} = O(\epsilon)$ and that direct elimination of $\cos 2(\omega t - \tilde{\beta})$ by the generating function W_1 will lead to the so-called "small divisor" which endangers the convergence of the perturbation development (for example, see Ref. 21, p. 296). To avoid the appearance of this small divisor, $\cos 2(\omega t - \tilde{\beta})$ should be treated in the same way as the secular terms in example 1.

By ignoring the additive long-period terms in W_1 and W_2, the second-order analysis leads to the following results:

$$W_1 = \bar{\alpha}\left[a_1 \sin 2\omega t - a_2 \sin 2(\omega t + \bar{\beta})\right] \tag{3.74a}$$

$$W_2 = \frac{\bar{\alpha}}{4}\left[a_2 \sin 2\bar{\beta} + 8a_3 \sin 2(2\omega t + \bar{\beta})\right.$$

$$\left. - 8a_4 \sin 2(2\omega t - \bar{\beta}) - 2a_1 a_2 \sin 4\omega t\right] \tag{3.74b}$$

$$K = \bar{\alpha} - \epsilon\frac{\bar{\alpha}}{4}\cos 2(\omega t - \bar{\beta}) - \frac{\epsilon^2}{4}\, a_2\bar{\alpha} \tag{3.74c}$$

$$\begin{aligned}\tilde{\alpha} = {} & \bar{\alpha} + 2a_2\bar{\alpha}\epsilon\cos 2(\omega t + \bar{\beta}) \\ & + \epsilon^2\bar{\alpha}\Big[-2a_4\cos 2(2\omega t - \bar{\beta}) + (a_1a_2 - 2a_3)\cos 2(2\omega t + \bar{\beta}) \\ & - \left(a_1a_2 + \frac{a_2}{4}\right)\cos 2\bar{\beta} + 2a_2^2\Big]\end{aligned} \tag{3.74d}$$

$$\begin{aligned}\tilde{\beta} = {} & \bar{\beta} + \epsilon\left[a_1\sin 2\omega t - a_2\sin 2(\omega t + \bar{\beta})\right] \\ & + \frac{\epsilon^2}{2}\Big[-2a_4\sin 2(2\omega t - \bar{\beta}) + (2a_3 - a_1a_2)\sin 2(2\omega t + \bar{\beta}) \\ & + a_2^2\sin 4(\omega t + \bar{\beta}) - \frac{a_1a_2}{2}\sin 4\omega t \\ & + \left(a_1a_2 + \frac{a_2}{4}\right)\sin 2\bar{\beta}\Big]\end{aligned} \tag{3.74e}$$

$$\begin{aligned}\bar{\alpha} = {} & \tilde{\alpha} - 2a_2\tilde{\alpha}\epsilon\cos 2(\omega t + \tilde{\beta}) \\ & + \epsilon^2\tilde{\alpha}\Big[2a_4\cos 2(2\omega t - \tilde{\beta}) + (a_1a_2 + 2a_3)\cos 2(2\omega t + \tilde{\beta}) \\ & - \left(a_1a_2 - \frac{a_2}{4}\right)\cos 2\tilde{\beta} + 2a_2^2\Big]\end{aligned} \tag{3.74f}$$

$$\begin{aligned}\bar{\beta} = {} & \tilde{\beta} - \epsilon\left[a_1\sin 2\omega t - a_2\sin 2(\omega t + \tilde{\beta})\right] \\ & + \frac{\epsilon^2}{2}\Big[2a_4\sin 2(2\omega t - \tilde{\beta}) - (a_1a_2 + 2a_3)\sin 2(\omega t + \tilde{\beta}) \\ & + a_2^2\sin 4(\omega t + \tilde{\beta}) + \frac{a_1a_2}{2}\sin 4\omega t \\ & + \left(a_1a_2 - \frac{a_2}{4}\right)\sin 2\tilde{\beta}\Big]\end{aligned} \tag{3.74g}$$

$$q = (2\bar{\alpha})^{1/2} \sin \bar{\beta} + \epsilon(2\bar{\alpha})^{1/2}\left[\left(\frac{a_1}{2} - a_2\right) \sin (2\omega t + \bar{\beta})\right.$$

$$\left. + \frac{a_1}{2} \sin (2\omega t - \bar{\beta})\right] + \frac{\epsilon^2}{2} (2\bar{\alpha})^{1/2}\left[-\left(2a_4 + \frac{a_1 a_2}{4} + \frac{a_1^2}{4}\right)\right.$$

$$\sin (4\omega t - \bar{\beta}) + \left(2a_3 - \frac{a_1 a_2}{4} + \frac{1}{4} a_1^2\right) \sin (4\omega t + \bar{\beta})$$

$$\left. + \left(a_2^2 + \frac{a_2}{4} - \frac{a_1^2}{2}\right) \sin \bar{\beta}\right] \tag{3.74h}$$

where

$$a_1 = 1/4\omega$$

$$a_2 = 1/16(1+\omega)$$

$$a_3 = (2a_2 - a_1)/16(2\omega+1)$$

$$a_4 = a_1/16(1-2\omega)$$

To solve the transformed equations, time t is first eliminated from (3.74c) by using a generating function S of the form

$$S = \alpha^*(\bar{\beta} - \omega t) \tag{3.75}$$

where α^* is the new momentum. With reference to Eq. (2.30), the new coordinate β^* and the old momentum $\bar{\alpha}$ are

$$\beta^* = S_{\alpha^*} = \bar{\beta} - \omega t \tag{3.76a}$$

$$\bar{\alpha} = S_{\bar{\beta}} = \alpha^* \tag{3.76b}$$

The transformed Hamiltonian K^* then is defined by Eq. (2.29) as

$$\begin{aligned} K^* &= K + S_t \\ &= -\eta\alpha^* - \epsilon \frac{\alpha^*}{4} \cos 2\beta^* - \frac{\epsilon^2}{4} a_2 \alpha^* \\ &= -\left(\eta + \frac{\epsilon^2}{4} a_2\right)\alpha^* - \epsilon \frac{\alpha^*}{4} \cos 2\beta^* \end{aligned} \tag{3.77}$$

where the definition of the detuning η is used.

Because K^* does not depend explicitly on t, it is a constant of the motion. The stability or instability of a system governed by Hamiltonians of the form (3.77) can be established by a comparison of the magnitude of the coefficients of α^* and $\alpha^* \cos 2\beta^*$. The relative magnitudes required of these instability or stability coefficients can be determined as follows.

From (3.77), the differential equations for α^* and β^* are obtained:

$$\dot{\alpha}^* = -K^*_{\beta^*} = -\frac{\alpha^* \epsilon}{2} \sin 2\beta^* \tag{3.78a}$$

$$\dot{\beta}^* = K^*_{\alpha^*} = -\left(\eta + \frac{\epsilon^2}{4} a_2 + \frac{\epsilon}{4} \cos 2\beta^*\right) \tag{3.78b}$$

Squaring (3.78a) yields

$$\dot{\alpha}^{*2} = \frac{\alpha^{*2} \epsilon^2}{4} \left\{ 1 - \left[\frac{4K_0^*}{\epsilon \alpha^*} + \frac{4\left(\eta + \frac{\epsilon^2}{4} a_2\right)\alpha^*}{\epsilon \alpha^*} \right]^2 \right\} \tag{3.79}$$

after $\sin^2 2\beta^*$ is replaced from (3.77) and the constancy of K^* is used.

The condition necessary for $\dot{\alpha}^*$ to vanish is obtained by setting the right-hand side of the above equation equal to zero, i.e., at the intersection of the two lines

$$y = \pm \alpha^* \tag{3.80a}$$

and

$$y = \frac{4}{\epsilon} K_0^* + \frac{(4\eta + \epsilon^2 a_2)}{\epsilon} \alpha^* \tag{3.80b}$$

$$= -\alpha^* \cos 2\beta^*$$

as shown in Fig. 5. From this sketch and with the help of Eqs. (3.78) and (3.80), at point 1,

$$\dot{\alpha}^* = 0 \qquad \dot{\beta}^* < 0$$

$$\alpha_1^* = \alpha_0^* \frac{4\eta + \epsilon^2 a_2 + \epsilon \cos 2\beta_0^*}{4\eta + \epsilon + \epsilon^2 a_2}$$

$$\beta_1^* = n\,\frac{\pi}{2} \qquad |n| = 1,3,5,\ \ldots \tag{3.81}$$

at point 2,

$$\dot{\alpha}^* = \pm\,\alpha_2^*\,\epsilon/2 \qquad \dot{\beta}^* < 0$$

$$\alpha_2^* = \alpha_0^* \frac{4\eta + \epsilon^2 a_2 + \epsilon \cos 2\beta_0^*}{4\eta + \epsilon^2 a_2} \tag{3.82}$$

$$\beta_2^* = n\,\frac{\pi}{4} \qquad |n| = 1,3,5,\ \ldots$$

at point 3,

$$\dot{\alpha}^* = 0 \qquad \dot{\beta}^* < 0$$

$$\alpha_3^* = \alpha_0^* \frac{4\eta + \epsilon^2 a_2 + \epsilon \cos 2\beta_0^*}{4\eta - \epsilon + \epsilon^2 a_2}$$

$$\beta_3^* = n\,\frac{\pi}{2} \qquad |n| = 0,2,4,\ \ldots \tag{3.83}$$

and at point 4,

$$\dot{\alpha}^* = 0 \qquad \dot{\beta}^* > 0$$

$$\alpha_4^* = \alpha_0^* \frac{4\eta + \epsilon^2 a_2 + \epsilon \cos 2\beta_0^*}{4\eta - \epsilon + \epsilon^2 a_2}$$

$$\beta_4^* = n\,\frac{\pi}{2} \qquad |n| = 1,3,5,\ \ldots \tag{3.84}$$

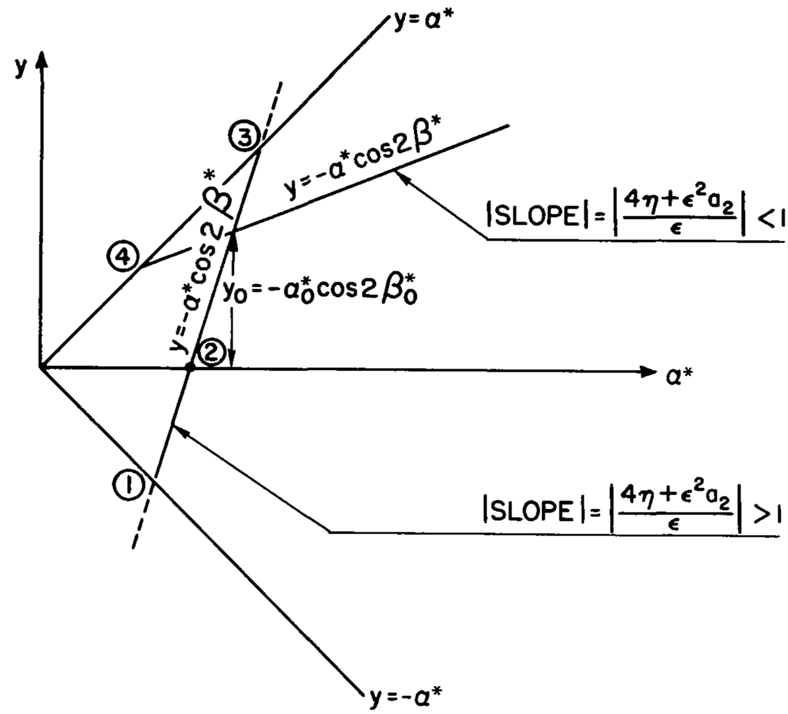

Fig. 5. STABILITY CONDITIONS FOR MATHIEU-TYPE HAMILTONIANS.

On the line from 1 to 3, β^* is monotonically increasing or decreasing depending on whether the slope is negative or positive, respectively; on the other line, α^* is monotonically increasing.

For some initial conditions α_0^* and β_0^*, the variation of α^* is bounded by the lines $y = \pm\alpha^*$ if $|(4\eta + \epsilon^2 a_2/\epsilon| > 1$, thus implying a stable motion; if $|(4\eta + \epsilon^2 a_2)/\epsilon| < 1$, α^* grows without limit. This leads to the conclusion that the motion is unstable if, in the Hamiltonian K^*, the magnitude of the coefficient of α^* is smaller than the magnitude of $\alpha^* \cos 2\beta^*$.

The transition curve between the stable and unstable regions noted above is defined by $|\text{slope}| = |(4\eta + \epsilon^2 a_2)/\epsilon| = 1$. On this boundary, unstable motion occurs if β_0^* is not a multiple of $\pi/2$. For slope = 1 and $\beta_0^* = \pi/2, 3\pi/2, \ldots$, or slope = -1 and $\beta_0^* = 0, \pi, 2\pi, \ldots$, a stationary solution is obtained (i.e., $\dot{\alpha}^* = \dot{\beta}^* = 0$). In these cases and referring to Eqs. (3.74h) and (3.76), q represents a family of periodic orbits whose period is twice the period of the perturbing excitation. These orbits, however, are unstable because, for any disturbance in phase β^*, unstable motion is obtained similar to that when $|\text{slope}| < 1$.

E. Perturbation Theory Based on the Lie Series

This section derives the general formulas of Hori's perturbation theory based on the Lie series. Hori [27] chose W and f in Eqs. (3.7) and (3.9) so that they do not depend explicitly on η. In this case,

$$f(x,X)\Big|_{\eta=\epsilon} = f(y,Y) + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!}\left[\frac{d^n f(x,X)}{d\eta^n}\right]_{\eta=0} \tag{3.85}$$

and

$$\frac{df(x,X)}{d\eta} = L_W\, f(x,X) \tag{3.86}$$

Because the right-hand side of the above equation does not depend explicitly on η,

$$\begin{aligned}\frac{d^2 f(x,X)}{d\eta^2} &= \frac{d}{d\eta}\left[\frac{df(x,X)}{d\eta}\right] \\ &= L_W\left[L_W\, f(x,X)\right] \\ &= L_W^2\, f(x,X)\end{aligned} \tag{3.87a}$$

can be written, in general, for $k \geq 1$,

$$\frac{d^k f(x,X)}{d\eta^k} = L_W^k\, f(x,X) \tag{3.87b}$$

To construct Hori's expansion, choose

$$\epsilon W(x,X) = \sum_{n=1}^{\infty} W_n(x,X) \tag{3.88a}$$

$$f(x,X) = \sum_{n=0}^{\infty} f_n(x,X) \tag{3.88b}$$

Equations (3.86) and (3.88) now lead to

$$\epsilon \frac{df(x,X)}{d\eta} = \sum_{n=0}^{\infty} f_n^{(1)}(x,X) \tag{3.89}$$

where

$$f_n^{(1)}(x,X) = \sum_{m=0}^{n} L_{m+1}\, f_{n-m}(x,X)$$

In general, for $k \geq 1$,

$$\epsilon^k \frac{df^{(k)}(x,X)}{d\eta^k} = \sum_{n=0}^{\infty} f_n^{(k)}(x,X) \tag{3.90}$$

where

$$f_n^{(k)}(x,X) = \sum_{m=0}^{n} L_{m+1}\, f_{n-m}^{(k-1)}(x,X)$$

Therefore,

$$\epsilon^k \left(\frac{d^k f(x,X)}{d\eta^k}\right)_{\eta=0} = \sum_{n=0}^{\infty} f_n^{(k)}(y,Y) \tag{3.91}$$

where

$$f_n^{(k)}(y,Y) = \sum_{m=0}^{n} L_{m+1}\, f_{n-m}^{(k-1)}(y,Y)$$

Because the transformation is stationary (W does not depend explicitly on t), and with reference to Eq. (3.11c) and its definition,

$$H(x,X) = K(y,Y) \tag{3.92a}$$

or

$$\sum_{n=0}^{\infty} H_n(x,X) = \sum_{n=0}^{\infty} K_n(y,Y) \tag{3.92b}$$

substituting $f = H(x,X)$ in Eq. (3.85) and making use of (3.91) obtain

$$K_0(y,Y) = H_0(y,Y) \tag{3.93a}$$

$$K_n(y,Y) = H_n(y,Y) + \sum_{m=0}^{n-1} \frac{1}{(n-m)!}\, H_m^{(n-m)}(y,Y) \qquad n \geq 1 \tag{3.93b}$$

where

$$H_m^{(i)}(y,Y) = \sum_{j=0}^{m} L_{j+1}\, H_{m-j}^{(i-1)}(y,Y) \tag{3.94}$$

These equations provide the general formulas needed to construct the transformed Hamiltonian for Hori's method. They can be written in the form,

$$K_n(y,Y) = H_n(y,Y) + \sum_{m=0}^{n-2} \left[L_{m+1} H_{n-m-1}(y,Y) + \frac{1}{(n-m)!} H_m^{(n-m)}(y,Y) \right]$$

$$+ L_n H_0(y,Y) \tag{3.95}$$

In computing $H_m^{(n-m)}$ from (3.94) one may take

$$H_i^{(1)} = K_{i+1} - H_{i+1} - \sum_{m=0}^{i-1} \frac{1}{(i-m+1)!} H_m^{(i-m+1)} \tag{3.96}$$

The fourth-order expansions of (3.93) and (3.95) confirm those of Hori [27].

It should be noted that the expansions based on Lie transforms can be obtained from the expansions based on the Lie series by replacing W and f of Eqs. (3.88) by

$$f(x,X) = \sum \frac{1}{n!} f_n(x,X) \tag{3.97a}$$

$$\epsilon W(x,X) = \sum_{n=1}^{\infty} \frac{1}{n!} W'_n(x,X) \tag{3.97b}$$

where

$$W'_1 = W_1$$

$$W'_2 = W_2$$

$$W'_3 = W_3 + \frac{1}{2}(W_2;W_1)$$

$$W'_4 = W_4 + (W_3;W_1)$$

and so on.

It was shown [29] that expansions similar to those obtained in this chapter for treating nonlinear oscillation problems with non-Hamiltonian formulation can be obtained. In this case, one constructs the mapping $x \to \bar{x}$ so that the given system of differential equations

$$\dot{x} = f(x;\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} f_n(x) \tag{3.98}$$

reduces to a simpler form of

$$\dot{\bar{x}} = g(\bar{x};\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} g_n(\bar{x}) \tag{3.99}$$

in which $g(\bar{x};\epsilon)$ contains only some desirable terms.

In the same spirit as (3.1), this transformation can be obtained by introducing a generating vector W and the differential equation

$$\frac{dx}{d\eta} = W \tag{3.100}$$

with initial conditions $x = \bar{x}(t;\epsilon)$.

According to the choice of W [$W = W(x;\eta)$, $W(\bar{x};\eta)$, or $W(x;\epsilon)$], some related formulas are obtainable. For $W = W(x;\eta)$, the analyses were carried out in full detail, see Ref. 29. In the case when $W = W(\bar{x};\eta) = \sum_{n=1}^{\infty} \eta^{n-1}/(n-1)!\, W^{(n)}(\bar{x})$, the obtained expansions serve as general formulas for the well-known Krylov-Bogoliubov method of averaging [30] in which

$$x = \bar{x} + \sum_{n=1}^{\infty} \frac{\eta^n}{n!} W^{(n)}(\bar{x}) \tag{3.101}$$

In the following chapters, some of the formulas obtained here, based on *Lie transforms*, will be used in a fourth-order analysis of the motion of a particle in the neighborhood of L_4 of the earth-moon system in the presence of the sun.

Chapter IV

THE HAMILTONIAN R OF A PARTICLE NEAR L_4

In this chapter, Newton's gravitational law is used to obtain the Lagrangian of a particle moving in the gravitational fields of the sun, earth, and the moon. After being normalized, this Lagrangian is used to obtain the Hamiltonian of such a particle in the neighborhood of the L_4 libration point in a rotating frame. Because Newton's law is only valid in an inertial frame of reference, the following calculations begin with such an inertial frame in which the positions of the earth, moon, sun, and particle P are designated by the numbers 1, 2, 3, and 4, respectively, and by the position vectors $\vec{r}_i (i = 1,2,3,4)$, as shown in Fig. 6. Newton's law then takes the form of

$$\ddot{\vec{r}}_4 = \sum_{i=1}^{3} \mu_i \frac{\vec{r}_{4i}}{r_{4i}^3} \tag{4.1}$$

$$\ddot{\vec{r}}_1 = \sum_{i=2}^{4} \mu_i \frac{\vec{r}_{1i}}{r_{1i}^3} \tag{4.2}$$

where $\mu_i = Gm_i$; G is the gravitational constant, and m_i is the mass of the i^{th} body.

From vector addition,

$$\vec{r}_{14} = \vec{r}_4 - \vec{r}_1 \tag{4.3}$$

and

$$\ddot{\vec{r}}_{14} = \ddot{\vec{r}}_4 - \ddot{\vec{r}}_1 \tag{4.4a}$$

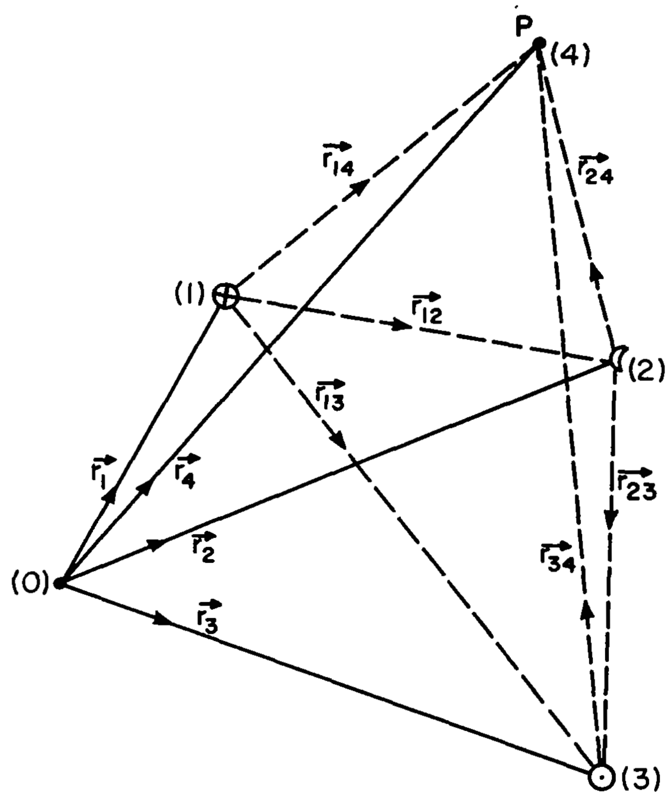

Fig. 6. GEOMETRY FOR THE PROBLEM OF FOUR BODIES.

which, in view of Eqs. (4.1) and (4.2), implies

$$\ddot{\vec{r}}_{14} = - \frac{(\mu_1 + \mu_4)}{r_{14}^3} \vec{r}_{14} + \sum_{i=2}^{3} \mu_i \left(\frac{\vec{r}_{4i}}{r_{4i}^3} - \frac{\vec{r}_{1i}}{r_{1i}^3} \right) \tag{4.4b}$$

This equation can be written as

$$\ddot{\vec{r}}_{14} = - \frac{\partial V}{\partial \vec{r}_{14}} \tag{4.5}$$

where V is the potential energy of the particle P defined by

$$V = -\frac{(\mu_1 + \mu_4)}{r_{14}} - \sum_{i=2}^{3} \mu_i \left(\frac{1}{r_{4i}} - \frac{\vec{r}_{1i} \cdot \vec{r}_{14}}{r_{1i}^3} \right) \tag{4.6}$$

Equation (4.5) can be derived from a Lagrangian $\mathfrak{L}$ of the form [see Eq. (2.1)]

$$\mathfrak{L} = \frac{1}{2} \dot{\vec{r}}_{14} \cdot \dot{\vec{r}}_{14} - V \tag{4.7}$$

Because $\mu_4 \ll \mu_1$, $\vec{r}_{12}(t)$ and $\vec{r}_{13}(t)$ are known essentially from lunar theory. Neglecting μ_4 in Eq. (4.6) and substituting for V in (4.7) lead to the Lagrangian

$$\mathfrak{L} = \frac{1}{2} \dot{\vec{r}}_{14} \cdot \dot{\vec{r}}_{14} + \frac{\mu_1}{r_{14}} + \mu_2 \left(\frac{1}{r_{24}} - \frac{\vec{r}_{12} \cdot \vec{r}_{14}}{r_{12}^3} \right)$$

$$+ \mu_3 \left(\frac{1}{r_{34}} - \frac{\vec{r}_{13} \cdot \vec{r}_{14}}{r_{13}^3} \right) \tag{4.8}$$

Before the Hamiltonian is obtained, it is convenient to nondimensionalize all the quantities. For this purpose, reference frequency n and length D are chosen, defined by

$$n = \sqrt{\frac{\mu_1 + \mu_2}{a^3}} = \text{mean angular velocity of the moon (see Fig. 7)} \cong 0.23 \text{ rad/day} \tag{4.9a}$$

$$D \cong \langle r_{12} \rangle = \text{mean distance between earth and moon}$$

$$\cong 2.4 \times 10^5 \text{ mi} \tag{4.9b}$$

It should be noted that the physical quantity that can be measured most accurately is n. Here length a is actually a computed rather than a

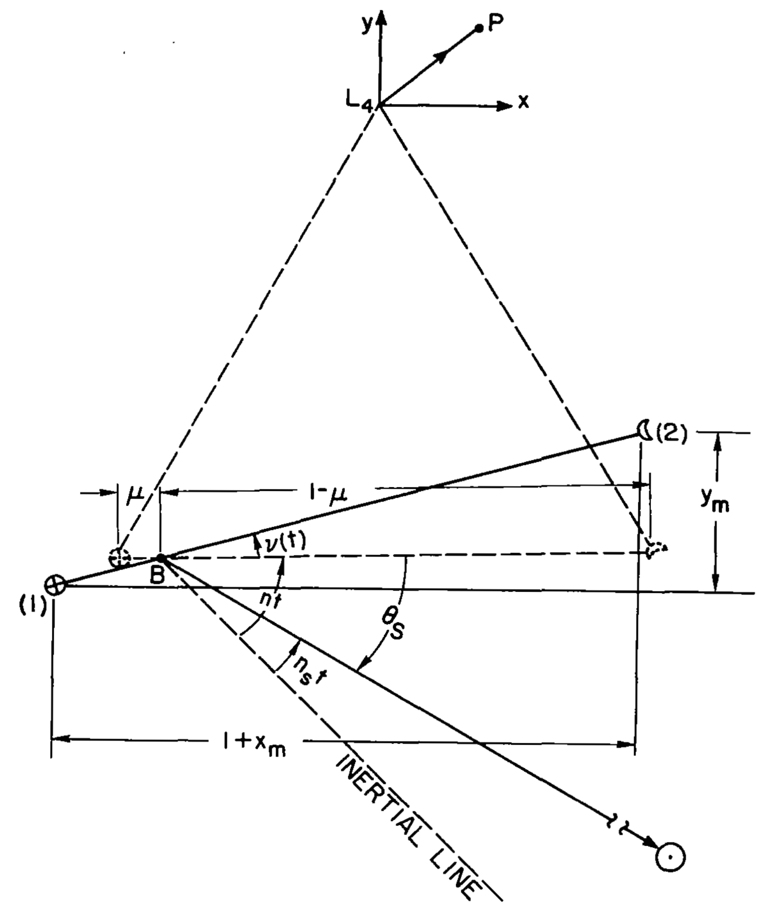

Fig. 7. ADOPTED COPLANAR MODEL FOR THE DESCRIPTION OF THE MOTION OF A PARTICLE P NEAR THE SUN-PERTURBED EARTH-MOON L_4 LIBRATION POINT.

natural quantity and is defined by Eq. (4.9a). Note also that a is not exactly equal to D; the difference is caused by the presence of the sun and the lunar eccentricity and is obtainable from the lunar theory. To be precise, Fig. 7 defines L_4 as an equilateral triangle with a "mean earth" and a "mean moon" that revolve about their barycenter B at uniform rate n and mutual distance $D \equiv 1$.

Two basic dimensionless quantities to appear often in the following equations are

$$m = \frac{n_s}{n} \cong \sqrt{\frac{\mu_3}{r_{3B}^3} \frac{a^3}{\mu_1 + \mu_2}} = 0.074801 \tag{4.10a}$$

and

$$\mu = \frac{\mu_2}{\mu_1 + \mu_2} \cong \frac{1}{82.3} \tag{4.10b}$$

where r_{3B} is the mean distance between the sun and the barycenter B, and n_s is the mean angular velocity of the mass center around the sun. Now, the Lagrangian $\mathfrak{L}$ in the dimensionless form but retaining the old symbols can be written as

$$\mathfrak{L} = \frac{1}{2} \dot{\vec{r}}_{14} \cdot \dot{\vec{r}}_{14} - V_{EM} - V_S \tag{4.11}$$

where

$$V_{EM} = -\bar{C}^3 \left[\frac{1}{r_{14}} + \mu \left(\frac{1}{r_{24}} - \frac{1}{r_{14}} - \frac{\vec{r}_{12} \cdot \vec{r}_{14}}{r_{12}^3} \right) \right] \tag{4.12a}$$

$$V_S = -m^2 \left(\frac{r_{3B}}{r_{13}} \right)^3 \left(\frac{1}{r_{34}} - \frac{\vec{r}_{13} \cdot \vec{r}_{14}}{r_{13}^3} \right) \tag{4.12b}$$

and $\bar{C}$ denotes a/D.

Just as n was the basic quantity selected in the nondimensionalization of the equations, m is selected as the basic quantity that defines the order of magnitude; $O(1)$ will denote a quantity of zeroth-order, $O(m)$ a quantity of first-order, $O(m^2)$ of second-order, etc.

Because of Schechter's conclusion concerning the out-of-plane motion, this research will be limited to a coplanar model, as shown in

Fig. 7. In this model, $r_{12}(t)$ and $\nu(t)$ are obtainable from De Pontecoulant's lunar theory [31,32] as

$$\frac{\bar{C}}{r_{12}} = 1 + e\left(1 - \frac{1}{8}e^2\right)C_{\theta_e} + e^2\left(1 - \frac{1}{3}e^2\right)C_{2\theta_e} + \frac{9}{8}e^3 C_{3\theta_e}$$

$$+ \frac{4}{3}e^3 C_{4\theta_e} + \frac{1}{6}m^2 - \frac{179}{288}m^4$$

$$+ \left(m^2 + \frac{19}{6}m^3 + \frac{131}{18}m^4\right)C_{2\theta_s} + \frac{7}{8}m^4 C_{4\theta_s}$$

$$- \left(\frac{15}{16}m + \frac{81}{16}m^2\right)\frac{1}{r_{3B}}C_{\theta_s}$$

$$+ \frac{25}{64}m^2\frac{1}{r_{3B}}C_{3\theta_s} - \frac{7}{12}m^2 eC_{\theta_e} + \frac{15}{8}meC_{2\theta_s-\theta_e}$$

$$+ \frac{33}{16}m^2 eC_{2\theta_s+\theta_e} + O(m^5) \tag{4.13}$$

$$\nu(t) = 2e\left(1 - \frac{1}{8}e^2\right)S_{\theta_e} + \frac{5}{4}\left(1 - \frac{11}{30}e^2\right)S_{2\theta_e} + \frac{13}{12}e^3 S_{3\theta_e}$$

$$+ \frac{103}{96}e^4 S_{4\theta_e} + \left(\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4\right)S_{2\theta_s}$$

$$+ \frac{201}{256}m^4 S_{4\theta_s} + \frac{15}{4}meS_{2\theta_s-\theta_e} + \frac{17}{8}m^2 eS_{2\theta_s+\theta_e}$$

$$- \left(\frac{15}{8}m + \frac{93}{8}m^2\right)\frac{1}{r_{3B}}S_{\theta_s} + \frac{15}{32}m^2\frac{1}{r_{3B}}S_{3\theta_s} + O(m^5) \tag{4.14}$$

where

$S \equiv \sin$

$C \equiv \cos$

$e \simeq 0.0549 = O(m)$

$$\frac{1}{r_{3B}} \cong 0.002559 = O(m^2)$$

$$\theta_s = \omega_s t$$

$$\theta_e = \omega_e t - \bar{\theta}_e$$

$$\omega_s = 1-m$$

$$\omega_e = 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3$$

and $\bar{\theta}_e$ denotes the initial longitude of the moon perigee relative to the inertial line of Fig. 7.

The Lagrangian $\mathcal{L}$ of Eq. (4.11) now can be written in terms of the displacements and velocities measured in the L_4-centered xy frame by writing the dimensionless vector relations as

$$\vec{r}_{14} = \left(\frac{1}{2} + x + \mu x_m\right)\hat{i}_x + \left(\sqrt{3}/2 + y + \mu y_m\right)\hat{i}_y \tag{4.15a}$$

$$\begin{aligned}\dot{\vec{r}}_{14} &= {}^{R}\!\left(\frac{d}{dt}\right) \vec{r}_{14} + \hat{i}_z \times \vec{r}_{14} \\ &= (\dot{x} + \mu\dot{x}_m - \sqrt{3}/2 - y - \mu y_m)\hat{i}_x \\ &\quad + \left(\dot{y} + \mu\dot{y}_m + \frac{1}{2} + x + \mu x_m\right)\hat{i}_y\end{aligned} \tag{4.15b}$$

$$\begin{aligned}\vec{r}_{24} &= \left(- \frac{1}{2} + x - (1-\mu)x_m\right)\hat{i}_x \\ &\quad + (\sqrt{3}/2 + y - (1-\mu)y_m)\hat{i}_y\end{aligned} \tag{4.15c}$$

$$\vec{r}_{12} = (1 + x_m)\hat{i}_x + y_m\hat{i}_y \tag{4.15d}$$

$$\begin{aligned}\vec{r}_{13} &= \left[r_{3B}C_{\theta_s} + \mu(1 + x_m)\right]\hat{i}_x \\ &\quad + \left(-r_{3B}S_{\theta_s} + \mu y_m\right)\hat{i}_y\end{aligned} \tag{4.15e}$$

where $\hat{i}_x$ and $\hat{i}_y$ are unit vectors along the x and y directions, $\hat{i}_z = \hat{i}_x \times \hat{i}_y$, R denotes the rotating frame, and

$$x_m = -1 + r_{12} \cos \nu(t) \tag{4.16a}$$

$$y_m = r_{12} \sin \nu(t) \tag{4.16b}$$

The quantities x, y, p_x, and p_y will be treated as being of $O(m)$; p_x and p_y are the momenta conjugates to x and y and are introduced through the relations

$$p_x - \sqrt{3}/2 = \mathfrak{L}_{\dot{x}} \tag{4.17a}$$

$$p_y + \frac{1}{2} = \mathfrak{L}_{\dot{y}} \tag{4.17b}$$

which yields a Hamiltonian,

$$R = (p_x - \sqrt{3}/2)\dot{x} + \left(p_y + \frac{1}{2}\right)\dot{y} - \mathfrak{L} \tag{4.18}$$

Writing $\mathfrak{L}$ in terms of $\dot{x}$ and $\dot{y}$ by using Eq. (4.15b), and substituting for $\dot{x}$ and $\dot{y}$ in terms of p_x and p_y from (4.17) will yield

$$R = \frac{1}{2}\left(p_x^2 + p_y^2\right) + (yp_x - xp_y) - \frac{1}{2}(x + \sqrt{3}y)$$

$$+ \mu(y_m - \dot{x}_m)p_x - \mu(\dot{y}_m + x_m)p_y + V_{EM} + V_S \tag{4.19}$$

where V_S can be written as

$$V_s = -m^2\left\{\left[\frac{3}{2}\left(\frac{\vec{r}_{14}\cdot\vec{r}_{13}}{r_{13}}\right)^2 - \frac{1}{2}r_{14}^2\right] + \left(\frac{\vec{r}_{14}\cdot\vec{r}_{13}}{r_{13}^2}\right)\right.$$

$$\left.\left[\frac{5}{2}\left(\frac{\vec{r}_{14}\cdot\vec{r}_{13}}{r_{13}}\right)^2 - \frac{3}{2}r_{14}^2\right]\right\} + O(m^6) \tag{4.20}$$

The first pair of square brackets encloses the gravity-gradient terms.

Using Eqs. (4.13) and (4.14), x_m and y_m and their derivatives $\dot{x}_m$ and $\dot{y}_m$ can be expressed up to fourth-order as (the eccentricity is taken up to third order only)

$$x_m = x_{m1} + x_{m2} + x_{m3} + x_{m4} \tag{4.21}$$

where

$$x_{m1} = -eC_{\theta_e}$$

$$x_{m2} = -\frac{1}{2} e^2 + \frac{1}{2} e^2 C_{2\theta_e} - \frac{15}{8} emC_{2\theta_s - \theta_e} - m^2 C_{2\theta_s}$$

$$x_{m3} = \frac{1}{8} e^3 C_{\theta_e} - \frac{9}{8} e^3 C_{3\theta_e} + \frac{3}{4} em^2 C_{\theta_e}$$

$$+ \frac{5}{16} em^2 C_{2\theta_s + \theta_e} - \frac{3}{8} em^2 C_{2\theta_s - \theta_e}$$

$$- \frac{9}{16} m^3 C_{2\theta_s} + \frac{15}{11} \frac{m}{r_{3B}} C_{\theta_s}$$

$$x_{m4} = \frac{25}{256} m^4 C_{4\theta_s} - \frac{64}{9} m^4 C_{2\theta_s} + \frac{159}{256} m^4$$

$$+ \frac{81}{16} \frac{m^2}{r_{3B}} C_{\theta_s} - \frac{25}{64} \frac{m^2}{r_{3B}} C_{3\theta_s}$$

$$y_m = y_{m1} + y_{m2} + y_{m3} + y_{m4} \tag{4.22}$$

where

$$y_{m1} = 2eS_{\theta_e}$$

$$y_{m2} = \frac{1}{4} e^2 S_{2\theta_e} + \frac{15}{4} emS_{2\theta_s - \theta_e} + \frac{11}{8} m^2 S_{2\theta_s}$$

$$y_{m3} = \frac{13}{12} e^3 S_{3\theta_e} - \frac{1}{4} e^3 S_{\theta_e} + \frac{7}{16} em^2 S_{2\theta_s + \theta_e}$$

$$+ \frac{5}{16} em^2 S_{2\theta_s - \theta_e} + \frac{59}{12} m^3 S_{2\theta_s} - \frac{15}{8} \frac{m}{r_{3B}} S_{\theta_s}$$

$$y_{m4} = \frac{25}{256} m^4 S_{4\theta_s} + \frac{893}{72} m^4 S_{2\theta_s} + \frac{15}{32} \frac{m^2}{r_{3B}} S_{3\theta_s}$$

$$- \frac{93}{8} \frac{m^2}{r_{3B}} S_{\theta_s}$$

$$\dot{x}_m = \dot{x}_{m1} + \dot{x}_{m2} + \dot{x}_{m3} + \dot{x}_{m4} \tag{4.23}$$

where

$$\dot{x}_{m1} = eS_{\theta_e}$$

$$\dot{x}_{m2} = -e^2 S_{2\theta_e} + \frac{15}{8} emS_{2\theta_s - \theta_e} + 2m^2 S_{2\theta_s}$$

$$\dot{x}_{m3} = \frac{27}{8} e^3 S_{3\theta_e} - \frac{1}{8} e^3 S_{\theta_e} - \frac{15}{16} em^2 S_{2\theta_s + \theta_e}$$

$$- \frac{27}{8} em^2 S_{2\theta_s - \theta_e} - \frac{3}{2} em^2 S_{\theta_e}$$

$$+ \frac{13}{3} m^3 S_{2\theta_s} - \frac{15}{16} \frac{m}{r_{3B}} S_{\theta_s}$$

$$\dot{x}_{m4} = - \frac{25}{64} m^4 S_{4\theta_s} + \frac{71}{9} m^4 S_{2\theta_s} + \frac{75}{64} \frac{m^2}{r_{3B}} S_{3\theta_s}$$

$$- \frac{33}{8} \frac{m^2}{r_{3B}} S_{\theta_s}$$

and

$$\dot{y}_m = \dot{y}_{m1} + \dot{y}_{m2} + \dot{y}_{m3} + \dot{y}_{m4} \tag{4.24}$$

where

$$\dot{y}_{m1} = 2eC_{\theta_e}$$

$$\dot{y}_{m2} = \frac{1}{2} e^2 C_{2\theta_e} + \frac{15}{4} emC_{2\theta_s - \theta_e} + \frac{11}{4} m^2 C_{2\theta_s}$$

$$\dot{y}_{m3} = -\frac{1}{4} e^3 C_{\theta_e} + \frac{13}{4} e^3 C_{3\theta_e} - \frac{3}{2} em^2 C_{\theta_e}$$

$$+ \frac{21}{16} m^2 e C_{2\theta_s+\theta_e} - \frac{115}{16} em^2 C_{2\theta_s-\theta_e}$$

$$+ \frac{85}{12} m^3 C_{2\theta_s} - \frac{15}{8} \frac{m}{r_{3B}} C_{\theta_s}$$

$$\dot{y}_{m4} = \frac{25}{64} m^4 C_{4\theta_s} + \frac{539}{36} m^4 C_{2\theta_s}$$

$$- \frac{39}{4} \frac{m^2}{r_{3B}} C_{\theta_s} + \frac{45}{32} \frac{m^2}{r_{3B}} C_{3\theta_s}$$

Substituting Eqs. (4.15) and (4.21) to (4.24) into (4.19), approximating $\bar{C}$ by $1 + m^2/6$ to eliminate the constant coefficients of $(x + \sqrt{3}y)$ terms, and keeping up to sixth-order terms lead to the following expansion for the Hamiltonian R:

$$R = \sum_{n=0}^{4} R_n \tag{4.25}$$

where

$$R_0 = \frac{1}{2} (p_x^2 + p_y^2) + (yp_x - xp_y) + \frac{1}{8} (x^2 - 5y^2 - 6\sqrt{3}Uxy)$$

$$- \frac{3}{16} m^2 (x^2 + 3y^2 + 2\sqrt{3}Uxy)$$

$$R_1 = \frac{3\sqrt{3}}{16} (x^2 y + y^3) + \frac{U}{16} (33xy^2 - 7x^3)$$

$$- \frac{3m^2}{4} \left[(x - \sqrt{3}y)C_{\theta_s} - (y + \sqrt{3}x)S_{\theta_s}\right]$$

$$+ \mu \left[y_{m1}p_x - x_{m1}p_y - (\dot{x}_{m1}p_x + \dot{y}_{m1}p_y) \right.$$

$$\left. + \left(y_{m1} - \frac{3\sqrt{3}}{2} x_{m1}\right) y - \left(2x_{m1} + \frac{3\sqrt{3}}{2} y_{m1}\right)x\right]$$

$$R_2 = \frac{5\sqrt{3}}{32} U(5x^3y - 9xy^3) + \frac{37}{128} x^4 - \frac{123}{64} x^2y^2 - \frac{3}{128} y^4$$

$$- \frac{3}{4} m^2 \left[(x^2-y^2)C_{2\theta_s} - 2xyS_{2\theta_s}\right]$$

$$+ \mu \left\{(y_{m2}p_x - x_{m2}p_y) - (\dot{x}_{m2}p_x + \dot{y}_{m2}p_y)\right.$$

$$+ \left(y_{m2} - \frac{3\sqrt{3}}{2} x_{m2}\right)y - \left(2x_{m2} + \frac{3\sqrt{3}}{2} y_{m2}\right) x$$

$$+ \frac{3}{8} \left[(11y^2 - 7x^2)x_{m1} + 22xyy_{m1}\right]$$

$$\left. + \frac{3\sqrt{3}}{2} \mu\left[xy_{m1} + yx_{m1}\right]\right\}$$

$$R_3 = \frac{1}{256} \left[\sqrt{3}(960x^2y^3 - 285x^4y - 33y^5)\right.$$

$$\left. + 23x^5 - 430x^3y^2 + 555xy^4\right]$$

$$- \frac{3}{16} \frac{m^2}{r_{3B}} \left[(3x + \sqrt{3}y)C_{\theta_s} - (\sqrt{3}x + 5y)S_{\theta_s}\right]$$

$$+ \mu \left\{(y_{m3}p_x - x_{m3}p_y) - (\dot{x}_{m3}p_x + \dot{y}_{m3}p_x)\right.$$

$$+ \left(y_{m3} - \frac{3\sqrt{3}}{2} x_{m3}\right)y - \left(2x_{m3} + \frac{3\sqrt{3}}{2} y_{m3}\right)x$$

$$+ \frac{3}{8} \left[(11y^2 - 7x^2)x_{m2} + 22xyy_{m2}\right]$$

$$+ \frac{3\sqrt{3}}{2} \mu(xy_{m2} + yx_{m2})$$

$$\left. + \frac{3}{8} \mu\left[(7x^2 - 11y^2)x_{m1} - 22xyy_{m1}\right]\right\}$$

$$+ \frac{m^2}{2} \left[\frac{3\sqrt{3}}{16} (x^2y + y^3) + \frac{1}{16} (33xy^2 - 7x^3)\right]$$

$$R_4 = \frac{1}{1024}\Big[\sqrt{3}(294x^5y - 420x^3y^3 - 714xy^5) - 331x^6 + 6105x^4y^2$$

$$- 7965x^2y^4 + 383y^6\Big]$$

$$- \frac{\mu m^2}{8r_{3B}}\left[(x - 5\sqrt{3}y)C_{\theta_s} - 3(\sqrt{3}x + y)S_{\theta_s}\right]$$

$$+ \mu\Big\{(y_{m4}p_x - x_{m4}p_y) - (\dot{x}_{m4}p_x + \dot{y}_{m4}p_y)$$

$$+ \left(y_{m4} - \frac{3\sqrt{3}}{2}x_{m4}\right)y - \left(2x_{m4} + \frac{3\sqrt{3}}{2}y_{m4}\right)x$$

$$+ \frac{3}{8}\Big[(11y^2 - 7x^2)x_{m3} + 22xyy_{m3}$$

$$+ \frac{3\sqrt{3}}{2}\mu\Big[xy_{m3} + yx_{m3}\Big]$$

$$+ \frac{3}{8}\mu\left[(7x^2 - 11y^2)x_{m2} - 22xyy_{m2}\right]\Big\}$$

$$+ \frac{m^2}{2}\left[\frac{5\sqrt{3}}{32}(5x^3y - 9xy^3) + \frac{37}{128}x^4 - \frac{123}{64}x^2y^2 - \frac{3}{128}y^4\right]$$

The "unperturbed" Hamiltonian R_0 in the above equations includes, for convenience, all terms quadratic in x, y, p_x, p_y with constant coefficients. The corresponding terms from the solar gravity-gradient potential were left in the perturbing Hamiltonian in Ref. 12.

Because R involves the mass ratio μ in its coefficients both by itself and in combination $(1-2\mu) \equiv U$ and because R_0 involves only $(1-2\mu)$, this combination is treated as a numerical value of zeroth-order, and μ is treated as a numerical multiple of m [i.e., O(m)] as is e. Note that the region of convergence in the above expansion is the interior of the intersection of the two circles

$$\left(x + \frac{1}{2}\right)^2 + (y + \sqrt{3}/2)^2 \cong 2 \qquad (4.26a)$$

$$\left(x - \frac{1}{2}\right)^2 + (y + \sqrt{3}/2)^2 \cong 2 \tag{4.26b}$$

Finally, the equations of motion take the canonical forms

$$\dot{x} = R_{p_x} \qquad \dot{p}_x = -R_x$$

$$\dot{y} = R_{p_y} \qquad \dot{p}_y = -R_y \tag{4.27}$$

Chapter V

TRANSFORMATION TO THE STANDARD FORM

Chapter II described a method of solution that will be applied to the present problems. In applying this method, a new set of variables must be used to reduce Eq. (4.27) to a standard form similar to Eq. (2.39) or (2.41). To obtain this standard form and to solve the Hamilton-Jacobi equation (2.37), it is necessary to separate, by modes, the two harmonic oscillators that make up the expansion for R_0 of Eq. (4.25). Also, to introduce the small parameter m in the expansion of the Hamiltonian, it is desirable to rescale the physical coordinates (x,y) and the generalized momenta (p_x,p_y) so that the rescaled variables are of $O(1)$ instead of $O(m)$ and, thus, satisfy the Hamilton canonical equations with respect to a new Hamiltonian $H = R/m^2$. This rescaling process gives rise to the appearance of the factor m^i as a coefficient of $H_i = R_i/m^2$.

First, consider the system described by the "unperturbed" Hamiltonian

$$R_0 = \frac{1}{2}(p_x^2 + p_y^2) + (yp_x - xp_y) + \frac{1}{8}(x^2 - 5y^2 - 6\sqrt{3}Uxy)$$

$$- \frac{3}{16} m^2 (x^2 + 3y^2 + 2\sqrt{3}Uxy) \tag{5.1}$$

The corresponding equations of motion are

$$\dot{x} = R_{0p_x} = p_x + y \tag{5.2a}$$

$$\dot{y} = R_{0p_y} = p_y - x \tag{5.2b}$$

$$\dot{p}_x = -R_{0x} = p_y - \frac{1}{4}x + \frac{3\sqrt{3}}{4}Uy + \frac{3}{8}m^2(x + \sqrt{3}Uy) \tag{5.2c}$$

$$\dot{p}_y = -R_{0y} = -p_x + \frac{5}{4}y + \frac{3\sqrt{3}}{4}Ux + \frac{3\sqrt{3}}{8}m^2(Ux + \sqrt{3}Uy) \tag{5.2d}$$

which are equivalent to the system of second-order differential equations

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \tag{5.3a}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \tag{5.3b}$$

where $U^* = 3/8\ (1 + m^2/2)(x^2 + 3y^2 + 2\sqrt{3}Uxy)$.

The characteristic equation is

$$\omega^4 - \left(1 - \frac{3}{2}m^2\right)\omega^2 + \frac{27}{4}\mu(1-\mu)\left(1 + \frac{m^2}{2}\right)^2 = 0 \tag{5.4}$$

in which the solutions are the eigenvalues

$$\omega_1 = \pm\ 0.949313 \tag{5.5a}$$

$$\omega_2 = \pm\ 0.300684 \tag{5.5b}$$

therefore, the linearized solution is stable. Note that the stability of these equations occurs even though the effective potential energy $[-U^*(x,y)]$ has a maximum rather than a minimum at L_4; this stability depends on the influence of the Coriolis acceleration which gave rise to the terms $\dot{x}$ and $\dot{y}$ in Eqs. (5.3).

To obtain the standard form of equations, the stationary canonical transformations

$$\bar{q} = (\bar{q}_1, \bar{q}_2) \tag{5.6a}$$

$$\bar{p} = (\bar{p}_1, \bar{p}_2) \tag{5.6b}$$

are introduced to satisfy the Hamilton equation of motion with respect to a Hamiltonian $H = R/m^2$, and they represent uncoupled motion in the form of independent simple harmonic oscillations having the eigenvalues ω_1 and ω_2 as frequencies. The linear equations of transformation can be written as

$$\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} = J \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} \tag{5.7}$$

where the Jacobian J is a 4×4 matrix that satisfies the condition

$$J \Phi_0 J^T = \frac{\Phi_0}{m^2} \tag{5.8}$$

In view of Eqs. (2.22), (2.23), and (5.8), the new set of variables satisfies Hamilton's equations of motion with respect to a Hamiltonian $H = R/m^2$, as desired.

The inverse of the Jacobian J was found to have the form

$$J^{-1} = \begin{bmatrix} 0 & 0 & \frac{K_1}{\omega_1}(\omega_1^2 + b) & -\frac{K_2}{\omega_2}(\omega_2^2 + b) \\ -2K_1\omega_1 & -2K_2\omega_2 & -\frac{K_1}{\omega_1}\eta & \frac{K_2}{\omega_2}\eta \\ -K\omega_1(\omega_1^2 + b - 2) & -K_2\omega_2(\omega_2^2 + b - 2) & \frac{K_1}{\omega_1}\eta & -\frac{K_2}{\omega_2}\eta \\ K_1\eta\omega_1 & K_2\eta\omega_2 & \frac{K_1}{\omega_1}(b - \omega_1^2) & -\frac{K_2}{\omega_2}(b - \omega_2^2) \end{bmatrix} \tag{5.9}$$

where

$$a = \frac{3}{4}\left(1 + \frac{m^2}{2}\right)$$

$$b = \frac{9}{4}\left(1 + \frac{m^2}{2}\right)$$

$$\eta = \frac{3\sqrt{3}}{4} U\left(1 + \frac{m^2}{2}\right)$$

$$K_i = \frac{m}{\sqrt{\left(\omega_1^2 - \omega_2^2\right)\left(\omega_i^2 + b\right)}} \qquad i = 1,2$$

In terms of $\bar{q}$ and $\bar{p}$, the Hamiltonian H_0 becomes

$$H_0 = \frac{1}{2}\left(\bar{p}_1^2 + \omega_1^2\bar{q}_1^2\right) - \frac{1}{2}\left(\bar{p}_2^2 + \omega_2^2\bar{q}_2^2\right) \tag{5.10}$$

and its nondefinite form is associated with the fact that L_4 is a maximum rather than a minimum of $[-U^*(x,y)]$, as noted earlier.

With reference to Eq. (2.37), the solution for the two harmonic oscillators that make up the expression for H_0 can be given by

$$\bar{q}_1 = \sqrt{\frac{2\tilde{\alpha}_1}{\omega_1}} \sin \tilde{\beta}_1 \tag{5.11a}$$

$$\bar{q}_2 = -\sqrt{\frac{2\tilde{\alpha}_2}{\omega_2}} \sin \tilde{\beta}_2 \tag{5.11b}$$

$$\bar{p}_1 = \sqrt{2\omega_1\tilde{\alpha}_1} \cos \tilde{\beta}_1 \tag{5.11c}$$

$$\bar{p}_2 = -\sqrt{2\omega_2\tilde{\alpha}_2} \cos \tilde{\beta}_2 \tag{5.11d}$$

where $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are the action and angle variables, respectively, associated with H. Substituting these equations into (5.10) obtains the simple expression

$$H_0 = \omega_1\tilde{\alpha}_1 - \omega_2\tilde{\alpha}_2 \tag{5.12}$$

Therefore,

$$\dot{\tilde{\beta}}_1 = H_{0\tilde{\alpha}_1} = \omega_1 \qquad \dot{\tilde{\alpha}}_1 = -H_{0\tilde{\beta}_1} = 0 \tag{5.13a}$$

$$\dot{\tilde{\beta}}_2 = H_{0\tilde{\alpha}_2} = -\omega_2 \qquad \dot{\tilde{\alpha}}_2 = -H_{0\tilde{\beta}_2} = 0 \tag{5.13b}$$

The solutions to these equations are simply

$$\tilde{\beta}_1 = \omega_1 t + \beta_1 \qquad \tilde{\alpha}_1 = \alpha_1 \tag{5.14a}$$

$$\tilde{\beta}_2 = -\omega_2 t + \beta_2 \qquad \tilde{\alpha}_2 = \alpha_2 \tag{5.14b}$$

where α_i and β_i are the four constants of integration.

The solution of Eqs. (5.3) is obtained through (5.7) and (5.11). After evaluating the numerical coefficients in J^{-1}, this solution is given in terms of $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ as

$$\begin{bmatrix} \frac{x}{m} \\ \frac{y}{m} \\ \frac{p_x}{m} \\ \frac{p_y}{m} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2.078779 & 5.658032 \\ -1.249989 & 1.44993 & -0.83679 & -3.064483 \\ -0.723424 & 0.251348 & 0.83679 & 3.064483 \\ 0.794376 & -0.921442 & 0.892148 & 5.22062 \end{bmatrix} \begin{bmatrix} \sqrt{2\omega_1\tilde{\alpha}_1} \sin \tilde{\beta}_1 \\ \sqrt{2\omega_2\tilde{\alpha}_2} \sin \tilde{\beta}_2 \\ \sqrt{2\omega_1\tilde{\alpha}_1} \cos \tilde{\beta}_1 \\ \sqrt{2\omega_2\tilde{\alpha}_2} \cos \tilde{\beta}_2 \end{bmatrix} \tag{5.15}$$

The particle trajectories in the xy plane, corresponding to each of the two coplanar modes are ellipses with major axes at right angles to the vector $\vec{r}_{1L_4}$ and with a thickness ratio (minor axis/major axis) of approximately 1:2 for mode 1 and of approximately 1:5 for mode 2, as shown in Fig. 8. Motion proceeds in a retrograde direction. The complete unperturbed xy motion consists of a weighted superposition of these two normal modes, and, in general, is quasi-periodic.

The standard form of equations now can be obtained by substituting x, y, p_x, and p_y of Eq. (5.15) into $H = R/m^2$. Use of (5.14) yields

$$\dot{\alpha}_i = -\sum_{n=1}^{4} \frac{m^n}{n!} H_{n\beta_i} \tag{5.16a}$$

$$\dot{\beta}_i = \sum_{n=1}^{4} \frac{m^n}{n!} H_{n\alpha_i} \qquad (5.16b)$$

Appendix A defines $H_n/n! = R_n/m^2$, in which A_i denotes $\sqrt{2\omega_i\alpha_i}$, B_1 denotes $\tilde{\beta}_1$, and B_2 denotes $-\tilde{\beta}_2$. Also, the substitutions

$$\mu = 0.162440 \text{ m} \qquad (5.17a)$$

$$\frac{1}{r_{3B}} = 0.457357 \text{ m}^2 \qquad (5.17b)$$

have been used. The ratio e/m is retained, however, as an adjustable parameter for later comparison with Ref. 14.

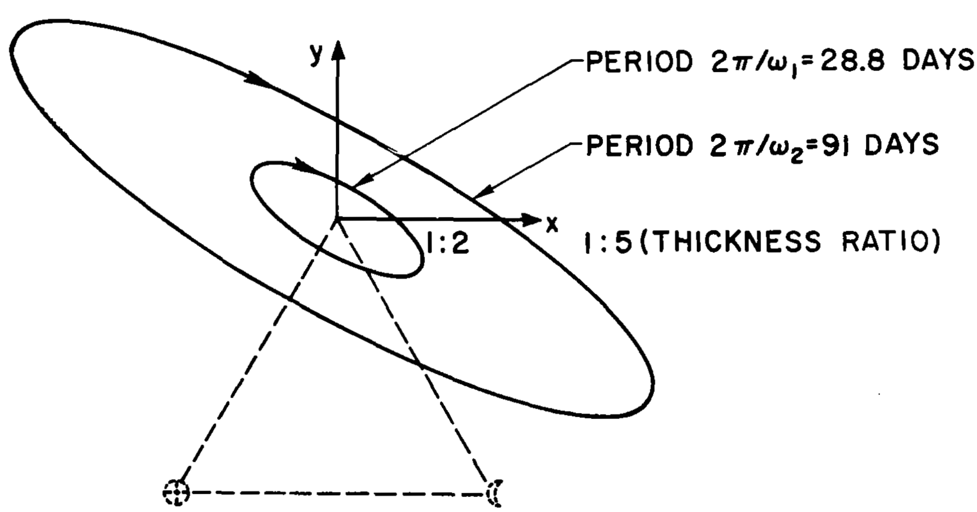

Fig. 8. TRAJECTORIES OF NORMAL MODES.

Chapter VI

THE LONG-PERIOD HAMILTONIAN AND THE ELIMINATION OF TIME t

This chapter carries out the second step in the method of solution outlined in Chapter II. For this purpose, Eqs. (3.37) and (3.38) are used to define the W_n successively to eliminate all short-period terms from the Hamiltonian K. This long-period Hamiltonian will contain slowly varying trigonometric terms of angular frequencies $\omega_1-\omega_s$, $3\omega_2-\omega_1$, $\omega_1-\omega_e$, and combinations thereof.

Because of the presence of two distinct external frequencies ω_s and ω_e, the resulting differential equations will be complicated; therefore, an attempt to investigate the eccentricity effect is made by eliminating all trigonometric terms containing ω_e. This elimination was suggested by the fact that the response of the linear system to a small eccentricity-forcing term is an exact imitation of the moon's fluctuation about the "mean moon"; i.e., a possible motion is such as to form an equilateral triangle with the instantaneous earth and moon in their plane of relative motion. This well-known fact [Ref. 8, pp. 587-602] is not confined to only small eccentricities. The justification for removal of these additional trigonometric terms lies in the nonappearance of excessively large coefficients of $(e/m)^n$ in the resulting computations. It should be added, however, that because of the limitations on computer storage, the inclusion of (e/m) was terminated at $(e/m)^3$ and at K_3 instead of K_4.

Adopting the above policy, the long-period Hamiltonian K was found to be

$$K = \sum_{n=1}^{4} \frac{m^n}{n!} K_n \tag{6.1}$$

where

$$K_1 = 0$$

$$\frac{1}{2!} K_2 = 0.028726\, \bar{A}_1^4 - 1.816528\, \bar{A}_1^2 \bar{A}_2^2 + 0.276550\, \bar{A}_2^4$$

$$+ \left[1.213685 \; S_{2(\bar{B}_1-\theta_s)} - 2.148727 \; C_{2(\bar{B}_1-\theta_s)}\right] \bar{A}_1^2$$

$$- \left[0.959853 \; S_{3\bar{B}_2-\bar{B}_1} + 7.168701 \; C_{3\bar{B}_2-\bar{B}_1}\right] \bar{A}_1 \bar{A}_2^3$$

$$\frac{1}{3!} K_3 = - \left[0.146701 \; S_{\bar{B}_1-\theta_s} + 0.398984 \; C_{\bar{B}_1-\theta_s}\right] \bar{A}_1$$

$$+ \left[0.271807 \; S_{2(\bar{B}_1-\theta_s)} - 0.672102 \; C_{2(\bar{B}_1-\theta_s)}\right] \bar{A}_1^2$$

$$+ 0.081621 \left(\frac{e}{m}\right)^2 \bar{A}_1^2 + 1.487332 \left(\frac{e}{m}\right)^2 \bar{A}_2^2$$

$$\frac{1}{4!} K_4 = -0.136967 \; \bar{A}_1^6 + 1.276611 \; \bar{A}_1^4 \bar{A}_2^2 - 4509.578 \; \bar{A}_1^2 \bar{A}_2^4$$

$$+ 104.9922 \; \bar{A}_2^6 + 20.0769 \; \bar{A}_2^2 - 29.62918 \; \bar{A}_1^2$$

$$+ 1.04752 \; \bar{A}_1^4 + 38.85197 \; \bar{A}_1^2 \bar{A}_2^2 + 110.3419 \; \bar{A}_2^4$$

$$+ \left[958.7131 \; S_{3\bar{B}_2-\bar{B}_1} + 6217.148 \; C_{3\bar{B}_2-\bar{B}_1}\right] \bar{A}_1^3 \bar{A}_2^3$$

$$+ \left[2659.185 \; S_{3\bar{B}_2-\bar{B}_1} - 3148.168 \; C_{3\bar{B}_2-\bar{B}_1}\right] \bar{A}_1 \bar{A}_2^5$$

$$- \left[1.579912 \; S_{\bar{B}_1-\theta_s} - 0.5720766 \; C_{\bar{B}-\theta_s}\right] \bar{A}_1$$

$$+ \left[0.82831 \; S_{2(\bar{B}_1-\theta_s)} - 2.639881 \; C_{2(\bar{B}_1-\theta_s)}\right] \bar{A}_1^2$$

$$- \left[38.82468 \; S_{2(\bar{B}_1-\theta_s)} - 15.97099 \; C_{2(\bar{B}_1-\theta_s)}\right] \bar{A}_1^4$$

$$+ \left[185.4023 \; S_{3\bar{B}_2+\bar{B}_2-2\theta_s} + 115.3682 \; C_{3\bar{B}_2-\bar{B}_1-2\theta_s}\right] \bar{A}_1 \bar{A}_2^3$$

$$+ \left[1.8914\ S_{3\bar{B}_2-\bar{B}_1} + 36.89038\ C_{3\bar{B}_2-\bar{B}_1}\right] \bar{A}_1 \bar{A}_2^3$$

and the corresponding W_n are defined in Appendix B. In the above relations, $\bar{A}_i$ denotes $\sqrt{2\omega_i\bar{\alpha}_i}$ and $\bar{B}_1$, $\bar{B}_2$ denote $\omega_1 t + \bar{\beta}_1$ and $\omega_2 t - \bar{\beta}_2$, respectively.

Equation (6.1) shows that the explicit dependence of K on time t comes about through the presence of the slowly varying trigonometric terms with angular frequencies $\omega_1-\omega_s$, $2(\omega_1-\omega_s)$, $3\omega_2-\omega_1$, and $3\omega_2+\omega_1-2\omega_s$. This explicit appearance of the time terms can be eliminated by means of a transformation to a new canonical set of variables α_i^* and β_i^*, defined by

$$\beta_1^* = \bar{B}_1 - \theta_s = (\omega_1-\omega_s)t + \bar{\beta}_1 \tag{6.2a}$$

$$3\beta_2^* = \theta_s - 3\bar{B}_2 = (\omega_s-3\omega_2)t + 3\bar{\beta}_2 \tag{6.2b}$$

so that

$$\bar{B}_1 - 3\bar{B}_2 = \beta_1^* + 3\beta_2^* \tag{6.3a}$$

and

$$\bar{B}_1 + 3\bar{B}_2 - 2\theta_s = \beta_1^* - 3\beta_2^* \tag{6.3b}$$

The conjugate momenta α_i^* are obtained by the introduction of the generating function

$$S = \alpha_1^*\left(\sigma_1 t + \bar{\beta}_1\right) + \alpha_2^*\left(\sigma_2 t + \bar{\beta}_2\right) \tag{6.4}$$

where

$$\sigma_1 = \omega_1 - \omega_s$$

$$\sigma_2 = \frac{\omega_s - 3\omega_2}{3}$$

As a result,

$$\bar{\alpha}_i = \frac{\partial S}{\partial \bar{\beta}_i} = \alpha_i^* \tag{6.5}$$

The transformed time-independent Hamiltonian K^* is given by

$$K^* = K + S_t = K + \sigma_1 \alpha_1^* + \sigma_2 \alpha_2^* \tag{6.6}$$

and α_i^*, β_i^* are determined by

$$\dot{\alpha}_i^* = -K^*_{\beta_i^*} \tag{6.7a}$$

$$\dot{\beta}_i^* = K^*_{\alpha_i^*} \tag{6.7b}$$

However, these equations, in general, cannot be integrated in terms of elementary functions and, except for some special cases, their solutions must be found with the help of numerical methods.

Equation (4.4b) can be directly integrated by employing these numerical methods; however, this is a complicated task, requiring much time. It is also difficult to perform because of the possible accumulation of large systematic errors.

The numerical integration of Eqs. (6.7) does not present difficulties because only amplitude and phase appear and not the oscillatory functions of x and y. To obtain a complete picture of the process, it is sufficient to calculate a small number of points along a comparatively "smooth" curve, which simplifies the integration. On the other hand, in the direct integration of (4.4b), not only an envelope but direct sinusoids must be determined.

This investigation is interested only in obtaining special solutions of Eqs. (6.7) which lead to periodic and quasi-periodic orbits. These orbits are defined by equilibrium points. Such points in (α_i^*, β_i^*)-space are determined by looking for solutions to (6.7) in the form of $\dot{\alpha}_i^* = 0$ and $\dot{\beta}_i^* = 0$. Once such points are located, it is necessary to investigate the type of equilibrium that exists and to identify those that are stable and unstable.

The search for these equilibria is facilitated by switching over to a set of rectangular coordinates defined by

$$Q_i = \sqrt{2\omega_i \alpha_i^*} \sin \beta_i^* \tag{6.8a}$$

and

$$P_i = \sqrt{2\omega_i \alpha_i^*} \cos \beta_i^* \tag{6.8b}$$

where $i = 1,2$. Note that Q_i/ω_i and P_i define a canonical set of coordinates and momenta with respect to K^*.

Substituting $\bar{\alpha}_i$ and $\bar{\beta}_i$ in terms of P_i and Q_i in Eq. (6.1) and using (6.6) lead to the following expression for K^*:

$$K^* = \sum_{n=2}^{4} \frac{m^n}{n!} K_n^* \tag{6.9}$$

where

$$\begin{aligned}\frac{1}{2!} K_2^* = {} & 4.418708\, Q_1^2 + 2.42737\, Q_1 P_1 + 0.121254\, P_1^2 \\ & + 2.292988\, Q_2^2 + 2.292988\, P_2^2 \\ & + 0.028726\, Q_1^4 + 0.057451\, Q_1^2 P_1^2 \\ & + 0.028726\, P_1^4 \\ & + 0.2765502\, Q_2^4 - 7.168701\, Q_2^3 Q_1 \\ & - 0.959853\, Q_2^3 P_1 + 0.5531008\, Q_2^2 P_2^2 - 2.879588\, Q_2^2 P_2 Q_1 \\ & + 21.5061\, Q_2^2 P_2 P_1 - 1.816528\, Q_2^2 Q_1^2 - 1.816528\, Q_2^2 P_1^2\end{aligned}$$

$$+ 21.5061\ Q_2 P_2^2 Q_1 + 2.879558\ Q_2 P_2^2 P_1 + 0.2765502\ P_2^4$$

$$+ 0.959853\ P_2^3 Q_1 - 7.168701\ P_2^3 P_1 - 1.816528\ P_2^2 Q_1^2$$

$$- 1.816528\ P_2^2 P_1^2$$

$$\frac{1}{3!} K_3^* = + 0.672102\ Q_1^2 + 0.543614\ Q_1 P_1 - 0.672102\ P_1^2$$

$$+ \left(\frac{e}{m}\right)^2 \left[0.081621\ (Q_1^2 + P_1^2) + 1.487332\ (Q_2^2 + P_2^2)\right]$$

$$- 0.146701\ Q_1 - 0.398984\ P_1$$

$$\frac{1}{4!} K_4^* = -26.98928\ Q_1^2 + 1.65662\ Q_1 P_1 - 32.26905\ P_1^2$$

$$+ 20.07690\ (Q_2^2 + P_2^2) - 1.579912\ Q_1 + 0.572077\ P_1$$

$$+ 104.9922\ Q_2^6 - 3148.167\ Q_2^5 Q_1 + 2659.185\ Q_2^5 P_1$$

$$+ 314.9765\ Q_2^4 P_2^2 + 7977.554\ Q_2^4 P_2 Q_1$$

$$+ 9444.503\ Q_2^4 P_2 P_1 - 4509.578\ Q_2^4 Q_1^2$$

$$- 4509.578\ Q_2^4 P_1^2 + 110.3419\ Q_2^4$$

$$+ 6296.335\ Q_2^3 P_2^2 Q_1 - 5318.367\ Q_2^3 P_2^2 P_1 + 6217.148\ Q_2^3 Q_1^3$$

$$+ 958.7131\ Q_2^3 Q_1^2 P_1 + 6217.148\ Q_2^3 Q_1 P_1^2 - 78.47181\ Q_2^3 Q_1$$

$$+ 958.7131\ Q_2^3 P_1^3 + 187.2936\ Q_2^3 P_1 + 314.9765\ Q_2^2 P_2^4$$

$$+ 5318.367\ Q_2^2 P_2^2 Q_1 + 6296.335\ Q_2^2 P_2^3 P_1 - 9019.156\ Q_2^2 P_2^2 Q_1^2$$

$$- 9019.156\ Q_2^2 P_2^2 P_1^2 + 220.6838\ Q_2^2 P_2^2$$

$$+ 2876.139\ Q_2^2P_2Q_1^3 - 18651.44\ Q_2^2P_2Q_1^2P_1$$

$$+ 2876.139\ Q_2^2P_2Q_1P_1^2 - 550.5324\ Q_2^2P_2Q_1$$

$$- 18651.44\ Q_2^2P_2P_1^3 - 456.7756\ Q_2^2P_2P_1$$

$$+ 1.276611\ Q_2^2Q_1^4 + 2.553222\ Q_2^2Q_1^2P_1^2$$

$$+ 140.0567\ Q_2^2Q_1^2 - 521.3286\ Q_2^2Q_1P_1$$

$$+ 1.276611\ Q_2^2P_1^4 - 62.35282\ Q_2^2P_1^2$$

$$+ 9444.503\ Q_2P_2^4Q_1 - 7977.554\ Q_2P_2^4P_1$$

$$- 18651.44\ Q_2P_2^2Q_1^3 - 2876.139\ Q_2P_2^2Q_1^2P_1$$

$$- 18651.44\ Q_2P_2^2Q_1P_1^2 + 235.4333\ Q_2P_2^2Q_1$$

$$- 2876.139\ Q_2P_2^2P_1^3 - 561.8808\ Q_2P_2^2P_1$$

$$+ 104.9922\ P_2^6 - 2659.185\ P_2^5Q_1 - 3148.167\ P_2^5P_1$$

$$- 4509.578\ P_2^4P_1^2 + 110.3419\ P_2^4 - 958.7131\ P_2^3Q_1^3$$

$$+ 6217.148\ P_2^3Q_1P_1^2 + 183.5108\ P_2^3Q_1 + 6217.148\ P_2^3P_1^3$$

$$+ 152.2585\ P_2^3P_1 + 1.276611\ P_2^2Q_1^4 + 2.553222\ P_2^2Q_1^2P_1^2$$

$$+ 140.0567\ P_2^2Q_1^2 - 521.3286\ P_2^2Q_1P_1 + 1.276611\ P_2^2P_1^4$$

$$- 62.35282\ P_2^2P_1^2 - 0.136966\ Q_1^6 - 0.410900\ Q_1^4P_1^2$$

$$+ 17.0185\ Q_1^4 - 77.64935\ Q_1^3P_1 - 0.410900\ Q_1^2P_1^4$$

$$+ 2.095039\ Q_1^2P_1^2 - 77.64935\ Q_1P_1^3 - 0.136967\ P_1^6 - 14.92347\ P_1^4$$

Chapter VII

PERIODIC AND QUASI-PERIODIC ORBITS

This chapter carries out special solutions to the main problem, as a partial completion of the last step in the method of solution. Section A obtains the stationary solutions (or equilibrium points), and Section B identifies the stable from the unstable ones. In Section C, the physical coordinates x and y are computed as functions of time, and the resulting stable periodic orbits are plotted.

A. Determination of Equilibrium Points

The equilibrium points $(Q_i, P_i)_j$ are obtained from the solution of the algebraic equations

$$K^*_{P_1} = K^*_{P_2} = K^*_{Q_1} = K^*_{Q_2} = 0 \tag{7.1}$$

Because of the terms in K_3, which are linear in P_1, Q_1 (arising from the higher order solar perturbation term with the factor $1/r_{3B}$), the origin is no longer an equilibrium, but a nearby equilibrium exists. In search of this nearby equilibrium solution, the linear and quadratic terms in P_i and Q_i of Eq. (6.9) are kept, which leads to

$$K^* = C_1 Q_1^2 + C_2 Q_1 P_1 + C_3 P_1^2 + C_4 Q_2^2 + C_5 P_2^2 + C_6 Q_1 + C_7 P_1 \tag{7.2}$$

where

$$C_1 = 4.418708\ m^2 + (0.672102\ m^3 + 0.081621\ m^2 e) - 26.98928\ m^4$$

$$C_2 = 2.42737\ m^2 + 0.543614\ m^3 + 1.65662\ m^4$$

$$C_3 = 0.121254\ m^2 + (-0.672102\ m^3 + 0.081621\ m^2 e) - 32.26905\ m^4$$

$$C_4 = 2.292988\ m^2 + 1.487332\ me^2 + 20.0769\ m^4$$

$$C_5 = C_4$$

$$C_6 = -0.146701\ m^3 - 1.579912\ m^4$$

$C_7 = -0.398984\ m^3 + 0.572077\ m^4$

In view of (7.2), Eq. (7.1) yields the equilibrium solution $P_2 = Q_2 = 0$ and $(P_1, Q_1)_1$ as given in Table 1.

Table 1

FIRST EQUILIBRIUM SOLUTION FOR VARIOUS ORDERS AND ECCENTRICITIES

	Second Order	Third Order		Fourth Order
		e = 0	e = 0.0549	
P_1	0	-0.049689	-0.050364	-0.022543
Q_1	0	0.014951	0.0151235	0.008761

With reference to Eqs. (6.2), (6.8), and (7.2), the first equilibrium corresponds to a one-month periodic orbit whose stability is essentially the Mathieu-type Hamiltonian of Eq. (3.77). This Hamiltonian is not definite as a quadratic form in (P_1, Q_1); as a result, mode 1 (or a faster mode) is parametrically excited, and the periodic motion corresponding to this small (P_1, Q_1) is necessarily unstable. The monthly position fluctuation in response to the higher order solar perturbation is comparable in size to the direct gravity-gradient twice-monthly fluctuation (see Table 4).

In addition to the first equilibrium, eight others were found and constructed as a power series in m in the form

$$(P_i, Q_i)_j = (P_{i0}, Q_{i0})_j + m(P_{i1}, Q_{i1})_j + m^2 (P_{i2}, Q_{i2})_j \qquad \begin{matrix} i = 1,2 \\ 2 \le j \le 9 \end{matrix} \tag{7.3}$$

Substituting Eq. (7.3) into (7.1) and equating to zero the coefficients of m^n lead to the following equilibrium solutions:

(1) $e = 0$

$$(P_1, Q_1)_2 = (1.7946, -0.4718) + m(3.7021, -0.9113) + m^2(-18.0202, 60.9256)$$

$$(P_2, Q_2)_2 = 0$$

$$(P_1, Q_1)_3 = (-1.79468, 0.4718) + m(-2.8380, 0.7349)$$

$$+ m^2(12.6714,\ -59.2286)$$

$$(P_2, Q_2)_3 = 0$$

$$(P_1, Q_1)_4 = (2.3026,\ -0.6053) + m(6.6419,\ -1.6738)$$

$$+ m^2(15002.25,\ -3811.372)$$

$$(P_2, Q_2)_4 = (-0.3076,\ -0.0127) + m(-1.3886,\ -0.0544)$$

$$+ m^2(-4782.425,\ -209.4033)$$

$$(P_1, Q_1)_5 = (-2.3026,\ 0.6053) + m(-5.4080,\ 1.3996)$$

$$+ m^2(-15011.63,\ 3814.135)$$

$$(P_2, Q_2)_5 = (0.3076,\ 0.0127) + m(1.1314,\ 0.0459)$$

$$+ m^2(4784.648,\ 209.5115)$$

$$(P_1, Q_1)_6 = (P_1, Q_1)_4$$

$$(P_2, Q_2)_6 = \sqrt{(P_2)_4^2 + (Q_2)_4^2}\left(+\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$$

$$(P_1, Q_1)_7 = (P_1, Q_1)_4$$

$$(P_2, Q_2)_7 = \sqrt{(P_2)_4^2 + (Q_2)_4^2}\left(+\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$$

$$(P_1, Q_1)_8 = (P_1, Q_1)_5$$

$$(P_2, Q_2)_8 = \sqrt{(P_2)_5^2 + (Q_2)_5^2}\left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$$

$$(P_1, Q_1)_9 = (P_1, Q_1)_5$$

$$(P_2, Q_2) = \sqrt{(P_2)_5^2 + (Q_2)_5^2}\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

(2) $e = 0.0549$

$$(P_1, Q_1)_2 = (1.7946, -0.4718) + m(3.5027, -0.8589) + 0(m^2)$$

$$(P_2, Q_2)_2 = 0$$

$$(P_1, Q_1)_3 = (-1.7946, 0.4718) + m(-2.6381, 0.6825) + 0(m^2)$$

$$(P_2, Q_2)_3 = 0$$

$$(P_1, Q_1)_4 = (2.3026, -0.6053) + m(-1.5310, -1.2441) + 0(m^2)$$

$$(P_2, Q_2)_4 = (-0.3078, -0.0127) + m(-1.2441, -0.0484) + 0(m^2)$$

$$(P_1, Q_1)_5 = (-2.3026, 0.6053) + m(-4.8615, 1.2567) + 0(m^2)$$

$$(P_2, Q_2)_5 = (0.3076, 0.0127) + m(0.9874, 0.0399) + 0(m^2)$$

With reference to Eqs. (6.2) and (6.8), the first term in the above solutions resulting from the second-order analysis yields, for equilibria 2 and 3, a monthly motion in phase or 180° out of phase with the sun. Both have the same size and are of larger value than those in Ref. 12. The difference must arise from the inclusion of the m^2 terms in R_0 of Eq. (5.1).

Equilibria 4 through 9 correspond to three-month periodic orbits, 60° apart. Each series of three (with spacing of 120°) forms a family of the same size, as shown in Fig. 9; therefore, only one from each family is considered as a candidate for this analysis.

The inclusion of the first power of m in the calculation of $(P_i, Q_i)_j$ now gives two slightly different magnitudes that correspond to two slightly different sizes for the semimajor axis, depending on motion in or out of phase with the sun. The inclusion of the second power of m yields a slight decrease in the magnitude of equilibria 2 and 3 and a drastic increase in 4 through 9, indicating the nonexistence of such equilibria. This is possibly caused by the fact that the corresponding orbits are so large that sections of them lie outside the region of convergence defined by Eq. (4.26).

B. Stability of Equilibrium Points

The stability of the slow variations around the periodic and quasi-periodic equilibrium motions can be determined by setting up the expression for δK^* by taking small displacements δP and δQ around the equilibrium values $(P_i, Q_i)_j$. Because fourth-order corrections for equilibria 4 and 5 are too large, the stability is investigated using only third-order corrections. Because $(P_i, Q_i)_j$ are equilibrium points, the coefficients of the linear terms in δP and δQ must vanish, and the resulting δK^*, for the first three equilibria, takes the form

$$\delta K^* = m^2\left(C_1\delta P_1^2 + C_2\delta P_1\delta Q_1 + C_3\delta Q_1^2 + C_4\delta P_2^2 + C_5\delta Q_2^2\right) \tag{7.4}$$

where C_i are given for each equilibria, as seen in Table 2.

Table 2

NUMERICAL VALUES OF C_i FOR DIFFERENT EQUILIBRIA E_j

	C_1	C_2	C_3	C_4	C_5
E_1	0.07098	2.468033	4.468982	4.405318	4.405318
E_2	0.813874	2.215314	4.760572	-5.884246	-5.884246
E_3	0.7731718	2.227761	4.745097	-5.440188	-5.440188

For the equilibrium to be stable, δK^* should be definite, i.e.,

$$4C_1C_3 > C_2^2 \tag{7.5a}$$

$$C_4C_5 > 0 \tag{7.5b}$$

In this case, any disturbance in mode 1 $(P_1,Q_1)_j$ will result in oscillation around the equilibrium of a period

$$T_1 = \frac{\pi}{m^2\omega_1\sqrt{C_1C_3 - C_2^2/4}} \tag{7.6a}$$

and any disturbance in mode 2 $(P_2,Q_2)_j$ will result in oscillation with

$$T_2 = \frac{\pi}{m^2\omega_2\sqrt{C_4C_5}} \tag{7.6b}$$

From Eqs. (7.5) and (7.6),

E_1: unstable

E_2: stable $T_1 = 47$ mo $T_2 = 42$ mo

E_3: stable $T_1 = 49$ mo $T_2 = 45$ mo

For equilibria E_4 and E_5 (if they exist),

$$\begin{aligned} E_4:\ \delta K^* &= 1.120477\ \delta P_1^2 + 2.012351\ \delta P_1\delta Q_1 + 4.705913\ \delta Q_1^2 \\ &+ 4.62852\ \delta P_1\delta P_2 - 1.355016\ \delta P_2\delta Q_1 + 12.44027\ \delta P_2^2 \\ &+ 1.061331\ \delta Q_2\delta P_1 + 3.28021\ \delta Q_2\delta Q_1 + 4.045434\ \delta Q_2\delta P_2 \\ &- 36.9593\ \delta Q_2^2 \end{aligned}$$

$$E_5:\ \delta K^* = -34.52581\ \delta Q_2^2 + 3.808855\ \delta Q_2 \delta P_2$$

$$+ 3.033715\ \delta Q_1 \delta Q_2 + 0.9892793\ \delta Q_2 \delta P_1$$

$$+ 11.61883\ \delta P_2^2 - 1.277582\ \delta P_2 \delta Q_1$$

$$+ 4.351618\ \delta P_1 \delta P_2 + 4.698642\ \delta Q_1^2 + 2.036076\ \delta P_1 \delta Q_1$$

$$+ 1.067101\ \delta P_1^2$$

As a result, δK^*, for both E_4 and E_5, is not definite in mode 2, i.e., P_2 and Q_2; therefore, they are unstable equilibria (if they exist).

It is possible to take advantage of the fact that the stable and unstable equilibrium points are close to each other (by third-order analysis) and that instability occurs through mode 2, i.e., P_2 and Q_2. It is, thus, of particular interest to determine the extent of the stable region around E_2 and E_3 by expanding K^* up to cubic powers in δP^* and δQ^* around E_2 and E_3:

$$\delta K^* = C_7\left(\delta P_2^2 + \delta Q_2^2\right) - \frac{2}{3}\, C_8\left(\delta P_2^3 - 3\delta P_2 \delta Q_2^2\right) \tag{7.7}$$

where, for E_2,

$$C_7 = -5.884246 \qquad C_8 = 23.05297 \tag{7.8a}$$

and, for E_3,

$$C_7 = -5.440188 \qquad C_8 = -22.33889 \tag{7.8b}$$

In addition to the stable equilibrium at the origin, the associated contours in the δP_2-δQ_2 plane have unstable equilibria at $(C_7/C_8, 0)$ and $(-1/2\ C_7/C_8, \pm \sqrt{3}/2\ C_7/C_8)$, which form the vertices of an equilateral triangle, as shown in Fig. 9. The straight lines through these vertices are the separatrices corresponding to $\delta K^* = 1/3\ C_7^3/C_8^2$ so that only the interior of this triangle leads to bounded $(\delta P_2, \delta Q_2)$ motion.

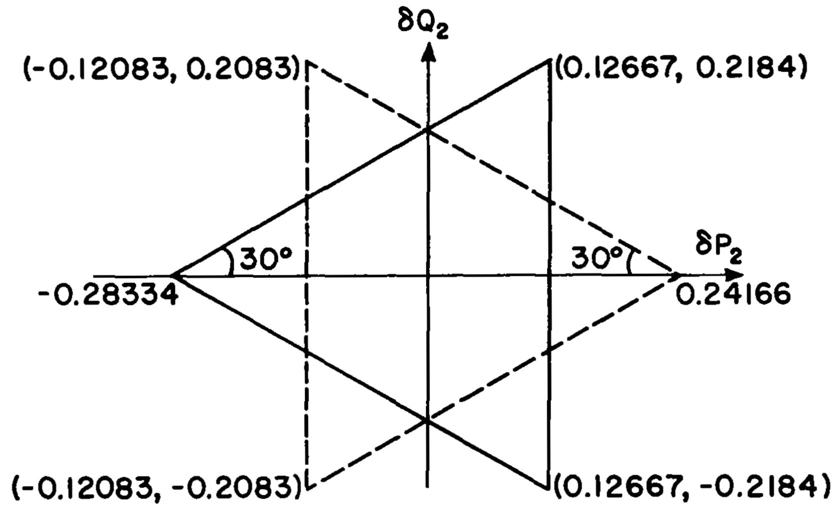

Fig. 9. STABILITY REGIONS IN THE $\delta P_2, \delta Q_2$ PLANE NEAR THE EQUILIBRIUM POINTS.

This approximate analysis has not taken into account that the unstable $(\delta P_1, \delta Q_1)$ equilibrium values are larger than the stable ones; nevertheless, some idea is obtained about how small δP_2 and δQ_2 of mode 2 must be to avoid the eventual wild divergence of the orbit.

C. The Physical Coordinates

The physical xy coordinates can be obtained in terms of $\bar{\alpha}_i$ and $\bar{\beta}_i$ by using Eqs. (3.6), (3.7), (3.26a), and (5.15). Substituting $\bar{\alpha}_i$ and $\bar{\beta}_i$ in terms of P_i and Q_i and keeping only the influence of W_1 and W_2 obtain, for $f = x, y$,

$$f = \sum_{i=0}^{9} \left[a_i C_{\frac{i}{3}\theta_s} + b_i S_{\frac{i}{3}\theta_s} \right]$$

$$+ a_{10} C_{\theta_e} + b_{10} S_{\theta_e} + a_{11} C_{2\theta_e} + b_{11} S_{2\theta_e}$$

$$+ a_{12} C_{2\theta_s - \theta_e} + b_{12} S_{2\theta_s - \theta_e} \qquad (7.9)$$

where

$$a_0 = m^2\left[(P_1^2 + Q_1^2)c_4 + c_8(P_2^2 + Q_2^2)\right] + e^2mc_{26}$$

$$a_1 = m(c_2P_2 - s_2Q_2) + m^3\Big\{- s_{14}Q_2(P_1^2 + Q_1^2) + c_{14}P_2(P_1^2 + Q_1^2)$$

$$+ c_{17}\left[P_1(P_2^2 - Q_2^2) - 2Q_1P_2Q_2\right] - s_{17}\left[Q_1(P_2^2 - Q_2^2) + 2P_1P_2Q_2\right]$$

$$- s_{21}Q_2(P_2^2 + Q_2^2) + c_{21}(P_2^2 + Q_2^2)P_2\Big\}$$

$$b_1 = m(c_2Q_2 + s_2P_2) + m^3\Big\{s_{14}P_2(P_1^2 + Q_1^2)$$

$$+ c_{14}Q_2(P_1^2 + Q_1^2) - s_{17}\left[P_1(P_2^2 - Q_2^2) - 2Q_1P_2Q_2\right]$$

$$- c_{17}\left[Q_1(P_2^2 - Q_2^2) + 2P_1P_2Q_2\right] + s_{21}(P_2^2 + Q_2^2)P_2 + c_{21}Q_2(P_2^2 + Q_2^2)\Big\}$$

$$a_2 = m^2\Big[- (P_1Q_2 + Q_1P_2)s_5 + c_5(P_1P_2 - Q_1Q_2) - 2s_7P_2Q_2$$

$$+ c_7(P_2^2 - Q_2^2)\Big]$$

$$b_2 = m^2\Big[-c_5(P_1Q_2 + Q_1P_2) - s_5(P_1P_2 - Q_1Q_2)$$

$$+ s_7(P_2^2 - Q_2^2) + 2c_7P_2Q_2\Big]$$

$$a_3 = m(c_1P_1 + s_1Q_1) + m^3\Big[s_{10}Q_1(P_1^2 + Q_1^2) + c_{10}P_1(P_1^2 + Q_1^2)$$

$$+ s_{16}Q_1(P_2^2 + Q_2^2) + c_{16}P_1(P_2^2 + Q_2^2) - Q_1s_{19}$$

$$+ P_1c_{19} - s_{20}(3Q_2P_2^2 - Q_2^3) + c_{20}(P_2^3 - 3P_2Q_2^2)\Big]$$

$$b_3 = (-c_1Q_1 + s_1P_1)m + m^3\Big[s_{10}P_1(P_1^2 + Q_1^2) - c_{10}Q_1(P_1^2 + Q_1^2)$$

$$+ (P_1s_{16} - Q_1c_{16})(P_2^2 + Q_2^2) + s_{19}P_1 + Q_1c_{19}$$

$$+ (P_2^3 - 3P_2Q_2^2)s_{20} + (3Q_2P_2^2 - Q_2^3)c_{20}\Big]$$

$$a_4 = m^2\Big[s_6(Q_1P_2 - P_1Q_2) + c_6(P_1P_2 + Q_1Q_2)\Big]$$

$$b_4 = m^2\Big[s_6(P_1P_2 + Q_1Q_2) - c_6(Q_1P_2 - P_1Q_2)\Big]$$

$$a_5 = m^3\Big\{-\Big[2P_1Q_1P_2 + (P_1^2 - Q_1^2)Q_2\Big]s_{13} + \Big[(P_1^2 - Q_1^2)P_2 - 2P_1Q_1Q_2\Big]c_{13}$$

$$+ \Big[Q_1(P_2^2 - Q_2^2) - 2P_2Q_2P_1\Big]s_{15} + \Big[P_1(P_2^2 - Q_2^2) + 2P_2Q_2Q_1\Big]c_{16}$$

$$+ Q_2s_{23} + P_2c_{23}\Big\}$$

$$b_5 = m^3\Big\{-s_{13}\Big[(P_1^2 - Q_1^2)P_2 - 2P_1Q_1Q_2\Big] - c_{13}\Big[2P_1Q_1P_2 + (P_1^2 - Q_1^2)Q_2\Big]$$

$$+ s_{15}\Big[P_1(P_2^2 - Q_2^2) + 2P_2Q_2Q_1\Big] - c_{15}\Big[Q_1(P_2^2 - Q_2^2) - 2P_2Q_2P_1\Big]$$

$$+ P_2s_{23} - Q_2c_{23}\Big\}$$

$$a_6 = m^2\Big[2P_1Q_1s_3 + (P_1^2 - Q_1^2)c_3 + c_9\Big] + m^3c_{24}$$

$$b_6 = m^2\Big[(P_1^2 - Q_1^2)s_3 - 2P_1Q_1c_3 + s_9\Big] + m^2s_{24}$$

$$a_7 = m^3\Big\{s_{12}\Big[(P_1^2 - Q_1^2)P_2 + 2P_1Q_1Q_2\Big] + c_{12}\Big[(P_1^2 - Q_1^2)P_2 + 2P_1Q_1P_2\Big]$$

$$- s_{22}Q_2 + P_2c_{22}\Big\}$$

$$b_7 = m^3\Big\{s_{12}\Big[(P_1^2 - Q_1^2)P_2 + 2P_1Q_1Q_2\Big] + c_{12}\Big[(P_1^2 - Q_1^2)Q_2 - 2P_1Q_1P_2\Big]$$

$$+ P_2s_{22} + Q_2c_{22}\Big\}$$

$$a_8 = b_8 = 0$$

$$a_9 = m^3 \left[c_{11}(P_1^3 - 3P_1Q_1^2) + s_{11}(3Q_1P_1^2 - Q_1^3) + s_{18}Q_1 + c_{18}P_1\right]$$

$$b_9 = m^3\left[s_{11}(P_1^3 - 3P_1Q_1^2) - Q_1(3P_1^2 - Q_1^2)c_{11} + P_1s_{18} - Q_1c_{18}\right]$$

$$a_{10} = mec_{25} + m^2e(c_{33} + c_{28}P_1 + s_{28}Q_1 + c_{29}P_1 + s_{29}Q_1 + c_{30}P_2$$

$$+ s_{30}Q_2 + c_{31}P_2 + s_{31}Q_2)$$

$$b_{10} = mes_{25} + m^2e(s_{33} - c_{28}Q_1 + s_{28}P_1 + c_{29}Q_1 - s_{29}P_1 - c_{30}Q_2$$

$$+ s_{30}P_2 + c_{31}P_2 + c_{31}Q_2 - s_{31}P_2)$$

$$a_{11} = e^2m\ c_{27}$$

$$b_{11} = e^2m\ s_{27}$$

$$a_{12} = em^2\ c_{32}$$

$$b_{12} = em^2\ s_{32} \tag{7.10}$$

The coefficients c_i and s_i for x and y are presented in Table 3.

Substitution of the equilibrium values $(P_i,Q_i)_j$ into Eqs. (7.10) leads to the periodic and quasi-periodic orbits in the form of Eq. (7.9). The coefficients a_i and b_i for various orders and equilibria are listed in Tables 4 through 8. Note that, as expected, the effect of the eccentricity is a close imitation of the moon motion; the secular motion of the moon's perigee, caused by the sun, somewhat distorts the exact eccentricity imitation. The features of the obtained periodic and quasi-periodic orbits are tabulated in Table 9, and the stable orbits are plotted in Fig. 10. Allowing for a 3 percent difference in size, these orbits are almost indistinguishable from those in Fig. 4 of Ref. 14.

Table 3

NUMERICAL VALUES FOR c_i AND s_i USED IN EQS. (7.10)

i	f = x		f = y	
	c_i	s_i	c_i	s_i
1	2.078779	0	-0.836790	-1.244989
2	5.658032	0	-3.064482	-1.449433
3	-0.477376	0.054201	-0.410524	-0.42068
4	-2.754069	0	0.7989711	0
5	6.189117	-4.99475	-5.846259	7.468903
6	-4.556632	1.353424	-2.25725	-1.495887
7	9.066064	-3.217712	-15.86619	-7.255554
8	-27.45219	0	4.785202	0
9	-1.032939	1.335706	1.330059	0.5061397
10	0.984043	12.25133	12.18598	-3.564763
11	-0.036963	-0.502484	0.5260429	-0.1406562
12	-0.662384	-4.659913	5.024582	-1.261261
13	-7.857452	4.552246	0.864563	6.483002
14	3.188846	-98.85581	-12.43194	51.90156
15	14.17146	-1.021435	5.959321	-6.845459
16	13.11474	78.98347	87.73532	-62.48818
17	-167.1086	-322.519	152.3395	182.4546
18	0.128523	-0.620743	0.5178337	0.200539
19	19.75492	13.52622	-3.383034	-15.30651
20	-35.05274	-140.9546	-108.5904	51.57765
21	19.64869	-1268.607	-197.7113	70.90593
22	-1.359009	-3.329499	0.7789154	1.865039
23	-11.87335	-9.426957	-1.157608	11.04512
24	0.1790439	-0.1758309	-0.083248	-0.112715
25	-6.571491	-23.33596	-11.60792	13.16193
26	3.455698	0	-1.853139	0
27	0.3103889	-0.202524	-0.2781192	-0.012889
28	4.010418	9.30537	-7.196863	12.37255
29	-31.43699	-17.25999	23.17124	-8.922399
30	17.66468	28.2631	-16.48526	44.58725
31	118.3920	10.57441	-66.68749	-11.49231
32	8.68036	31.37417	14.52541	-18.30503
33	1.069179	3.78897	1.883865	-2.138359

Table 4

NUMERICAL VALUES FOR a_i AND b_i USED IN EQ. (7.9) CORRESPONDING TO EQUILIBRIUM E_1

	e = 0												e = .0549			
	Second Order				Third Order				Fourth Order				Third Order			
	f = x		f = y		f = x		f = y		f = x		f = y		f = x		f = y	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	0	0	0	0	-.000041	0	.000012	0	-0.000009	0	.000003	0	0.000736	0	-.000405	
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	-0.00822	-0.00248	.001878	.005879	-0.00374	-0.001418	.00068	.002788	-0.008333	-0.002512	.001906	.00595
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	-0.0057	0.0074	.007407	.002785	-0.0057	0.0074	.007405	.002776	-0.0057	0.0074	.007407	.002783	-0.0057	0.0074	.007405	.002776
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	-0.00007	0.000012	-.00001	-.000007	-0.000003	0.000005	-.000004	-.000004	-0.000007	0.000012	-.00001	-.000008
10	0	0	0	0	0	0	0	0	0	0	0	0	-0.026271	-0.095243	-.047321	.053208
11	0	0	0	0	0	0	0	0	0	0	0	0	0.00007	-0.000046	-.000063	-0.000003
12	0	0	0	0	0	0	0	0	0	0	0	0	0.002666	0.009637	0.004462	-.005623

Table 5

NUMERICAL VALUES FOR a_i AND b_i USED IN EQ. (7.9) CORRESPONDING TO EQUILIBRIUM E_2

	e = 0												e = .0549			
	Second Order				Third Order				Fourth Order				Third Order			
	f = x		f = y		f = x		f = y		f = x		f = y		f = x		f = y	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	-.05306	0	0.015393	0	-0.07062	0	.020487	0	-.060458	0	.017539	0	-0.068827	0	.019775	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	.29078	.111977	-.03984	-.20909	0.333521	0.140924	-.033457	-.241539	0.323038	0.080434	-.068205	-.21662	0.331245	0.139196	-.03396	-.2397
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	-.014226	.003786	.004506	-.008162	-0.017066	0.002637	.003485	-.011769	-0.01621	0.00647	.000424	-.008066	-0.016903	0.002706	.00354	-.01156
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	00	0
9	.00108	-.00147	.00162	.00964	0.001571	-0.002103	.002362	.00137	0.000529	-0.002097	.00218	.000281	0.001541	-0.002065	.002317	.001345
10	0	0	0	0	0	0	0	0	0	0	0	0	-0.042675	-.072047	-.037866	.06184
11	0	0	0	0	0	0	0	0	0	0	0	0	0.00007	-0.000046	-.000063	-.000003
12	0	0	0	0	0	0	0	0	0	0	0	0	0.002666	0.009637	.004462	-.005623

Table 6

NUMERICAL VALUES FOR a_i AND b_i USED IN EQ. (7.9) CORRESPONDING TO EQUILIBRIUM E_3

	e = 0												e = .0549			
	Second Order				Third Order				Fourth Order				Third Order			
	f = x		f = y		f = x		f = y		f = x		f = y		f = x		f = y	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	-0.05306	0	.015393	0	-0.066341	0	.019246	0	-0.058346	0	.016926	0	-.064577	0	.018542	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	-0.290779	-0.111976	.03984	.209093	-0.32357	-0.134154	.035131	.234055	-0.317379	-.077617	.068415	.212537	-.321277	-.132489	.035567	.23229
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	-0.014226	0.003786	.004506	-.008162	-0.016363	0.00289	.00377	-.010899	-.015843	0.006504	.000666	-.007685	-0.016204	0.002959	.003824	-.01069
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	-0.001085	0.001473	-.00162	-.000964	-0.001453	0.001938	-.002169	-.001274	-.000507	0.002006	-.002081	-.000273	-.001425	.001902	-.002126	-.00125
10	0	0	0	0	0	0	0	0	0	0	0	0	-.011154	-.116615	-.05631	.045241
11	0	0	0	0	0	0	0	0	0	0	0	0	.00007	-.000046	-.00063	-.000003
12	0	0	0	0	0	0	0	0	0	0	0	0	.002666	.009637	.004462	-.005623

Table 7

NUMERICAL VALUES FOR a_i AND b_i USED IN EQ. (7.9) CORRESPONDING TO EQUILIBRIUM E_4

	e = 0												e = .0549			
	Second Order				Third Order				Fourth Order				Third Order			
	f = x		f = y		f = x		f = y		f = x		f = y		f = x		f = y	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	-.101902	0	.02787	0	-.155029	0	.041959	0	-	-	-	-	-.149179	0	.04026	0
1	-.157192	.109399	.099365	-.016458	-.231881	.23157	.156606	-.058934	-	-	-	-	-.223608	.217287	.150139	-.05397
2	-.015475	-.026744	.008795	.030522	-.024402	-.043655	.013038	.048854	-	-	-	-	-.023517	-.041864	.012676	.046972
3	.367798	.179783	-.012072	-.273109	.439894	.266656	.037075	-.337118	-	-	-	-	.43817	.258185	.031608	-.331643
4	.019491	.000189	.007044	.008584	.031685	.000232	.011483	.013927	-	-	-	-	.030404	.000222	.011019	.013364
5	.006467	.007078	-.001882	.002078	.012061	.01329	-.003605	.004942	-	-	-	-	.011433	.012602	-.003423	.00463
6	-.019733	.001451	.002632	-.015236	-.02645	-.00131	.00026	-.0238	-	-	-	-	-.025853	-.001058	.000464	-.023033
7	-.001226	.003582	-.003694	-.001397	-.002497	.00681	-.007248	-.002574	-	-	-	-	-.002355	.006456	-.006858	-.00244
8	0	0	0	0	0	0	0	0	-	-	-	-	0	0	0	0
9	.00211	-.002746	.003132	.001826	.0036	-.004619	.005377	.003061	-	-	-	-	.003455	-.00444	.005161	.00294
10	0	0	0	0	0	0	0	0	0	0	0	0	-.06508	-.066999	-.024246	.058075
11	0	0	0	0	0	0	0	0	0	0	0	0	.00007	-.000046	-.000063	-.000003
12	0	0	0	0	0	0	0	0	0	0	0	0	.002666	.009637	.004462	-.005623

Table 8

NUMERICAL VALUES FOR a_i AND b_i USED IN EQ. (7.9) CORRESPONDING TO EQUILIBRIUM E_5

	e = 0												e = .0549			
	Second Order				Third Order				Fourth Order				Third Order			
	f = x		f = y		f = x		f = y		f = x		f = y		f = x		f = y	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	-.101902	0	.027877	0	-.144349	0	0.039135	0	-	-	-	-	-.138685	0	.03745	0
1	.157192	-.109399	-.099365	.016458	.216966	-.20432	-.14475	.04908	-	-	-	-	.208987	-.191175	-.138611	.04459
2	-.015475	-.02674	.008795	.030522	-.02258	-.040246	.012174	.045169	-	-	-	-	-.02173	-.038523	.01182	.043358
3	-.367798	-.17978	.012072	.273109	-.426638	-.248888	-.026058	.32499	-	-	-	-	-.420854	-.240878	-.02108	.31958
4	.01949	.000189	.007044	.008584	.029217	.000262	.010568	.01286	-	-	-	-	.027985	.000251	.010122	.01232
5	-.00647	-.007078	.001882	-.002078	-.010829	-.011957	.003242	-.004275	-	-	-	-	-.010246	-.011315	.00307	-.00401
6	-.01973	.0045	.002632	-.015236	-.025095	-.000798	.000782	-.022105	-	-	-	-	-.024517	-.000553	.00098	-.02136
7	.001226	-.00358	.003694	.001397	.00223	-.006103	.006467	.002334	-	-	-	-	.002098	-.005774	.006104	.002211
8	0	0	0	0	0	0	0	0	-	-	-	-	0	0	0	0
9	-.00211	.002746	-.003132	-.001826	-.003294	.004211	-.00489	-.002812	-	-	-	-	-.003157	.004044	-.0047	-.0027
10	0	0	0	0	0	0	0	0	0	0	0	0	.010224	-.12148	-.069	.0488
11	0	0	0	0	0	0	0	0	0	0	0	0	.00007	-.000046	-.00006	-.000003
12	0	0	0	0	0	0	0	0	0	0	0	0	.002666	.009637	.00466	-.00562

Table 9

NUMERICAL VALUES FOR SEMIMAJOR AXES, SEMIMINOR AXES, INCLINATION TO THE x-AXIS, AND ECCENTRICITIES OF ORBITS FOR VARIOUS ORDERS AND EQUILIBRIA

Equilibrium E_i		Period in Lunar Month	Semimajor Axis			Semiminor Axis			Inclination to the x-Axis			Eccentricity of the Orbit		
			Second Order	Third Order	Fourth Order	Second Order	Third Order	Fourth Order	Second Order	Third Order	Fourth Order	Second Order	Third Order	Fourth Order
E_1	First Harmonic	1	0	2.29×10^3	1.07×10^3	0	1.1×10^3	0.5×10^3	-	-29.6°	-29.6°	-	0.88	0.88
	Second Harmonic	$\frac{1}{2}$	0	2.40×10^3	2.4×10^3	0	1.7×10^3	1.7×10^3	-	-30.2°	-30.2°	-	0.71	0.71
E_2	First Harmonic	1	81×10^3	94×10^3	87×10^3	40×10^3	46×10^3	43×10^3	-26.7°	-25.8°	-26.5°	0.87	0.87	0.87
E_3	First Harmonic	1	81×10^3	91×10^3	85×10^3	40×10^3	45×10^3	42×10^3	-26.7°	-26°	-26.6°	0.87	0.87	0.87
E_4	First Harmonic	3	51×10^3	87×10^3	∞	9×10^3	15×10^3	∞	-26.4°	-25.8°	-	0.98	0.98	-
	Third Harmonic	1	105×10^3	130×10^3	∞	54×10^3	70×10^3	∞	-24.5°	-22.3°	-	0.86	0.84	-
E_5	First Harmonic	3	51×10^3	79×10^3	∞	9×10^3	14×10^3	∞	-26.4°	-25.8°	-	0.98	0.98	-
	Third Harmonic	1	105×10^3	125×10^3	∞	54×10^3	67×10^3	∞	-24.5°	-22.7°	-	0.86	0.85	-

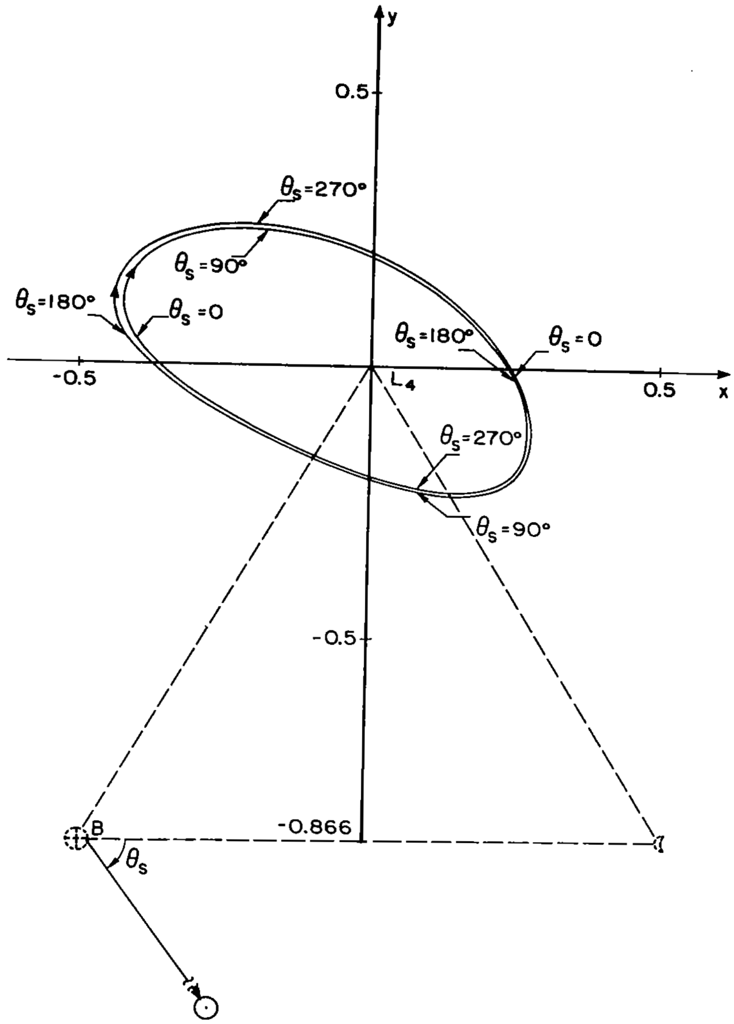

Fig. 10. ONE-MONTH STABLE PERIODIC ORBITS.

Chapter VIII

SUMMARY AND CONCLUSIONS

Simplified general expansions in the theory of perturbation based on Lie transforms have been developed. Application of these expansions to the theory of nonlinear oscillations was outlined, and two simple examples were presented for clarification.

General formulas for the theory of perturbation based on the Lie series were also obtained and discussed in relation to those based on Lie transforms. Computerized symbolic manipulation of some of these formulas was employed in a canonical perturbation treatment of sun-perturbed motion near the earth-moon L_4 libration point up to fourth order, idealized as coplanar.

In the absence of the mean lunar eccentricity, periodic orbits were found by searching for equilibrium solutions. Because of the higher order nongravity-gradient solar-perturbation terms, the origin is no longer an equilibrium. A nearby equilibrium exists and corresponds to unstable one-month periodic motion.

The search for other equilibrium solutions led to the possibility (as in Ref. 12) of stable one-month periodic orbits in phase or 180° out of phase with the sun. For these orbits, the second-order analysis yielded semimajor axes approximately 10 percent lower than that in Ref. 14, and a third-order analysis yielded a substantial improved agreement. Depending on motion in or out of phase with the sun, the analysis resulted in two slightly different sizes for the semimajor axis, both only about 3 percent larger than in Ref. 14. The extent of the region of stability around these orbits was found approximately to be the interior of equilateral triangles (see Fig. 9). The fourth-order analysis, however, failed to produce further improved agreement but, instead, yielded orbits about 3 percent too small.

The existence of slightly larger unstable orbits, as in Ref. 12, is more questionable. The third-order analysis led to three-month unstable periodic orbits again in or out of phase with the sun but with semimajor axes about 45 percent larger than the stable periodic orbits for mode 1, and about 80,000 mi for mode 2 (see Table 9). The fourth-order analysis, however, yielded a totally unreasonable change in these unstable orbits.

In the presence of lunar eccentricity, the stable orbits became quasi-periodic. The effect of this eccentricity on the size of these orbits was found to be small.

Appendix A

THE HAMILTONIAN H OF EQUATION (5.16)

$$H_1 = \Big(-1.8483\ S_{B_1-2\theta_s} - 1.791798\ C_{B_1-2\theta_s}$$

$$+ 0.224518\ S_{B_1+2\theta_s} - 0.854307\ C_{B_1+2\theta_s}\Big)A_1$$

$$+ \Big(-3.467575\ S_{B_2-2\theta_s} - 4.655926\ C_{B_2-2\theta_s}$$

$$+ 1.584059\ S_{B_2+2\theta_s} - 3.568476\ C_{B_2+2\theta_2}\Big)A_2$$

$$+ \Big(0.616987\ S_{B_1-\theta_e} - 0.173164\ C_{B_1-\theta_e}$$

$$-1.273558\ S_{B_1+\theta_e} + 0.350445\ C_{B_1+\theta_e}\Big)\frac{e}{m}\ A_1$$

$$+ \Big(2.255962\ S_{B_2-\theta_e} - 0.40764\ C_{B_2-\theta_e}$$

$$- 3.01758\ S_{B_2+\theta_e} + 0.104242\ C_{B_2+\theta_e}\Big)\frac{e}{m}\ A_2$$

$$+\Big(1.060361\ S_{B_1} - 0.386893\ C_{B_1}$$

$$+ 1.694639\ S_{3B_1} - 1.883111\ C_{3B_1}\Big)A_1^3$$

$$+\Big(2.459942\ S_{B_2} - 0.771045\ C_{B_2}$$

$$+ 7.684611\ S_{2B_1-B_2} - 3.896135\ C_{2B_1-B_2}$$

$$+ 12.35176\ S_{2B_1+B_2} - 10.00034\ C_{2B_1+B_2}\Big) A_1^2\ A_2$$

$$+ \Big(30.11885\ S_{B_1} - 10.11404\ C_{B_1}$$

$$+ 4.718869\ S_{B_1-2B_2} - 0.984863\ C_{B_1-2B_2}$$

$$+ 27.96021\ S_{B_1+2B_2} - 16.20978\ C_{B_1+2B_2}\Big) A_1\ A_2^2$$

$$+ \Big(17.46826\ S_{B_2} - 4.283916\ C_{B_2}$$

$$+ 18.45821\ S_{3B_2} - 7.31473\ C_{3B_2}\Big) A_2^3$$

$$\frac{1}{2!} H_2 = \Big(-0.260072\ S_{2(B_1+\theta_s)} + 0.002499\ C_{2(B_1+\theta_s)}$$

$$+ 1.04454\ S_{2(B_1-\theta_s)} - 1.946338\ C_{2(B_1-\theta_s)}$$

$$- 1.304626\ S_{2\theta_s} - 0.771988\ C_{2\theta_s}\Big) A_1^2$$

$$+ \Big(0.362402\ S_{B_1-2\theta_s} - 0.068577\ C_{B_1-2\theta_s}$$

$$- 1.115752\ S_{B_1+2\theta_s} + 0.137168\ C_{B_1+2\theta_s}\Big) A_1$$

$$+ \Big(6.055805\ S_{B_1+B_2-2\theta_s} - 7.911159\ C_{B_1+B_2-2\theta_s}$$

$$- 3.18288\ S_{B_1-B_2+2\theta_s} - 1.247505\ C_{B_1-B_2+2\theta_s}$$

$$+ 5.14584\ S_{B_1-B_2-2\theta_s} - 4.291294\ C_{B_1-B_2-2\theta_s}$$

$$- 2.272914\ S_{B_1+B_2+2\theta_s} - 0.3462371\ C_{B_1+B_2+2\theta_s}\Big) A_1 A_2$$

$$+ \Big(-2.576375\ S_{B_2+2\theta_s} - 0.3206347\ C_{B_2+2\theta_s}$$

$$+ 1.70252\ S_{B_2-2\theta_s} - 0.6189656\ C_{B_2-2\theta_s}\Big) A_2$$

$$+ \Big(-4.835863\ S_{2(B_2+\theta_s)} - 1.559442\ C_{2(B_2+\theta_s)}$$

$$+ 8.168333\ S_{2(B_2-\theta_s)} - 7.712266\ C_{2(B_2-\theta_s)}$$

$$- 13.00419\ S_{2\theta_s} - 7.69498\ C_{2\theta_s}\Big) A_2^2$$

$$+ \Big(-1.516013\ S_{2B_1+\theta_e} + 2.346225\ C_{2B_1+\theta_e}$$

$$+ 0.815141\ S_{2B_1-\theta_e} - 1.136029\ C_{2B_1-\theta_e}$$

$$- 2.331154\ S_{\theta_e} + 0.163241\ C_{\theta_e}\Big) \frac{e}{m} A_1^2$$

$$+ \Big(6.564164\ S_{B_1-B_2-\theta_e} - 1.678081\ C_{B_1-B_2-\theta_e}$$

$$- 9.130888\ S_{B_1+B_2+\theta_e} + 9.014361\ C_{B_1+B_2+\theta_e}$$

$$+ 5.751182\ S_{B_1+B_2-\theta_e} - 4.502922\ C_{B_1+B_2-\theta_e}$$

$$- 8.317907\ S_{B_1-B_2+\theta_e} + 3.760673\ C_{B_1-B_2+\theta_e}\Big) \frac{e}{m} A_1\ A_2$$

$$+ \Big(10.12954\, S_{2B_2-\theta_e} - 3.305373\, C_{2B_2-\theta_e}$$

$$- 13.10683\, S_{2B_2+\theta_e} + 7.68872\, C_{2B_2+\theta_e}$$

$$- 23.23638\, S_{\theta_e} + 2.974668\, C_{\theta_e}\Big) \frac{e}{m} A_2^2$$

$$+ \Big(-0.1992477\, S_{B_1} + 0.233561\, C_{B_1}$$

$$+ 0.1090638\, S_{B_1-2\theta_e} - 0.251837\, C_{B_1-2\theta_e}$$

$$+ 0.025665\, S_{B_1+2\theta_e} - 0.054184\, C_{B_1+2\theta_e}\Big)\left(\frac{e}{m}\right)^2 A_1$$

$$+ \Big(0.127725\, S_{B_2-2\theta_e} - 0.615312\, C_{B_2-2\theta_e}$$

$$+ 0.028554\, S_{B_2+2\theta_e} - 0.505394\, C_{B_2+2\theta_e}$$

$$- 0.231118\, S_{B_2} + 0.696572\, C_{B_2}\Big)\left(\frac{e}{m}\right)^2 A_2$$

$$+ \Big(1.156848\, S_{B_1-2\theta_s+\theta_e} - 0.3246819\, C_{B_1-2\theta_s+\theta_e}$$

$$- 2.387921\, S_{B_1+2\theta_s-\theta_e} + 0.657084\, C_{B_1+2\theta_s-\theta_e}$$

$$- 0.099664\, S_{B_1-\theta_e} + 0.028683\, C_{B_1-\theta_e}$$

$$+ 0.185356\, S_{B_1+\theta_e} + 0.028683\, C_{B_1+\theta_e}\Big) \frac{e}{m} A_1$$

$$
\begin{aligned}
&+ \Big(4.22931\ S_{B_2-2\theta_s+\theta_e} - 0.7643255\ C_{B_2-2\theta_s+\theta_e} \\
&- 5.657922\ S_{B_2+2\theta_s-\theta_e} + 0.195455\ C_{B_2+2\theta_s-\theta_e} \\
&- 0.338184\ S_{B_2-\theta_e} + 0.105042\ C_{B_2-\theta_e} \\
&+ 0.437584\ S_{B_2+\theta_e} + 0.105042\ C_{B_2+\theta_e}\Big) \frac{e}{m} A_2 \\
&+ \Big(-3.598858\ S_{4B_1} - 1.720395\ C_{4B_1} \\
&- 2.450025\ S_{2B_1} - 3.700764\ C_{2B_1} - 2.037587\Big) A_1^4 \\
&+ \Big(-28.85189\ S_{3B_1+B_2} - 20.00634\ C_{3B_1+B_2} \\
&- 14.99572\ S_{3B_1-B_2} - 16.31672\ C_{3B_1-B_2} \\
&- 4.810668\ S_{B_1-B_2} - 26.80242\ C_{B_1-B_2} \\
&- 13.33645\ S_{B_1+B_2} - 30.22656\ C_{B_1+B_2}\Big) A_1^3\ A_2 \\
&+ \Big(-84.10031\ S_{2B_1} - 128.5832\ C_{2B_1} \\
&- 84.51579\ S_{2(B_1+B_2)} - 81.57057\ C_{2(B_1+B_2)} \\
&- 21.04994\ S_{2B_2} - 85.30888\ C_{2B_2} \\
&- 18.74859\ S_{2(B_1-B_2)} - 51.44641\ C_{2(B_1-B_2)} - 81.33701\Big) A_1^2\ A_2^2
\end{aligned}
$$

$$+\left(-106.6698\ S_{B_1+3B_2} - 143.2789\ C_{B_1+3B_2}\right.$$

$$+ 5.702514\ S_{B_1-3B_2} - 88.11817\ C_{B_1-3B_2}$$

$$- 50.80462\ S_{B_1-B_2} - 284.7509\ C_{B_1-B_2}$$

$$\left.- 148.3573\ S_{B_1+B_2} - 339.5546\ C_{B_1+B_2}\right) A_1 A_2^3$$

$$+\left(-48.53103\ S_{4B_2} - 94.67042\ C_{4B_2}\right.$$

$$\left.- 77.00639\ S_{2B_2} - 314.6975\ C_{2B_2} - 220.1308\right) A_2^4$$

$$\frac{1}{3!} H_3 = \left(-1.151769\ S_{2(B_2+\theta_s)} + 1.802122\ C_{2(B_1+\theta_s)}\right.$$

$$+ 0.450898\ S_{2(B_1-\theta_s)} - 0.591926\ C_{2(B_1-\theta_s)}$$

$$\left.- 1.602668\ S_{2\theta_s} + 0.163242\ C_{2\theta_s}\right) A_1^2$$

$$+\left(3.425862\ S_{B_1+B_2-2\theta_s} - 2.390847\ C_{B_1+B_2-2\theta_s}\right.$$

$$- 5.992586\ S_{B_1-B_2+2\theta_s} + 2.910867\ C_{B_1-B_2+2\theta_s}$$

$$+ 4.238843\ S_{B_1-B_2-2\theta_s} - 0.828276\ C_{B_1-B_2-2\theta_s}$$

$$\left.- 6.805567\ S_{B_1+B_2+2\theta_s} + 6.902285\ C_{B_1+B_2+2\theta_s}\right) A_1\ A_2$$

$$+ \Big(0.6138827\ S_{B_1+\theta_s} - 0.087533\ C_{B_1+\theta_s}$$

$$+ 1.363265\ S_{B_1-2\theta_s} - 0.3070833\ C_{B_1-2\theta_s}$$

$$- 0.146701\ S_{B_1-\theta_s} - 0.398984\ C_{B_1-\theta_s}$$

$$- 3.539188\ S_{B_1+2\theta_s} + 0.759784\ C_{B_1+2\theta_s}\Big)A_1$$

$$+ \Big(1.164738\ S_{B_2+\theta_s} - 0.2340664\ C_{B_2+\theta_s}$$

$$- 8.210705\ S_{B_2+2\theta_s} - 0.2075681\ C_{B_2+2\theta_s}$$

$$+ 5.68673\ S_{B_2-2\theta_s} - 1.38939\ C_{B_2-2\theta_s}$$

$$- 0.622828\ S_{B_2-\theta_s} - 0.636276\ C_{B_2-\theta_s}\Big)A_2$$

$$+ \Big(-9.476148\ S_{2(B_2+\theta_s)} + 5.970895\ C_{2(B_2+\theta_s)}$$

$$+ 6.498857\ S_{2(B_2-\theta_s)} - 1.587549\ C_{2(B_2-\theta_s)}$$

$$- 15.975\ S_{2\theta_s} + 2.974667\ C_{2\theta_s}\Big)A_2^2$$

$$+ 0.081621\ \left(\frac{e}{m}\right)^2 A_1^2 + 1.487332\left(\frac{e}{m}\right)^2 A_2^2$$

$$+ \Big(-1.056931\ S_{5B_1} + 6.118156\ C_{5B_1}$$

$$- 5.411887\ S_{3B_1} + 6.277364\ C_{3B_1}$$

$$- 3.673614\ S_{B_1} + 1.256851\ C_{B_1}\Big) A_1^5$$

$$+ \Big(-22.13612\ S_{4B_1+B_2} + 64.31532\ C_{4B_1+B_2}$$

$$- 12.7837\ S_{B_2} + 3.324871\ C_{B_2}$$

$$- 54.86528\ S_{2B_1+B_2} + 46.19636\ C_{2B_1+B_2}$$

$$- 22.28373\ S_{4B_1-B_2} + 36.11276\ C_{4B_1-B_2}$$

$$- 38.27754\ S_{2B_1-B_2} + 19.9862\ C_{2B_1-B_2}\Big) A_1^4\ A_2$$

$$+ \Big(-219.69\ S_{B_1} + 77.37804\ C_{B_1}$$

$$- 190.4786\ S_{B_1+2B_2} + 115.2084\ C_{B_1+2B_2}$$

$$- 242.069\ S_{3B_1} + 279.9619\ C_{3B_1}$$

$$- 140.5677\ S_{3B_1+2B_2} + 262.394\ C_{3B_1+2B_2}$$

$$- 91.53829\ S_{3B_1-2B_2} + 67.79426\ C_{3B_1-2B_2}$$

$$- 28.41589\ S_{B_1-2B_2} + 7.7686\ C_{B_1-2B_2}\Big) A_1^3\ A_2^2$$

$$+ \Big(-392.9128\ S_{2B_1+3B_2} + 515.0043\ C_{2B_1+3B_2}$$

$$- 254.8306\ S_{B_2} + 72.28784\ C_{B_2}$$

$$- 254.2159\ S_{3B_2} + 107.1796\ C_{3B_2}$$

$$- 110.3925\ S_{2B_1-3B_2} + 32.1358\ C_{2B_1-3B_2}$$

$$- 910.5273\ S_{2B_1+B_2} + 765.5522\ C_{2B_1+B_2}$$

$$- 616.7583\ S_{2B_1-B_2} + 323.3112\ C_{2B_1-B_2}\Big) A_1^2\ A_2^3$$

$$+ \Big(-1187.942\ S_{B_1} + 421.497\ C_{B_1}$$

$$- 1397.546\ S_{B_1+2B_2} + 845.052\ C_{B_1+2B_2}$$

$$- 503.5476\ S_{B_1+4B_2} + 478.8732\ C_{B_1+4B_2}$$

$$+ 83.62292\ S_{B_1-4B_2} + 27.1759\ C_{B_1-4B_2}$$

$$- 216.489\ S_{B_1-2B_2} + 58.93139\ C_{B_1-2B_2}\Big) A_1\ A_2^4$$

$$+ \Big(-236.9616\ S_{5B_2} + 163.6965\ C_{5B_2}$$

$$+ 697.7133\ S_{3B_2} + 294.5327\ C_{3B_2}$$

$$- 459.3208\ S_{B_2} + 133.1324\ C_{B_2}\Big) A_2^5$$

+ other eccentricity terms

$$
\begin{aligned}
\frac{1}{4!} H_4 = & \left(1.75664 \; S_{2(B_1-\theta_s)} - 2.364125 \; C_{2(B_1-\theta_s)}\right) A_1^2 \\
& + \left(-1.579912 \; S_{B_1-\theta_s} + 0.572076 \; C_{B_1-\theta_s}\right) A_1 \\
& + 5.387558 \; A_1^6 + 512.545 \; A_1^4 \; A_2^2 \\
& + \left(-68.7732 \; S_{B_1-3B_2} + 110.5201 \; C_{B_1-3B_2}\right) A_1^3 A_2^3 \\
& + 534.5015 \; A_1^2 \; A_2^4 \\
& + \left(-360.6216 \; S_{B_1-3B_2} + 5770.786 \; C_{B_1-3B_2}\right) A_1 A_2^5 \\
& + 6252.216 \; A_2^6 \\
& + \text{other short-period terms}
\end{aligned}
$$

Appendix B

THE GENERATING FUNCTION W

$$W_1 = \Big(1.988491\ S_{\overline{B}_1-2\theta_s} - 2.051196\ C_{\overline{B}_1-2\theta_s}$$

$$- 0.3051412\ S_{\overline{B}_1+2\theta_s} - 0.080193\ C_{\overline{B}_1+2\theta_s}\Big)\ \overline{A}_1$$

$$+ \Big(3.00438\ S_{\overline{B}_2-2\theta_s} - 2.23756\ C_{\overline{B}_2-2\theta_s}$$

$$- 1.658921\ S_{\overline{B}_2+2\theta_s} - 0.736401\ C_{\overline{B}_2+2\theta_s}\Big)\ \overline{A}_2$$

$$+ \Big(3.724728\ S_{\overline{B}_1-\theta_e} + 13.2713\ C_{\overline{B}_1-\theta_e}$$

$$+ 0.1801664\ S_{\overline{B}_1+\theta_e} + 0.654746\ C_{\overline{B}_1+\theta_e}\Big)\ \frac{e}{m}\ \overline{A}_1$$

$$+ \Big(0.586432\ S_{\overline{B}_2-\theta_e} + 3.245429\ C_{\overline{B}_2-\theta_e}$$

$$+ 0.080404\ S_{\overline{B}_2+\theta_e} + 2.327486\ C_{\overline{B}_2+\theta_e}\Big)\ \frac{e}{m}\ \overline{A}_2$$

$$+ \Big(-0.4075506\ S_{\overline{B}_1} - 1.116976\ C_{\overline{B}_1}$$

$$- 0.6612187\ S_{3\overline{B}_1} - 0.59504\ C_{3\overline{B}_1}\Big)\ \overline{A}_1^3$$

$$+ \Big(-2.564302\ S_{\overline{B}_2} - 8.181146\ C_{\overline{B}_2}$$

$$- 2.438221\ S_{2\overline{B}_1-\overline{B}_2} - 4.809069\ C_{2\overline{B}_1-\overline{B}_2}$$

$$- 4.547033\ S_{2\bar{B}_1+\bar{B}_2} - 5.616196\ C_{2\bar{B}_1+\bar{B}_2}\Big)\ \bar{A}_1^2\ \bar{A}_2$$

$$+ \Big(-10.65406\ S_{\bar{B}_1} - 31.72697\ C_{\bar{B}_1}$$

$$- 2.820515\ S_{\bar{B}_1-2\bar{B}_2} - 13.56212\ C_{\bar{B}_1-2\bar{B}_2}$$

$$- 10.45333\ S_{\bar{B}_1+2\bar{B}_2} - 18.03091\ C_{\bar{B}_1+2\bar{B}_2}\Big)\ \bar{A}_1\ \bar{A}_2^2$$

$$+ \Big(-14.24722\ S_{\bar{B}_2} - 58.09502\ C_{\bar{B}_2}$$

$$- 8.108981\ S_{3\bar{B}_2} - 20.46243\ C_{3\bar{B}_2}\Big)\ \bar{A}_2^3$$

$$\frac{1}{2!}W_2 = \Big(0.038859\ S_{2(\bar{B}_1+\theta_s)} + 1.113381\ C_{2(\bar{B}_1+\theta_s)}$$

$$+ 0.106328\ S_{2\theta_s} + 2.122627\ C_{2\theta_s}\Big)\ \bar{A}_1^2$$

$$+ \Big(0.076105\ S_{\bar{B}_1-2\theta_s} + 0.402185\ C_{\bar{B}_1-2\theta_s}$$

$$+ 0.048994\ S_{\bar{B}_1+2\theta_s} + 0.398525\ C_{\bar{B}_1+2\theta_s}\Big)\ \bar{A}_1$$

$$+ \Big(14.60495\ S_{\bar{B}_1+\bar{B}_2-2\theta_s} + 13.83925\ C_{\bar{B}_1+\bar{B}_2-2\theta_s}$$

$$- 2.934155\ S_{\bar{B}_1-\bar{B}_2+2\theta_s} + 5.44868\ C_{\bar{B}_1-\bar{B}_2+2\theta_s}$$

$$+ 7.253594\ S_{\bar{B}_1-\bar{B}_2-2\theta_s} + 2.562956\ C_{\bar{B}_1-\bar{B}_2-2\theta_s}$$

$$- 0.716004\ S_{\overline{B}_1+\overline{B}_2+2\theta_s} + 6.381915\ C_{\overline{B}_1+\overline{B}_2+2\theta_s}\Big)\ \overline{A}_1\overline{A}_2$$

$$+ \Big(0.399407\ S_{\overline{B}_2-2\theta_s} + 1.098604\ C_{\overline{B}_2-2\theta_s}$$

$$- 0.149057\ S_{\overline{B}_2+2\theta_s} + 1.197711\ C_{\overline{B}_2+2\theta_s}\Big)\ \overline{A}_2$$

$$+ \Big(-4.242274\ S_{2(\overline{B}_2+\theta_s)} + 8.922069\ C_{2(\overline{B}_2+\theta_s)}$$

$$+ 16.21637\ S_{2(\overline{B}_2-\theta_s)} + 8.293832\ C_{2(\overline{B}_2-\theta_s)}$$

$$- 1.51004\ S_{2\theta_s} + 24.24634\ C_{2\theta_s}\Big)\ \overline{A}_2^2$$

$$+ \Big(13.47869\ S_{\theta_e} - 0.0427323\ C_{\theta_e}$$

$$+ 13.73593\ S_{2\overline{B}_1+\theta_e} - 7.791509\ C_{2\overline{B}_1+\theta_e}$$

$$- 6.786963\ S_{2\overline{B}_1-\theta_e} + 3.444001\ C_{2\overline{B}_1-\theta_e}\Big)\ \frac{e}{m}\ \overline{A}_1^2$$

$$+ \Big(-36.5925\ S_{\overline{B}_1-\overline{B}_2-\theta_e} + 18.59109\ C_{\overline{B}_1-\overline{B}_2-\theta_e}$$

$$+ 79.85246\ S_{\overline{B}_1+\overline{B}_2+\theta_e} - 29.59782\ C_{\overline{B}_1+\overline{B}_2+\theta_e}$$

$$- 62.41576\ S_{\overline{B}_1+\overline{B}_2-\theta_e} + 14.10607\ C_{\overline{B}_1+\overline{B}_2-\theta_e}$$

$$+ 63.38106\ S_{\overline{B}_1-\overline{B}_2+\theta_e} - 6.04423\ C_{\overline{B}_1-\overline{B}_2+\theta_e}\Big)\ \frac{e}{m}\ \overline{A}_1\overline{A}_2$$

$$
\begin{aligned}
&+ \Big(-65.34663\ S_{2\bar{B}_2-\theta_e} + 32.67532\ C_{2\bar{B}_2-\theta_e} \\
&- 19.8062\ S_{2\bar{B}_2+\theta_e} + 118.8623\ C_{2\bar{B}_2+\theta_e} \\
&+ 190.2572\ S_{\theta_e} - 0.1087173\ C_{\theta_e}\Big) \frac{e}{m}\, \bar{A}_2^2 \\
&+ \Big(0.2460315\ S_{\bar{B}_1} + 0.2098861\ C_{\bar{B}_1} \\
&+ 0.051986\ S_{\bar{B}_1-2\theta_e} + 0.104638\ C_{\bar{B}_1-2\theta_e} \\
&- 0.085632\ S_{\bar{B}_1+2\theta_e} - 0.008727\ C_{\bar{B}_1+2\theta_e}\Big) \left(\frac{e}{m}\right)^2 \bar{A}_1 \\
&+ \Big(2.316622\ S_{\bar{B}_2} + 0.7686418\ C_{\bar{B}_2} \\
&+ 0.363892\ S_{\bar{B}_2-2\theta_e} + 0.075536\ C_{\bar{B}_2-2\theta_e} \\
&- 0.220475\ S_{\bar{B}_2+2\theta_e} - 0.012456\ C_{\bar{B}_2+2\theta_e}\Big) \left(\frac{e}{m}\right)^2 \bar{A}_2 \\
&+ \Big(-3.427811\ S_{\bar{B}_1-2\theta_s+\theta_e} - 12.21336\ C_{\bar{B}_1-2\theta_s+\theta_e} \\
&+ 0.364256\ S_{\bar{B}_1+2\theta_s-\theta_e} + 1.32375\ C_{\bar{B}_1+2\theta_s-\theta_e} \\
&- 0.616966\ S_{\bar{B}_1-\theta_e} - 2.143755\ C_{\bar{B}_1-\theta_e} \\
&+ 0.014746\ S_{\bar{B}_1+\theta_e} - 0.095293\ C_{\bar{B}_1+\theta_e}\Big) \frac{e}{m}\, \bar{A}_1
\end{aligned}
$$

$$+ \Big(1.379874\ S_{\bar{B}_2-2\theta_s+\theta_e} + 7.636506\ C_{\bar{B}_2-2\theta_s+\theta_e}$$

$$+ 0.169184\ S_{\bar{B}_2+2\theta_s-\theta_e} + 4.897457\ C_{\bar{B}_2+2\theta_s-\theta_e}$$

$$- 0.151114\ S_{\bar{B}_2-\theta_e} - 0.486513\ C_{\bar{B}_2-\theta_e}$$

$$+ 0.081021\ S_{\bar{B}_2+\theta_e} - 0.337515\ C_{\bar{B}_2+\theta_e}\Big) \frac{e}{m}\, \bar{A}_2$$

$$+ \Big(-0.355317\ S_{4\bar{B}_1} + 0.79958\ C_{4\bar{B}_1}$$

$$+ 0.734842\ S_{2\bar{B}_1} + 2.918539\ C_{2\bar{B}_1}\Big)\, \bar{A}_1^4$$

$$+ \Big(-10.03061\ S_{3\bar{B}_1-\bar{B}_2} + 1.82702\ C_{3\bar{B}_1-\bar{B}_2}$$

$$- 4.207251\ S_{3\bar{B}_1+\bar{B}_2} + 8.173185\ C_{3\bar{B}_1+\bar{B}_2}$$

$$- 4.694077\ S_{\bar{B}_1-\bar{B}_2} + 10.41624\ C_{\bar{B}_1-\bar{B}_2}$$

$$+ 1.223805\ S_{\bar{B}_1+\bar{B}_2} + 15.83876\ C_{\bar{B}_1+\bar{B}_2}\Big)\, \bar{A}_1^3\bar{A}_2$$

$$+ \Big(-48.5735\ S_{2\bar{B}_1} + 32.24146\ C_{2\bar{B}_1}$$

$$- 23.95198\ S_{2(\bar{B}_1+\bar{B}_2)} + 20.41893\ C_{2(\bar{B}_1+\bar{B}_2)}$$

$$- 2.841673\ S_{2\bar{B}_2} + 3.439699\ C_{2\bar{B}_2}$$

$$- 34.77553\ S_{2(\bar{B}_1-\bar{B}_2)} - 17.13252\ C_{2(\bar{B}_1-\bar{B}_2)}\Big)\ \bar{A}_1^2\ \bar{A}_2^2$$

$$+ \Big(-56.42697\ S_{\bar{B}_1+3\bar{B}_2} - 10.34366\ C_{\bar{B}_1+3\bar{B}_2}$$

$$- 94.4205\ S_{\bar{B}_1-\bar{B}_2} + 47.20834\ C_{\bar{B}_1-\bar{B}_2}$$

$$- 64.41205\ S_{\bar{B}_1+\bar{B}_2} + 122.7702\ C_{\bar{B}_1+\bar{B}_2}\Big)\ \bar{A}_1\ \bar{A}_2^3$$

$$+ \Big(-42.08241\ S_{4\bar{B}_2} - 63.64653\ C_{4\bar{B}_2}$$

$$+ 9.753145\ S_{2\bar{B}_2} + 135.4698\ C_{2\bar{B}_2}\Big)\ \bar{A}_2^4$$

$$\frac{1}{3!}\ W_3 = \Big(0.427447\ S_{2(\bar{B}_1+\theta_s)} + 0.170889\ C_{2(\bar{B}_1+\theta_s)}$$

$$+ 0.109831\ S_{2\theta_s} + 0.395711\ C_{2\theta_s}\Big)\ \bar{A}_1^2$$

$$+ \Big(5.28391\ S_{\bar{B}_1+\bar{B}_2-2\theta_s} + 3.421687\ C_{\bar{B}_1+\bar{B}_2-2\theta_s}$$

$$+ 1.676418\ S_{\bar{B}_1-\bar{B}_2-2\theta_s} + 3.170164\ C_{\bar{B}_1-\bar{B}_2-2\theta_s}$$

$$+ 1.362291\ S_{\bar{B}_1-\bar{B}_2+2\theta_s} + 2.45448\ C_{\bar{B}_1-\bar{B}_2+2\theta_s}$$

$$+ 2.060082\ S_{\bar{B}_1+\bar{B}_2+2\theta_s} + 1.370538\ C_{\bar{B}_1+\bar{B}_2+2\theta_s}\Big)\ \bar{A}_1\ \bar{A}_2$$

$$+ \Big(0.538896\ S_{\bar{B}_1} - 9.854652\ C_{\bar{B}_1}$$

$$- 0.202583\ S_{\bar{B}_1+4\theta_s} - 0.235259\ C_{\bar{B}_1+4\theta_s}$$

$$+ 3.411282\ S_{\bar{B}_1-4\theta_s} + 3.8675\ C_{\bar{B}_1-4\theta_s}$$

$$- 0.046696\ S_{\bar{B}_1+\theta_s} - 0.3274896\ C_{\bar{B}_1+\theta_s}$$

$$+ 0.271379\ S_{\bar{B}_1+2\theta_s} + 1.264127\ C_{\bar{B}_1+2\theta_s}$$

$$\left. + 0.340793\ S_{\bar{B}_1-2\theta_s} + 1.512917\ C_{\bar{B}_1-2\theta_s}\right) \bar{A}_1$$

$$+ \left(-13.53088\ S_{\bar{B}_2} + 35.12306\ C_{\bar{B}_2}\right.$$

$$+ 1.95702\ S_{\bar{B}_2+4\theta_s} + 2.987282\ C_{\bar{B}_2+4\theta_s}$$

$$+ 2.815152\ S_{\bar{B}_2-4\theta_s} - 6.72634\ C_{\bar{B}_2-4\theta_s}$$

$$- 0.1909372\ S_{\bar{B}_2+\theta_s} - 0.950122\ C_{\bar{B}_2+\theta_s}$$

$$+ 1.018833\ S_{\bar{B}_2-\theta_s} - 0.997301\ C_{\bar{B}_2-\theta_s}$$

$$- 0.096495\ S_{\bar{B}_2+2\theta_s} + 3.81702\ C_{\bar{B}_2+2\theta_s}$$

$$\left. + 0.896547\ S_{\bar{B}_2-2\theta_s} + 3.669539\ C_{\bar{B}_2-2\theta_s}\right) \bar{A}_2$$

$$+ \left(1.876573\ S_{2(\bar{B}_2+\theta_s)} + 3.825366\ C_{2(\bar{B}_2+\theta_s)}\right.$$

$$+ 2.207245\ S_{2(\bar{B}_2-\theta_s)} + 4.67143\ C_{2(\bar{B}_2-\theta_s)}$$

$$\left. + 1.52836\ S_{2\theta_s} + 3.582391\ C_{2\theta_s}\right)\bar{A}_2^2$$

$$+ \left(-178.5942\ S_{3\bar{B}_2-2\theta_s} - 59.58496\ C_{3\bar{B}_2-2\theta_s}\right.$$

$$- 44.24309\ S_{3\bar{B}_2+2\theta_s} + 219.6777\ C_{3\bar{B}_2+2\theta_s}$$

$$+ 99.69841\ S_{\bar{B}_2+2\theta_s} - 189.6082\ C_{\bar{B}_2+2\theta_s}$$

$$\left. + 53.77713\ S_{\bar{B}_2-2\theta_s} + 102.0551\ C_{\bar{B}_2-2\theta_s}\right)\bar{A}_2^3$$

$$+ \left(95.50406\ S_{\bar{B}_1-2\bar{B}_2+2\theta_s} - 72.9011\ C_{\bar{B}_1-2\bar{B}_2+2\theta_s}\right.$$

$$+ 11.67558\ S_{\bar{B}_1+2\bar{B}_2-2\theta_s} - 96.85298\ C_{\bar{B}_1+2\bar{B}_2-2\theta_s}$$

$$+ 50.12306\ S_{\bar{B}_1+2\theta_s} - 113.4417\ C_{\bar{B}_1+2\theta_s}$$

$$+ 257.5142\ S_{\bar{B}_1-2\theta_s} - 234.9083\ C_{\bar{B}_1-2\theta_s}$$

$$+ 153.5865\ S_{\bar{B}_1-2\bar{B}_2-2\theta_s} - 32.67642\ C_{\bar{B}_1-2\bar{B}_2-2\theta_s}$$

$$\left. - 4.564745\ S_{\bar{B}_1+2\bar{B}_2+2\theta_s} + 120.0289\ C_{\bar{B}_1+2\bar{B}_2+2\theta_s}\right)\bar{A}_1\ \bar{A}_2^2$$

$$+ \left(68.98198\ S_{2\bar{B}_1+\bar{B}_2-2\theta_s} - 86.27562\ C_{2\bar{B}_1+\bar{B}_2-2\theta_s}\right.$$

$$+ 162.1803\ S_{2\overline{B}_1-\overline{B}_2-2\theta_s} - 10.23446\ C_{2\overline{B}_1-\overline{B}_2-2\theta_s}$$

$$- 128.6843\ S_{\overline{B}_2+2\theta_s} - 5.666478\ C_{\overline{B}_2+2\theta_s}$$

$$+ 92.26918\ S_{\overline{B}_2-2\theta_s} - 31.6273\ C_{\overline{B}_2-2\theta_s}$$

$$+ 10.47955\ S_{2\overline{B}_1-\overline{B}_2+2\theta_s} - 46.36234\ C_{2\overline{B}_1-\overline{B}_2+2\theta_s}$$

$$+ 2.130544\ S_{2\overline{B}_1+\overline{B}_2+2\theta_s} + 6.775098\ C_{2\overline{B}_1+\overline{B}_2+2\theta_s}\Big)\ \overline{A}_1^2\ \overline{A}_2$$

$$+ \Big(12.64884\ S_{3\overline{B}_1-2\theta_s} - 13.46989\ S_{3\overline{B}_1-2\theta_s}$$

$$- 0.5897111\ S_{3\overline{B}_1+2\theta_s} - 3.286213\ C_{3\overline{B}_1+2\theta_s}$$

$$- 22.79636\ S_{\overline{B}_1+2\theta_s} + 0.137468\ C_{\overline{B}_1+2\theta_s}$$

$$+ 12.80432\ S_{\overline{B}_1-2\theta_s} - 20.0037\ C_{\overline{B}_1-2\theta_s}\Big)\ \overline{A}_1^3$$

$$+ \Big(1.026707\ S_{\overline{B}_1} - 6.885755\ C_{\overline{B}_1}$$

$$- 1.980824\ S_{3\overline{B}_1} - 3.694381\ C_{3\overline{B}_1}$$

$$+ 0.2066327\ S_{5\overline{B}_1} - 6.996066\ C_{5\overline{B}_1}\Big)\ \overline{A}_1^5$$

$$+ \Big(33.099\ S_{4\overline{B}_1-\overline{B}_2} - 74.70222\ C_{4\overline{B}_1-\overline{B}_2}$$

$$+ 3.058436\, S_{4\bar{B}_1+\bar{B}_2} - 49.44173\, C_{4\bar{B}_1+\bar{B}_2}$$

$$+ 14.97404\, S_{2\bar{B}_1+\bar{B}_2} - 57.32628\, C_{2\bar{B}+\bar{B}_2}$$

$$+ 34.44356\, S_{2\bar{B}_1-\bar{B}_2} - 44.65163\, C_{2\bar{B}_1-\bar{B}_2}$$

$$- 6.59136\, S_{\bar{B}_2} - 65.21121\, C_{\bar{B}_2}\Big) \bar{A}_1^4\, \bar{A}_2$$

$$+ \Big(122.2825\, S_{\bar{B}_1} - 331.9351\, C_{B_1}$$

$$+ 53.84464\, S_{\bar{B}_1+2\bar{B}_2} - 87.91312\, C_{\bar{B}_1+2\bar{B}_2}$$

$$+ 198.9554\, S_{\bar{B}_1-2\bar{B}_2} + 134.0697\, C_{\bar{B}_1-2\bar{B}_2}$$

$$+ 168.1948\, S_{3\bar{B}_1} - 435.9854\, C_{3\bar{B}_1}$$

$$+ 181.7902\, S_{3\bar{B}_1-2\bar{B}_2} - 60.3284\, C_{3\bar{B}_1-2\bar{B}_2}$$

$$- 78.25591\, S_{3\bar{B}_1+2\bar{B}_2} - 6.245646\, C_{3\bar{B}_1+2\bar{B}_2}\Big) \bar{A}_1^3\, \bar{A}_2^2$$

$$+ \Big(426.8296\, S_{\bar{B}_2} - 1120.0283\, C_{\bar{B}_2}$$

$$+ 363.363\, S_{2\bar{B}_1-3\bar{B}_2} + 374.5361\, C_{2\bar{B}_1-3\bar{B}_2}$$

$$- 185.7916\, S_{3\bar{B}_2} + 335.7813\, C_{3\bar{B}_2}$$

$$- 712.1592\ S_{2\bar{B}_1+3\bar{B}_2} + 652.1155\ C_{2\bar{B}_1+3\bar{B}_2}$$

$$+ 110.779\ S_{2\bar{B}_1+\bar{B}_2} - 1080.308\ C_{2\bar{B}_1+\bar{B}_2}$$

$$\left. - 151.9454\ S_{2\bar{B}_1-\bar{B}_2} + 100.4452\ C_{2\bar{B}_1-\bar{B}_2}\right) \bar{A}_1^2\ \bar{A}_2^3$$

$$+ \Big(-1768.28\ S_{\bar{B}_1} - 595.311\ C_{\bar{B}_1}$$

$$+ 463.7324\ S_{\bar{B}_1-4\bar{B}_2} + 495.5015\ C_{\bar{B}_1-4\bar{B}_2}$$

$$- 1944.027\ S_{\bar{B}_1+4\bar{B}_2} + 1671.95\ C_{\bar{B}_1+4\bar{B}_2}$$

$$+ 417.3494\ S_{\bar{B}_1+2\bar{B}_2} - 1894.879\ C_{\bar{B}_1+2\bar{B}_2}$$

$$\left. - 2069.625\ S_{\bar{B}_1-2\bar{B}_2} + 2190.171\ C_{\bar{B}_1-2\bar{B}_2}\right) \bar{A}_1\ \bar{A}_2^4$$

$$+ \Big(3861.27\ S_{\bar{B}_2} - 3638.23\ C_{\bar{B}_2}$$

$$+ 2636.735\ S_{3\bar{B}_2} - 1952.747\ C_{3\bar{B}_2}$$

$$\left. - 1724.758\ S_{5\bar{B}_2} + 1252.506\ C_{5\bar{B}_2}\right) \bar{A}_2^5$$

REFERENCES

1. J. M. A. Danby, "Stability of the Triangular Points in the Elliptic Restricted Problem of Three Bodies," Astron. J., 69, 2, 1964, pp. 165-172.

2. A. Bennett, "Characteristic Exponents of the Five Equilibrium Solutions in the Elliptically Restricted Problem," Icarus, 4, 1965, pp. 177-187.

3. A. Bennett, "Analytical Determination of Characteristic Exponents," in Progress in Astronautics and Aeronautics, Vol. 17 (R. L. Duncombe and V. G. Szebehely, eds), Academic Press, New York, 1966.

4. K. T. Alfriend and R. M. Rand, "The Stability of the Triangular Points in the Elliptic Restricted Problem of Three Bodies," Paper 68-090, AAS/AIAA, Jackson, Wyoming, 1968.

5. A. H. Nayfeh and A. A. Kamel, "On the Stability of the Triangular Points in the Elliptic Restricted Problem of Three Bodies," AIAA, to appear in 1970.

6. W. B. Klemperer and E. T. Benedikt, "Selenoidal Satellites," Astronautica Acta, 4, 1958, pp. 25-30.

7. K. Kordylewski, "Photographic Investigations of the Libration Point L_5 in the Earth-Moon System," Acta Astronomica, 11, 3, 1961, pp. 165-169; also Research Transl. E-T-G-64-35, Cambridge Research Laboratories, Office of Aerospace Research, 1961.

8. V. Szebehely, Theory of Orbits: the Restricted Problem of Three Bodies, Academic Press, New York, 1967.

9. L. Steg and E. Shoemaker (eds), "A Review of Space Research," Natl. Acad. of Sci. Natl. Research Publication 1079, State University of Iowa, 17 Jun-10 Aug 1962.

10. R. W. Farquhar, "The Control and Use of Libration-Point Satellites," SUDAAR-350, Dept. of Aeronautics and Astronautics, Stanford University, Stanford, Calif., Jul 1968.

11. J. V. Breakwell and R. Pringle, "Resonances Affecting Motion near Earth-Moon Equilateral Libration Points," in Progress in Astronautics and Aeronautics, Vol. 17, Academic Press, New York, 1966, pp. 55-73.

12. H. B. Schechter, "Three-Dimensional Nonlinear Stability Analysis of Sun-Perturbed Earth-Moon Equilateral Points," AIAA, Aug 1968.

13. A. Deprit, J. Henrard, and A. R. M. Rom, "Trojan Orbits. II. Birkhoff's Normalization," Icarus, 6, 1967, pp. 381-406.

14. R. Kolenkiewicz and L. Carpenter, "Stable Periodic Orbits about the Sun-Perturbed Earth-Moon Triangular Points," AIAA, Aug 1968.

15. A. Deprit, "Canonical Transformations Depending on a Small Parameter," Celestial Mechanics, 1, 1, 1969.

16. A. A. Kamel, "Expansion Formulae in Canonical Transformations Depending on a Small Parameter," Celestial Mechanics, 1, 2, 1969.

17. A. C. Hearn, REDUCE Users' Manual, Stanford Computation Center, Stanford University, Stanford, Calif., 1967.

18. A. Rom, "MAO," Boeing Document D1-82-0840, submitted for publication in Celestial Mechanics, 1969.

19. B. Garfinkel, "The Lagrange-Hamilton-Jacobi Mechanics," in Space Mathematics, Part 1, Vol. 5. Amer. Math. Soc., Providence, R.I., 1966.

20. A. Wintner, The Analytical Foundation of Celestial Mechanics, Princeton University Press, Princeton, N.J., 1941.

21. D. Brouwer and G. M. Clemence, Methods of Celestial Mechanics, Academic Press, New York, 1961.

22. W. M. Smart, Celestial Mechanics, John Wiley & Sons, Inc., New York, 1953.

23. F. R. Moulton, An Introduction to Celestial Mechanics, The Macmillan Co., New York, 1958.

24. H. Goldstein, Classical Mechanics, Addison-Wesley Publishing Co., Inc., Reading, Mass. and London, Jun 1959.

25. A. Deprit, "A Third Order Canonical Approximation of the General Solution of the Restricted Problem of Three Bodies in the Neighborhood of L_4," Proc. of the IAU, Symposium No. 25, Academic Press, 1966, pp. 170-175.

26. E. Leimanis, The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point, Springer-Verlag, New York, 1965, pp. 121-128.

27. G. Hori, "Theory of General Perturbations with Unspecified Canonical Variables," Astron. Soc., Japan, 18, 4, 1966, pp. 287-296.

28. G. Hori, "Nonlinear Coupling of Two Harmonic Oscillations," Astron. Soc., Japan, 19, 2, 1967, pp. 229-241.

29. A. A. Kamel, "Perturbation Method in the Theory of Nonlinear Oscillations," NASA CR-1554, March 1970; also Celestial Mechanics, Vol. 3, No. 1.

30. N. N. Bogoliubov and Y. A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordan and Breach, New York, 1961.

31. G. DePontecoulant, Theorie Analytique du Systeme du Monde, Vol. 4, Bachelier, Paris, 1846.

32. E. W. Brown, An Introductory Treatise on the Lunar Theory, Dover Publications, New York, 1960.