207

yes

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MSC INTERNAL NOTE NO. 68-FM-91

May 27, 1968

VERIFICATION OF THE KALMAN FILTER EQUATIONS USED IN THE LM AGS

By Richard E. Eckelkamp and Troy J. Blucker,
Mathematical Physics Branch,
and
Samuel Pines,
Analytical Mechanics Associates, Inc.

MISSION PLANNING AND ANALYSIS DIVISION

MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

MSC INTERNAL NOTE NO. 68-FM-91

PROJECT APOLLO

VERIFICATION OF THE KALMAN FILTER EQUATIONS USED IN THE LM AGS

By Richard E. Eckelkamp and Troy J. Blucker,
Mathematical Physics Branch, and
Samuel Pines, Analytical Mechanics Associates, Inc.

May 27, 1968

MISSION PLANNING AND ANALYSIS DIVISION
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

Approved: James C. McPherson
James C. McPherson, Chief
Mathematical Physics Branch

Approved: John P. Mayer
John P. Mayer, Chief
Mission Planning and Analysis Division

PRECEDING PAGE BLANK NOT FILMED.

CONTENTS

Section	Page
SUMMARY	1
INTRODUCTION.	1
The Kalman Filter	2
SYMBOLS	3
APPLICATION OF THE KALMAN FILTER TO THE AGS	4
Linearity Assumption in the State and Error Transition Matrices.	4
Incorporation of Range Measurement.	5
Incorporation of Range-Rate Measurement	11
CONCLUSIONS AND RECOMMENDATION.	13
REFERENCE	14

VERIFICATION OF THE KALMAN FILTER EQUATIONS

USED IN THE LM AGS

By Richard E. Eckelkamp and Troy J. Blucker, MPB, and Samuel Pines, AMA, Inc.

SUMMARY

The coasting-flight navigation equations used for processing rendezvous radar measurements in the lunar module abort guidance system (AGS) computer (AGC) are shown to be soundly based upon Kalman filter theory. Explanation and implications are given for the assumptions of linearity and perfect range-rate data and for the time-delayed incorporation of measurements.

INTRODUCTION

Reference to the AGS rendezvous radar navigation program as a "simplied Kalman filter" has left many uneasy as to its theoretical validity and numerical accuracy. An exposition of the theoretical and analytical consequences of the simplifications, originally made to save storage space in the AGC, has been needed. This report examines the theoretical consequences. The analytical or numeric consequences are being explored by the Mathematical Physics Branch and TRW (Los Angeles).

Closer observance of the filter equations leads to their simple derivation from basic Kalman theory, the theory on which the primary navigation systems in both the CM and LM are built. This paper derives the Kalman equations from theory and from the assumptions made in the AGS. The results are compared to the equations programed in the AGC.

Before beginning the derivation, a brief explanation and outline of the Kalman filter is given to aid in the understanding of this text.

The Kalman Filter

The current position and velocity relative to a fixed reference system is computed onboard the spacecraft by integrating the vehicle's state vector and processing navigational observations. The Kalman filter statistically weights the observation and uses it to modify the state vector.

The process used to correct the state at time t_i with the Kalman filter is as follows:

1. Make an observation γ_i at t_i.

2. Integrate the state vector as stored in the vehicle computer to t_i:

$$\int \begin{bmatrix} r \\ \\ \dot{r} \end{bmatrix}_{i-1} = \begin{bmatrix} r \\ \\ \dot{r} \end{bmatrix}_i .$$

3. Compute from this state vector the estimated observation γ_{ic} for t_i.

4. Find the residual $\gamma_i - \gamma_{ic} = \Delta\gamma_i$.

5. Compute the weighting factor K_i for the residual for t_i. This factor is obtained by propagating the covariance matrix of state uncertainty to t_i.

6. Correct the state

$$\begin{bmatrix} r \\ \\ \dot{r} \end{bmatrix}_i^+ = \begin{bmatrix} r \\ \\ \dot{r} \end{bmatrix}_i + K_i \, \Delta\gamma_i .$$

7. Also correct, or update, P_i to P_i^+ .

This process of statistically weighting single observations for vector correction works well. The number of observations required to reduce initial errors to the noise level and maintain this level of accuracy varies with mission phase and requirements.

SYMBOLS

F_{CSM}	total force on the CSM
F_{LM}	total force on the LM
K_i	statistical weight placed on observational residual at t_i
M_i	$\frac{\partial \text{ computed observation}}{\partial \text{ computed state}}$ at t_i
P_i	covariance matrix of the state vector uncertainty at time t_i.
r_i	relative position vector from the CSM to the LM
r_{ic}	computed relative position vector from the CSM to the LM
$\dot{r}_i$	relative velocity vector from the CSM to the LM
$\dot{r}_{ic}$	computed relative velocity vector from the CSM to the LM
$\ddot{r}_i$	relative acceleration vector from the CSM to the LM
$\ddot{r}_{ic}$	computed relative acceleration vector from the CSM to the LM
R_i	range measurement at t_i
R_{ic}	computed range at t_i
$\dot{R}_i$	range-rate measurement at t_i
$\dot{R}_{ic}$	computed range rate at t_i
t_i	time, where i denotes any instant from the initial time to the final time
α	uncertainties
γ_i	observation at t_i
γ_{ic}	computed observation at t_i
Ω	error transition matrix
$+$	the superscript $^+$ indicates an update has occurred

APPLICATION OF THE KALMAN FILTER TO THE AGS

Let us see how the AGS rendezvous radar filter equations follow from the Kalman equations.

Linearity Assumption in the State and Error Transition Matrices

The basic assumption made by the AGS filter is that the relative velocity between the CSM and LM is constant; i.e.,

$$\dot{r}_i = \dot{r}_o. \tag{1}$$

This implies

$$\ddot{r} = 0$$

$$F_{CSM} = F_{LM} .$$

Since the vehicles are coasting, the principal force on them is gravity. Thus, we assume that the two vehicles are along equipotential gravitational lines.

The equation for integration of the state vector is

$$r_i = r_o + \dot{r}_o \, \Delta t. \tag{2}$$

Equations (1) and (2) are combined to get a state transition matrix representation:

$$\begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i = \begin{bmatrix} I & \Delta t I \\ 0 & I \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \end{bmatrix}_{i-1}$$

where I and 0 are the 3 × 3 identity and zero matrices, respectively.

The error transition matrix, Ω, also follows from equations (1) and (2).

$$\Omega = \frac{\partial \, \mathrm{STATE}_i}{\partial \, \mathrm{STATE}_{i-1}} = \begin{bmatrix} \dfrac{\partial r_i}{\partial r_o} & \dfrac{\partial r_i}{\partial \dot{r}_o} \\ \dfrac{\partial \dot{r}_i}{\partial r_o} & \dfrac{\partial \dot{r}_i}{\partial \dot{r}_o} \end{bmatrix} = \begin{bmatrix} I & \Delta t I \\ 0 & I \end{bmatrix}.$$

So the error propagates as

$$\begin{bmatrix} \delta r \\ \delta \dot{r} \end{bmatrix}_i = \begin{bmatrix} I & \Delta t I \\ 0 & I \end{bmatrix} \begin{bmatrix} \delta r \\ \delta \dot{r} \end{bmatrix}_{i-1} . \tag{3}$$

Incorporation of Range Measurement

Take a radar range measurement R_i at t_i. To modify the state vector at t_i with this measurement, the full Kalman equation is

$$\begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i^+ = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i + K_i \; \Delta R_i$$

where

$$K_i = P_i M_i^T \left(M_i P_i M_i^T + \alpha^2\right)^{-1} ,$$

$$P_i = \begin{bmatrix} \sigma_{rr} & \sigma_{r\dot{r}} \\ \sigma_{\dot{r}r} & \sigma_{\dot{r}\dot{r}} \end{bmatrix}_i ,$$

where

$$E[\Delta r, \Delta \dot{r}] = \sigma_{r\dot{r}}$$

and

$$E[\Delta r, \Delta \dot{r}] = E[\Delta \dot{r}, \Delta r]$$

$$P_i = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} ,$$

where

$$P_{1,1} = \sigma_{rr}$$

$$P_{1,2} = P_{2,1} = \sigma_{r\dot{r}}$$

$$P_{2,2} = \sigma_{\dot{r}\dot{r}}$$

$$M_i = [1 \quad 0] .$$

Before explaining these equations, we should note that, except in the case of the state vector itself, all matrices have been reduced to scalar matrices to save storage space in the AGC. This reduction implies that the uncertainty in position and velocity is the same in each of their respective components. The covariance matrix, P, which is normally a 6 × 6, can be reduced to a 2 × 2. (Ordinary algebraic operations with scalar matrices are valid. Closure for addition and multiplication, as well as other properties, may be easily proved.)

The covariance matrix, P, represents a numerical estimate of the uncertainties associated with the mathematical description space. These uncertainties include noises and biases on the state and prediction models and limitations of their representation for programing.

The term α^2 represents the uncertainties of the observation space. Noises and biases of the observing instruments and the observer are included.

Now

$$M_i = \frac{\partial \text{ computed observation } (R_{ic})}{\partial \text{ computed state}_i} ,$$

i.e., M gives a measure of the effect of a particular measurement on the state vector. For a range measurement, R_i,

$$M_i = \begin{bmatrix} \dfrac{\partial R_{ic}}{\partial r_i} & \dfrac{\partial R_{ic}}{\partial \dot{r}_i} \end{bmatrix}$$

$$= [1 \quad 0].$$

The position component is unity since the observation measures the relative range between the two vehicles. Recall that the state in the above equation is the relative state.

To incorporate R_i, we first must find K_i. This requires P_i. P propagates as

$$P_i = \Omega P_{i-1} \Omega^T$$

$$= \begin{bmatrix} I & \Delta t I \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix} \begin{bmatrix} I & 0 \\ \Delta t I & I \end{bmatrix}$$

$$\begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix} = \begin{bmatrix} P_{1,1} + 2P_{1,2}\,\Delta t + P_{2,2}\,(\Delta t)^2 & P_{1,2} + P_{2,2}\Delta t \\ P_{1,2} + P_{2,2}\Delta t & P_{2,2} \end{bmatrix}_{i-1}$$

or

$$P_{1,1} = P_{1,1} + 2P_{1,2}\Delta t + P_{2,2}(\Delta t)^2 \tag{4}$$

$$P_{1,2} = P_{1,2} + P_{2,2}\Delta t \tag{5}$$

$$P_{2,2} = P_{2,2} \quad . \tag{6}$$

Now

$$\Delta t = t_i - t_{i-1} \; .$$

Comparing equations (4), (5), and (6) to the following equations found in the AGC (ref. 1),

$$\left.\begin{aligned} P_{1,1} &= P_{1,1} + 2P_{1,2}\Delta t + P_{2,2}(\Delta t)^2 + K_9^2 \\ P_{1,2} &= P_{1,2} + P_{2,2}\Delta t \\ P_{2,2} &= P_{2,2} + K_{10}{}^2\Delta t \end{aligned}\right\} \tag{7}$$

it is seen that the constant K_9^2 has been added to keep the matrix positive definite during updating. The constant K_{10}^2 has been added to model the assumption that the variance in velocity increases with Δt. This is meant to compensate for the basic assumption of linear motion, equation (1).

To get K_i,

$$K_i = P_i M_i^T (M_i P_i M_i^T + \alpha^2)^{-1}$$

$$= \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left\{ [1 \quad 0] \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha^2 \right\}^{-1}$$

$$K_i = \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix} \left[P_{11} + \alpha^2 \right]^{-1}$$

$$K_i = \begin{bmatrix} \dfrac{P_{11}}{P_{11} + \alpha^2} \\ \\ \dfrac{P_{12}}{P_{11} + \alpha^2} \end{bmatrix} .$$

Again, comparing with the equations in the AGC,

$$K_i = \begin{bmatrix} \dfrac{P_{11}}{\sigma^2} \\ \\ \dfrac{P_{12}}{\sigma^2} \end{bmatrix}$$

where

$$\sigma^2 = P_{11} + K_3^3 R^2 + K_7^3 \quad .$$

So, the equations are identical if we set

$$\alpha^2 = K_3^3 R^2 + K_7^3 .$$

The constants K_3^3 and K_7^3 represent the noise on the range measurement.

If R_i were incorporated into the state at t_i, the correction would be:

$$\begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i^+ = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \Delta R_i$$

where

$$W_1 = \frac{P_{11}}{\sigma^2} \quad , \quad W_2 = \frac{P_{12}}{\sigma^2} \quad ,$$

$$\Delta R = r_{ic} - R_i \, z_i$$

where z is a unit vector along the computed line of sight as derived from angle data.

Due to hardware limitations, however, the procedure does not occur until $t_i + n\delta t$, where δt represents a computer cycle time and n is some positive integer.

The updates at t_i would be

$$r_i^+ = r_i + W_1 \, \delta r_i \tag{8}$$

$$\dot{r}_i^+ = \dot{r}_i + W_2 \, \delta r_i . \tag{9}$$

Propagating to $t_i + \delta t$ (n = 1 for simplicity) according to (2) gives

$$r_{i+\delta t}^+ = r_i^+ + \delta t \, \dot{r}_i^+ \quad .$$

From (8) and (9)

$$r_{i+\delta t}^+ = r_i + W_1 \, \delta r_i + \delta t(\dot{r}_i + W_2 \, \delta r_i)$$

$$= r_i + \delta t \dot{r}_i + (W_1 + \delta t W_2) \, \delta r_i .$$

$$r^{+}_{i+\delta t} = r_{i+\delta t} + W_{1+\delta t}\, \delta r_i \quad , \tag{10}$$

where

$$W_{1+\delta t} = W_1 + \delta t W_2 \quad .$$

The incorporation at $t_i + \delta t$, is thus equivalent to (8).

Since velocity is assumed linear,

$$\dot{r}^{+}_{i} = \dot{r}_i + W_2 \delta r_i$$

remains unchanged.

Our range measurement has successfully altered the state vector according to equations (9) and (10).

Finally P_i must be updated before its propagation to the next observation.

The Kalman update procedure is

$$P^{+}_{i} = [I - K_i M_i]\, P_i$$

$$P^{+}_{i} = \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} \right\} P_i$$

$$= \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} W_1 & 0 \\ W_2 & 0 \end{bmatrix} \right\} P_i$$

$$= \begin{bmatrix} I - W_1 & 0 \\ - W_2 & I \end{bmatrix} \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix}_i$$

$$\begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix}_i^+ = \begin{bmatrix} P_{1,1} - P_{1,1}W_1 & P_{1,2} - P_{1,2}W_1 \\ P_{1,2} - P_{1,1}W_2 & P_{2,2} - P_{1,2}W_2 \end{bmatrix}_i .$$

The $P_{1,2}$ and $P_{2,1}$ elements are equivalent:

$$P_{1,2} - P_{1,2}W_1 = P_{1,2} - P_{1,1}\left(\frac{P_{1,2}}{\sigma^2}\right)$$

$$= P_{1,2} - P_{1,2}\left(\frac{P_{1,1}}{\sigma^2}\right)$$

$$P_{1,2} - P_{1,2}W_1 = P_{1,2} - P_{1,2}W_1 .$$

So at t_i,

$$P^+_{1,1} = P_{1,1}(1 - W_1)$$

$$P^+_{1,2} = P_{1,2}(1 - W_1)$$

$$P^+_{2,2} = P_{2,2} - P_{1,2}W_2$$

which agree with the equations found in the AGS. We therefore conclude that the equations used to process range in the AGC are theoretically sound.

Incorporation of Range-Rate Measurement

The rendezvous radar on the LM can also measure range rate $\dot{R}$. To modify the state vector with the measurement, the Kalman equation is:

$$\begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i^+ = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}_i + K_i \, \Delta\dot{R}_i .$$

To find K we need M. Again recalling that we are measuring the relative state,

$$\frac{\partial \dot{R}_{ic}}{\partial \dot{r}_i} = 1 .$$

Since

$$\ddot{r}_i = 0 ,$$

$$\frac{\partial \dot{R}_i}{\partial r_i} = 0 .$$

Then,

$$M_i = [0, 1]$$

and

$$K_i = P_i M_i^T (M_i P_i M_i^T + \alpha^2)^{-1}$$

$$= \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left\{ [0 \quad 1] \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2} & P_{2,2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha^2 \right\}^{-1}$$

$$K_i = \begin{bmatrix} \dfrac{P_{1,2}}{P_{2,2} + \alpha^2} \\ \\ \dfrac{P_{2,2}}{P_{2,2} + \alpha^2} \end{bmatrix} .$$

To alter the state with $\dot{R}_i$

$$r_i^+ = r_i + \frac{P_{1,2}}{P_{2,2} + \alpha^2} \Delta\dot{R}_i , \; \Delta\dot{R}_i = \left(|\dot{r}_{ic}| - \dot{R}_i \right) z_i \tag{12}$$

and

$$\dot{r}_i^+ = \dot{r}_i + \frac{P_{2,2}}{P_{2,2} + \alpha^2} \Delta\dot{R}_i . \tag{13}$$

Correction with (12) and (13) is possible. Present limitation of storage in AGC requires a simpler solution. Looking closer at (13), one can drop the noise, α^2, so that

$$\dot{r}_i^+ = \dot{r}_i + \frac{P_{2,2}}{P_{2,2}} \Delta\dot{R}_i$$

$$\dot{r}_i^+ = \dot{r}_i + \Delta\dot{R}_i .$$

This assumption of perfect $\dot{R}_i$ data is used onboard the LM in order to use the range-rate Doppler data. Equation (12) is not utilized; therefore, no position update occurs.

CONCLUSIONS AND RECOMMENDATION

We have shown that the AGS radar equations follow directly from Kalman theory after applying the assumption of linearity and perfect range-rate data, and after compensations for time-delayed incorporation of measurements.

Having explored the theoretical grounds for the AGS and found them to be based soundly upon the given assumptions, one can suggest that the proper incorporation of range-rate data to update position and velocity could notably improve the accuracies obtained. This improvement used with proper adjustment of the compensating constants would increase the rendezvous capability of the AGS.

REFERENCE

1. TRW: LM AGS Coasting Flight Navigation Simulation Computer Program Compensation. TRW Note 66-FMT-543, August 24, 1967.