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AN ANALYSIS OF THE RELATIVE MERITS OF VARIOUS PCM CODE FORMATS

[redacted]

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AN ANALYSIS OF THE RELATIVE MERITS OF VARIOUS PCM CODE FORMATS

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LIST OF ACRONYMS, ABBREVIATIONS, AND SYMBOLS

ASK	Amplitude Shift Keyed
Biϕ	Bi-Phase
Biϕ-L	Bi-Phase Level (or Split-Phase)
Biϕ-M	Bi-Phase Mark
Biϕ-S	Bi-Phase Space
$E(X_{t1})$	Expected value of the random variable X_{t1}
f	Frequency
FSK	Frequency Shift Keyed
IRIG	Inter-Range Instrumentation Group
NRZ	Non Return to Zero
NRZ-L	Non Return to Zero Level
NRZ-M	Non Return to Zero Mark
NRZ-S	Non Return to Zero Space
PCM	Pulse Code Modulation
PSK	Phase Shift Keyed
P_T	The total power of a random PCM code
$p_{X_{t1}}(\mathfrak{X}_{t1})$	The probability density function of the random variable X_{t1}
$p_{X_{t1},X_{t2}}(\mathfrak{X}_{t1},\mathfrak{X}_{t2})$	Joint probability density function of X_{t1} and X_{t2}
$P(X_{t1}=E)$	The unconditional probability that the random variable X at time t_1 assumes the value E.
$P(X_{t2}=E \mid X_{t1}=E)$	The probability that $X_{t2}=E$, *given* that $X_{t1}=E$.
$P_{2\omega_B}$	The amount of power of a random PCM code contained in the frequency band extending from $-\omega_B$ to $+\omega_B$.
$\frac{P_{2\omega_B}}{P_T} \times 100$	The percentage of total code power contained in the frequency band extending from $-\omega_B$ to $+\omega_B$.

R	Bit Rate of a PCM Code
rad	radian
RF	Radio Frequency
RZ	Return to Zero
$R(\tau)$	Ensemble-average autocorrelation function
$Si(X)$	The sine integral of X, $Si(X) = \int_0^X \frac{\sin \xi \, d\xi}{\xi}$
$S(\omega)$	Power Spectral Density
T_1	PCM Code Bit Period ($T_1 = \frac{1}{R}$)
t_1, t_2	Times at which the values of the members of an ensemble of random functions are sampled
X_{t1}, X_{t2}	Values which the members of an ensemble of random functions assume at the sampling times t_1 and t_2
$\delta(X)$	Unit impulse function
τ	Time-shift
ϕ_c	Initial carrier phase (relative to the modulating sequence)
ω	Angular frequency ($\omega = 2\pi f$)
ω_B	Single-sided bandwidth of an ideal low-pass filter
$2\omega_B$	Double-sided bandwidth of an ideal low-pass filter
ω_c	Carrier frequency
ω_0	Bit rate angular frequency ($\omega_0 = \frac{2\pi}{T_1}$)
$\oplus$	Exclusive "or" operation
$\cdot$	Logical "and" operation
$\bar{A}$	The *inverse* of A

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SECTION 1

INTRODUCTION

1.1 GENERAL

Pulse Code Modulation (PCM) telemetry utilizes a series of binary digits (ones and zeros) to describe the analog level of a sample taken from a data channel. The output signal of a PCM encoder is some voltage waveform that represents the "1" and "0" bits of the data sample. Several different waveforms have been used in the formulation of the bits of a PCM bit stream.

The Inter-Range Instrumentation Group (IRIG) of the Range Commanders Council recognizes seven permissible digital formats for PCM (Ref. 1). Figure 1-1 illustrates these various formats. The "+E" and "-E" levels indicated in Figure 1-1 represent the actual voltage levels of the PCM bit stream, while the "1's" and "0's" indicate binary logic levels. It should be noted that each of the formats illustrated in Figure 1-1 is actually a variant of one of the three following basic configurations:

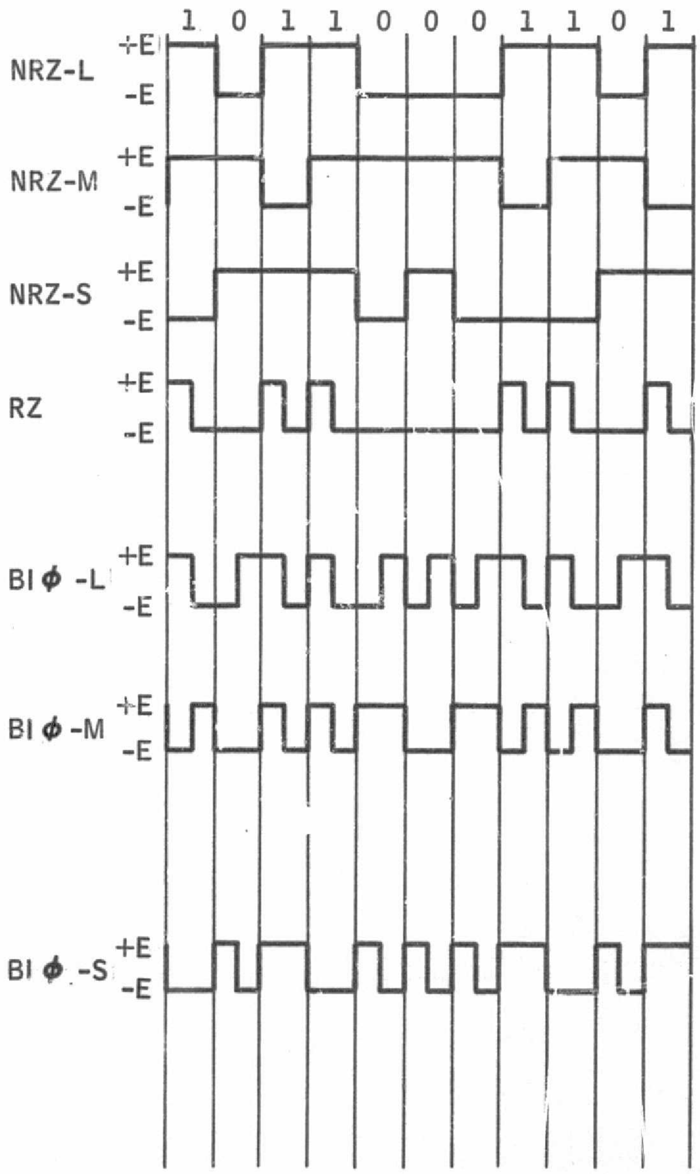
A. Non-return-to-zero level (NRZ-L)

B. Split-phase or bi-phase level (Biø-L)

C. Return-to-zero (RZ).

One important difference among the three basic PCM code configurations is the required transmission bandwidth. The bandwidths associated with the various NRZ formats are the same, as are those of each of the split-phase formats. However, the transmission bandwidth required for an NRZ code is not, in general, the same as that for a split-phase code or an RZ code.

Prior to transmission, the PCM bit stream is used to modulate some parameter (phase, frequency, or amplitude) of an RF carrier. In some systems, the PCM signal first modulates a subcarrier, which subsequently is used to modulate the main carrier. Such systems are capable of transmitting other channels of information, such as voice, in addition to the PCM telemetry channel.



NRZ - LEVEL (OR NRZ - CHANGE)
"ONE" IS REPRESENTED BY ONE LEVEL
"ZERO" IS REPRESENTED BY THE OTHER LEVEL

NRZ-MARK
"ONE" IS REPRESENTED BY A CHANGE IN LEVEL
"ZERO" IS REPRESENTED BY NO CHANGE IN LEVEL

NRZ-SPACE
"ONE" IS REPRESENTED BY NO CHANGE IN LEVEL
"ZERO" IS REPRESENTED BY A CHANGE IN LEVEL

RZ
"ONE" IS REPRESENTED BY A HALF-BIT WIDE PULSE
"ZERO" IS REPRESENTED BY NO PULSE CONDITION

BI-PHASE-LEVEL (OR SPLIT PHASE, MANCHESTER 11 + 180°)
"ONE" IS REPRESENTED BY A 10
"ZERO" IS REPRESENTED BY A 01

BI-PHASE-MARK (OR MANCHESTER 1)
A TRANSITION OCCURS AT THE BEGINNING OF EVERY BIT PERIOD
"ONE" IS REPRESENTED BY A SECOND TRANSITION 1/2 BIT PERIOD LATER
"ZERO" IS REPRESENTED BY NO SECOND TRANSITION

BI-PHASE-SPACE
A TRANSITION OCCURS AT THE BEGINNING OF EVERY BIT PERIOD
"ONE" IS REPRESENTED BY NO SECOND TRANSITION
"ZERO" IS REPRESENTED BY A SECOND TRANSITION 1/2 BIT PERIOD LATER

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Figure 1-1 PCM Code Formats

The functions of the data receiving system are to detect the PCM signal and regenerate the bit stream in a noise-free form. The successful recovery of a PCM signal from a noise background is dependent upon the ability of the receiving device to reconstruct the bit rate clock. Normally, the bit rate clock is recovered by locking a phase-locked loop device onto state transitions of the incoming bit stream. Such a device has an inherent flywheel effect, and a certain minimum bit transition density is required for continuous operation. By definition, PCM receiving equipment is synchronized, or "in sync," when it is locked in frequency and phase on the transmitted format. The efficiency with which bit synchronization can be acquired and maintained is dependent upon the type of bit frequency and phase information available to the receiver synchronizer.

For certain telemetry systems employing phase-coherent modulation and demodulation techniques, it is possible to recover the bit rate clock by counting down the locally-regenerated subcarrier frequency. This is not a commonly employed technique, however, and, for purposes of comparing the synchronization efficiency of the various PCM code formats, it will be assumed that "flywheel bit synchronization" is to be employed.

1.2 PURPOSE

The purpose of this document is to determine the various characteristics of each of the basic PCM code formats and, from a comparison of these characteristics, to present the relative advantages and disadvantages of each format.

1.3 SCOPE

This document contains calculations of the autocorrelation functions and power density spectra of the three basic PCM code formats (NRZ-L, split-phase, RZ). These characteristics are used to determine bandwidth requirements [the (baseband) bandwidth required to pass a given percentage of total code power] and to determine some measure of synchronization efficiency (the ease with which the bit rate clock may be recovered at the receiver). In addition, some discussion is provided regarding the actual channel (RF) bandwidth required for transmission of a carrier which is modulated by a PCM code of some type.

SECTION 2

SUMMARY AND CONCLUSIONS

2.1 GENERAL

Various characteristics, such as autocorrelation functions, power spectral densities, bandwidths, and bit synchronization efficiency, are determined for each of the basic PCM code formats. These basic formats include NRZ-L, split-phase, and RZ.

2.2 CONCLUSIONS

The relative bandwidth requirements for the various PCM code formats may be determined from Figure 2-1, which contains plots of the percentage of total code power contained within a given *baseband* (premodulation or postdetection) bandwidth. It is evident that the NRZ-L code requires the *least* bandwidth for a given percentage of total code power. The split-phase code requires the most bandwidth.

The relative RF or *transmission* bandwidth required is likewise least for the NRZ-L code and most for the split-phase code. The actual amount of transmission bandwidth required depends on the nature of the carrier modulation involved. For noncoherent phase-shift keying, the baseband code power spectrum is merely shifted in frequency to appear about the carrier. This is true for any code format. The required transmission bandwidth for a given percentage of total code power may then be obtained from Figure 2-1. For coherent phase-shift-keying of the carrier, the shape of the power spectrum changes somewhat, and, therefore, so does the required transmission bandwidth. To illustrate this point, the power spectra for a carrier which is coherently and noncoherently phase-shift-keyed by an NRZ-L code are calculated in Section 6.

Table 2-1 contains an overall comparison of various characteristics of the basic PCM code formats, for a completely random bit pattern (equally likely "ones" and "zeros"). Each format is given a number indicating the order of preference under each characteristic.

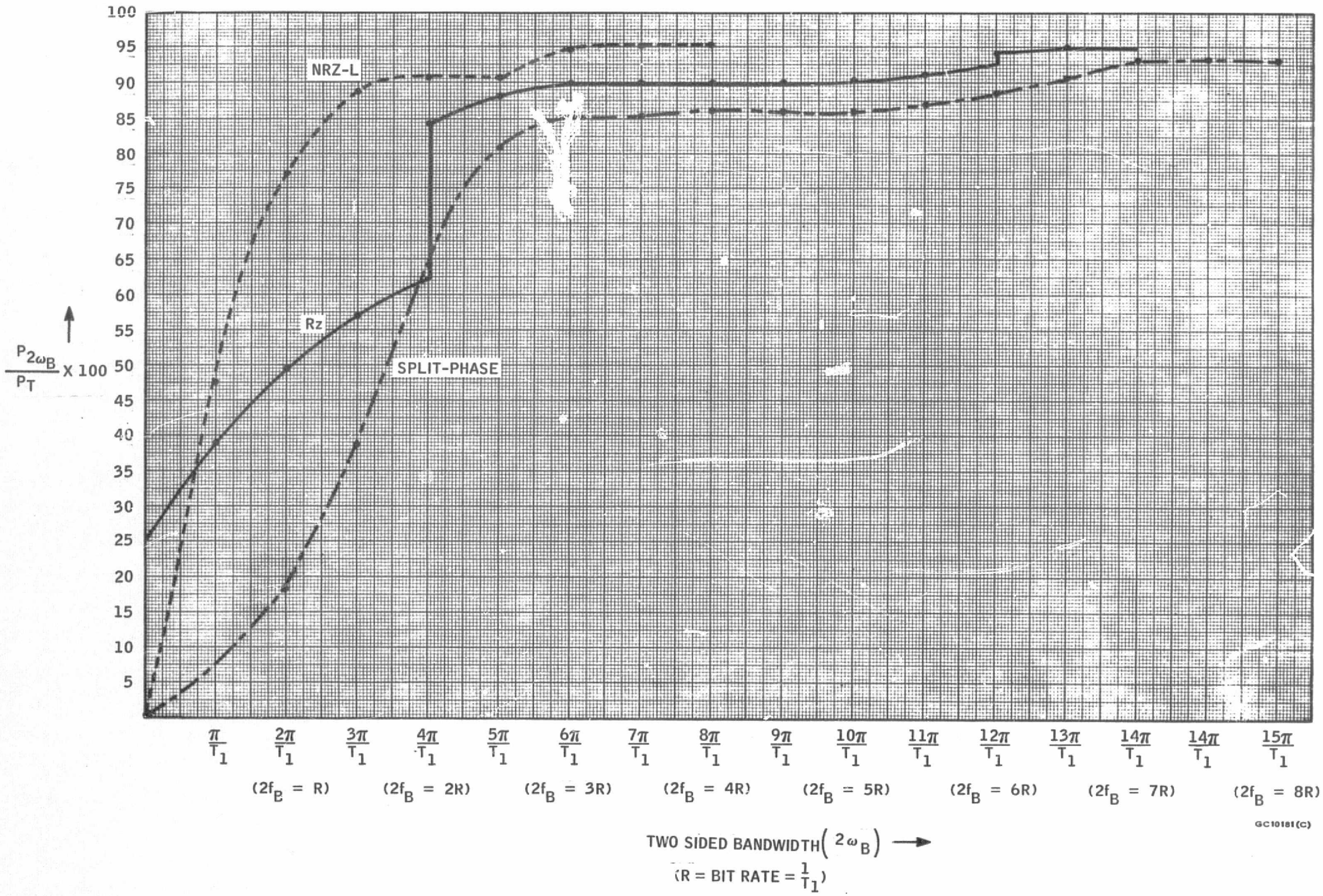

Figure 2-1 Percentage of Total Power of a PCM Code Contained in the Frequency Band Extending from $-\omega_B$ to $+\omega_B$

TABLE 2-1

COMPARISON OF IMPORTANT CHARACTERISTICS OF THE BASIC PCM CODE FORMATS

PCM CODE FORMAT	RELATIVE BANDWIDTH EFFICIENCY	RELATIVE BIT TRANSITION DENSITY (RANDOM BIT PATTERN) *	RELATIVE BIT SYNCHRONIZATION EFFICIENCY (USING A "FLY-WHEEL" TYPE BIT SYNCHRONIZER)	RELATIVE BIT SYNCHRONIZATION EFFICIENCY (USING FILTERING OF DISCRETE CODE COMPONENTS)
NRZ-L	1	3	3	Not Applicable
Split-Phase	3	1	1	Not Applicable
RZ	2	2	2	1

*For a random bit pattern, with equally likely "ones" and "zeros," there is an average of 1/2 transition per bit for the NRZ-L code, 1-1/2 transitions per bit for the split-phase code, and 1 transition per bit for the RZ code.

The same number given to two systems indicates that the systems are equally preferable *with respect to that characteristic.*

It should be noted that the relative bit synchronization efficiency using a "flywheel" type bit synchronizer is directly related to the relative bit transition density of the code (i.e., the more bit transitions per second, the greater the probability of maintaining synchronization). Table 2-1 indicates that the split-phase code is superior, in this respect, to the other code formats whenever "flywheel" bit synchronization is employed. However, if bit synchronization information is obtained by direct filtering of discrete frequency components from the code, the RZ code is unquestionably superior. This is because the RZ code is the *only* code that contains such discrete components.

If other than random bit patterns occur, then the comparison given in Table 2-1 is not necessarily valid. The split-phase code format is the only format which *guarantees* bit transitions for all possible bit patterns. Long strings of "ones" or "zeros" will result in no transitions for the NRZ-L format, while long strings of "zeros" will result in no transitions for the RZ format.

Synchronization may also be obtained, in some instances, by "counting down" or dividing the frequency of the telemetry carrier or subcarrier. It is, of course, necessary that the bit rate clock and the carrier or subcarrier frequency be coherently related (i.e., related in frequency and phase). If this type of synchronization is employed, the three basic code formats are equally efficient with respect to synchronization.

SECTION 3

CHARACTERISTICS OF NRZ CODES

3.1 GENERAL

The NRZ-L PCM code format is perhaps the most widely used of all the formats illustrated in Figure 1-1. For this format, a one-to-one correspondence exists between binary ones and zeros (logic levels) and PCM ones and zeros (voltage levels).

Although NRZ-L coding is frequently used in telemetry systems employing phase modulation (PCM/PSK), it is ideally suited for frequency modulation (PCM/FSK) or amplitude modulation (PCM/ASK) techniques. This is because the detection process consists merely of a determination of which level is being received during any given bit period. Standard frequency discriminators or envelope detectors may, therefore, be used for detection of PCM/FSK and PCM/ASK signals, respectively.

On the other hand, the NRZ-M and NRZ-S code formats are particularly well-suited for phase modulation schemes. This is because it is only necessary to detect a change of phase rather than the actual phase. Thus, any particular bit provides the phase reference for the succeeding bit.

3.2 POWER SPECTRAL DENSITY

As previously noted, the required transmission bandwidths are the same for each of the NRZ code formats. The actual required bandwidth is determined by the power spectrum, or power spectral density, of the NRZ code. In order to determine the power spectral density of a random process, such as an NRZ code with a random bit pattern, it is first necessary to determine the ensemble-average autocorrelation function[1] of that process. The Wiener-Khintchine

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[1] In statistical communication theory there are two kinds of averages, namely, time averages and ensemble averages. Ensemble averages are statistical in nature and depend on the probability densities of the random process. The elements of an ensemble are referred to as sample functions, and time averages may be calculated for each sample function. In general, these time averages of a single sample function are not necessarily equal to the corresponding ensemble averages. When time averages and ensemble averages are equal, the process is called *ergodic*.

theorem (Ref. 2) states that the power spectral density and the ensemble-average autocorrelation function are Fourier transforms of one another, or

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau \qquad (1)$$

where $R(\tau)$ is the ensemble-average autocorrelation function of the random process

$\omega = 2\pi f$ is the angular frequency

$S(\omega)$ is the power spectral density of the process

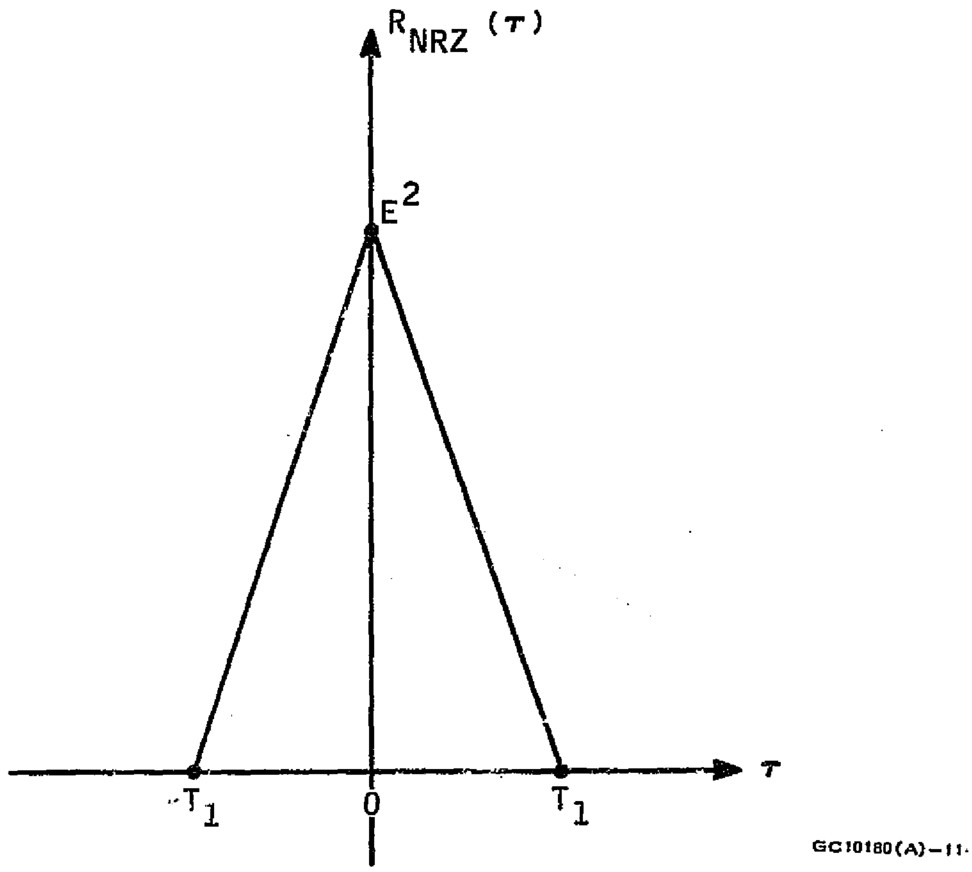
The initial problem involved in finding the power spectral density of the NRZ code, then, is that of determining its ensemble-average autocorrelation function. This function is well-known for the NRZ code, but its determination is outlined in Appendix A for purposes of information. The result is shown in Figure 3-1, where T_1 is the bit period and $\pm E$ is the amplitude of an NRZ pulse. The ensemble-average autocorrelation function, which is merely the expected value of the product of two samples of each member of an ensemble of NRZ codes, should be maximum (E^2) if both samples are taken at the same instant of time ($\tau = 0$). The expected value of the product decreases to zero as the time between samples increases to T_1, and it is zero for all $|\tau| > T_1$.

The Fourier transform of a triangular pulse (Ref. 3) is a function of the form

$$K \frac{\sin^2 X}{X^2}$$

For the triangular pulse of Figure 3-1, then, the Fourier transform is given by

$$S_{NRZ}(\omega) = \frac{E^2 T_1}{2\pi} \left(\frac{\sin \frac{\omega T_1}{2}}{\frac{\omega T_1}{2}} \right)^2 \qquad (2)$$



$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} = \frac{1}{R}\right)$

Figure 3-1 Autocorrelation Function of an NRZ Code (Random Bit Pattern)

The power density spectrum for the random NRZ code is shown in Figure 3-2. The values in parentheses correspond to frequencies which are multiples of the NRZ bit rate, R.

The actual bandwidth, B, required for transmission of an NRZ code depends on the desired signal fidelity (i.e., a narrow transmission bandwidth results in a train of distorted pulses, rather than the square pulses originally present at the output of the PCM encoder). The allowable bandwidth is limited, however, by noise introduced in the channel. The wider the bandwidth of the receiver, the higher the noise power present in the telemetry channel and, hence, the lower the probability of correctly reconstructing the received PCM bit stream. A tradeoff between faithful signal shape at the receiver and noise power present in the telemetry channel is reached when a bandwidth is defined.

Appendix B contains a calculation of the percentage of total NRZ code power present in a given bandwidth (this corresponds to the fraction of total NRZ power passed by an ideal rectangular low pass filter of one-sided bandwidth ω_B). The results of this calculation are summarized in Figure 3-3. The bandwidth required to pass any desired percentage of total power may be easily determined. It is evident, however, that increasing the two-sided bandwidth past three times the bit rate does not significantly increase the percentage of total power (this corresponds to a baseband, or one-sided, bandwidth of 1-1/2 times the bit rate).

It should be noted that the required RF *transmission* bandwidth for an NRZ PCM channel is not necessarily equal to $2\omega_B$. The required transmission bandwidth is affected by the type of modulation/demodulation scheme utilized by the system. For example, the bandwidth of a wideband frequency modulated PCM signal could be several times that of an amplitude-modulated PCM signal. If phase modulation is employed, the required transmission bandwidth is dependent on whether the modulation is coherent or noncoherent (i.e., on whether the PCM bit transitions always occur at zero crossings of the carrier). In general, noncoherent phase modulation results in a larger required transmission bandwidth than that for coherent phase modulation with bit transitions occurring at zero crossings of the carrier (see Section 6).

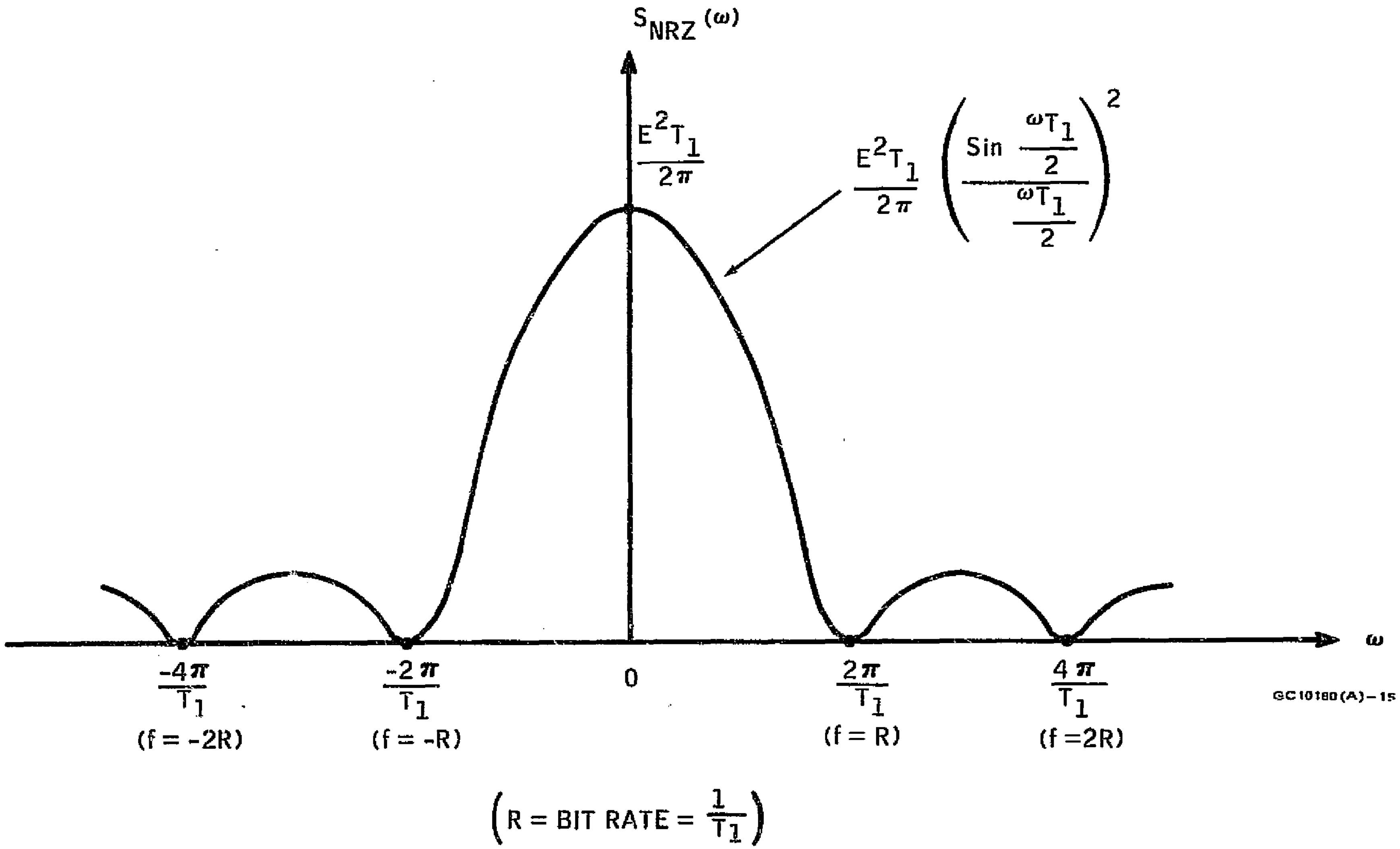


Figure 3-2 Power Density Spectrum of an NRZ Code (Random Bit Pattern)

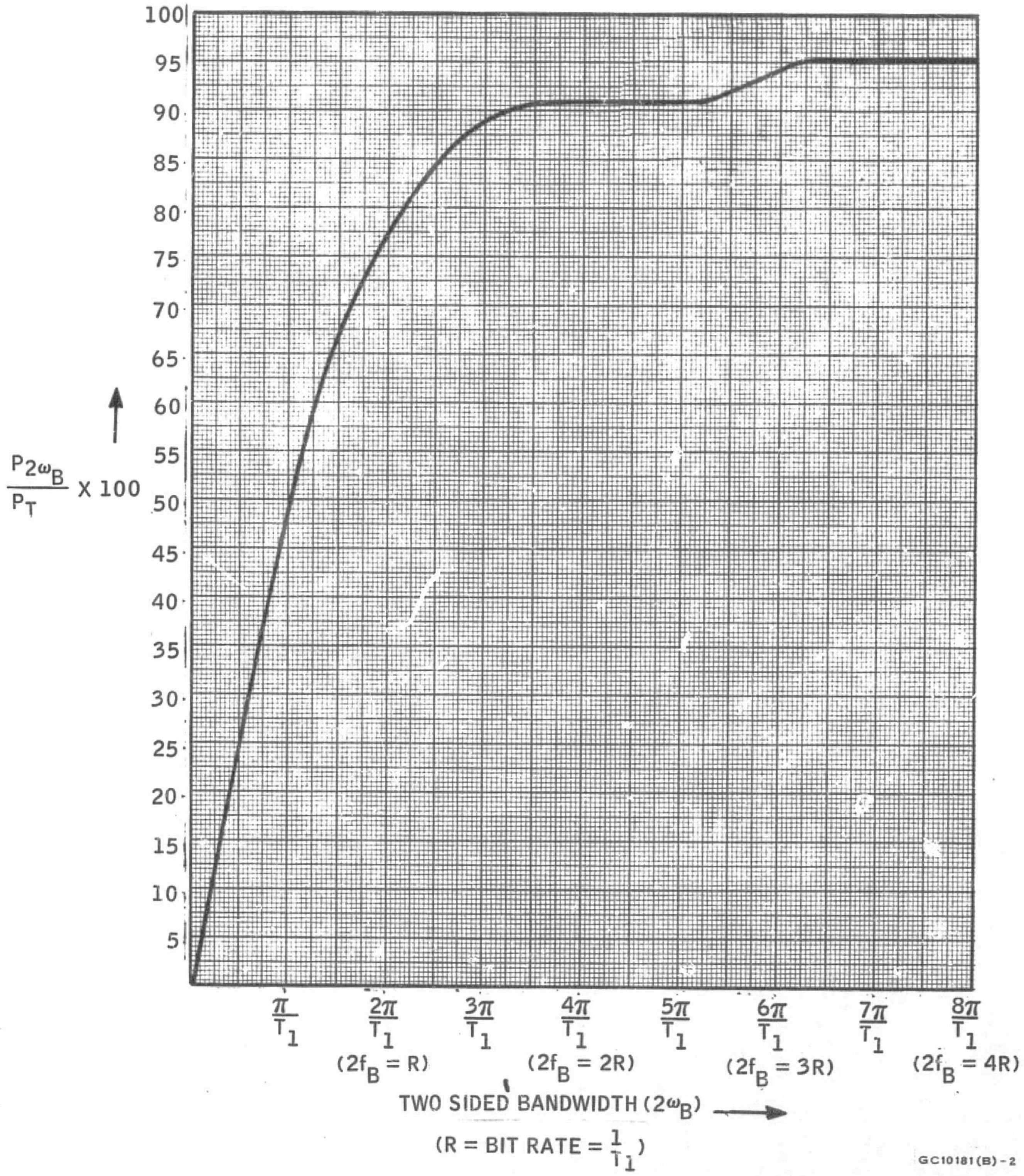


Figure 3-3 Percentage of Total Power of an NRZ Code Contained in the Frequency Band Extending from $-\omega_B$ to $+\omega_B$

3.3 BIT SYNCHRONIZATION EFFICIENCY

Except during transmission of known patterns, the level transitions of an NRZ code occur in an unpredictable fashion (although some degree of periodicity generally occurs). Since there is one transition for each change of binary state, the average transition density is 1/2 for a completely random bit stream (there is an average of 1/2 transition per bit). As a finite probability exists that no transitions will occur over any given time interval, some means must be available to maintain the bit transition density above a tolerable minimum if "flywheel" bit synchronization is to be employed at the receiver.

Proper system design should ensure that an adequate number of bit transitions per unit time is available to assure proper functioning of the receiver bit synchronizer. In particular, the various analog data channels could be designed so that at least one transition would occur for each telemetry word and so that all unallocated or failed channels would be represented by non-zero words. In addition, since difficulties can be caused by long strings of digital words of all ones or all zeros, practices such as interleaving analog and digital words should be observed.

Observance of the above design practices is not necessary if "flywheel" bit synchronization is not employed at the receiver. If the bit rate clock can be obtained by some other means, such as by counting down the received or locally generated carrier or subcarrier frequency, then synchronization becomes independent of bit transition density.

SECTION 4

CHARACTERISTICS OF SPLIT-PHASE CODES

4.1 GENERAL

As shown in Figure 1-1, a split-phase PCM code utilizes a binary "10" to represent a "1" and a binary "01" to represent a zero. The term "split-phase" applies to the code structure of the PCM modulating signal and not to the modulation scheme used (i.e., PSK, FSK or ASK). A split-phase PCM code may be used to modulate the amplitude, frequency, or phase of a carrier signal.

A split-phase code can be formed by combining an NRZ code in an appropriate manner with a clock (which has a period equal to the bit period of the NRZ code). Specifically, a split-phase code may be obtained by using an NRZ code to amplitude-modulate a square-wave subcarrier with a frequency equal to the bit rate. Appendix C shows that the logical operations required for the proper combination of the NRZ code and the square-wave clock depend on whether the system employs positive or negative logic. For either case, however, it is shown that the equivalent analog operation consists of a simple multiplication of the clock and the NRZ code.

4.2 POWER SPECTRAL DENSITY

The determination of the ensemble-average autocorrelation function for a split-phase PCM code is outlined in Appendix D. The result is illustrated in Figure 4-1, where T_1 is the code bit period and $\pm E$ is the amplitude of a code pulse.

The power spectral density of the split-phase code is equal to the Fourier transform of the ensemble-average autocorrelation function of the code. The autocorrelation function shown in Figure 4-1 is not of an elementary form, so its Fourier transform must be calculated, rather than obtained from a standard table of transform pairs (as was the power spectral density of the NRZ code in Paragraph 3.2).

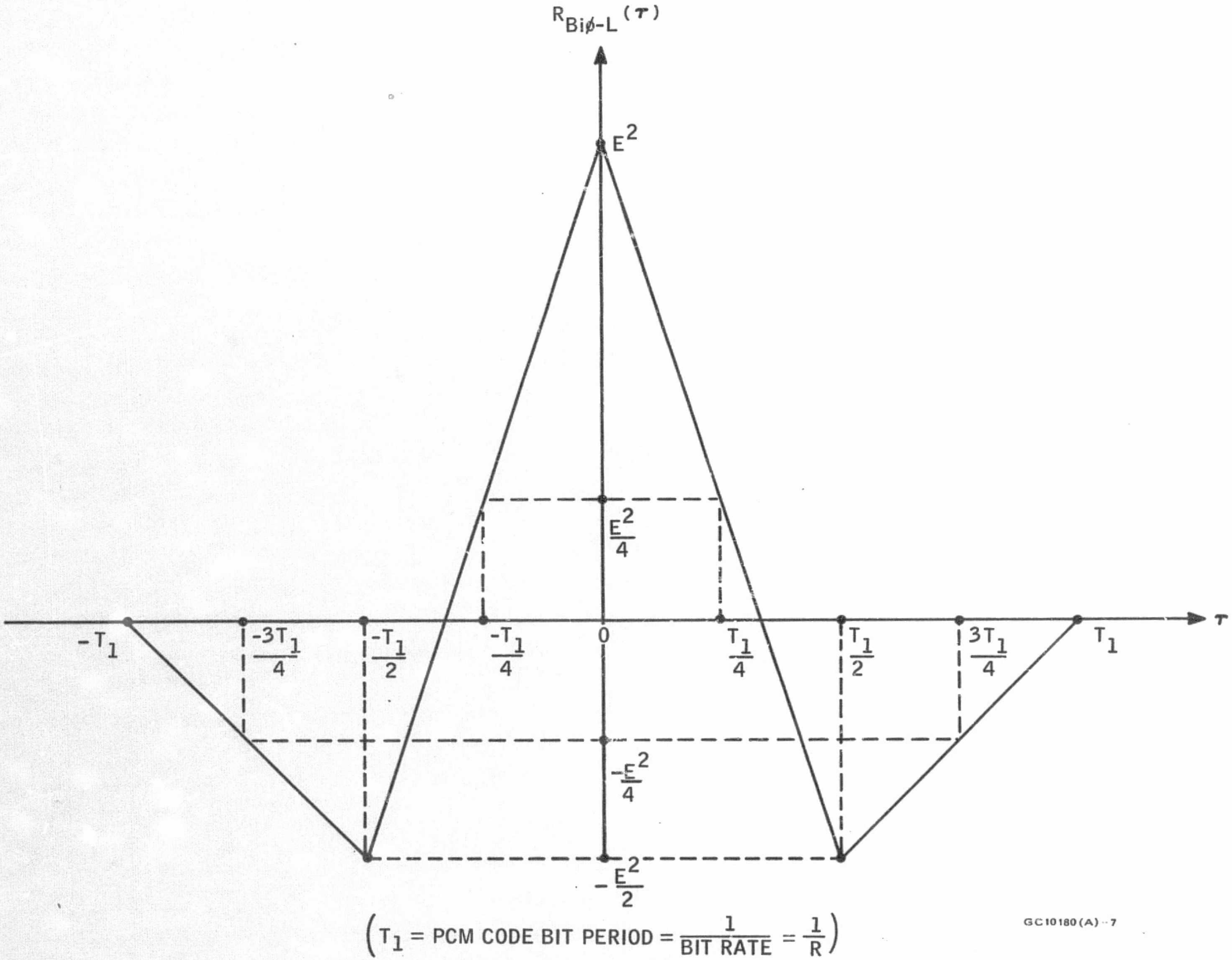


Figure 4-1 Autocorrelation Function of a Split-Phase Code (Random Bit Pattern)

Before the Fourier transform of a function can be determined, an exact expression for that function must be obtained. The auto-correlation function of Figure 4-1 may be expressed by linear equations over certain intervals of τ. A complete representation of $R_{Bi\phi-L}(\tau)$, then, is

$$R_{Bi\phi-L}(\tau) = 0 \qquad \text{for } \tau < -T_1 \tag{3}$$

$$\begin{aligned} R_{Bi\phi-L}(\tau) &= -\frac{E^2}{T_1}\tau - E^2 = -E^2\left(\frac{\tau}{T_1} + 1\right) \\ &= -\frac{E^2}{T_1}(\tau + T_1) \quad \text{for } -T_1 \leq \tau \leq -\frac{T_1}{2} \end{aligned} \tag{4}$$

$$R_{Bi\phi-L}(\tau) = \frac{3E^2}{T_1}\tau + E^2 = \frac{E^2}{T_1}(3\tau + T_1) \quad \text{for } -\frac{T_1}{2} \leq \tau \leq 0 \tag{5}$$

$$R_{Bi\phi-L}(\tau) = -\frac{3E^2}{T_1}\tau + E^2 = \frac{E^2}{T_1}(-3\tau + T_1) \quad \text{for } 0 \leq \tau \leq \frac{T_1}{2} \tag{6}$$

$$R_{Bi\phi-L}(\tau) = \frac{E^2}{T_1}\tau - E^2 = \frac{E^2}{T_1}(\tau - T_1) \quad \text{for } \frac{T_1}{2} \leq \tau \leq T_1 \tag{7}$$

$$R_{Bi\phi-L}(\tau) = 0 \qquad \text{for } \tau > T_1 \tag{8}$$

The power density spectrum of the split-phase code is, therefore, given by the following.

$$S_{Bi\phi-L}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{Bi\phi-L}(\tau)e^{-j\omega\tau}d\tau$$

$$= \cancelto{0}{\frac{1}{2\pi}\int_{-\infty}^{-T_1}(0)e^{-j\omega\tau}d\tau} + \frac{1}{2\pi}\int_{-T_1}^{-\frac{T_1}{2}} -\frac{E^2}{T_1}(\tau+T_1)e^{-j\omega\tau}d\tau$$

$$+ \frac{1}{2\pi}\int_{-\frac{T_1}{2}}^{0} \frac{E^2}{T_1}(3\tau+T_1)e^{-j\omega\tau}d\tau + \frac{1}{2\pi}\int_{0}^{\frac{T_1}{2}} \frac{E^2}{T_1}(-3\tau+T_1)e^{-j\omega\tau}d\tau$$

$$+ \frac{1}{2\pi}\int_{\frac{T_1}{2}}^{T_1} \frac{E^2}{T_1}(\tau-T_1)e^{-j\omega\tau}d\tau + \cancelto{0}{\frac{1}{2\pi}\int_{T_1}^{\infty}(0)e^{-j\omega\tau}d\tau}$$

$$= \frac{E^2}{2\pi T_1}\Bigg[-\int_{-T_1}^{-\frac{T_1}{2}}\tau e^{-j\omega\tau}d\tau - T_1\int_{-T_1}^{-\frac{T_1}{2}} e^{-j\omega\tau}d\tau + 3\int_{-\frac{T_1}{2}}^{0}\tau e^{-j\omega\tau}d\tau$$

$$+ T_1\int_{-\frac{T_1}{2}}^{0} e^{-j\omega\tau}d\tau - 3\int_{0}^{\frac{T_1}{2}}\tau e^{-j\omega\tau}d\tau + T_1\int_{0}^{\frac{T_1}{2}} e^{-j\omega\tau}d\tau$$

$$+\int_{\frac{T_1}{2}}^{T_1}\tau e^{-j\omega\tau}d\tau - T_1\int_{\frac{T_1}{2}}^{T_1} e^{-j\omega\tau}d\tau\Bigg] \qquad (9)$$

The preceding integral expression for $S_{Bi\phi\text{-}L}(\omega)$ contains only elementary integrals, which are evaluated in various mathematical handbooks. Reference 4 shows that

$$\int x\ e^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) \tag{10}$$

and

$$\int e^{ax}dx = \frac{1}{a}\ e^{ax} \tag{11}$$

Then

$$\int \tau e^{-j\omega\tau}d\tau = \frac{e^{-j\omega\tau}}{-\omega^2}(-j\omega\tau-1) = \frac{e^{-j\omega\tau}}{\omega^2}(1+j\omega\tau) \tag{12}$$

and

$$\int e^{-j\omega\tau}d\tau = \frac{e^{-j\omega\tau}}{-j\omega} \tag{13}$$

Evaluating the integrals of equation (9) according to equations (12) and (13), the following expressions are obtained

$$S_{Bi\phi\text{-}L}(\omega) = \frac{E^2}{2\pi T_1}\left\{ -\left[\frac{e^{-j\omega\tau}}{\omega^2}(1+j\omega\tau)\right]_{\tau=-T_1}^{\tau=-\frac{T_1}{2}} -T_1\left[\frac{e^{-j\omega\tau}}{-j\omega}\right]_{\tau=-T_1}^{\tau=-\frac{T_1}{2}}\right.$$

$$+\ 3\left[\frac{e^{-j\omega\tau}}{\omega^2}(1+j\omega\tau)\right]_{\tau=-\frac{T_1}{2}}^{\tau=0} +T_1\left[\frac{e^{-j\omega\tau}}{-j\omega}\right]_{\tau=-\frac{T_1}{2}}^{\tau=0}$$

$$- 3\left[\frac{e^{-j\omega\tau}}{\omega^2}(1+j\omega\tau)\right]_{\tau=0}^{\tau=\frac{T_1}{2}} + T_1\left[\frac{e^{-j\omega\tau}}{-j\omega}\right]_{\tau=0}^{\tau=\frac{T_1}{2}}$$

$$+ \left[\frac{e^{-j\omega\tau}}{\omega^2}(1+j\omega\tau)\right]_{\tau=\frac{T_1}{2}}^{\tau=T_1} - T_1\left[\frac{e^{-j\omega\tau}}{-j\omega}\right]_{\tau=\frac{T_1}{2}}^{\tau=T_1}\Bigg\}$$

$$= \frac{E^2}{2\pi T_1}\Bigg\{-\left[\frac{e^{+j\omega\frac{T_1}{2}}}{\omega^2}\left(1-j\omega\frac{T_1}{2}\right) - \frac{e^{+j\omega T_1}}{\omega^2}(1-j\omega T_1)\right] + T_1\left[\frac{e^{+j\omega\frac{T_1}{2}} - e^{+j\omega T_1}}{j\omega}\right]$$

$$+ 3\left[\frac{1}{\omega^2} - \frac{e^{+j\omega\frac{T_1}{2}}}{\omega^2}\left(1-j\omega\frac{T_1}{2}\right)\right] - T_1\left[\frac{1-e^{+j\omega\frac{T_1}{2}}}{j\omega}\right]$$

$$- 3\left[\frac{e^{-j\omega\frac{T_1}{2}}}{\omega^2}\left(1+j\omega\frac{T_1}{2}\right) - \frac{1}{\omega^2}\right] - T_1\left[\frac{e^{-j\omega\frac{T_1}{2}} - 1}{j\omega}\right]$$

$$+ \left[\frac{e^{-j\omega T_1}}{\omega^2}(1+j\omega T_1) - \frac{e^{-j\omega\frac{T_1}{2}}}{\omega^2}\left(1+j\omega\frac{T_1}{2}\right)\right] + T_1\left[\frac{e^{-j\omega T_1} - e^{-j\omega\frac{T_1}{2}}}{j\omega}\right]\Bigg\}$$

$$= \frac{E^2}{2\pi T_1}\Bigg\{-\frac{e^{+j\omega\frac{T_1}{2}}}{\omega^2} + \left(j\omega\frac{T_1}{2}\right)\frac{e^{+j\omega\frac{T_1}{2}}}{\omega^2} + \frac{e^{+j\omega T_1}}{\omega^2} - \left(j\omega T_1\right)\frac{e^{+j\omega T_1}}{\omega^2}$$

$$+ T_1\frac{e^{+j\omega\frac{T_1}{2}}}{j\omega} - T_1\frac{e^{+j\omega T_1}}{j\omega} + \frac{3}{\omega^2} - \frac{3e^{+j\omega\frac{T_1}{2}}}{\omega^2} + \left(3j\omega\frac{T_1}{2}\right)\frac{e^{+j\omega\frac{T_1}{2}}}{\omega^2}$$

$$
- \cancel{\frac{T_1}{j\omega}} + \frac{T_1 e^{+j\omega\frac{T_1}{2}}}{j\omega} - \frac{3e^{-j\omega\frac{T_1}{2}}}{\omega^2} - \left(3j\omega\frac{T_1}{2}\right)\frac{e^{-j\omega\frac{T_1}{2}}}{\omega^2} + \frac{3}{\omega^2}
$$

$$
- \frac{T_1 e^{-j\omega\frac{T_1}{2}}}{j\omega} + \cancel{\frac{T_1}{j\omega}} + \frac{e^{-j\omega T_1}}{\omega^2} + \left(j\omega T_1\right)\frac{e^{-j\omega T_1}}{\omega^2} - \frac{e^{-j\omega\frac{T_1}{2}}}{\omega^2}
$$

$$
- \left(j\omega\frac{T_1}{2}\right)\frac{e^{-j\omega\frac{T_1}{2}}}{\omega^2} + \frac{T_1 e^{-j\omega T_1}}{j\omega} - \frac{T_1 e^{-j\omega\frac{T_1}{2}}}{j\omega} \Bigg\} \tag{14}
$$

After collecting and rearranging terms, the above expression becomes

$$
\begin{aligned}
S_{Bi\phi-L}(\omega) = \frac{E^2}{2\pi T_1}\Bigg[& \frac{6}{\omega^2} + e^{+j\omega\frac{T_1}{2}}\left(-\frac{1}{\omega^2} + \frac{j\omega T_1}{2\omega^2} + \frac{T_1}{j\omega} - \frac{3}{\omega^2} + \frac{3j\omega T_1}{2\omega^2} + \frac{T_1}{j\omega}\right) \\
& + e^{-j\omega\frac{T_1}{2}}\left(-\frac{3}{\omega^2} - \frac{3j\omega T_1}{2\omega^2} - \frac{T_1}{j\omega} - \frac{1}{\omega^2} - \frac{j\omega T_1}{2\omega^2} - \frac{T_1}{j\omega}\right) \\
& + e^{+j\omega T_1}\left(\frac{1}{\omega^2} - \frac{j\omega T_1}{\omega^2} - \frac{T_1}{j\omega}\right) \\
& + e^{-j\omega T_1}\left(\frac{1}{\omega^2} + \frac{j\omega T_1}{\omega^2} + \frac{T_1}{j\omega}\right)\Bigg]
\end{aligned} \tag{15}
$$

But

$$
\frac{T_1}{j\omega} = \left(\frac{j\omega}{j\omega}\right)\left(\frac{T_1}{j\omega}\right) = \frac{j\omega T_1}{-\omega^2} = -\frac{j\omega T_1}{\omega^2} \tag{16}
$$

Equation (15) may be reduced to

$$S_{Bi\phi-L}(\omega) = \frac{E^2}{2\pi T_1}\left[\frac{6}{\omega^2} + e^{+j\omega\frac{T_1}{2}}\left(\frac{-2+\cancel{j\omega T_1}-2\cancel{j\omega T_1}-6+3\cancel{j\omega T_1}-2\cancel{j\omega T_1}}{2\omega^2}\right)\right.$$

$$+ e^{-j\omega\frac{T_1}{2}}\left(\frac{-6-3\cancel{j\omega T_1}+2\cancel{j\omega T_1}-2-\cancel{j\omega T_1}+2\cancel{j\omega T_1}}{2\omega^2}\right)$$

$$\left. + e^{+j\omega T_1}\left(\frac{1-\cancel{j\omega T_1}+\cancel{j\omega T_1}}{\omega^2}\right) + e^{-j\omega T_1}\left(\frac{1+\cancel{j\omega T_1}-\cancel{j\omega T_1}}{\omega^2}\right)\right]$$

$$= \frac{E^2}{\pi\omega^2 T_1}\left[3-4\left(\frac{e^{+j\omega\frac{T_1}{2}} + e^{-j\omega\frac{T_1}{2}}}{2}\right) + \left(\frac{e^{+j\omega T_1} + e^{-j\omega T_1}}{2}\right)\right] \qquad (17)$$

But

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2} \qquad (18)$$

So

$$S_{Bi\phi-L}(\omega) = \frac{E^2}{\pi\omega^2 T_1}\left[3-4\cos\left(\frac{\omega T_1}{2}\right) + \cos(\omega T_1)\right] \qquad (19)$$

The above expression can be simplified considerably by a series of simple trigonometric substitutions. Substituting

$$\cos(2\beta) = 1-2\sin^2\beta \qquad (20)$$

into equation (19) gives

$$S_{Bi\phi-L}(\omega) = \frac{E^2}{\pi\omega^2 T_1}\left\{3-4\left[1-2\sin^2\left(\frac{\omega T_1}{4}\right)\right] + 1-2\sin^2\left(\frac{\omega T_1}{2}\right)\right\}$$

$$= \frac{E^2}{\pi\omega^2 T_1}\left[8\sin^2\left(\frac{\omega T_1}{4}\right) - 2\sin^2\left(\frac{\omega T_1}{2}\right)\right] \qquad (21)$$

But $$\sin(2\beta) = 2 \sin\beta \cos\beta \tag{22}$$

and $$\sin^2(2\beta) = 4 \sin^2\beta \cos^2\beta \tag{23}$$

So

$$S_{Bi\phi\text{-}L}(\omega) = \frac{E^2}{\pi\omega^2 T_1}\left[8 \sin^2\left(\frac{\omega T_1}{4}\right) - 8 \sin^2\left(\frac{\omega T_1}{4}\right) \cos^2\left(\frac{\omega T_1}{4}\right)\right]$$

$$= \frac{8E^2}{\pi\omega^2 T_1} \sin^2\left(\frac{\omega T_1}{4}\right)\left[1 - \cos^2\left(\frac{\omega T_1}{4}\right)\right]$$

$$= \frac{8E^2}{\pi\omega^2 T_1} \sin^2\left(\frac{\omega T_1}{4}\right) \sin^2\left(\frac{\omega T_1}{4}\right) = \frac{8E^2}{\pi\omega^2 T_1} \sin^4\left(\frac{\omega T_1}{4}\right)$$

$$= \frac{E^2 T_1}{2\pi}\left[\frac{\sin^4\left(\frac{\omega T_1}{4}\right)}{\left(\frac{\omega T_1}{4}\right)^2}\right] \tag{24}$$

The power density spectrum for the random split-phase code is shown in Figure 4-2. Frequencies which are multiples of the PCM bit rate, R, are indicated in parentheses.

The split-phase spectrum contains no components about DC ($\omega = 0$) and reaches a peak at a frequency ($\omega = \frac{3\pi}{2T_1}$) equal to 3/4 of the bit rate R.

Appendix E contains a calculation of the percentage of total split-phase power present in a given bandwidth. The results of this calculation are summarized in Figure 4-3. It is evident that increasing the two-sided bandwidth past seven times the bit rate does not significantly increase the percentage of total power present. In practice, the bandwidth utilized is usually much smaller than seven times the bit rate.

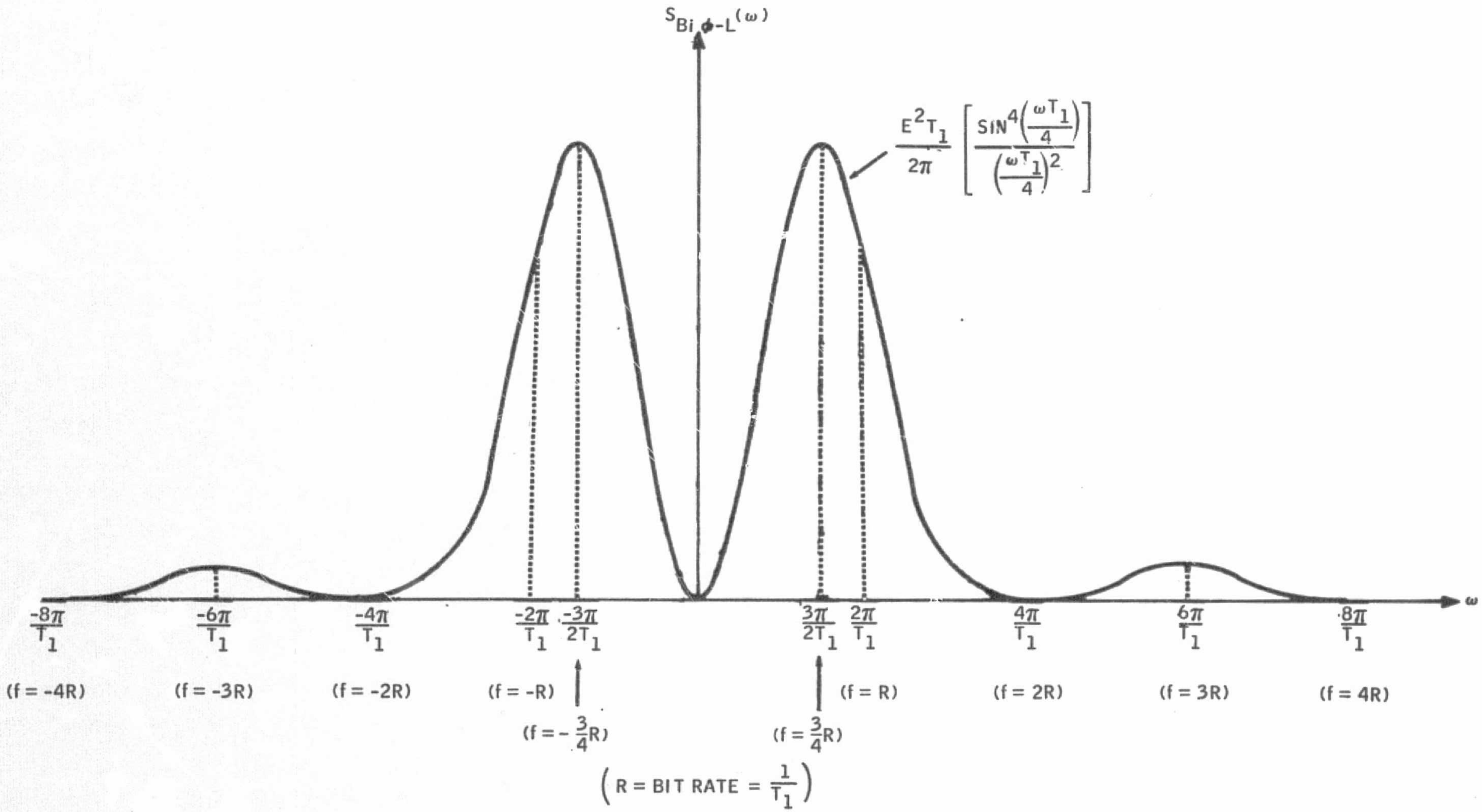


Figure 4-2 Power Density Spectrum of a Split-Phase Code
(Random Bit Pattern)

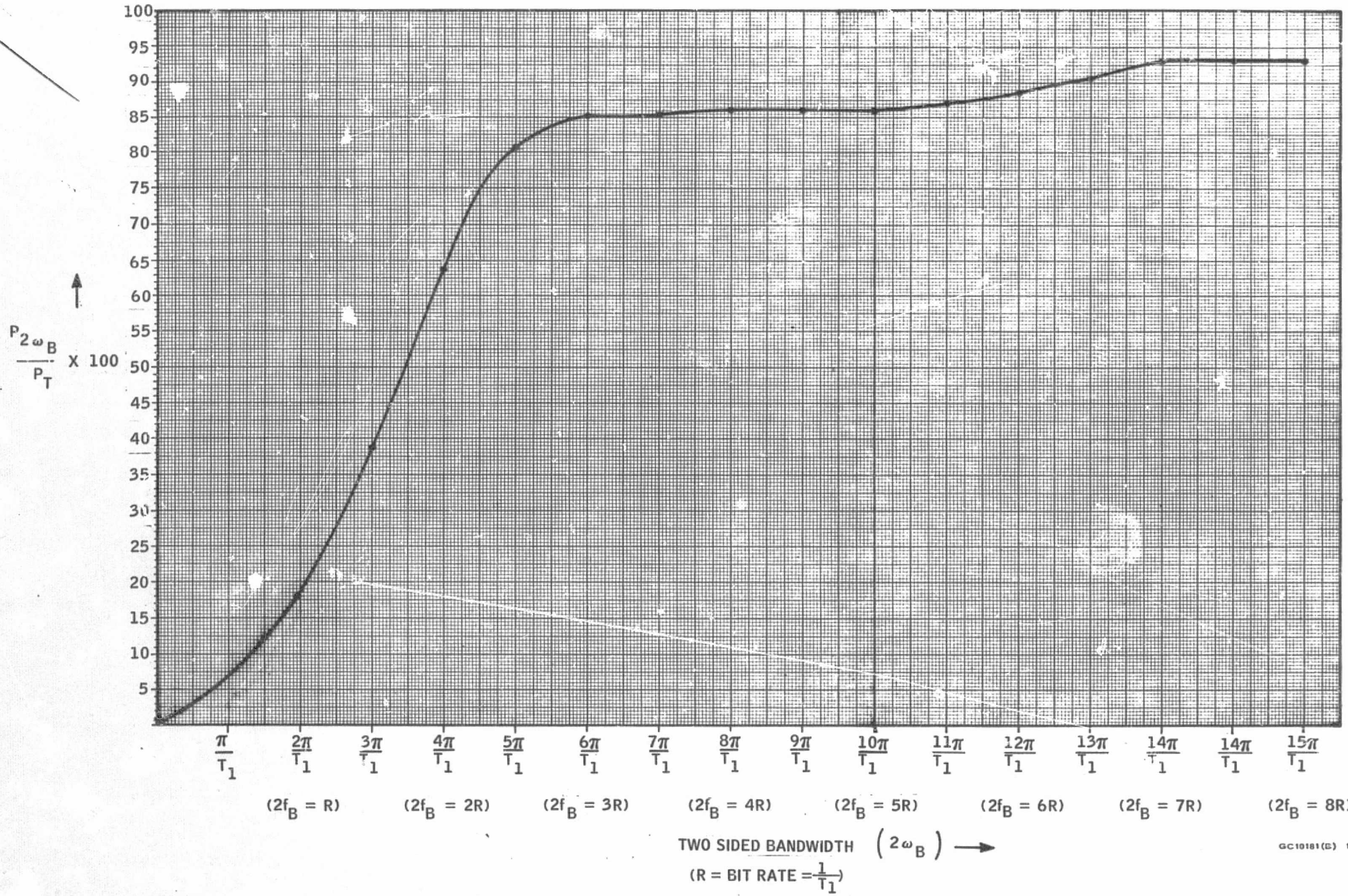


Figure 4-3 Percentage of Total Power of a Split-Phase Code Contained in the

4.3 BIT SYNCHRONIZATION EFFICIENCY

The synchronization problem is apparently solved when split-phase coding is employed. The bit transition density for a split-phase PCM code is higher than that of an NRZ code because *at least* one transition occurs during each bit period. Transmission of all zeros or all ones results in two transitions per bit, while transmission of alternating ones and zeros results in one transition per bit. The transition density for a random split-phase PCM bit stream is 1.5 (there is an average of 1-1/2 transitions per bit). A split-phase PCM code, then, is ideally suited for systems employing "flywheel" bit synchronizers at the receiver.

SECTION 5

CHARACTERISTICS OF RZ CODES

5.1 GENERAL

As shown in Figure 1-1, an RZ PCM code utilizes a binary "10" to represent a "1" and a binary "00" to represent a zero. An RZ code can be formed by combining an NRZ code and a clock, according to the logical "and" function, as shown in Appendix F.

The RZ PCM code possesses a basic periodicity, as evidenced by the fact that the second half of any bit period is always at the "0" level regardless of the type of message being encoded. It is to be expected, then, that this periodicity will appear in the autocorrelation function of the code and will result in discrete spectral lines being present in the RZ power density spectrum.

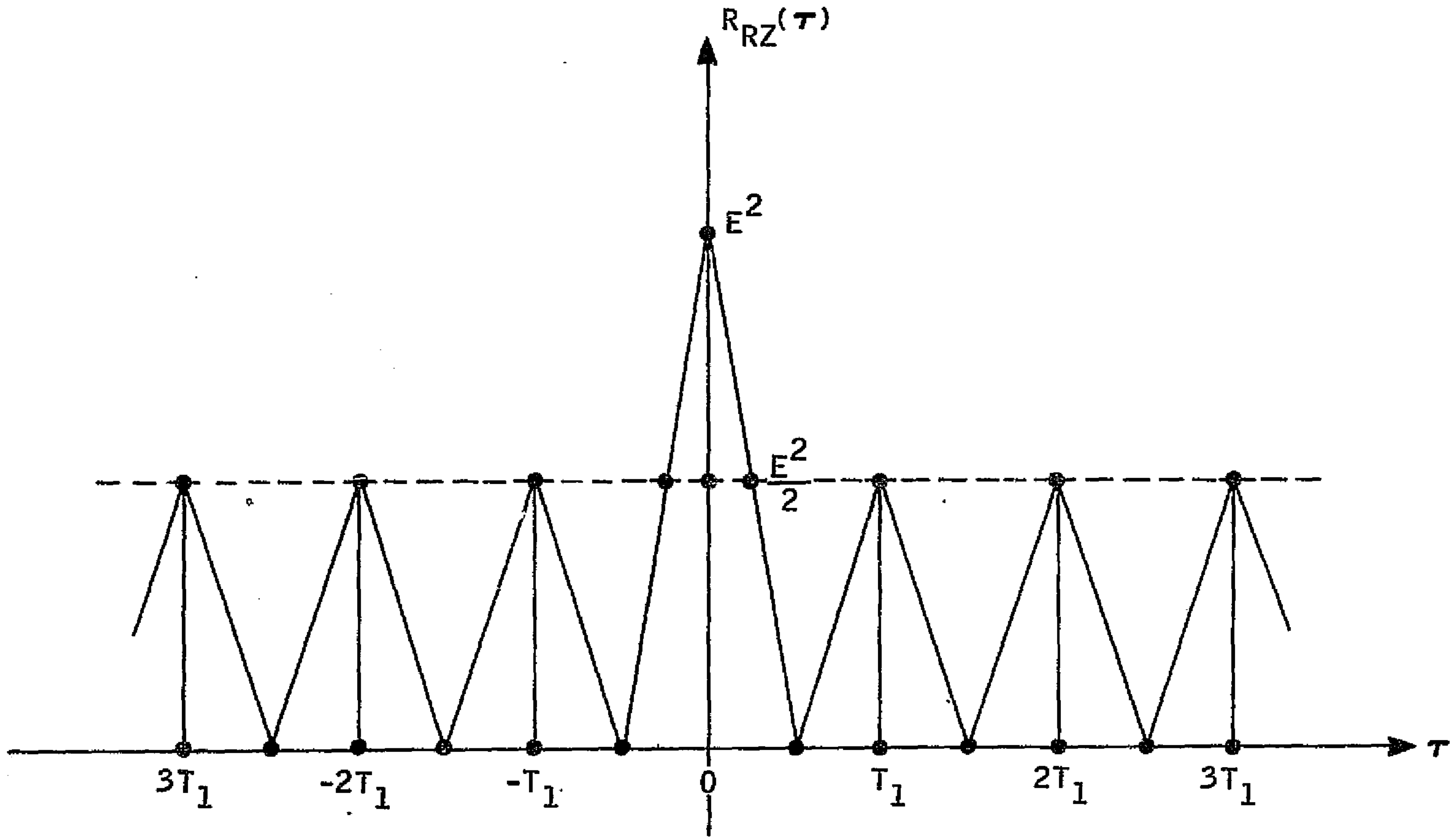
5.2 POWER SPECTRAL DENSITY

The ensemble-average autocorrelation function for an RZ PCM code is outlined in Appendix G. The result is shown in Figure 5-1, where T_1 is the code bit period and $\pm E$ is the amplitude of a code pulse.

The power spectrum of the RZ code may be found by taking the Fourier transform of the ensemble-average autocorrelation function. The autocorrelation function of Figure 5-1 can, however, be replaced by the sum of two simpler functions, as illustrated in Figure 5-2. One of these two functions is a simple triangular pulse, and its Fourier transform is given (see Paragraph 3.2) by

$$S_{RZ_1}(\omega) = \frac{E^2 T_1}{8\pi} \left(\frac{\sin \frac{\omega T_1}{4}}{\frac{\omega T_1}{4}} \right)^2 \qquad (25)$$

The above expression gives the component of the total RZ power spectral density due to the single triangular pulse of Figure 5-2.



$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} = \frac{1}{R} \right)$

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Figure 5-1 Autocorrelation Function of an RZ Code (Random Bit Pattern)

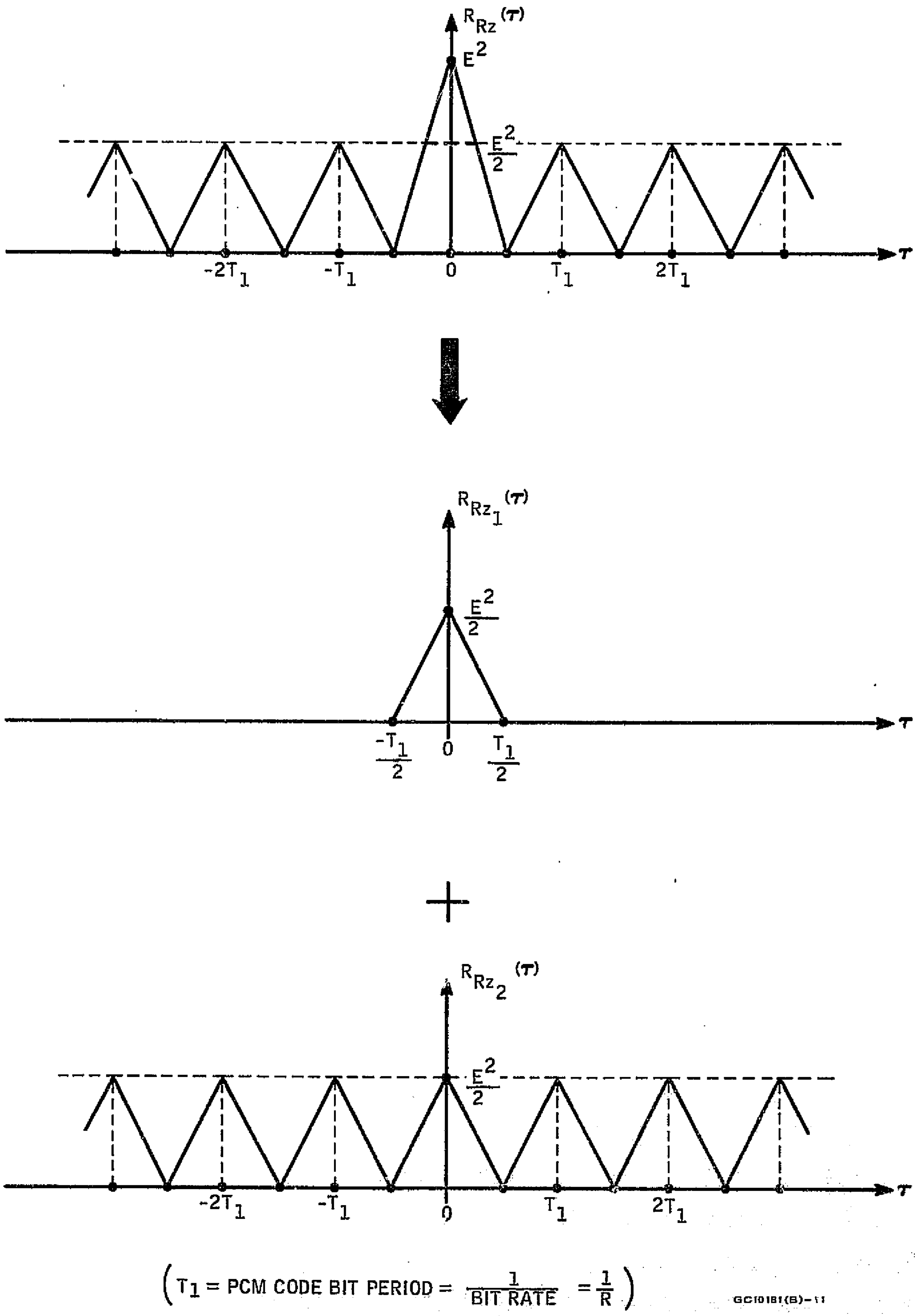


Figure 5-2 Simplification of the Autocorrelation Function of the RZ Code

The other function of Figure 5-2, however, is periodic (of period T_1), and the component of total RZ power spectral density due to it is provided by (Ref. 5)

$$S_{RZ_2}(\omega) = \sum_{n=-\infty}^{\infty} b_n \, \delta(\omega - n\omega_0) \tag{26}$$

where $\omega_0 = \frac{2\pi}{T_1}$, $\delta(\omega - n\omega_0)$ is the unit impulse[2] at $\omega = n\omega_0$, and

$$b_n = \frac{1}{T_1}\int_0^{T_1} R_{RZ_2}(\tau)e^{-jn\omega_0\tau}d\tau = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{+\frac{T_1}{2}} R_{RZ_2}(\tau)e^{-jn\omega_0\tau}d\tau$$

Before b_n can be evaluated, an expression for $R_{RZ_2}(\tau)$ must be obtained for the interval from $\tau = \frac{-T_1}{2}$ to $\tau = \frac{+T_1}{2}$. It is easily verified that the following representation is correct:

$$R_{RZ_2}(\tau) = \frac{E^2}{T_1}\tau + \frac{E^2}{2} = \frac{E^2}{2T_1}(2\tau + T_1) \quad \text{for} \quad \frac{-T_1}{2} \leq \tau \leq 0 \tag{27}$$

and

$$R_{RZ_2}(\tau) = -\frac{E^2}{T_1}\tau + \frac{E^2}{2} = -\frac{E^2}{2T_1}(2\tau - T_1) \quad \text{for } 0 \leq \tau \leq +\frac{T_1}{2} \tag{28}$$

Then

$$b_n = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{0} \frac{E^2}{2T_1}(2\tau+T_1)e^{-jn\omega_0\tau}d\tau + \frac{1}{T_1}\int_0^{\frac{T_1}{2}} -\frac{E^2}{2T_1}(2\tau-T_1)e^{-jn\omega_0\tau}d\tau$$

---

[2] The *unit impulse function* $\delta(X-X_0)$, also called the Dirac delta function, is defined to be infinite when its argument is zero, to be zero when its argument is nonzero, and to have a unit area.

$$= \frac{E^2}{T_1^2}\int_{-\frac{T_1}{2}}^{0} \tau e^{-jn\omega_0\tau} d\tau + \frac{E^2}{2T_1}\int_{-\frac{T_1}{2}}^{0} e^{-jn\omega_0\tau} d\tau$$

$$- \frac{E^2}{T_1^2}\int_{0}^{\frac{T_1}{2}} \tau e^{-jn\omega_0\tau} d\tau + \frac{E^2}{2T_1}\int_{0}^{\frac{T_1}{2}} e^{-jn\omega_0\tau} d\tau$$

$$= \frac{E^2}{T_1^2}\left[\frac{e^{-jn\omega_0\tau}}{n^2\omega_0^2}(1+jn\omega_0\tau)\right]_{\tau=-\frac{T_1}{2}}^{\tau=0} + \frac{E^2}{2T_1}\left[\frac{e^{-jn\omega_0\tau}}{-jn\omega_0}\right]_{\tau=-\frac{T_1}{2}}^{\tau=0}$$

$$- \frac{E^2}{T_1^2}\left[\frac{e^{-jn\omega_0\tau}}{n^2\omega_0^2}(1+jn\omega_0\tau)\right]_{\tau=0}^{\tau=\frac{T_1}{2}} + \frac{E^2}{2T_1}\left[\frac{e^{-jn\omega_0\tau}}{-jn\omega_0}\right]_{\tau=0}^{\tau=\frac{T_1}{2}}$$

$$= \frac{E^2}{T_1^2}\left[\frac{1}{n^2\omega_0^2} - \frac{e^{+jn\omega_0\frac{T_1}{2}}}{n^2\omega_0^2}\left(1-jn\omega_0\frac{T_1}{2}\right)\right] + \frac{E^2}{2T_1}\left[\frac{jn\omega_0}{n^2\omega_0^2} - \frac{jn\omega_0 e^{+jn\omega_0\frac{T_1}{2}}}{n^2\omega_0^2}\right]$$

$$- \frac{E^2}{T_1^2}\left[\frac{e^{-jn\omega_0\frac{T_1}{2}}}{n^2\omega_0^2}\left(1+jn\omega_0\frac{T_1}{2}\right) - \frac{1}{n^2\omega_0^2}\right] + \frac{E^2}{2T_1}\left[\frac{jn\omega_0 e^{-jn\omega_0\frac{T_1}{2}}}{n^2\omega_0^2} - \frac{jn\omega_0}{n^2\omega_0^2}\right]$$

$$= e^{+jn\omega_0\frac{T_1}{2}}\left[\frac{-2E^2 + \cancel{E^2 jn\omega_0 T_1} - \cancel{E^2 jn\omega_0 T_1}}{2T_1^2 n^2 \omega_0^2}\right]$$

$$+ e^{-jn\omega_0\frac{T_1}{2}}\left[\frac{-2E^2 - \cancel{E^2 jn\omega_0 T_1} + \cancel{E^2 jn\omega_0 T_1}}{2T_1^2 n^2 \omega_0^2}\right]$$

$$+ \frac{2E^2 + \cancel{E^2 jn\omega_0 T_1} + 2E^2 - \cancel{E^2 jn\omega_0 T_1}}{2T_1^2 n^2 \omega_0^2}$$

$$= \frac{2E^2}{T_1^2 n^2 \omega_0^2} - \left(\frac{2E^2}{T_1^2 n^2 \omega_0^2}\right)\left(\frac{e^{+jn\omega_0\frac{T_1}{2}} + e^{-jn\omega_0\frac{T_1}{2}}}{2}\right)$$

$$= \frac{2E^2}{T_1^2 n^2 \omega_0^2}\left[1 - \cos\left(n\omega_0\frac{T_1}{2}\right)\right] \tag{29}$$

But

$$\sin^2\alpha = \frac{1}{2}\,[1 - \cos 2\alpha] \tag{30}$$

so

$$b_n = \frac{4E^2}{T_1^2 n^2 \omega_0^2}\sin^2\left(n\omega_0\frac{T_1}{4}\right) = \frac{E^2}{4}\;\frac{\sin^2\left(n\omega_0\frac{T_1}{4}\right)}{\left(n\omega_0\frac{T_1}{4}\right)^2} \tag{31}$$

Then

$$S_{RZ_2}(\omega) = \sum_{n=-\infty}^{\infty} \frac{E^2}{4}\;\frac{\sin^2\left(\frac{n\omega_0 T_1}{4}\right)}{\left(\frac{n\omega_0 T_1}{4}\right)^2}\,\delta\,(\omega - n\omega_0) \tag{32}$$

and

$$S_{RZ}(\omega) = S_{RZ_1}(\omega)+S_{RZ_2}(\omega)= \frac{E^2T_1}{8\pi}\left(\frac{\sin\frac{\omega T_1}{4}}{\frac{\omega T_1}{4}}\right)^2 +\sum_{n=-\infty}^{\infty}\frac{E^2}{4}\left(\frac{\sin\frac{n\omega_0T_1}{4}}{\frac{n\omega_0T_1}{4}}\right)^2 \delta(\omega-n\omega_0) \tag{33}$$

Clearly, the power spectrum for the RZ code consists of a continuous component (corresponding to the random component of the RZ code) and a series of discrete frequencies (resulting from the periodic component of the RZ code). The power spectrum is illustrated in Figure 5-3. It should be noted that the continuous portion of the RZ spectrum occupies a wider band of frequencies than is occupied by the NRZ spectrum, although the two spectra have similar shapes.

The percentage of total RZ power present in a given bandwidth is calculated in Appendix H. The results of this calculation are summarized in Figure 5-4. By referring to Figure 3-3, it can be seen that the percentage of RZ power present in a certain bandwidth is less than the percentage of NRZ power present in that same bandwidth. For a desired signal fidelity, then, the required RZ bandwidth is greater (by a factor of approximately 2) than the required NRZ bandwidth.

5.3 BIT SYNCHRONIZATION EFFICIENCY

An RZ bit stream contains two transitions for each binary one and no transitions for each binary zero. The bit transition density for a random RZ PCM bit stream is 1.0 (there is an average of 1 transition per bit). The synchronization efficiency of an RZ code for systems employing "flywheel" bit synchronization is, therefore, higher than that of an NRZ code. However, a finite probability still exists that no transitions will occur over any given time interval (i.e., transmission of all zeros over any given interval will result in no transitions). Therefore, if "flywheel" synchronization is used, provisions must be made to insure that an adequate number of transitions is available per unit time.

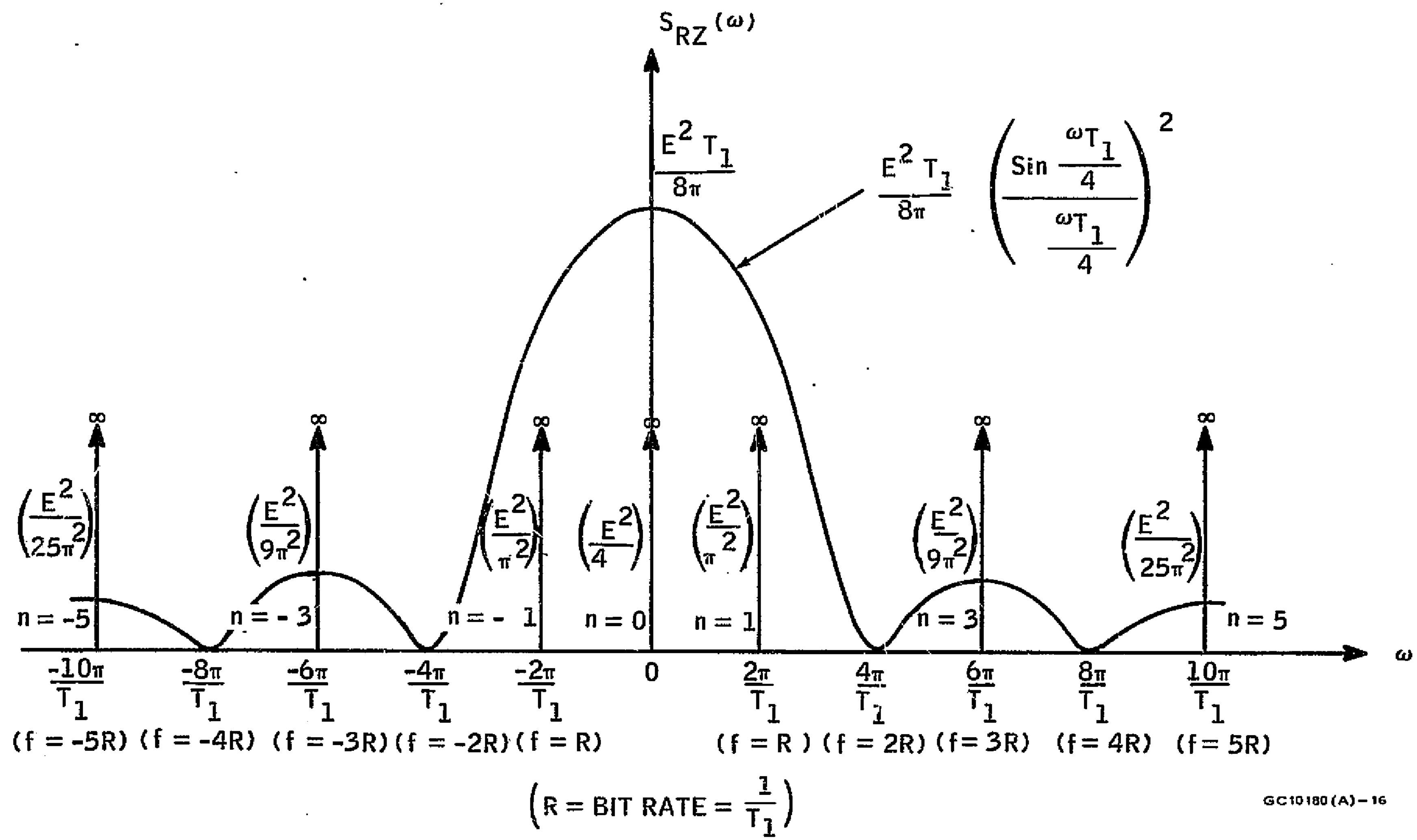


Figure 5-3 Power Spectrum of an RZ Code (Random Bit Pattern)

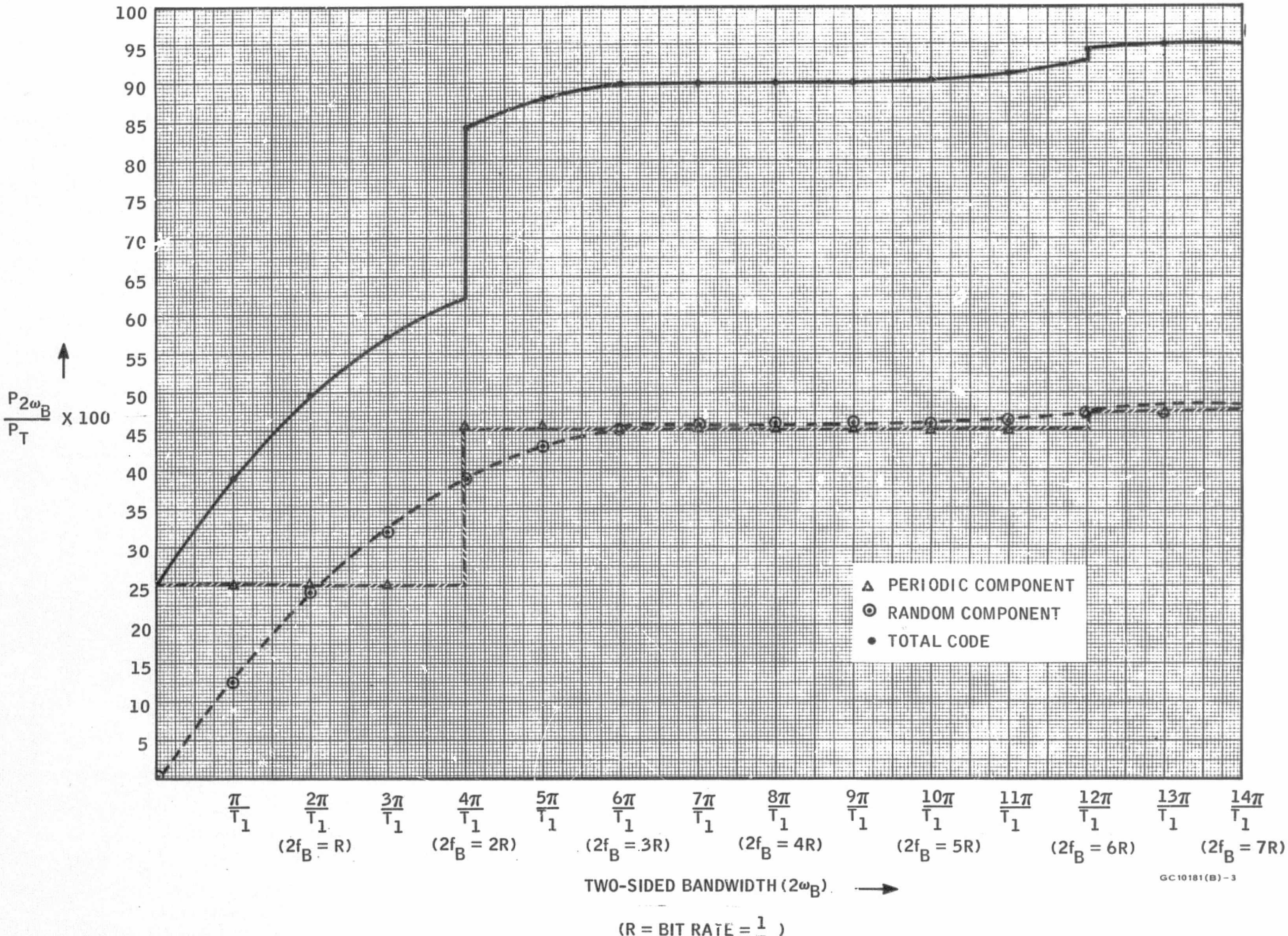


Figure 5-4 Percentage of Total Power of an RZ Code Contained in the Frequency Band Extending from $-\omega_B$ to $+\omega_B$

One distinct advantage of RZ coding becomes apparent upon inspection of the RZ power spectrum. That spectrum contains discrete frequency components at odd multiples of the bit rate, R (or at odd multiples of the clock frequency). Any one of these discrete frequencies may be isolated (by filtering) at the receiver. The fundamental clock frequency is, therefore, directly recoverable, and "flywheel" type bit synchronization is no longer required.

SECTION 6

POWER SPECTRA OF SIGNALS MODULATED BY RANDOM PCM CODES

6.1 GENERAL

Appendices B, E, and H contain calculations of bandwidth requirements for NRZ, split-phase, and RZ PCM codes, respectively. Those calculations, however, provide measures only of the percentage of total PCM code power present in given *baseband* bandwidths and not of the actual required *transmission* bandwidths for the various PCM channels. The required transmission bandwidth for a carrier that is modulated by a PCM bit stream is dependent on the nature of the carrier modulation employed.

When performing calculations of power spectra for signals modulated by binary sequences, i.e., PCM codes, it is usually assumed that the modulation is not coherent with the carrier. That is, the phase and frequency of the modulation is assumed to be independent of the phase and frequency of the carrier. For the case of noncoherent modulation, calculations of power spectral density can be made by assuming that the carrier and the modulation are statistically independent. This assumption is not valid for phase-coherent systems, however, and may lead to erroneous results (reference 6). To illustrate this point, the power spectra for a carrier which is noncoherently and coherently phase-shift-keyed by an NRZ PCM code will now be calculated.

6.2 NONCOHERENT PHASE MODULATION

A generalized expression for a sinusoidal carrier which is phase-modulated by a binary sequence is

$$e(t) = A \cos \left[\omega_c t + \phi_c + m(t) E \beta_c\right] \tag{32}$$

where A is the carrier amplitude,
ω_c is the carrier frequency,
ϕ_c is the initial phase of the carrier,
$m(t) = \pm 1$ is a switching function,
E represents the voltage level of the binary sequence, and
β_c is the carrier deviation sensitivity (rad/volt)

Expanding equation (32) trigonometrically, the following expression is obtained:

$$e(t) = A\cos(\omega_c t + \phi_c)\cos\left[m(t)E\beta_c\right] - A\sin(\omega_c t + \phi_c)\sin\left[m(t)E\beta_c\right] \tag{33}$$

But, since

$$\cos(\pm x) = \cos x \tag{34}$$

and

$$\sin(\pm x) = \pm\sin(x) \tag{35}$$

then

$$e(t) = A\cos(\omega_c t + \phi_c)\cos(E\beta_c) - A\left[\sin(\omega_c t + \phi_c)\right]\left[m(t)\ \sin(E\beta_c)\right] \tag{36}$$

Since $E\beta_c$ is a constant, the second term of the above expression represents double-sideband (suppressed-carrier) modulation of the carrier by the binary sequence, while the first term represents a discrete carrier component. The binary phase-shift-keyed (PSK) signal to be considered is one for which

$$E\beta_c = \frac{\pi}{2} \tag{37}$$

and

$$\begin{aligned} e_{PSK}(t) &= A\cos(\omega_c t + \phi_c)\cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0} - A\left[\sin(\omega_c t + \phi_c)\right]\left[m(t)\cancel{\sin\left(\frac{\pi}{2}\right)}^{\,1}\right] \\ &= -A\,m(t)\,\sin(\omega_c t + \phi_c) \end{aligned} \tag{38}$$

The two possible signal states of the PSK signal are

$$e_1(t) = +A\sin(\omega_c t + \phi_c) \tag{39}$$

and

$$e_2(t) = -A\sin(\omega_c t + \phi_c) \tag{40}$$

It can be seen that the PSK signal can be generated by multiplying the sinusoidal carrier by the binary sequence. This sequence could represent any of the various PCM code formats, but for purposes of illustration, it will be assumed that the NRZ format is utilized.

The ensemble-average autocorrelation function of the PSK signal discussed above is given by[3]

$$\begin{aligned} R_{PSK}(\tau) &= E[e_{PSK}(t_1)e_{PSK}(t_1 + \tau)] \\ &= E\left\{\left[-A\,m(t_1)\sin(\omega_c t_1 + \phi_c)\right]\left\{-A\,m(t_1 + \tau)\sin\left[\omega_c(t_1 + \tau) + \phi_c\right]\right\}\right\} \\ &= A^2E\left\{m(t_1)\,m(t_1 + \tau)\sin(\omega_c t_1 + \phi_c)\sin\left[\omega_c(t_1 + \tau) + \phi_c\right]\right\} \end{aligned} \tag{41}$$

If the modulation process is *noncoherent*, i.e., the phase and frequency of the PCM modulating signal are not related to the phase and frequency of the carrier, then the PCM signal and the carrier may be assumed to be statistically independent. Since the expected value of the product of statistically independent random variables is equal to the product of their expected values (reference 7), then

$$R_{PSK}(\tau) = A^2E\left[m(t_1)m(t_1 + \tau)\right]E\left\{\sin(\omega_c t_1 + \phi_c)\sin\left[\omega_c(t_1 + \tau) + \phi_c\right]\right\} \tag{42}$$

But

$$\begin{aligned} \sin(\omega_c t_1 + \phi_c)\sin\left[\omega_c(t_1 + \tau) + \phi_c\right] &= \frac{1}{2}\cos\omega_c\tau \\ &\quad - \frac{1}{2}\cos\left[\omega_c(2t_1 + \tau) + 2\phi_c\right] \end{aligned} \tag{43}$$

Therefore,

$$R_{PSK}(\tau) = A^2E\left[m(t_1)m(t_1 + \tau)\right]\left\{E\left[\frac{1}{2}\cos\omega_c\tau\right] - E\left\{\frac{1}{2}\cos\left[\omega_c(2t_1 + \tau) + 2\phi_c\right]\right\}\right\} \tag{44}$$

---

[3] Note that the E used in the following equations denotes "expected value of", rather than voltage levels of the PCM waveform.

If ϕ_c is assumed to be a random variable, uniformly distributed over the range 0 to 2π, then

$$E\left\{\frac{1}{2}\cos\left[\omega_c(2t_1+\tau)+2\phi_c\right]\right\}=0 \tag{45}$$

or the ensemble average of a sinusoid of random phase is the same as the time average of a single sinusoid. The expression for $R_{PSK}(\tau)$ reduces to

$$R_{PSK}(\tau) = E\left[m(t_1)m(t_1+\tau)\right]\frac{A^2}{2}\cos\omega_c\tau \tag{46}$$

But

$$E\left[m(t_1)m(t_1+\tau)\right] = R_{NRZ}(\tau) \tag{47}$$

and

$$\frac{A^2}{2}\cos\omega_c\tau = R_{CARRIER}(\tau) \tag{48}$$

For the case of noncoherent phase-shift-keying of a carrier by a PCM code, then, the autocorrelation function of the PSK signal is equal to the product of the autocorrelation function of the carrier and the autocorrelation function of the code. It is well known that multiplication of autocorrelation functions results in convolution of power spectra.

The power spectrum of the carrier consists of two impulses (of weight $\frac{A^2}{4}$) located at $\omega = \omega_c$ and at $\omega = -\omega_c$. Since convolution of a function with an impulse results in that function being shifted to the point at which the impulse occurs, then, from equation (2), it should be expected that the power spectral density of the noncoherent PSK signal would be of the form

$$S_{PSK}(\omega) = K\left[\frac{\sin\left(\frac{\omega-\omega_c}{2}T_1\right)}{\frac{\omega-\omega_c}{2}T_1}\right]^2 \tag{49}$$

That the above is actually the case is verified in Appendix I. Equations (I-8) and (I-14) show that

$$S_{PSK}(\omega) = \frac{A^2E^2T_1}{8\pi}\left[\frac{\sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)}{\left(\frac{\omega - \omega_c}{2} T_1\right)^2} + \frac{\sin^2\left(\frac{\omega + \omega_c}{2} T_1\right)}{\left(\frac{\omega + \omega_c}{2} T_1\right)^2}\right] \tag{50}$$

Inspection of equation (50) reveals that the power spectrum of the *noncoherent* PSK signal consists merely of the baseband spectrum of the modulating signal (in this case, the NRZ PCM code) translated to appear about plus and minus the carrier frequency. This spectrum is shown in Figure 6-1. It should be observed that there are, in general, two contributions to the spectral density of the PSK signal at any frequency ω. Each term of equation (50) contributes to $S_{PSK}(\omega)$ at any value of ω. For

$$\omega_c >> \frac{1}{T_1}, \tag{51}$$

however, the contribution from the "folded over" sideband is often negligible.

If the carrier frequency, ω_c, is related to the frequency of the modulating sequence, equation (50) can be reduced to a single term. Specifically, if

$$\omega_c = \frac{n\pi}{T_1}, \text{ with n integral} \tag{52}$$

then

$$\begin{aligned}\sin^2\left(\frac{\omega + \omega_c}{2} T_1\right) &= \sin^2\left(\frac{\omega - \omega_c}{2} T_1 + \omega_c T_1\right)\\ &= \sin^2\left(\frac{\omega - \omega_c}{2} T_1 + n\pi\right)\\ &= \sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)\end{aligned} \tag{53}$$

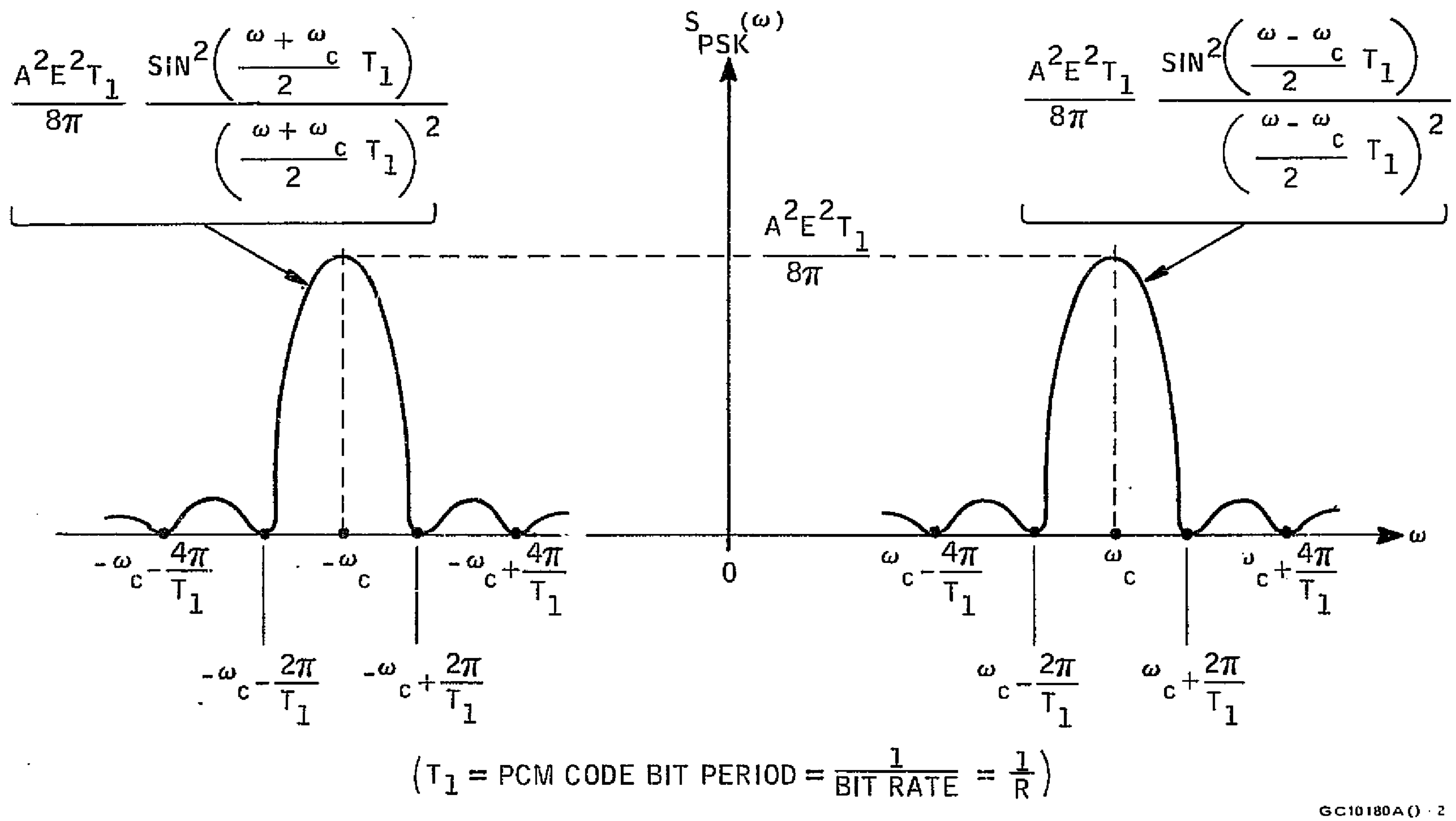


Figure 6-1 Power Density Spectrum for a Phase-Shift-Keyed Sinusoid (Noncoherent Modulation)

and

$$S_{PSK}(\omega) = \frac{A^2E^2T_1}{8\pi}\left[\frac{\sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)}{\left(\frac{\omega - \omega_c}{2} T_1\right)^2} + \frac{\sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)}{\left(\frac{\omega + \omega_c}{2} T_1\right)^2}\right]$$

$$= \frac{A^2E^2T_1}{8\pi}\left[\frac{\sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)}{\left(\frac{\omega - \omega_c}{2} T_1\right)^2}\right]\left[1 + \frac{\left(\frac{\omega - \omega_c}{2} T_1\right)^2}{\left(\frac{\omega + \omega_c}{2} T_1\right)^2}\right]$$

$$= \frac{A^2E^2T_1}{8\pi}\left[\frac{\sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)}{\left(\frac{\omega - \omega_c}{2} T_1\right)^2}\right] \frac{2\left[1 + \left(\frac{\omega}{\omega_c}\right)^2\right]}{\left(1 + \frac{\omega}{\omega_c}\right)^2}$$

$$= \frac{A^2E^2T_1}{4\pi}\left[\frac{\sin^2\left(\frac{\omega - \omega_c}{2} T_1\right)}{\left(\frac{\omega - \omega_c}{2} T_1\right)^2}\right]\left[\frac{1 + \left(\frac{\omega}{\omega_c}\right)^2}{\left(1 + \frac{\omega}{\omega_c}\right)^2}\right] \tag{54}$$

It should be noted that equation (52, neither imposes nor assumes any particular phase relationship between the modulating sequence and the carrier signal. Therefore, the assumption of noncoherent phase modulation should still be valid, and the simplification of equation (50) should likewise be valid.

6.3 COHERENT PHASE MODULATION

In Paragraph 6.2, it was shown that a sinusoidal carrier that is phase-shifted $\pm \frac{\pi}{2}$ radians by a binary sequence can be expressed as

$$e_{PSK}(t) = -A\, m(t) \sin(\omega_c t + \phi_c) \tag{55}$$

where A is the carrier amplitude
$m(t) = \pm 1$ is a switching function representing the binary sequence
ω_c is the carrier frequency
ϕ_c is the initial phase of the carrier (relative to the sequence).

The above expression is valid for both coherent and noncoherent phase-modulation of the carrier. The ensemble-average autocorrelation function of the PSK signal is given by

$$R_{PSK}(\tau) = E\left[e_{PSK}(t_1)e_{PSK}(t_1 + \tau)\right]$$
$$= A^2 E\left\{m(t_1)m(t_1 + \tau) \sin(\omega_c t_1 + \phi_c) \sin\left[\omega_c(t_1 + \tau) + \phi_c\right]\right\} \tag{56}$$

If the modulation process is *phase-coherent*, i.e., the bit transitions of the binary sequence always occur at the same point of a carrier cycle (ϕ_c is constant for all ensemble members), then the sequence and the carrier may *not* be assumed to be statistically independent. In fact, the samples of the carrier at time t_1 and and at time $t_1 + \tau$ are *constant* for all members of the ensemble. Since the expected value of the product of a constant and a random variable is equal to the constant times the expected value of the random variable, then

$$R_{PSK}(\tau) = A^2 \sin(\omega_c t_1 + \phi_c) \sin\left[\omega_c(t_1 + \tau) + \phi_c\right] E\left[m(t_1)m(t_1 + \tau)\right]$$
$$= \left(\frac{A^2}{2} \cos\omega_c\tau\right) E\left[m(t_1)m(t_1 + \tau)\right]$$
$$-\left\{\frac{A^2}{2} \cos\left[\omega_c(2t_1 + \tau) + 2\phi_c\right]\right\} E\left[m(t_1)m(t_1 + \tau)\right] \tag{57}$$

If, for instance, the binary sequence represents the NRZ PCM code, then

$$E\left[m(t_1)m(t_1 + \tau)\right] = R_{NRZ}(\tau) \tag{58}$$

and

$$R_{PSK}(\tau) = \left(\frac{A^2}{2} \cos\omega_c\tau\right) R_{NRZ}(\tau) - \left\{\frac{A^2}{2} \cos\left[\omega_c(2t_1 + \tau) + 2\phi_c\right]\right\} R_{NRZ}(\tau) \tag{59}$$

The ensemble-average autocorrelation function of the coherent PSK signal is a function of τ and t_1. Strictly speaking, then, the

PSK signal is not a stationary process, and the power spectral density cannot be evaluated by simply taking the Fourier transform of equation (59).

Titsworth and Welch (Reference 6) have shown that the power spectral density of the coherent PSK signal considered here is given by

$$S_{PSK}(\omega) = \frac{A^2E^2T_1}{2\pi}\left[\frac{\sin^2\left(\frac{\omega - \omega_c}{2}T_1\right)}{\left(\frac{\omega - \omega_c}{2}T_1\right)^2}\right]\left[\frac{\cos^2\phi_c + \left(\frac{\omega}{\omega_c}\right)^2\sin^2\phi_c}{\left(1 + \frac{\omega}{\omega_c}\right)^2}\right] \qquad (60)$$

The above expression reduces to that for noncoherent modulation, equation (54), if ϕ_c is equal to $\frac{\pi}{4}$. This is statistically equivalent to the case where ϕ_c is a random variable, uniformly distributed over the range 0 to 2π.

Equation (60) may be rewritten as follows:

$$S_{PSK}(\omega) = \frac{A^2E^2T_1}{2\pi}\left[\frac{\omega_c^2\cos^2\phi_c + \omega^2\sin^2\phi_c}{(\omega_c + \omega)^2}\right]\left[\frac{4}{(\omega - \omega_c)^2\,T_1^2}\right]\sin^2\left(\frac{\omega - \omega_c}{2}T_1\right)$$

$$= \frac{2A^2E^2}{\pi T_1}\left[\frac{\omega_c^2\cos^2\phi_c + \omega^2\sin^2\phi_c}{(\omega^2 - \omega_c^2)^2}\right]\sin^2\left(\frac{\omega - \omega_c}{2}T_1\right) \qquad (61)$$

The power spectral density envelope is given by

$$S_{env}(\omega) = \frac{2A^2E^2}{\pi T_1}\left[\frac{\omega_c^2\cos^2\phi_c + \omega^2\sin^2\phi_c}{(\omega^2 - \omega_c^2)^2}\right] \qquad (62)$$

and

$$S_{PSK}(\omega) = S_{env}(\omega)\,\sin^2\left(\frac{\omega - \omega_c}{2}T_1\right) \qquad (63)$$

Inspection of equation (62) reveals the following:

A. If ϕ_c is zero or a multiple of π, the envelope reduces to

$$S_{env}(\omega) = \frac{2A^2E^2}{\pi T_1}\left[\frac{\omega_c{}^2}{(\omega^2 - \omega_c{}^2)^2}\right] \tag{64}$$

and, for large ω,

$$S_{env}(\omega) \sim \frac{K}{\omega^4} \tag{65}$$

For large ω, then, the power spectrum falls off at 12 dB/ octave. Doubling the frequency results in dividing the envelope by 2^4 or in reducing the amplitude of the envelope by 12 dB. This case corresponds to coherent modulation of the carrier, with bit transitions occurring *at the zero crossings* of the carrier; this is illustrated in Figure 6-2, Case A. Intuitively, it would be expected that this is the minimum bandwidth case, because the modulated signal is never discontinuous (only the *slope* is discontinuous).

B. If ϕ_c is an odd multiple of $\frac{\pi}{2}$, the envelope becomes

$$S_{env}(\omega) = \frac{2A^2E^2}{\pi\ T_1}\left[\frac{\omega^2}{(\omega^2 - \omega_c{}^2)^2}\right] \tag{66}$$

and, for large ω,

$$S_{env}(\omega) \sim \frac{K}{\omega^2} \tag{67}$$

For large ω, the power spectrum falls off at only 6 dB/ octave. Doubling the frequency results in dividing the envelope by 2^2 or in reducing the envelope amplitude by 6 dB. This case corresponds to coherent modulation of the carrier, with bit transitions occurring *at the peaks* of the carrier. This is illustrated in Figure 6-2, Case B. This

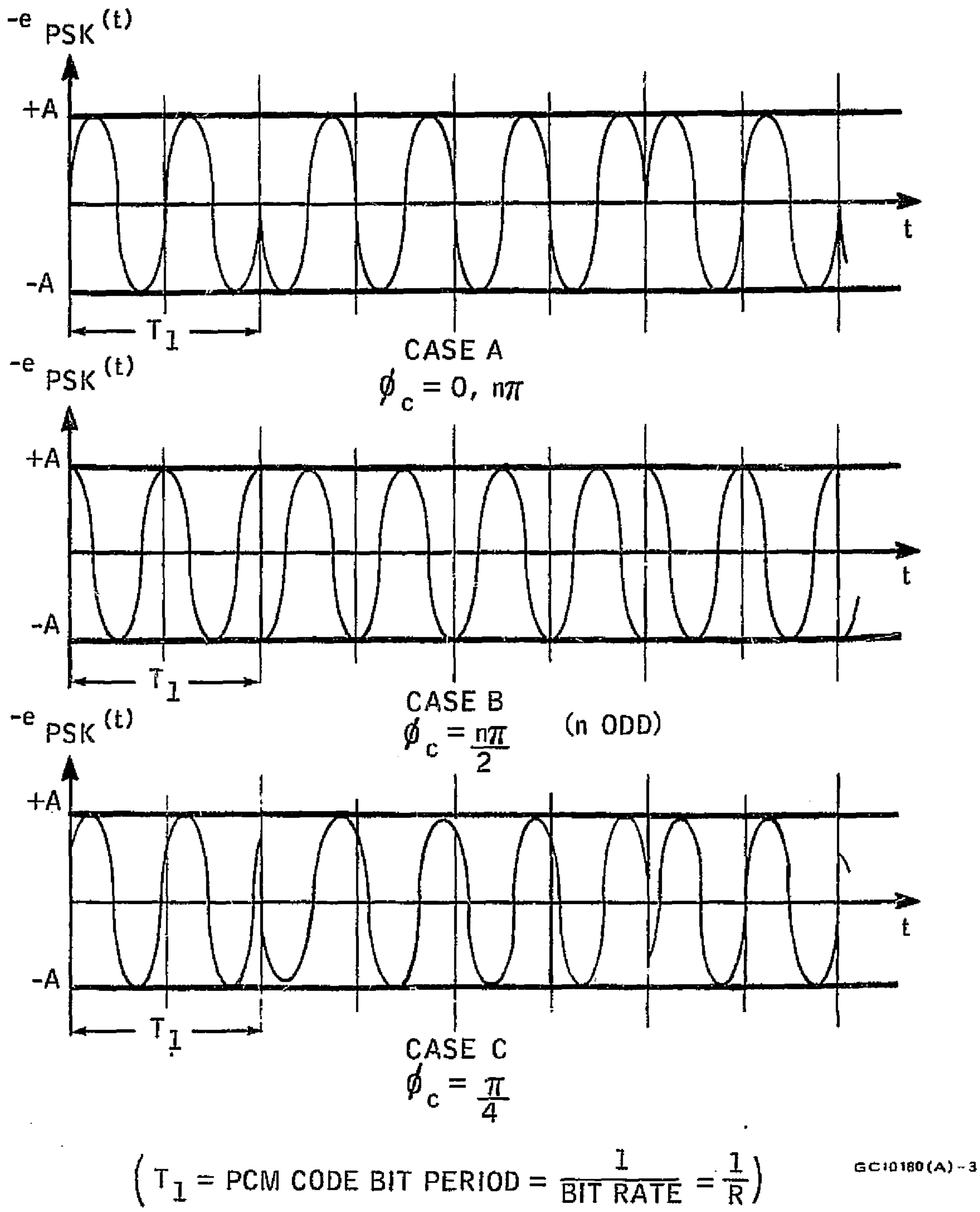


Figure 6-2 Coherent PSK Signals

would intuitively be expected to be the maximum bandwidth case, as bit transitions always result in large discontinuities in the modulated signal.

C. If ϕ_c is equal to $\frac{\pi}{4}$, the envelope becomes

$$S_{env}(\omega) = \frac{2A^2E^2}{\pi T_1}\left[\frac{\omega_c^{\,2}\left(\frac{1}{2}\right) + \left(\omega^2\right)\left(\frac{1}{2}\right)}{(\omega^2 - \omega_c^{\,2})^2}\right]$$

$$= \frac{A^2E^2}{\pi T_1}\left[\frac{\omega^2 + \omega_c^{\,2}}{(\omega^2 - \omega_c^{\,2})^2}\right] \tag{68}$$

For large ω, the power spectrum falls off at approximately 6 dB per octave. This case is equivalent to noncoherent modulation of the carrier, with bit transitions occurring *at random* with respect to carrier zero crossings and peaks; this is illustrated in Figure 6-2, case C. Intuitively, this would be expected to be an average bandwidth case.

Figure 6-3 illustrates the relative power spectra for cases A, B, and C, above. It appears that more power is concentrated in the frequency band immediately about ω_c for case A than for either of the other two cases. Thus, case A appears to be the minimum bandwidth case. Likewise, case B appears to be the maximum bandwidth case, while case C appears to be between case A and case B.

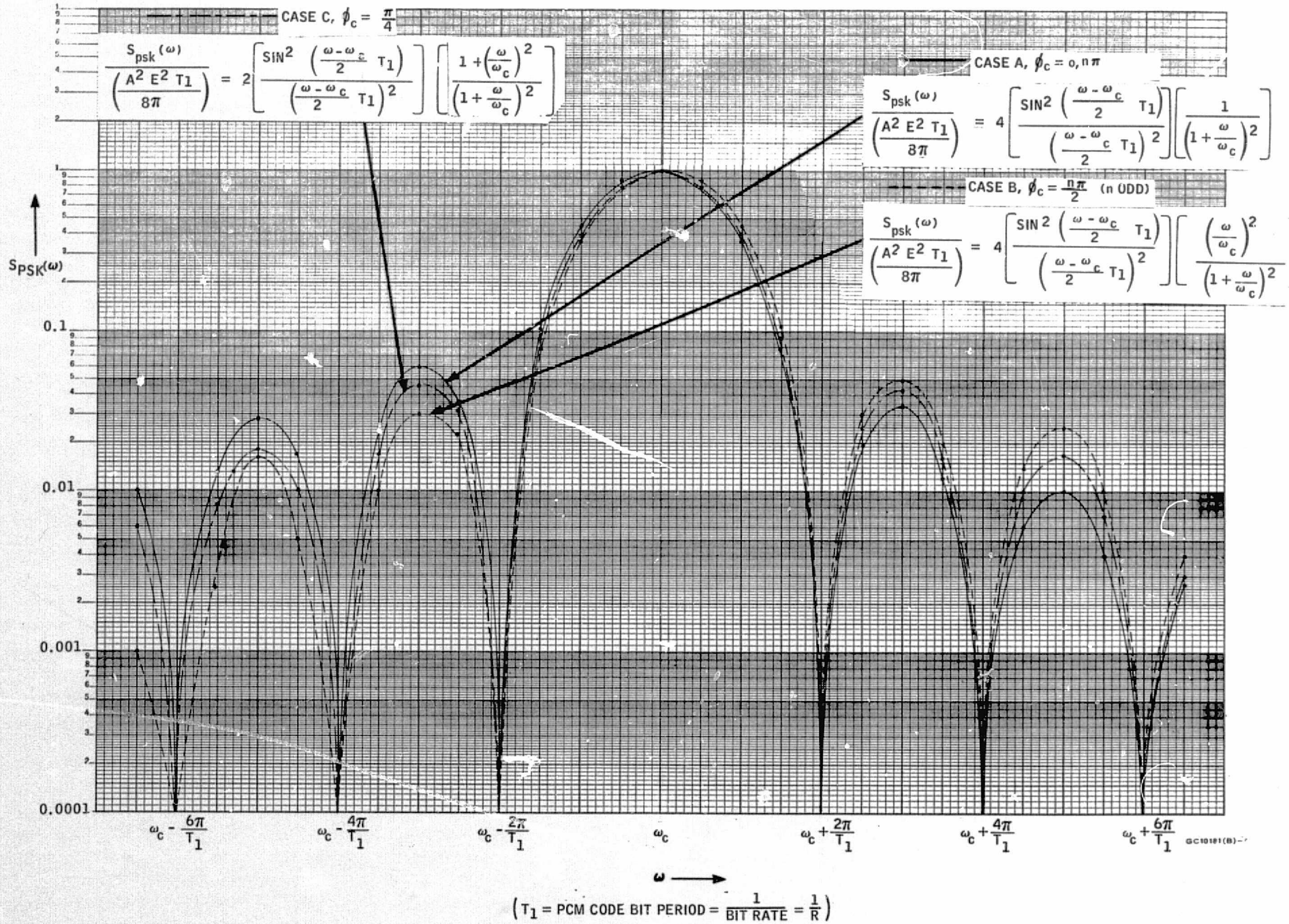


Figure 6-3 Power Density Spectra of PSK Signals

APPENDIX A

CALCULATION OF THE AUTOCORRELATION FUNCTION FOR RANDOM NRZ CODES

The members (sample functions) of an ensemble of random NRZ codes are illustrated in Figure A-1. In order to calculate the ensemble-average autocorrelation function, the following initial assumptions are made.

A. The process is at least wide-sense stationary (i.e., its autocorrelation function is dependent only on the time, τ, between successive samples and not on the actual sampling times t_1 and t_2).

B. The probability of occurrence of a "one" is equal to the probability of occurrence of a "zero," or

$$P(X_{t1}=E) = P(X_{t1}=-E) = P(X_{t2}=E) = P(X_{t2}=-E) = 1/2 \qquad \text{(A-1)}$$

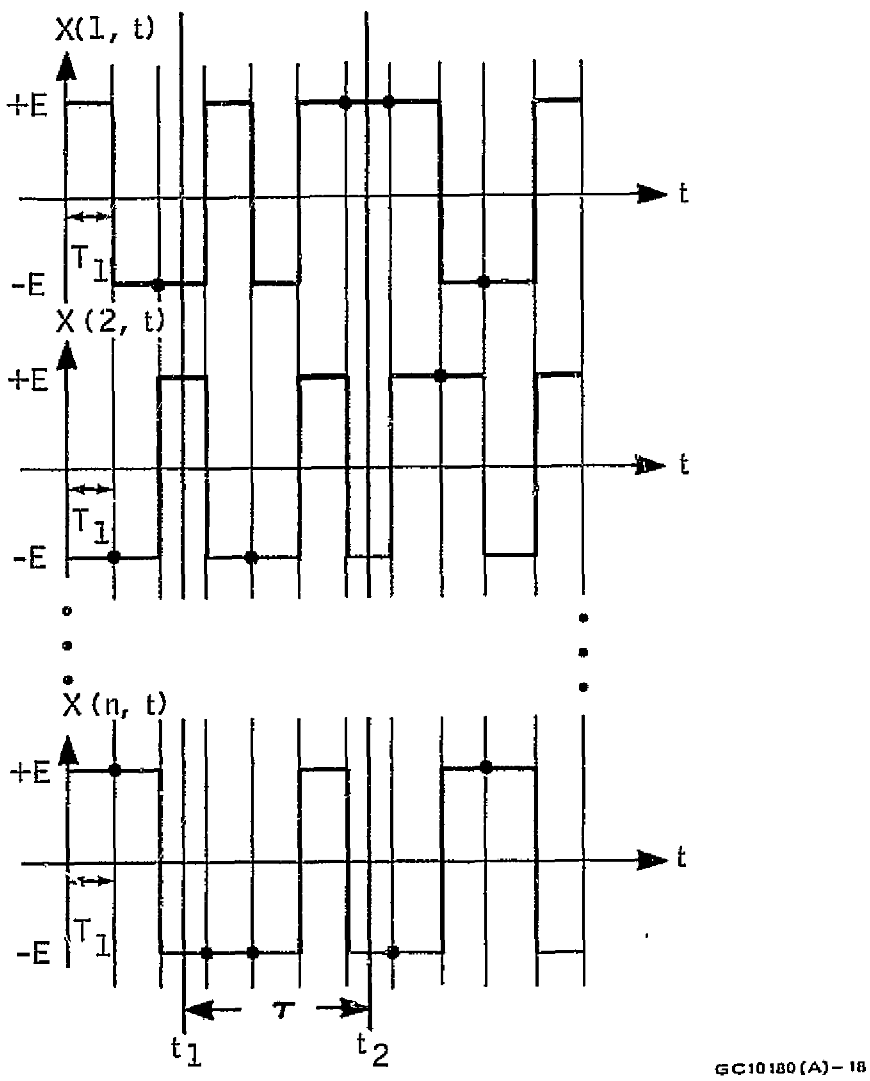
The ensemble-average autocorrelation function is equal to the expected value (mean) of the product of the samples X_{t1} and X_{t2} of each member of the ensemble, or

$$R_X(\tau) = E[X_{t1}\ X_{t2}] \qquad \text{(A-2)}$$

In general, the expected value of the product of two random variables X_{t1} and X_{t2} is given by (reference 5):

$$E[X_{t1}X_{t2}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{X}_{t1}\mathbb{X}_{t2} P_{X_{t1},X_{t2}}(\mathbb{X}_{t1},\mathbb{X}_{t2})\, d\mathbb{X}_1\, d\mathbb{X}_2 \qquad \text{(A-3)}$$

where $P_{X_{t1},X_{t2}}(\mathbb{X}_{t1},\mathbb{X}_{t2})$ is the joint probability density function of X_{t1} and X_{t2}.



$$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} = \frac{1}{R}\right)$$

Figure A-1 Ensemble of Sample Functions of a Random NRZ Code

For the case of *discrete* random variables (which take on only a finite number of values), the integrals in the above expression reduce to summations, and the expected value is given by

$$E[X_{t1}X_{t2}] = \sum_{\text{all } i} \sum_{\text{all } j} \mathcal{X}_{t1,i}\mathcal{X}_{t2,j}P\left(X_{t1}=\mathcal{X}_{t1,i}, X_{t2}=\mathcal{X}_{t2,j}\right) \qquad \text{(A-4)}$$

where the $\mathcal{X}_{t1,i}$ are the possible values that X_{t1} can assume,

the $\mathcal{X}_{t2,j}$ are the possible values that X_{t2} can assume,

and $P\left(X_{t1}=\mathcal{X}_{t1,i}, X_{t2}=\mathcal{X}_{t2,j}\right)$ is the probability of joint occurrence of $\mathcal{X}_{t1,i}$ and $\mathcal{X}_{t2,i}$.

The possible values of each of the discrete random variables X_{t1} and X_{t2}, for the case of the random NRZ PCM code, are +E volts and -E volts. The autocorrelation function, then, is

$$\begin{aligned} R_X(\tau) &= E[X_{t1}X_{t2}] \\ &= (E)(E)P(X_{t1}=E, X_{t2}=E) + (E)(-E)P(X_{t1}=E, X_{t2}=-E) \\ &\quad + (-E)(E)P(X_{t1}=-E, X_{t2}=E) + (-E)(-E)P(X_{t1}=-E, X_{t2}=-E) \\ &= E^2P(X_{t1}=E, X_{t2}=E) - E^2P(X_{t1}=E, X_{t2}=-E) \\ &\quad - E^2P(X_{t1}=-E, X_{t2}=E) + E^2P(X_{t1}=-E, X_{t2}=-E) \qquad \text{(A-5)} \end{aligned}$$

But the probability of the joint occurrence of the events $X_{t1} = E$ and $X_{t2} = E$ may be expressed as

$$P(X_{t1}=E, X_{t2}=E) = P(X_{t2}=E|X_{t1}=E)P(X_{t1}=E) \qquad \text{(A-6)}$$

where $P(X_{t2}=E|X_{t1}=E)$ is the probability of occurrence of the event $X_{t2}=E$, provided the event $X_{t1}=E$ has occurred,

and $P(X_{t1}=E)$ is the probability of occurrence of the event $X_{t1}=E$.

The other joint probabilities of equation (A-5) may be similarly expressed. Substitution of the conditional probability expressions of the form of equation (A-6), along with substitution of the expressions of equation (A-1) yields the following:

$$R_X(\tau) = \frac{E^2}{2}P(X_{t2}=E|X_{t1}=E) - \frac{E^2}{2}P(X_{t2}= -E|X_{t1}=E)$$

$$- \frac{E^2}{2}P(X_{t2}=E|X_{t1}= -E) + \frac{E^2}{2} P(X_{t2}= -E|X_{t2}= -E) \qquad \text{(A-7)}$$

Evaluation of the conditional probabilities of equation (A-7) is dependent upon the value of τ, the time difference between samples X_{t1} and X_{t2}. For instance, if $\tau = 0$, then $t_2 = t_1$ and the probability of the event $X_{t2} = E$, given that $X_{t1} = E$, is unity. On the other hand, for $\tau = 0$, the probability of $X_{t2} = -E$, given that $X_{t1} = E$, is zero. A complete evaluation of equation (A-7) for $\tau = 0$ is

$$R_X(0)=\left(\frac{E^2}{2}\right)(1)-\left(\frac{E^2}{2}\right)(0)-\left(\frac{E^2}{2}\right)(0)+\left(\frac{E^2}{2}\right)(1)=E^2 \qquad \text{(A-8)}$$

$R_X(\tau)$ is easily evaluated for $|\tau|\geq T_1$, as the conditional probabilities reduce to simple unconditional probabilities. For $|\tau|\geq T_1$, X_{t2} and X_{t1} are samples of different bit periods, and the value of X_{t2} is *independent* of the value of X_{t1}. Therefore,

$$P(X_{t2}=E|X_{t1}=E) = P(X_{t2}=E|X_{t1}= -E) = P(X_{t2}=E) = \frac{1}{2} \qquad \text{(A-9)}$$

and $$P(X_{t2}= -E|X_{t1}=E) = P(X_{t2}= -E|X_{t1}= -E) = P(X_{t2}= -E) = \frac{1}{2} \qquad \text{(A-10)}$$

For $|\tau|\geq T_1$, then, the autocorrelation function is

$$R_X(\tau) = \left(\frac{E^2}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{E^2}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{E^2}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{E^2}{2}\right)\left(\frac{1}{2}\right) = 0 \qquad \text{(A-11)}$$

$R_X(\tau)$ may be evaluated for values of $0<|\tau|<T_1$ by introducing additional conditional probabilities. For instance, if $\tau = \frac{T_1}{2}$ the determination of the conditional probability $P(X_{t2}=E|X_{t1}=E)$ is dependent upon whether X_{t1} is in the first or second half of a bit period. If X_{t1} is in the first half of a particular bit period, then X_{t2} must be a sample of that same bit period. Then

$$P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in first half})] = 1 \qquad \text{(A-12)}$$

However, if X_{t1} is in the second half of a bit period, then X_{t2} will be a sample of the next bit period and, therefore, is independent of X_{t1}. Then

$$P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{in second half})] = P(X_{t2}=E) = \frac{1}{2} \qquad \text{(A-13)}$$

The total conditional probability $P(X_{t2}=E|X_{t1}=E)$ is given by

$$\begin{aligned} P(X_{t2}=E|X_{t1}=E) &= P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in first half})]\, P(X_{t1} \text{ in first half}) \\ &+ P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in second half})]\, P(X_{t1} \text{ in second half}) \\ &= (1)(\tfrac{1}{2})+(\tfrac{1}{2})(\tfrac{1}{2}) = \frac{3}{4} \end{aligned} \qquad \text{(A-14)}$$

The other total conditional probabilities may be similarly computed for $\tau = \frac{T_1}{2}$:

$$\begin{aligned} P(X_{t2}= -E|X_{t1}=E) &= P[(X_{t2}= -E|X_{t1}=E)|(X_{t1} \text{ in first half})]\, P(X_{t1} \text{ in first half}) \\ &+ P[(X_{t2}= -E|X_{t1}=E)|(X_{t1} \text{ in second half})]\, P(X_{t1} \text{ in second half}) \\ &=(0)(\tfrac{1}{2})+(\tfrac{1}{2})(\tfrac{1}{2}) = \frac{1}{4} \end{aligned} \qquad \text{(A-15)}$$

$$P(X_{t2}=E|X_{t1}=-E) = P[(X_{t2}=E|X_{t1}=-E)|(X_{t1} \text{ in first half})]\,P(X_{t1} \text{ in first half})$$

$$+\ P[(X_{t2}=E|X_{t1}=-E)|(X_{t1} \text{ in second half})]\,P(X_{t1} \text{ in second half})$$

$$= (0)\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} \qquad \text{(A-16)}$$

$$P(X_{t2}=-E|X_{t1}=-E) = P[(X_{t2}=-E|X_{t1}=-E)|(X_{t1} \text{ in first half})]\,P(X_{t1} \text{ in first half})$$

$$+\ P[(X_{t2}=-E|X_{t1}=-E)|(X_{t1} \text{ in second half})]\,P(X_{t1} \text{ in second half})$$

$$= (1)\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{3}{4} \qquad \text{(A-17)}$$

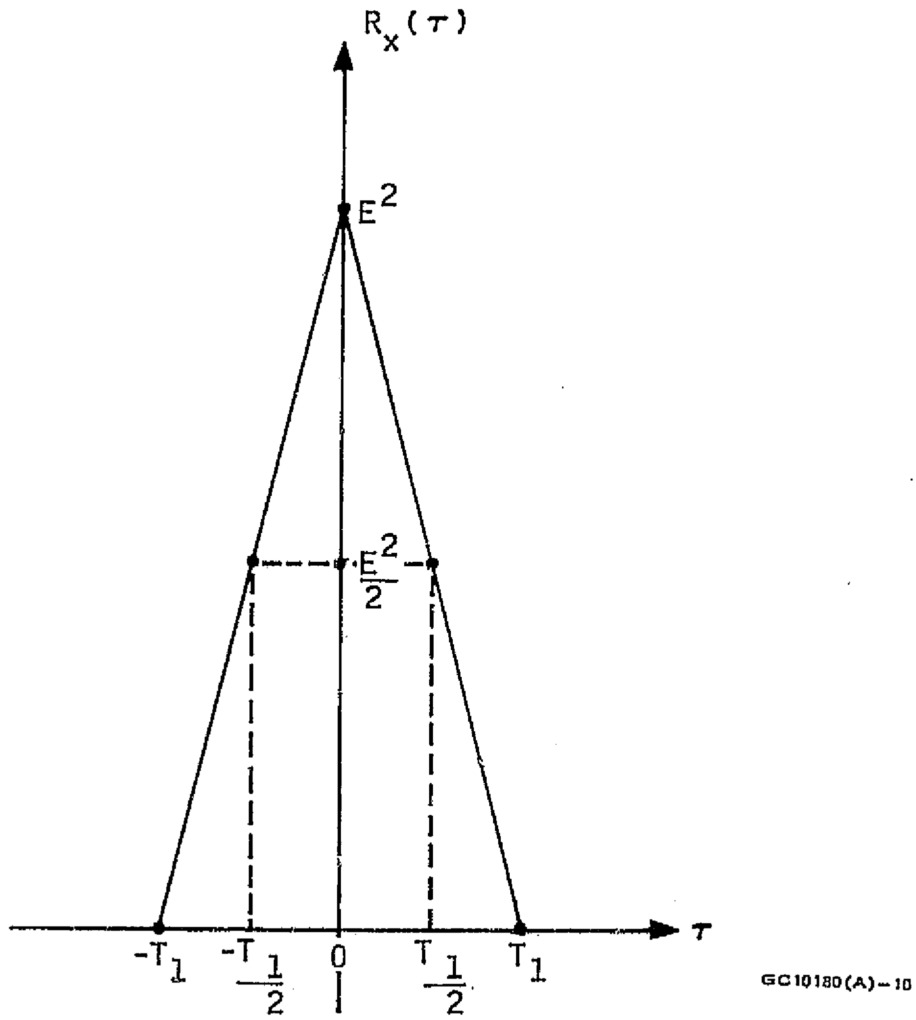
A complete evaluation of $R_X(\tau)$ for $\tau = \frac{T_1}{2}$, then, is obtained by substituting equations (A-14) through (A-17) into equation (A-7):

$$R_X\left(\frac{T_1}{2}\right) = \left(\frac{E^2}{2}\right)\left(\frac{3}{4}\right) - \left(\frac{E^2}{2}\right)\left(\frac{1}{4}\right) - \left(\frac{E^2}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{E^2}{2}\right)\left(\frac{3}{4}\right) = \frac{E^2}{2} \qquad \text{(A-18)}$$

Using a method similar to that followed above, it becomes simple to show that

$$R_X\left(-\frac{T_1}{2}\right) = \frac{E^2}{2} \qquad \text{(A-19)}$$

A plot of $R_X(\tau)$ for all τ is shown in Figure A-2.



$$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} = \frac{1}{R}\right)$$

Figure A-2 Ensemble-Average Autocorrelation Function for a Random NRZ PCM Code

APPENDIX B

BANDWIDTH REQUIREMENTS FOR NRZ CODES

The power density spectrum of an NRZ PCM code with a random bit pattern was derived in Paragraph 3.2. This spectrum is given by

$$S_{NRZ}(\omega) = \frac{E^2 T_1}{2\pi} \left(\frac{\text{Sin } \frac{T_1}{2}}{\frac{\omega T_1}{2}} \right)^2 \tag{B-1}$$

The total power of the NRZ code may be determined by integrating the above expression over all values of ω, or

$$P_T = \int_{-\infty}^{\infty} S_{NRZ}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{E^2 T_1}{2\pi} \left(\frac{\text{Sin } \frac{\omega T_1}{2}}{\frac{\omega T_1}{2}} \right)^2 d\omega$$

$$= \frac{E^2 T_1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\text{Sin } \frac{\omega T_1}{2}}{\frac{\omega T_1}{2}} \right)^2 d\omega \tag{B-2}$$

Equation (B-2) may be put in a standard integral form

$\left(\int \frac{\text{Sin}^2 x}{x^2}\, dx \right)$ by letting $x = \frac{\omega T_1}{2}$. Then,

$$dx = \frac{T_1}{2}\, d\omega$$

or

$$d\omega = \frac{2}{T_1}\, dx \tag{B-3}$$

Then

$$P_T = \frac{E^2 T_1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\text{Sin } x}{x}\right)^2 \frac{2}{T_1}\, dx = \frac{E^2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\text{Sin } x}{x}\right)^2 dx \qquad \text{(B-4)}$$

Since $\left(\frac{\text{Sin } x}{x}\right)^2$ is an even function, the above may be written

$$P_T = \frac{2E^2}{\pi} \int_{0}^{\infty} \left(\frac{\text{Sin } x}{x}\right)^2 dx = \left(\frac{2E^2}{\pi}\right)\left(\frac{\pi}{2}\right) = E^2 \qquad \text{(B-5)}$$

It can be seen, then, that the total power contained in the NRZ code is equal to the peak value of the autocorrelation function for that code (Figure 3-1) or, simply, to the square of the code amplitude (Figure A-1).

It is of interest to determine the percentage of total power present in any finite frequency band centered at $\omega = 0$. The total power present in any such frequency band is given by

$$P_{2\omega_B} = \int_{-\omega_B}^{+\omega_B} S_{NRZ}(\omega)\, d\omega = \frac{E^2 T_1}{2\pi} \int_{-\omega_B}^{+\omega_B} \left(\frac{\text{Sin } \frac{\omega T_1}{2}}{\frac{\omega T_1}{2}}\right)^2 d\omega \qquad \text{(B-6)}$$

where $\pm\omega_B$ are the limits of the frequency band,. as shown in Figure B-1. By substituting

$x = \frac{\omega T_1}{2}$, as before, equation (B-6) may be reduced to

$$P_{2\omega_B} = \frac{E^2}{\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \left(\frac{\text{Sin } x}{x}\right)^2 dx \qquad \text{(B-7)}$$

But

$$\text{Sin}^2 x = \frac{1}{2}(1-\text{Cos}\, 2x) \qquad \text{(B-8)}$$

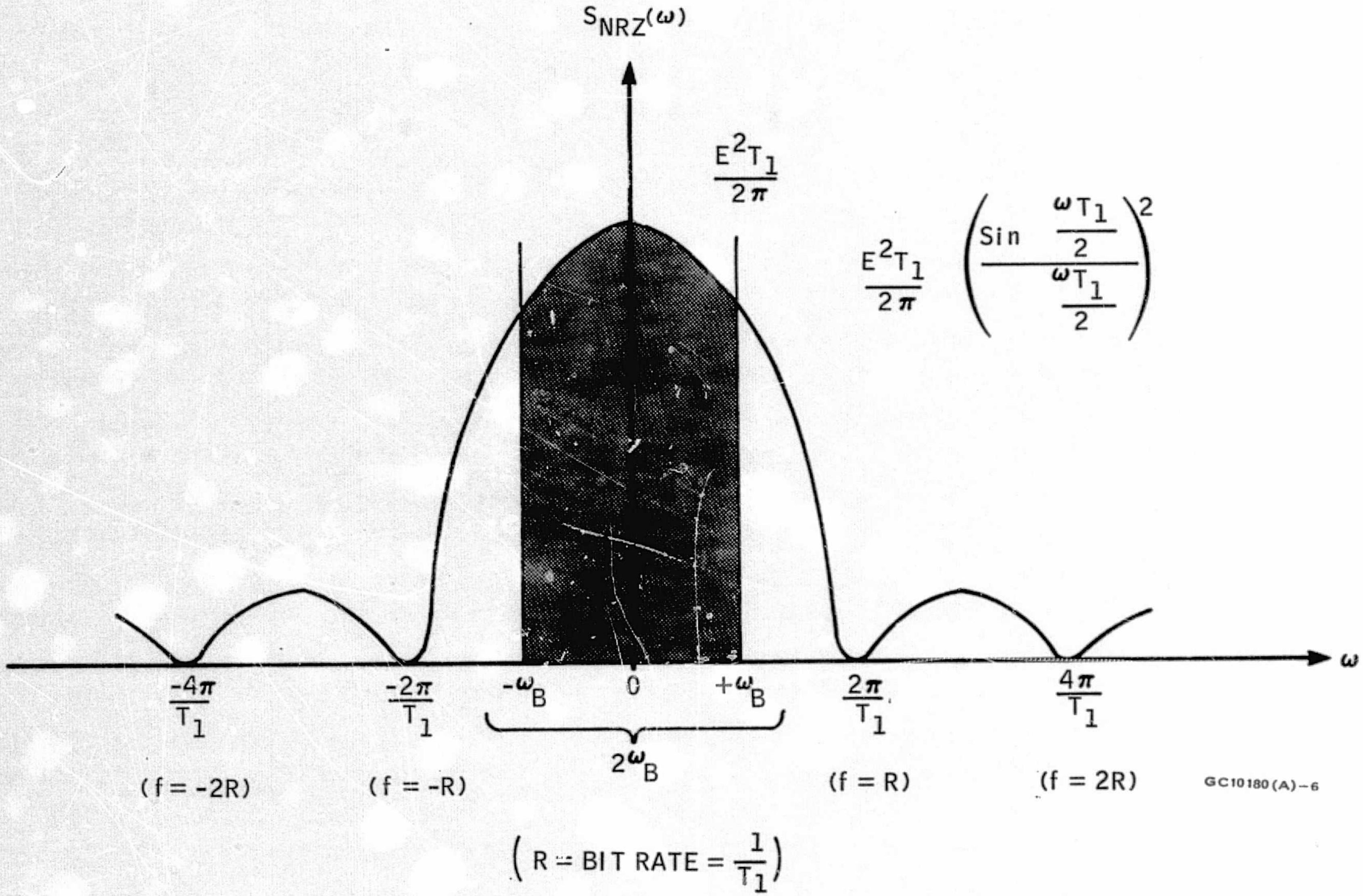


Figure B-1 Power Density Spectrum of an NRZ Code (Random Bit Pattern)

So

$$P_{2\omega_B} = \frac{E^2}{2\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{1-\cos 2x}{x^2}\, dx = \frac{E^2}{2\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{dx}{2} - \frac{E^2}{2\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\cos 2x}{x^2}\, dx$$

$$= \frac{E^2}{2\pi} \left[-\frac{1}{x}\right]_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} - \frac{E^2}{2\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\cos 2x}{x^2}\, dx$$

$$= -\frac{2E^2}{\omega_B T_1 \pi} - \frac{E^2}{2\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\cos 2x}{x^2}\, dx \qquad \text{(B-9)}$$

The second term of equation (B-9) can be simplified by using the following standard integral relationship (Reference 4):

$$\int \frac{\cos ax}{x^m}\, dx = -\frac{1}{m-1} \frac{\cos ax}{x^{m-1}} - \frac{a}{m-1} \int \frac{\sin ax\, dx}{x^{m-1}} \qquad \text{(B-10)}$$

or

$$\int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\cos 2x}{x^2}\, dx = - \left[\frac{\cos 2x}{x}\right]_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} - 2 \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\sin\ 2x\ dx}{x}$$

$$= - \frac{4\cos(\omega_B T_1)}{\omega_B T_1} - 2 \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\sin\ 2x\ d(2x)}{2x} \qquad \text{(B-11)}$$

Substituting equation (B-11) into equation (B-9) gives

$$P_{2\omega_B} = - \frac{2E^2}{\omega_B T_1 \pi} + \frac{2E^2 \cos(\omega_B T_1)}{\omega_B T_1 \pi} + \frac{E^2}{\pi} \int_{-\frac{\omega_B T_1}{2}}^{\frac{\omega_B T_1}{2}} \frac{\sin\ 2x\ d(2x)}{2x} \qquad \text{(B-12)}$$

The first two terms of the above expression may be combined trigonometrically as follows:

$$- \frac{2E^2}{\omega_B T_1 \pi} + \frac{2E^2 \cos(\omega_B T_1)}{\omega_B T_1 \pi} = - \frac{4E^2}{\omega_B T_1 \pi} \left(\frac{1}{2}\right) [1 - \cos(\omega_B T_1)] = - \frac{4E^2}{\omega_B T_1 \pi} \operatorname{Sin}^2 \left(\frac{\omega_B T_1}{2}\right) \qquad \text{(B-13)}$$

The third term of equation (B-12) may be simplified by letting $\xi = 2X$. Equation (B-12) may now be written

$$P_{2\omega_B} = - \frac{4E^2}{\omega_B T_1 \pi} \operatorname{Sin}^2 \left(\frac{\omega_B T_1}{2}\right) + \frac{2E^2}{\pi} \int_0^{\omega_B T_1} \frac{\operatorname{Sin}\xi\, d\xi}{\xi} \qquad \text{(B-14)}$$

The integral in the preceding expression is recognizable as being the well-known function $\mathrm{Si}(\omega_B T_1)$. This function is tabulated in various mathematical handbooks (Ref. 4).

Equation (B-14) may be expressed as

$$P_{2\omega_B} = -\frac{4E^2}{\omega_B T_1 \pi} \mathrm{Sin}^2\left(\frac{\omega_B T_1}{2}\right) + \frac{2E^2}{\pi} \mathrm{Si}(\omega_B T_1) \qquad \text{(B-15)}$$

The percentage of total power P_T present between $-\omega_B$ and $+\omega_B$ is given by

$$\frac{P_{2\omega_B}}{P_T} \times 100 = \frac{P_{2\omega_B}}{E^2} \times 100 = -\frac{400}{\omega_B T_1 \pi} \mathrm{Sin}^2\left(\frac{\omega_B T_1}{2}\right) + \frac{200}{\pi} \mathrm{Si}(\omega_B T_1) \qquad \text{(B-16)}$$

Letting $\omega_B = \dfrac{2\pi}{T_1}$, it is possible to calculate the percentage of P_T present between the first nulls of the power density spectrum. This corresponds to a total bandwidth ($-\omega_B$ to $+\omega_B$) equal to twice the bit rate R.

$$\frac{P_{2\left(\frac{2\pi}{P_T}\right)}}{P_T} \times 100 = \cancelto{0}{\frac{-400}{2\pi^2} \mathrm{Sin}^2(\pi)} + \frac{200}{\pi} \mathrm{Si}(2\pi)$$

$$= \left(\frac{200}{\pi}\right)(1.42)$$

$$= 90.5\%$$

Values of $\dfrac{P_{2\omega_B}}{P_T} \times 100$ are given in Table B-1 for various values of ω_B.

The two-sided bandwidth corresponding to any value of ω_B is equal to $2\omega_B$. Inspection of the values listed in Table B-1 shows that increasing the two-sided bandwidth past three times the bit rate has very little effect on the percentage of total power contained in that frequency band.

TABLE B-1

PERCENTAGE OF TOTAL POWER OF AN NRZ CODE CONTAINED IN THE FREQUENCY BAND EXTENDING FROM $-\omega_B$ TO $+\omega_B$

ω_B	TWO-SIDED BANDWIDTH $(2\omega_B)$		$\frac{P_{2\omega_B}}{P_T} \times 100$ (PERCENT)
$\frac{\pi}{2T_1}$	$\frac{\pi}{T_1}$	($\frac{1}{2}$ X Bit Rate)	47.9
$\frac{\pi}{T_1}$	$\frac{2\pi}{T_1}$	(Bit Rate)	77.4
$\frac{3\pi}{2T_1}$	$\frac{3\pi}{T_1}$	($1\frac{1}{2}$ X Bit Rate)	88.9
$\frac{2\pi}{T_1}$	$\frac{4\pi}{T_1}$	(2 X Bit Rate)	90.4
$\frac{5\pi}{2T_1}$	$\frac{5\pi}{T_1}$	($2\frac{1}{2}$ X Bit Rate)	90.5
$\frac{3\pi}{T_1}$	$\frac{6\pi}{T_1}$	(3 X Bit Rate)	93.5
$\frac{7\pi}{2T_1}$	$\frac{7\pi}{T_1}$	($3\frac{1}{2}$ X Bit Rate)	94.7
$\frac{4\pi}{T_1}$	$\frac{8\pi}{T_1}$	(4 X Bit Rate)	95.0

APPENDIX C

GENERATION OF SPLIT-PHASE CODES

A split-phase PCM code can be generated by combining an NRZ code in an appropriate manner with a clock. The period of the clock, as shown in Figure C-1, is equal to the bit period of the NRZ code. It can be seen that if the clock and the NRZ code are combined according to the exclusive "or" operation (modulo 2 addition), and the result inverted, the split-phase signal will result. The exclusive "or" operation is defined by Table C-1. Thus, whenever

TABLE C-1

THE EXCLUSIVE "OR" OPERATION

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

the NRZ code assumes the "one" state, the exclusive "or" combination of that state with the alternating clock produces an *inverted* replica of the clock; the inverse of this result is a replica of the clock. When the NRZ code assumes the "zero" state, the inverted exclusive "or" combination yields an inverted replica of the clock.

The logic levels of the various codes presented in Figure C-1 actually are represented by discrete voltage levels. For a system employing positive logic (a "one" is represented by a positive voltage level and a "zero" is represented by a negative voltage

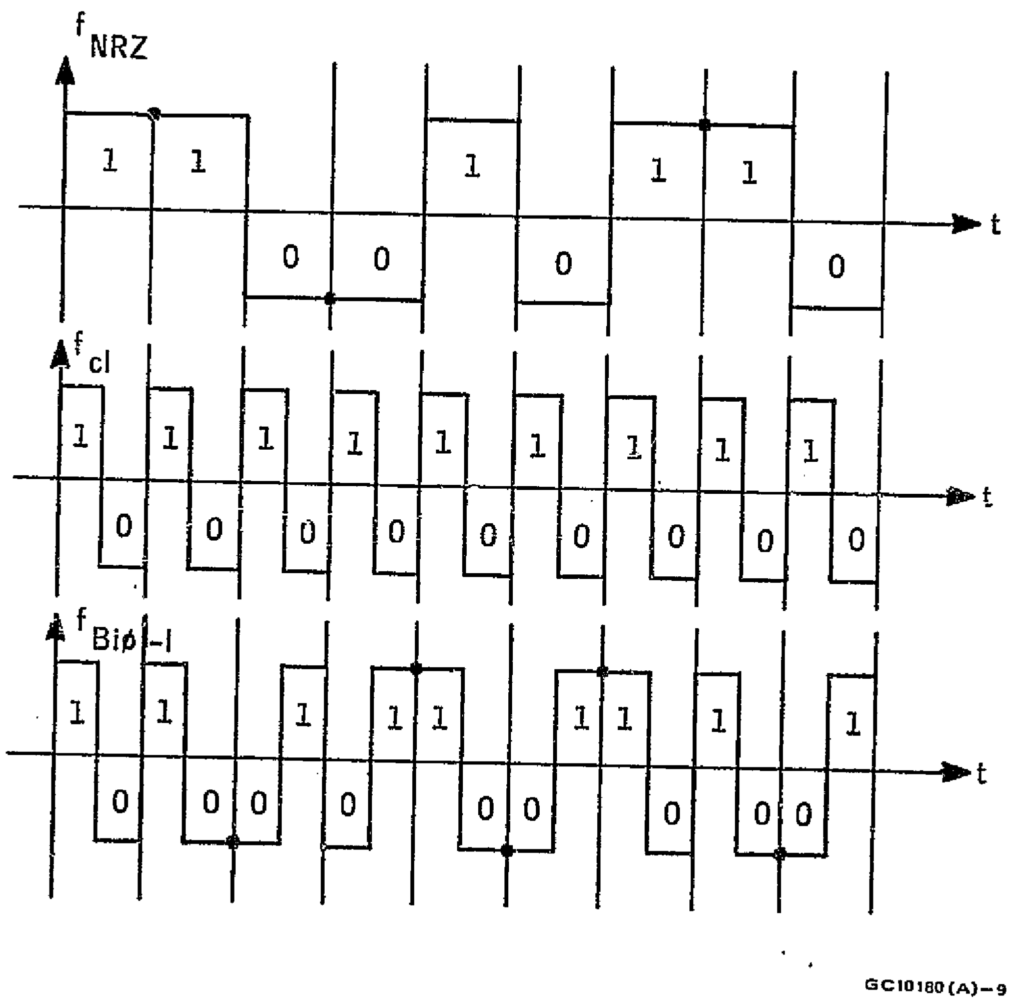
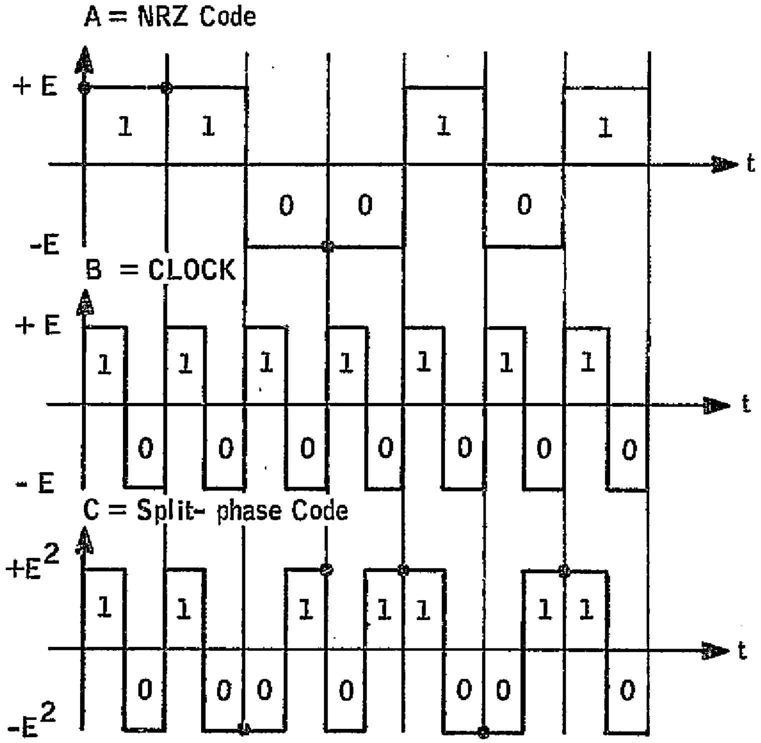
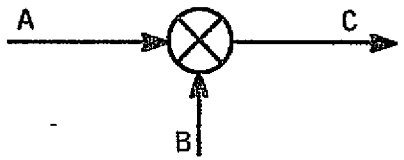


Figure C-1 Generation of a Split-Phase PCM Code level), it may be seen that the exclusive "or" and invert operation is equivalent to direct analog multiplication. That is, two positive voltage levels ("ones") multiplied together yield another positive voltage level (a "one"), and this is the same as obtained when combining two "ones" according to the exclusive "or" operation and inverting the result (Table C-1). Similarly, multiplication of two negative voltage levels ("zeros") yields a positive voltage level (a "one"), and this is the same result obtained when combining two "zeros" according to the exclusive "or" operation and then inverting. It is easily seen that analog multiplication of unlike voltage levels (a "one" and a "zero") gives the same result (a "zero") as when combining a "one" and a "zero" according to Table C-1 and then inverting.

Figure C-2 summarizes the preceding discussion regarding formation of a split-phase code by combining an NRZ code and a clock. It is evident that three completely equivalent schemes may be used in implementing the digital multiplier (i.e., the "invert" operation may be applied to either the NRZ code, the clock, or to the exclusive "or" combination of the NRZ code and the clock).

For a system employing negative logic (a "one" is represented by a negative voltage level and a "zero" by a positive voltage level), a split-phase code may still be generated by analog multiplication of an NRZ code and a clock. However, the equivalent logical operations involved are less complex. In fact, only a single operation (exclusive "or") is required. Figure C-3 summarizes the formation of a split-phase PCM code for a system employing negative logic.



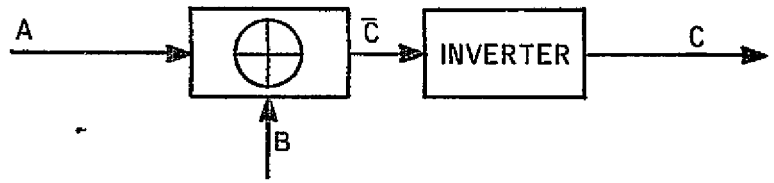


A	B	C
-E	-E	$+E^2$
-E	+E	$-E^2$
+E	-E	$-E^2$
+E	+E	$+E^2$

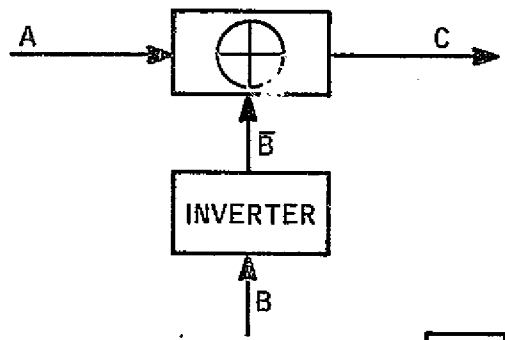
$+E, +E^2 \longleftrightarrow$ "ONE"

$-E, -E^2 \longleftrightarrow$ "ZERO"

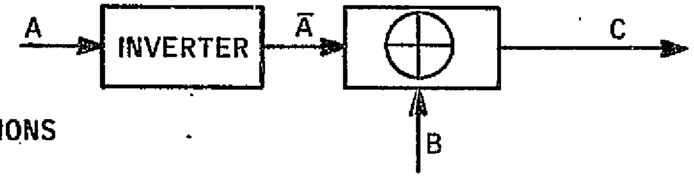
ANALOG OPERATIONS



A	B	$\overline{C}$	C
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1



A		$\overline{B}$	C
0	0	1	1
0	1	0	0
1	0	1	0
1	1	0	1



A	$\overline{A}$	B	C
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1

DIGITAL OPERATIONS

Figure C-2 Generation of a Split-Phase Code for a System Employing Positive Logic

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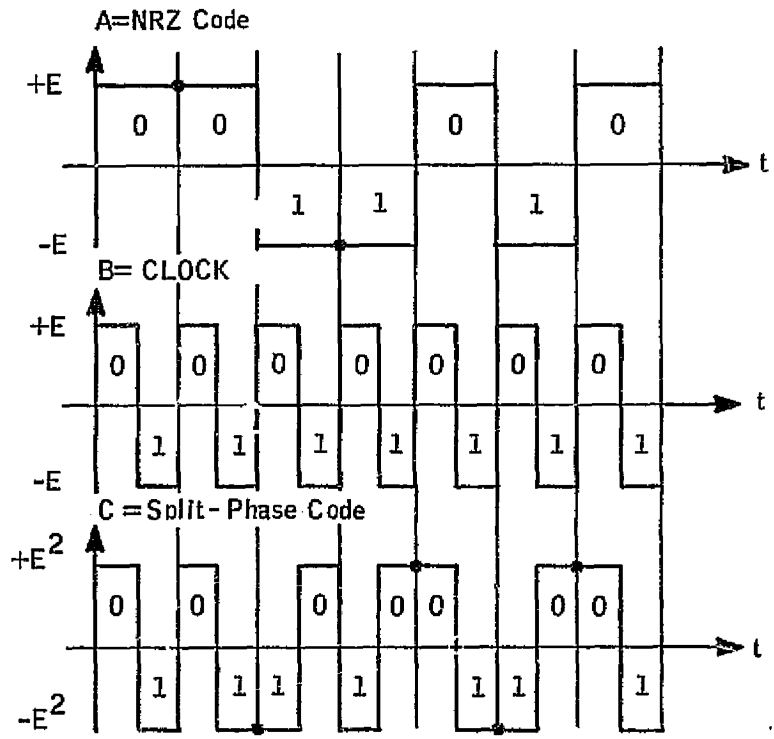
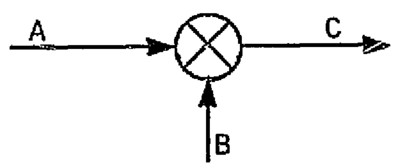


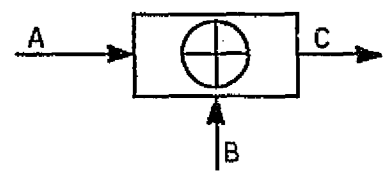


A	B	C
+E	+E	$+E^2$
+E	-E	$-E^2$
-E	+E	$-E^2$
-E	-E	$+E^2$

ANALOG OPERATIONS

$+E, +E^2 \leftrightarrow$ "ZERO"

$-E, -E^2 \leftrightarrow$ "ONE"



A	B	C
0	0	0
0	1	1
1	0	1
1	1	0

DIGITAL OPERATIONS

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Figure C-3 Generation of a Split-Phase Code for a System Employing Negative Logic

APPENDIX D

CALCULATION OF THE AUTOCORRELATION FUNCTION FOR RANDOM SPLIT-PHASE CODES

An ensemble of sample functions of a split-phase PCM code with random bit transitions is illustrated in Figure D-1. In order to calculate the ensemble—average autocorrelation function, the same initial assumptions are made as for the calculations contained in Appendix A (NRZ code), as follows:

A. The process is at least wide-sense stationary.

B. $P(X_{t1}=E) = P(X_{t1}= -E) = P(X_{t2}=E) = P(X_{t2}= -E) = \frac{1}{2}$ (D-1)

Following the procedure outlined in equations (A-2) through (A-7), it is evident that, as for the NRZ code,

$$R_X(\tau) = \frac{E^2}{2}P(X_{t2}=E|X_{t1}=E) - \frac{E^2}{2}P(X_{t2}= -E|X_{t1}=E)$$

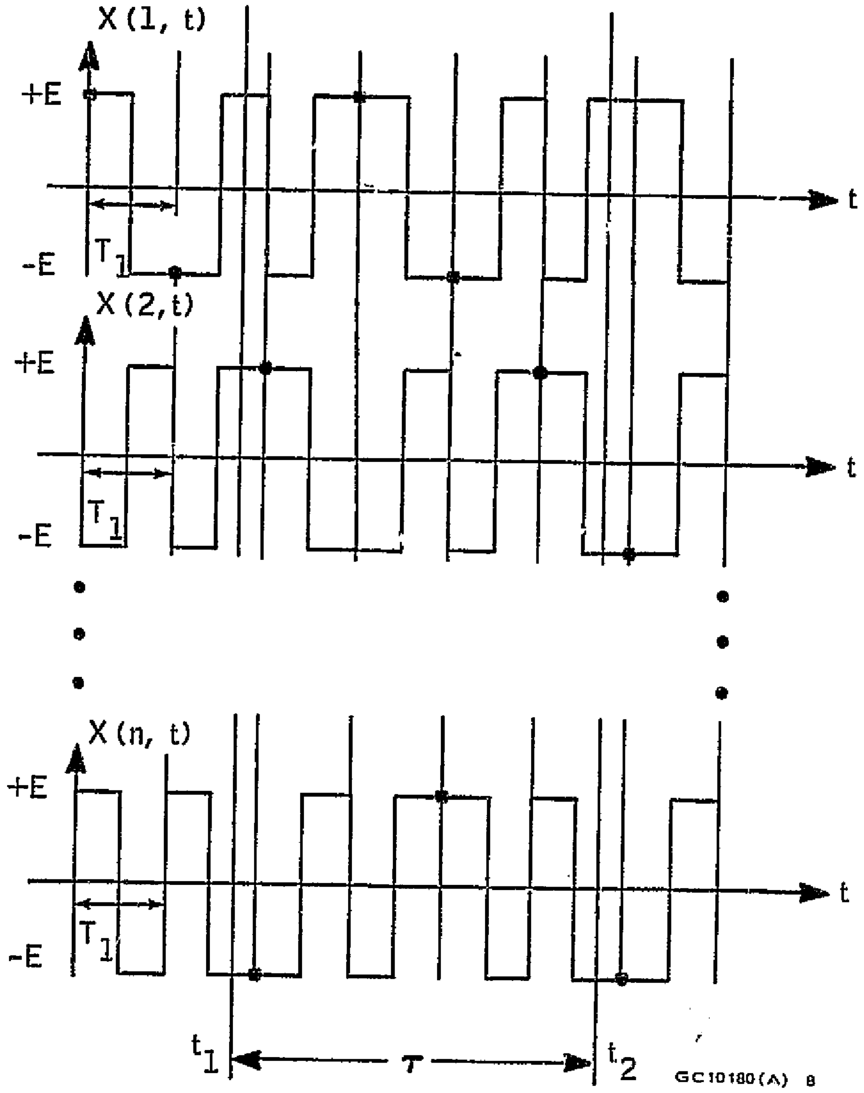
$$- \frac{E^2}{2}P(X_{t2}=E|X_{t1}= -E) + \frac{E^2}{2}P(X_{t2}= -E|X_{t1}= -E). \qquad \text{(D-2)}$$

Evaluation of the conditional probabilities is somewhat more complicated, however, for the split-phase code. For $\tau = 0$,

$$P(X_{t2}=E|X_{t1}=E) = P(X_{t2}= -E|X_{t1}= -E) = 1, \qquad \text{(D-3)}$$

$$P(X_{t2}= -E|X_{t1}=E) = P(X_{t2}=E|X_{t1}= -E) = 0, \qquad \text{(D-4)}$$

and $$R_X(0) = \left(\frac{E^2}{2}\right)(1) - \left(\frac{E^2}{2}\right)(0) - \left(\frac{E^2}{2}\right)(0) + \left(\frac{E^2}{2}\right)(1) = E^2 \qquad \text{(D-5)}$$



$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} = \frac{1}{R} \right)$

Figure D-1 Ensemble of Sample Functions of a Random Split-Phase Code

For $\tau \geq T_1$,

$$P(X_{t2}=E|X_{t1}=E) = P(X_{t2}=E|X_{t1}= -E) = P(X_{t2}=E) = \frac{1}{2} \qquad \text{(D-6)}$$

$$P(X_{t2}= -E|X_{t1}=E) = P(X_{t2}= -E|X_{t1}= -E) = P(X_{t2}= -E) = \frac{1}{2} \qquad \text{(D-7)}$$

and $$R_X(\tau) = \left(\frac{E^2}{2}\right)(\tfrac{1}{2}) - \left(\frac{E^2}{2}\right)(\tfrac{1}{2}) - \left(\frac{E^2}{2}\right)(\tfrac{1}{2}) + \left(\frac{E^2}{2}\right)(\tfrac{1}{2}) = 0 \qquad \text{(D-8)}$$

In order to evaluate $R_X(\tau)$ for $0<|\tau|<T_1$, it is again necessary to introduce additional conditional probabilities. For instance, if $\tau = \frac{T_1}{2}$, then $P(X_{t2}=E|X_{t1}=E)$ is dependent upon whether X_{t1} is in the first or second half of a bit period. If X_{t1} is in the first half of a bit period, then a sample of the second half of that same bit period must be of opposite polarity. Hence,

$$P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in first half})] = 0 \qquad \text{(D-9)}$$

However, if X_{t1} is in the second half of a bit period, the X_{t2} will be a sample function of the next bit period and, therefore, is independent of X_{t1}. Then,

$$P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in second half})] = P(X_{t2}=E) = \frac{1}{2} \qquad \text{(D-10)}$$

the total conditional probabilities for $\tau = \frac{T_1}{2}$ are

$$\begin{aligned} P(X_{t2}=E|X_{t1}=E) &= P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in first half})]P(X_{t1} \text{ in first half}) \\ &+P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in second half})]P(X_{t1} \text{ in second half}) \\ &=(0)(\tfrac{1}{2})+(\tfrac{1}{2})(\tfrac{1}{2}) = \frac{1}{4} \end{aligned} \qquad \text{(D-11)}$$

$P(X_{t2} = -E | X_{t1} = E) =$

$$P[(X_{t2} = -E | X_{t1} = E) | (X_{t1} \text{ in first half})] P(X_{t1} \text{ in first half})$$

$$+ P[(X_{t2} = -E | X_{t1} = E) | (X_{t1} \text{ in second half})] P(X_{t1} \text{ in second half})$$

$$= (1)(\tfrac{1}{2}) + (\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{3}{4} \tag{D-12}$$

$P(X_{t2} = E | X_{t1} = -E) =$

$$P[(X_{t2} = E | X_{t1} = -E) | (X_{t1} \text{ in first half})] P(X_{t1} \text{ in first half})$$

$$+ P[(X_{t2} = E | X_{t1} = -E) | (X_{t1} \text{ in second half})] P(X_{t1} \text{ in second half})$$

$$= (1)(\tfrac{1}{2}) + (\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{3}{4} \tag{D-13}$$

$P(X_{t2} = -E | X_{t1} = -E) =$

$$P[(X_{t2} = -E | X_{t1} = -E) | (X_{t1} \text{ in first half})] P(X_{t1} \text{ in first half})$$

$$+ P[(X_{t2} = -E | X_{t1} = -E) | (X_{t1} \text{ in second half})] P(X_{t1} \text{ in second half})$$

$$= (0)(\tfrac{1}{2}) + (\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{1}{4} \tag{D-14}$$

$R_X(\tau)$, then, for $\tau = \frac{T_1}{2}$ is given by

$$R_X\left(\frac{T_1}{2}\right) = \left(\frac{E^2}{2}\right)(\tfrac{1}{4}) - \left(\frac{E^2}{2}\right)(\tfrac{3}{4}) - \left(\frac{E^2}{2}\right)(\tfrac{3}{4}) + \left(\frac{E^2}{2}\right)(\tfrac{1}{4}) = -\frac{E^2}{2} \tag{D-15}$$

Similarly, it is easily shown that

$$R_X\left(-\frac{T_1}{2}\right) = -\frac{E^2}{2} \tag{D-16}$$

If $\tau = \frac{T_1}{4}$, then $P(X_{t2} = E | X_{t1} = E)$ is dependent upon which *quarter* of a bit period X_{t1} is located. If X_{t1} is in the first quarter of a bit period, then a sample of the second quarter of that bit period must be of the same polarity. So,

$$P[(X_{t2} = E | X_{t1} = E) | (X_{t1} \text{ in first quarter})] = 1 \tag{D-17}$$

However, if X_{t1} is in the second quarter of a bit period, then a sample of the third quarter of that bit period must be of opposite polarity. Then,

$$P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in second quarter})] = 0 \qquad \text{(D-18)}$$

Similarly,

$$P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in third quarter})] = 1 \qquad \text{(D-19)}$$

and $P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in fourth quarter})] = P(X_{t2}=E) = \frac{1}{2}$ (D-20)

The total conditional probabilities for $\tau = \frac{T_1}{4}$, then, are

$$\begin{aligned}
P(X_{t2}=E|X_{t1}=E) = \\
&P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in first quarter})]P(X_{t1} \text{ in first quarter}) \\
&+P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in second quarter})]P(X_{t1} \text{ in second quarter}) \\
&+P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in third quarter})]P(X_{t1} \text{ in third quarter}) \\
&+P[(X_{t2}=E|X_{t1}=E)|(X_{t1} \text{ in fourth quarter})]P(X_{t1} \text{ in fourth quarter}) \\
&= (1)(\tfrac{1}{4})+(0)(\tfrac{1}{4})+(1)(\tfrac{1}{4})+(\tfrac{1}{2})(\tfrac{1}{4}) = \tfrac{5}{8} \qquad \text{(D-21)}
\end{aligned}$$

$$\begin{aligned}
P(X_{t2}= -E|X_{t1}=E) = \\
&P[(X_{t2}= -E|X_{t1}=E)|(X_{t1} \text{ in first quarter})]P(X_{t1} \text{ in first quarter}) \\
&+P[(X_{t2}= -E|X_{t1}=E)|(X_{t1} \text{ in second quarter})]P(X_{t1} \text{ in second quarter}) \\
&+P[(X_{t2}= -E|X_{t1}=E)|(X_{t1} \text{ in third quarter})]P(X_{t1} \text{ in third quarter}) \\
&+P[(X_{t2}= -E|X_{t1}=E)|(X_{t1} \text{ in fourth quarter})]P(X_{t1} \text{ in fourth quarter}) \\
&= (0)(\tfrac{1}{4})+(1)(\tfrac{1}{4})+(0)(\tfrac{1}{4})+(\tfrac{1}{2})(\tfrac{1}{4}) = \tfrac{3}{8} \qquad \text{(D-22)}
\end{aligned}$$

$P(X_{t2}=E|X_{t1}= -E)=$

$P[(X_{t2}=E|X_{t1}= -E)|(X_{t1}$ in first quarter$)]P(X_{t1}$ in first quarter$)$

$+P[(X_{t2}=E|X_{t1}= -E)|(X_{t1}$ in second quarter$)]P(X_{t1}$ in second quarter$)$

$+P[(X_{t2}=E|X_{t1}= -E)|(X_{t1}$ in third quarter$)]P(X_{t1}$ in third quarter$)$

$+P[(X_{t2}=E|X_{t1}= -E)|(X_{t1}$ in fourth quarter$)]P(X_{t1}$ in fourth quarter$)$

$$=(0)\left(\tfrac{1}{4}\right)+(1)\left(\tfrac{1}{4}\right)+(0)\left(\tfrac{1}{4}\right)+\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \tfrac{3}{8} \qquad \text{(D-23)}$$

$P(X_{t2}= -E|X_{t1}= -E)=$

$P[(X_{t2}= -E|X_{t1}= -E)|(X_{t1}$ in first quarter$)]P(X_{t1}$ in first quarter$)$

$+P[(X_{t2}= -E|X_{t1}= -E)|(X_{t1}$ in second quarter$)]P(X_{t1}$ in second quarter$)$

$+P[(X_{t2}= -E|X_{t1}= -E)|(X_{t1}$ in third quarter$)]P(X_{t1}$ in third quarter$)$

$+P[(X_{t2}= -E|X_{t1}= -E)|(X_{t1}$ in fourth quarter$)]P(X_{t1}$ in fourth quarter$)$

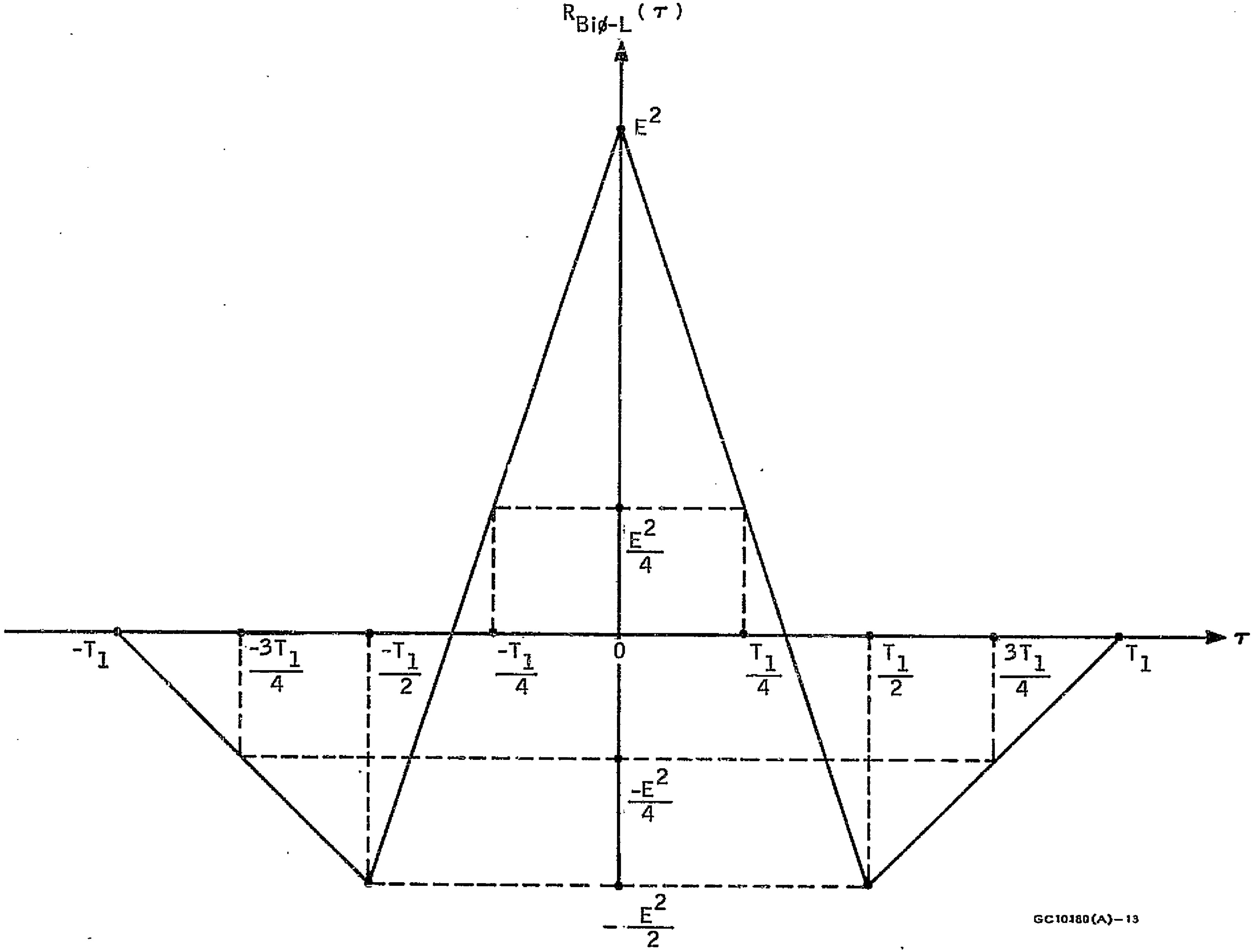
$$=(1)\left(\tfrac{1}{4}\right)+(0)\left(\tfrac{1}{4}\right)+(1)\left(\tfrac{1}{4}\right)+\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \tfrac{5}{8} \qquad \text{(D-24)}$$

Then, $R_X(\tau)$, for $\tau = \frac{T_1}{4}$, is given by

$$R_X\left(\frac{T_1}{4}\right) = \left(\frac{E^2}{2}\right)\left(\tfrac{5}{8}\right)-\left(\frac{E^2}{2}\right)\left(\tfrac{3}{8}\right)-\left(\frac{E^2}{2}\right)\left(\tfrac{3}{8}\right)+\left(\frac{E^2}{2}\right)\left(\tfrac{5}{8}\right) = \frac{E^2}{4} \qquad \text{(D-25)}$$

Again, it is easily shown that $R_X\left(-\frac{T_1}{4}\right) = \frac{E^2}{4}$ (D-25)

Figure D-2 contains a plot of $R_X(\tau)$ for all values of τ.



$\left(T_1 = \text{PCM CODE BIT PERIOD} = \frac{1}{\text{BIT RATE}} = \frac{1}{R} \right)$

Figure D-2 Ensemble-Average Autocorrelation Function for a Random Split-Phase PCM Code

APPENDIX E

BANDWIDTH REQUIREMENTS FOR SPLIT-PHASE CODES

The power density spectrum of a split-phase PCM co.e with a random bit pattern was determined in Paragraph 4.2 and is given by

$$S_{Bi\ \phi-L}(\omega) = \frac{E^2T_1}{2\pi}\left[\frac{\text{Sin}^4\left(\frac{\omega T_1}{4}\right)}{\left(\frac{\omega T_1}{4}\right)^2}\right] \qquad \text{(E-1)}$$

The amount of power present in a frequency band extending from $-\omega_B$ to $+\omega_B$, as illustrated in Figure E-1, is given by

$$P_{2\omega_B} = \int_{-\omega_B}^{+\omega_B} S_{Bi\phi-L}(\omega)\ d\omega$$

$$= \int_{-\omega_B}^{+\omega_B} \frac{E^2T_1}{2\pi}\left[\frac{\text{Sin}^4\left(\frac{\omega T_1}{4}\right)}{\left(\frac{\omega T_1}{4}\right)^2}\right] d\omega$$

$$= \frac{E^2T_1}{2\pi}\int_{-\omega_B}^{+\omega_B} \left[\frac{\text{Sin}^4\left(\frac{\omega T_1}{4}\right)}{\left(\frac{\omega T_1}{4}\right)^2}\right] d\omega \qquad \text{(E-2)}$$

The above expression may be simplified somewhat by letting $x = \frac{\omega T_1}{4}$. Then,

$$dx = \frac{T_1}{4}\ d\omega$$

or

$$d\omega = \frac{4}{T_1}\ dx \qquad \text{(E-3)}$$

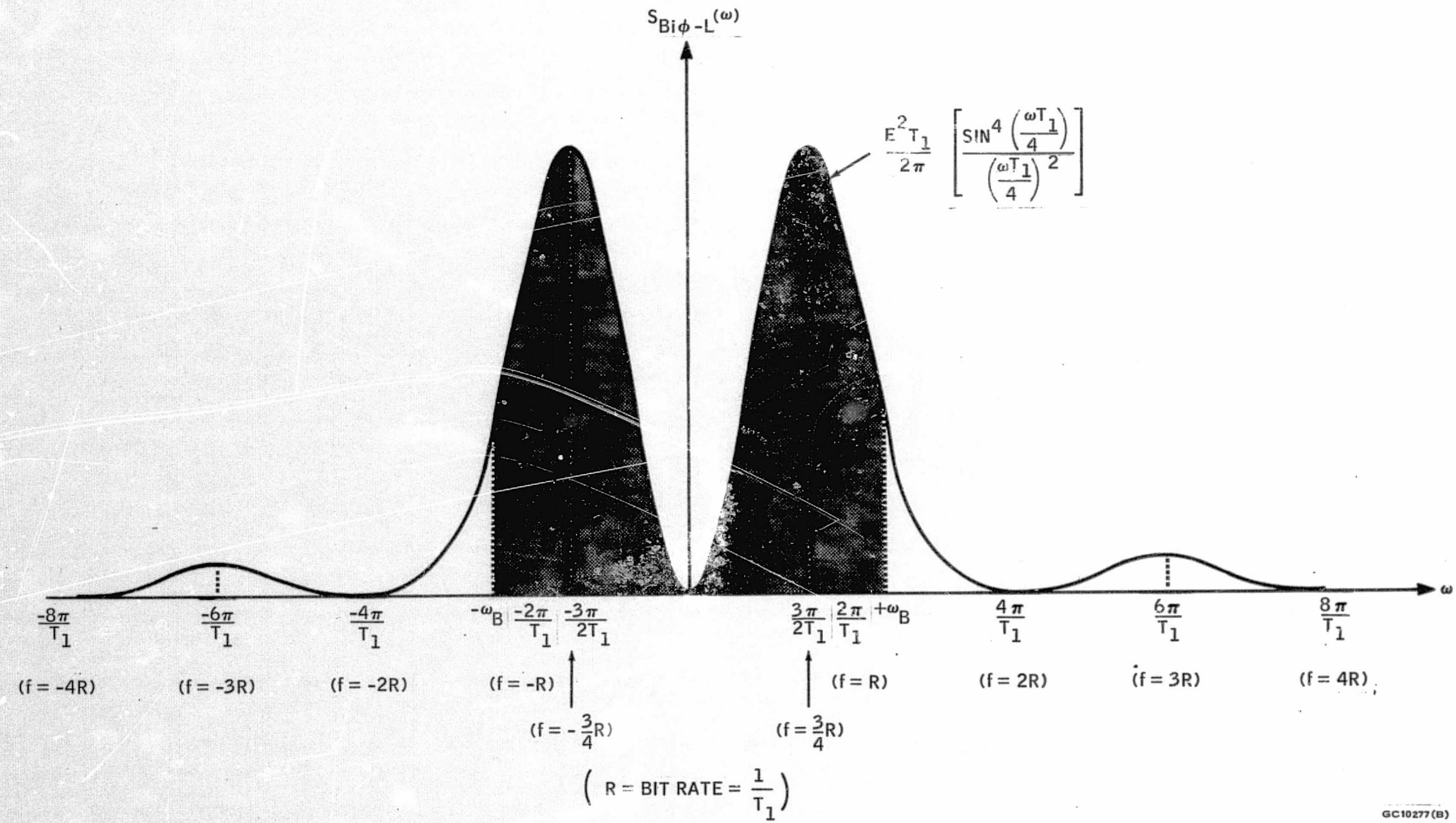


Figure E-1 Power Density Spectrum of a Split-Phase Code (Random Bit Pattern)

Then

$$P_{2\omega_B} = \frac{E^2 T_1}{2\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \left(\frac{\text{Sin}^4 x}{x^2}\right) \frac{4}{T_1}\, dx$$

$$= \frac{2E^2}{\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \frac{\text{Sin}^4 x}{x^2}\, dx \qquad \text{(E-4)}$$

Since $\text{Sin}^2 x = \frac{1}{2}(1-\cos 2x)$, (E-5)

equation (E-4) may be written

$$P_{2\omega_B} = \frac{2E^2}{\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \frac{\left[\frac{1}{2}(1-\cos 2x)\right]^2}{x^2}\, dx$$

$$= \frac{2E^2}{\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \left(\frac{\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x}{x^2}\right) dx$$

$$= \frac{E^2}{2\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \frac{dx}{x^2} - \frac{E^2}{\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \frac{\cos 2x}{x^2}\, dx + \frac{E^2}{2\pi} \int_{\frac{-\omega_B T_1}{4}}^{\frac{+\omega_B T_1}{4}} \frac{\cos^2 2x}{x^2}\, dx \qquad \text{(E-6)}$$

But $\cos^2 2x = \frac{1}{2}(1+\cos 4x)$ (E-7)

So

$$P_{2\omega_B} = \frac{E^2}{2\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{dx}{x^2} - \frac{E^2}{\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\cos 2x}{x^2}\,dx + \frac{E^2}{4\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{dx}{x^2}$$

$$+ \frac{E^2}{4\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\cos 4x}{x^2}\,dx$$

$$= \frac{3E^2}{4\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{dx}{x^2} - \frac{E^2}{\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\cos 2x}{x^2}\,dx + \frac{E^2}{4\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\cos 4x}{x^2}\,dx \quad \text{(E-8)}$$

The first term of equation (E-8) reduces to

$$\frac{3E^2}{4\pi}\left[-\frac{1}{x}\right]_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} = \frac{3E^2}{4\pi}\left[-\frac{4}{\omega_B T_1} - \frac{4}{\omega_B T_1}\right] = -\frac{6E^2}{\omega_B T_1 \pi} \quad \text{(E-9)}$$

and each of the last two terms may be expanded by using

$$\int \frac{\cos\, ax}{m}\,dx = -\frac{1}{m-1}\frac{\cos\, ax}{x^{m-1}} - \frac{a}{m-1}\int \frac{\sin\, ax\, dx}{x^{m-1}} \quad \text{(E-10)}$$

$P_{2\omega_B}$, then, is given by

$$P_{2\omega_B} = -\frac{6E^2}{\omega_B T_1 \pi} - \frac{E^2}{\pi}\left\{-\left[\frac{\cos 2x}{x}\right]_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} - 2\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\sin 2x\,dx}{x}\right\}$$

$$+ \frac{E^2}{4\pi}\left\{-\left[\frac{\cos 4x}{x}\right]_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} - 4\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\sin 4x\,dx}{x}\right\}$$

$$= -\frac{6E^2}{\omega_B T_1 \pi} + \frac{E^2}{\pi}\left[\frac{\cos\left(\frac{\omega_B T_1}{2}\right)}{\frac{\omega_B T_1}{4}} - \frac{\cos\left(-\frac{\omega_B T_1}{2}\right)}{-\frac{\omega_B T_1}{4}}\right]$$

$$- \frac{E^2}{4\pi}\left[\frac{\cos(\omega_B T_1)}{\frac{\omega_B T_1}{4}} - \frac{\cos\ (-\omega_B T_1)}{-\frac{\omega_B T_1}{4}}\right] + \frac{2E^2}{\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\sin 2x\,d(2x)}{2x}$$

$$- \frac{E^2}{\pi}\int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\sin\ 4x\ d(4x)}{4x}$$

$$= \frac{-6E^2}{\omega_B T_1 \pi} + \frac{8E^2}{\omega_B T_1 \pi}\cos\left(\frac{\omega_B T_1}{2}\right) - \frac{2E^2}{\omega_B T_1 \pi}\cos(\omega_B T_1)$$

$$+ \frac{2E^2}{\pi}\int_{-\frac{\omega_B T_1}{2}}^{+\frac{\omega_B T_1}{2}} \frac{\sin\xi\, d\xi}{\xi} - \frac{E^2}{\pi}\int_{-\omega_B T_1}^{+\omega_B T_1} \frac{\sin\zeta\, d\zeta}{\zeta} \qquad \text{(E-11)}$$

where $\xi = 2X$ and $\zeta = 4X$. Since $\frac{\text{Sin}X}{X}$ is an even function, equation (E-11) may be written

$$P_{2\omega_B} = -\frac{2E^2}{\omega_B T_1 \pi}\left[\cos(\omega_B T_1) - 4\cos\left(\frac{\omega_B T_1}{2}\right) + 3\right]$$

$$+ \frac{4E^2}{\pi}\int_0^{\frac{+\omega_B T_1}{2}} \frac{\sin\xi\, d\xi}{\xi} - \frac{2E^2}{\pi}\int_0^{+\omega_B T_1} \frac{\sin\zeta\, d\zeta}{\zeta}$$

$$= -\frac{2E^2}{\omega_B T_1 \pi}\left[\cos(\omega_B T_1) - 4\cos\left(\frac{\omega_B T_1}{2}\right) + 3\right]$$

$$+ \frac{2E^2}{\pi}\left[2\,\text{Si}\left(\frac{\omega_B T_1}{2}\right) - \text{Si}\,(\omega_B T_1)\right] \qquad \text{(E-12)}$$

The total power in the split-phase code may be found by letting ω_B approach infinity. Then, the first term of equation (E-12) becomes zero, and each of the Si integrals approaches $\frac{\pi}{2}$. The total power is given by

$$P_T = \frac{2E^2}{\pi}[(2)(\tfrac{\pi}{2}) - \tfrac{\pi}{2}]$$

$$= \frac{2E^2}{\pi}\left(\frac{\pi}{2}\right)$$

$$= E^2 \qquad \text{(E-13)}$$

The total power in the split-phase code, then, is equal to the square of the code amplitude (Figure D-1) or to the peak value of the autocorrelation function (Figure 4-1). This is, of course, the same result as for the NRZ code and was to be expected.

The percentage of total power present in the frequency band between $-\omega_B$ and $+\omega_B$ is

$$\frac{P_{2\omega_B}}{P_T} \times 100 = \frac{P_{2\omega_B}}{E^2} \times 100$$

$$= -\frac{200}{\omega_B T_1 \pi}\left[\cos(\omega_B T_1) - 4\cos\left(\frac{\omega_B T_1}{2}\right) + 3\right]$$

$$+ \frac{200}{\pi}\left[2\ \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right) - \mathrm{Si}(\omega_B T_1)\right] \qquad \text{(E-14)}$$

The first term in the above expression may be simplified considerably by noting that the same term appears in Section 4 (equation 19). Following the procedure outlined in Section 4, equation (E-14) may be reduced to

$$\frac{P_{2\omega_B}}{P_T} \times 100 = \left(-\frac{200}{\omega_B T_1 \pi}\right) 8\ \mathrm{Sin}^4\left(\frac{\omega_B T_1}{4}\right)$$

$$+ \frac{200}{\pi}\left[2\ \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right) - \mathrm{Si}(\omega_B T_1)\right]$$

$$= \frac{-1600}{\omega_B T_1 \pi}\ \mathrm{Sin}^4\left(\frac{\omega_B T_1}{4}\right)$$

$$+ \frac{200}{\pi}\left[2\ \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right) - \mathrm{Si}(\omega_B T_1)\right] \qquad \text{(E-15)}$$

Values of $\frac{P_{2\omega_B}}{P_T} \times 100$ are given in Table E-1 for various values of ω_B.

TABLE E-1

PERCENTAGE OF TOTAL POWER OF A SPLIT-PHASE CODE CONTAINED IN THE FREQUENCY BAND EXTENDING FROM $-\omega_B$ TO $+\omega_B$

ω_B	TWO-SIDED BANDWIDTH ($2\omega_B$)		$\frac{P_{2\omega_B}}{P_T} \times 100$ (PERCENT)
$\frac{\pi}{2T_1}$	$\frac{\pi}{T_1}$	($\frac{1}{2}$ X Bit Rate)	2.5
$\frac{\pi}{T_1}$	$\frac{2\pi}{T_1}$	(Bit Rate)	18.7
$\frac{3\pi}{2T_1}$	$\frac{3\pi}{T_1}$	($1\frac{1}{2}$ X Bit Rate)	38.8
$\frac{2\pi}{T_1}$	$\frac{4\pi}{T_1}$	(2 X Bit Rate)	63.9
$\frac{5\pi}{2T_1}$	$\frac{5\pi}{T_1}$	($2\frac{1}{2}$ X Bit Rate)	80.5
$\frac{3\pi}{T_1}$	$\frac{6\pi}{T_1}$	(3 X Bit Rate)	85.1
$\frac{7\pi}{2T_1}$	$\frac{7\pi}{T_1}$	($3\frac{1}{2}$ X Bit Rate)	85.5
$\frac{4\pi}{T_1}$	$\frac{8\pi}{T_1}$	(4 X Bit Rate)	85.7
$\frac{9\pi}{2T_1}$	$\frac{9\pi}{T_1}$	($4\frac{1}{2}$ X Bit Rate)	85.9
$\frac{5\pi}{T_1}$	$\frac{10\pi}{T_1}$	(5 X Bit Rate)	86.0

TABLE E-1 (CONT'D)

ω_B	TWO-SIDED BANDWIDTH ($2\omega_B$)		$\frac{P_{2\omega_B}}{P_T} \times 100$ (PERCENT)
$\frac{11\pi}{2T_1}$	$\frac{11\pi}{T_1}$	($5\frac{1}{2}$ X Bit Rate)	87.3
$\frac{6\pi}{T_1}$	$\frac{12\pi}{T_1}$	(6 X Bit Rate)	88.9
$\frac{13\pi}{2T_1}$	$\frac{13\pi}{T_1}$	($6\frac{1}{2}$ X Bit Rate)	90.7
$\frac{7\pi}{T_1}$	$\frac{14\pi}{T_1}$	(7 X Bit Rate)	93.0
$\frac{15\pi}{2T_1}$	$\frac{14\pi}{T_1}$	($7\frac{1}{2}$ X Bit Rate)	93.0
$\frac{8\pi}{T_1}$	$\frac{15\pi}{T_1}$	(8 X Bit Rate)	93.0

APPENDIX F

GENERATION OF RZ CODES

Appendix C shows that a split-phase PCM code can be generated by combining an NRZ code in an appropriate manner with a clock. Similarly, an RZ PCM code can be formed by logically combining an NRZ code with a clock. The period of the clock, as indicated in Figure F-1, is equal to the bit period of the NRZ code. It can be seen, by inspection, that if the clock and the NRZ code are combined according to the logical "and" operation, the RZ code will result. The logical "and" operation is defined in Table F-1. Thus,

TABLE F-1

THE LOGICAL "AND" OPERATION

A	B	A • B
0	0	0
0	1	0
1	0	0
1	1	1

whenever the NRZ signal assumes the "one" state, the logical "and" combination of that state with the alternating clock produces a replica of the clock. Whenever the NRZ signal assumes the "zero" state, the logical "and" operation results in a "zero" output.

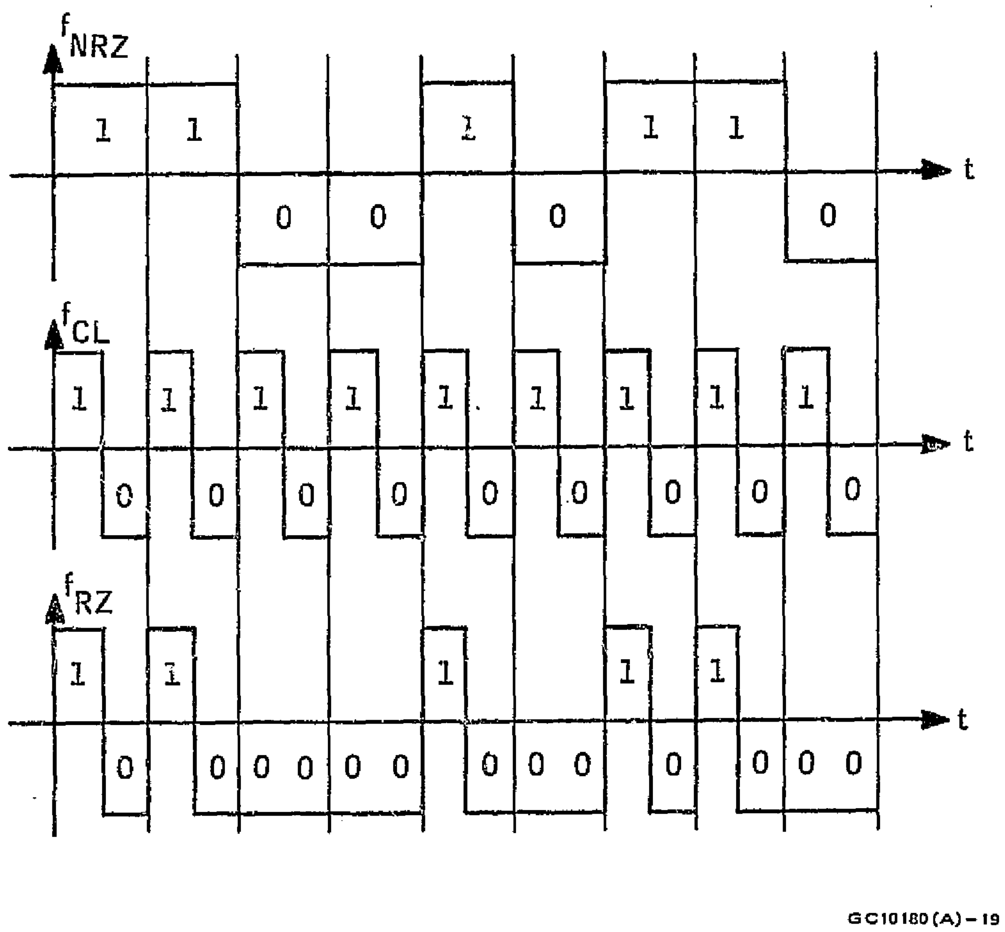


Figure F-1 Generation of an RZ PCM Code

APPENDIX G

CALCULATION OF THE AUTOCORRELATION FUNCTION FOR RANDOM RZ CODES

An ensemble of sample functions of an RZ PCM code with random bit transitions is illustrated in Figure G-1. The assumptions that must be made in order to calculate the ensemble-average autocorrelation function for the RZ code are no longer the same as were necessary for the calculations contained in Appendix A (NRZ code) and Appendix D (split-phase code). It is still necessary for the process to be at least stationary in the wide sense. However, if it is assumed that the probability of occurrence of a "one" is the same as the probability of occurrence of a "zero," then the +E level is only one-third as likely to occur as the -E level, and the (unconditional) probabilities of occurrences of these levels are given by

$$P(X_{t1} = E) = P(X_{t2} = E) = \frac{1}{4} \tag{G-1}$$

and

$$P(X_{t1} = -E) = P(X_{t2} = -E) = \frac{3}{4} \tag{G-2}$$

By equation (A-5), the autocorrelation function, $R_X(\tau)$, is

$$R_X(\tau) = E^2 P(X_{t1} = E, X_{t2} = E) - E^2 P(X_{t1} = E, X_{t2} = -E)$$
$$-E^2 P(X_{t1} = -E, X_{t2} = E) + E^2 P(X_{t1} = -E, X_{t2} = -E) \tag{G-3}$$

But, by equation (A-6), the joint probabilities in the above expression may be written as

$$P(X_{t1} = A, X_{t2} = B) = P(X_{t2} = B | X_{t1} = A) P(X_{t1} = A) \tag{G-4}$$

or

$$P(X_{t1} = E, X_{t2} = E) = \frac{1}{4} P(X_{t2} = E | X_{t1} = E) \tag{G-5}$$

$$P(X_{t1} = E, X_{t2} = -E) = \frac{1}{4} P(X_{t2} = -E | X_{t1} = E) \tag{G-6}$$

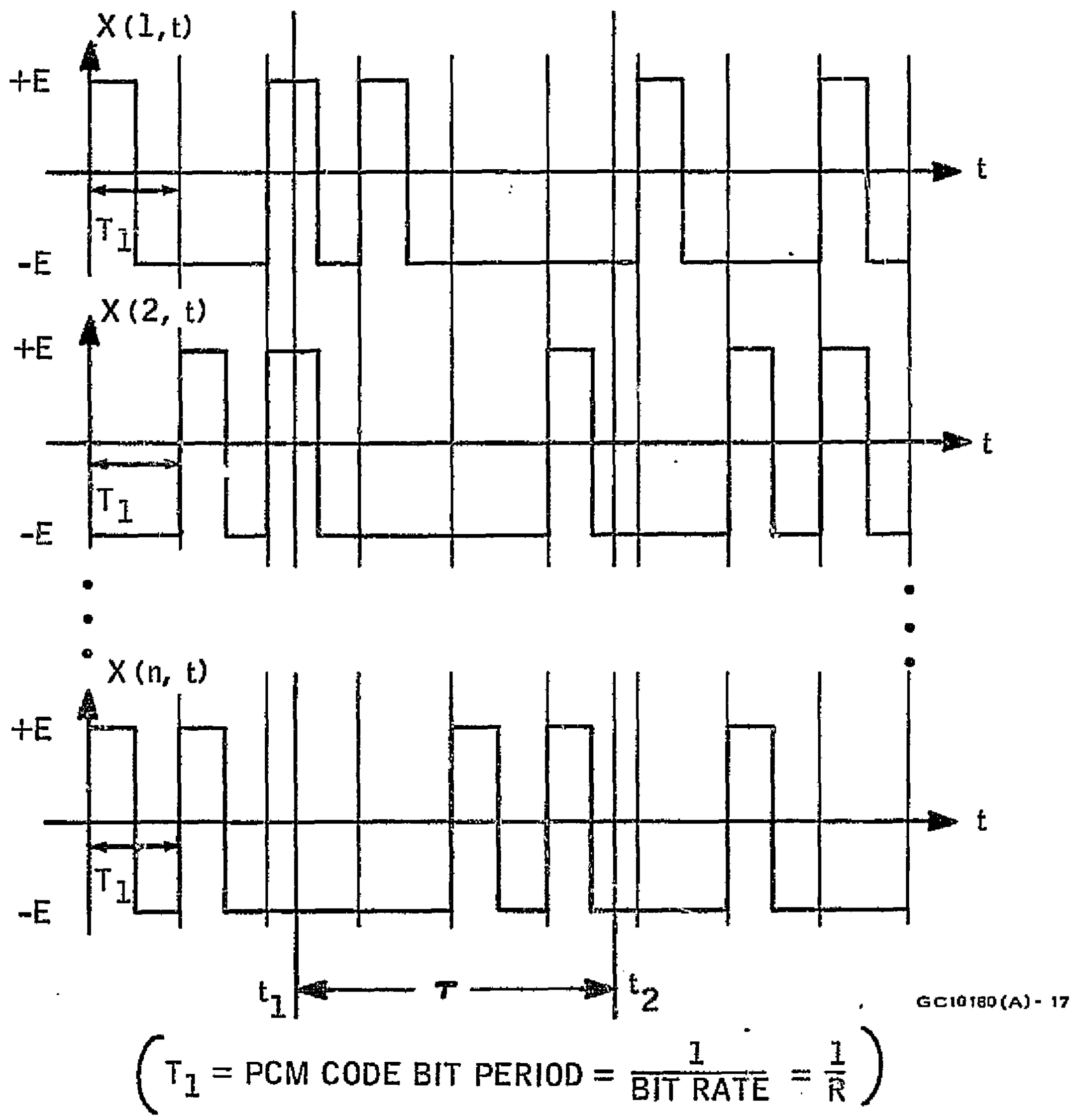


Figure G-1 Ensemble of Sample Functions of a Random RZ Code

$$P(X_{t1} = -E, X_{t2} = E) = \frac{3}{4} P(X_{t2} = E | X_{t1} = -E) \qquad \text{(G-7)}$$

$$P(X_{t1} = -E, X_{t2} = -E) = \frac{3}{4} P(X_{t2} = -E | X_{t1} = -E) \qquad \text{(G-8)}$$

Equations (G-5) through (G-8) may be substituted into equation (G-3), giving

$$R_X(\tau) = \frac{E^2}{4} P(X_{t2} = E | X_{t1} = E) - \frac{E^2}{4} P(X_{t2} = -E | X_{t1} = E)$$
$$\frac{-3E^2}{4} P(X_{t2} = E | X_{t1} = -E) + \frac{3E^2}{4} P(X_{t2} = -E | X_{t1} = -E) \qquad \text{(G-9)}$$

Again, evaluation of the conditional probabilities of equation (G-9) is dependent upon the value of τ, the time difference between samples X_{t1} and X_{t2}. For $\tau = 0$,

$$P(X_{t2} = E | X_{t1} = E) = P(X_{t2} = -E | X_{t1} = -E) = 1, \qquad \text{(G-10)}$$

$$P(X_{t2} = -E | X_{t1} = E) = P(X_{t2} = E | X_{t1} = -E) = 0, \qquad \text{(G-11)}$$

and

$$R_X(0) = \left(\frac{E^2}{4}\right)(1) - \left(\frac{E^2}{4}\right)(0) - \left(\frac{3E^2}{4}\right)(0) + \left(\frac{3E^2}{4}\right)(1) = E^2 \qquad \text{(G-12)}$$

Since a +E value for X_{t1} can occur only if X_{t1} is a sample of the first half of a bit period, then $X_{t2} = X_{t1} + \frac{T_1}{2}$ is a sample of the second half of a bit period and cannot assume a value of +E. So, for $\tau = \frac{T_1}{2}$,

$$P(X_{t2} = E | X_{t1} = E) = 0 \qquad \text{(G-13)}$$

By the same logic, for $\tau = \frac{T_1}{2}$, it is evident that

$$P(X_{t2} = -E | X_{t1} = E) = 1 \tag{G-14}$$

Since $X_{t1} = -E$ can be a sample of either half of a bit period, the remaining conditional probabilities for $\tau = \frac{T_1}{2}$ are:

$$P(X_{t2}=E|X_{t1}= -E) =$$

$$P\left[(X_{t2}=E|X_{t1}= -E)\,|\,\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix}\right] P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix}$$

$$+ P\left[(X_{t2}=E|X_{t1}= -E)\,|\,\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in second half}\end{pmatrix}\right] P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix}$$

$$= (0)\left[P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix}\right] + \left(\frac{1}{2}\right)\left[P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in second half}\end{pmatrix}\right] \tag{G-15}$$

Note that the probability of the sample $X_{t1} = -E$ occurring in the first half of a bit period is only one-half that of the probability that it will appear in the second half, or

$$P(\text{the sample } X_{t1} = -E \text{ occurs in first half}) = \frac{1}{3} \tag{G-16}$$

$$P(\text{the sample } X_{t1} = -E \text{ occurs in second half}) = \frac{2}{3} \tag{G-17}$$

Substituting equations (G-16) and (G-17) into equation (G-15):

$$P(X_{t2}=E|X_{t1}= -E) = (0)\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3} \tag{G-18}$$

Similarly,

$$P(X_{t2}= -E|X_{t1}= -E) =$$

$$P\left[(X_{t2}= -E|X_{t1}= -E)\,|\,\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix}\right] P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix}$$

$$+ P\left[(X_{t2}= -E|X_{t1}= -E)\,|\,\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in second half}\end{pmatrix}\right] P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in second half}\end{pmatrix}$$

$$= (1)\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{2}{3} \tag{G-19}$$

Then $R_X(\tau)$, for $\tau = \frac{T_1}{2}$, is given by

$$R_X\left(\frac{T_1}{2}\right) = \left(\frac{E^2}{4}\right)(0) - \left(\frac{E^2}{4}\right)(1) - \left(\frac{3E^2}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{3E^2}{4}\right)\left(\frac{2}{3}\right) = 0 \tag{G-20}$$

Similarly, it is easily shown that

$$R_X\left(-\frac{T_1}{2}\right) = 0 \tag{G-21}$$

If $\tau = \frac{T_1}{4}$, the conditional probabilities of equation (G-9) are dependent upon in which quarter of a bit period X_{t1} is located. If X_{t1} is in the first quarter of a bit period, a sample, X_{t2}, of the second quarter of that period must be of the same polarity. *It must be remembered that $X_{t1} = -E$ can be a sample for any part of a bit period, while $X_{t1} = +E$ must be a sample of the first half of a bit period.* The total conditional probabilities for $\tau = \frac{T_1}{4}$ are:

$$\begin{aligned}
P(X_{t2}=E \mid X_{t1}=E) = {} & P\left[(X_{t2}=E \mid X_{t1}=E) \mid (\text{the sample } X_{t1}=E \text{ occurs in first quarter})\right] P(\text{the sample } X_{t1}=E \text{ occurs in first quarter}) \\
& + P\left[(X_{t2}=E \mid X_{t1}=E) \mid (\text{the sample } X_{t1}=E \text{ occurs in second quarter})\right] P(\text{the sample } X_{t1}=E \text{ occurs in second quarter}) \\
& + P\left[(X_{t2}=E \mid X_{t1}=E) \mid (\text{the sample } X_{t1}=E \text{ occurs in third quarter})\right] P(\text{the sample } X_{t1}=E \text{ occurs in third quarter}) \\
& + P\left[(X_{t2}=E \mid X_{t1}=E) \mid (\text{the sample } X_{t1}=E \text{ occurs in fourth quarter})\right] P(\text{the sample } X_{t1}=E \text{ occurs in fourth quarter}) \\
= {} & (1)\left(\tfrac{1}{2}\right) + (0)\left(\tfrac{1}{2}\right) + (0)(0) + \left(\tfrac{1}{2}\right)(0) = \tfrac{1}{2}
\end{aligned} \tag{G-22}$$

$$
\begin{aligned}
&P(X_{t2}= -E|X_{t1}=E) = \\
&\quad P\left[(X_{t2}= -E|X_{t1}=E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}=E\\ \text{occurs in first quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}=E\\ \text{occurs in first quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= -E|X_{t1}=E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}=E\\ \text{occurs in second quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}=E\\ \text{occurs in second quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= -E|X_{t1}=E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}=E\\ \text{occurs in third quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}=E\\ \text{occurs in third quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= -E|X_{t1}=E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}=E\\ \text{occurs in fourth quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}=E\\ \text{occurs in fourth quarter}\end{pmatrix} \\
&\quad = (0)(\tfrac{1}{2})+(1)(\tfrac{1}{2})+(1)(0)+(\tfrac{1}{2})(0) = \tfrac{1}{2}
\end{aligned}
\tag{G-23}
$$

$$
\begin{aligned}
&P(X_{t2}= E|X_{t1}= -E) = \\
&\quad P\left[(X_{t2}= E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in first quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in first quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in second quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in second quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in third quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in third quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in fourth quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in fourth quarter}\end{pmatrix} \\
&\quad = (0)(\tfrac{1}{6})+(0)(\tfrac{1}{6})+(0)(\tfrac{2}{6})+(\tfrac{1}{2})(\tfrac{2}{6}) = \tfrac{1}{6}
\end{aligned}
\tag{G-24}
$$

$$
\begin{aligned}
&P(X_{t2}= -E|X_{t1}= -E) = \\
&\quad P\left[(X_{t2}= -E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in first quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in first quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= -E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in second quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in second quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= -E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in third quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in third quarter}\end{pmatrix} \\
&\quad + P\left[(X_{t2}= -E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E\\ \text{occurs in fourth quarter)}\end{matrix}\right]P\begin{pmatrix}\text{the sample } X_{t1}= -E\\ \text{occurs in fourth quarter}\end{pmatrix} \\
&\quad + (1)(\tfrac{1}{6})+(1)(\tfrac{1}{6})+(1)(\tfrac{2}{6})+(\tfrac{1}{2})(\tfrac{2}{6}) = \tfrac{5}{6}
\end{aligned}
\tag{G-25}
$$

Then $R_X(\tau)$, for $\tau = \frac{T_1}{4}$, is given by

$$R_X\left(\frac{T_1}{4}\right) = \left(\frac{E^2}{4}\right)(\tfrac{1}{2}) - \left(\frac{E^2}{4}\right)(\tfrac{1}{2}) - \left(\frac{3E^2}{4}\right)(\tfrac{1}{6}) + \left(\frac{3E^2}{4}\right)(\tfrac{5}{6}) = \frac{E^2}{2} \qquad \text{(G-26)}$$

A similar procedure can be followed for $\tau = -\frac{T_1}{4}$, yielding

$$R_X\left(-\frac{T_1}{4}\right) = \frac{E^2}{2} \qquad \text{(G-27)}$$

If τ is some multiple of the bit period T_1,

$$\tau = nT_1 \qquad , |n| \geq 1 \qquad \text{(G-28)}$$

then $R_X(\tau)$ is easily determined. The conditional probabilities of equation (G-9) are given by:

$$P(X_{t2}=E|X_{t1}=E) = \frac{1}{2}, \qquad \text{(G-29)}$$

since X_{t1} must be a sample of the first half of a bit period, and X_{t2} is therefore a sample of the first half of another bit period. Similarly,

$$P(X_{t2}= -E|X_{t1}=E) = \frac{1}{2} \qquad \text{(G-30)}$$

$$\begin{aligned}
&P(X_{t2}= E|X_{t1}= -E) = \\
&\quad P\left[(X_{t2}=E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E \\ \text{occurs in first half)}\end{matrix}\right] P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in first half}\end{pmatrix} \\
&\quad + P\left[(X_{t2}=E|X_{t1}= -E)\,\middle|\,\begin{matrix}\text{(the sample } X_{t1}= -E \\ \text{occurs in second half)}\end{matrix}\right] P\begin{pmatrix}\text{the sample } X_{t1}= -E \\ \text{occurs in second half}\end{pmatrix} \\
&\quad = (\tfrac{1}{2})(\tfrac{1}{3}) + (0)(\tfrac{2}{3}) = \tfrac{1}{6} \qquad \text{(G-31)}
\end{aligned}$$

$P(X_{t2}= -E|X_{t1}= -E) =$

$$P\left[(X_{t2}= -E|X_{t1}= -E)\,|\,(\text{the sample } X_{t1}= -E \text{ occurs in first half})\right]P(\text{the sample } X_{t1}= -E \text{ occurs in first half})$$

$$+ P\left[(X_{t2}= -E|X_{t1}= -E)\,|\,(\text{the sample } X_{t1}= -E \text{ occurs in second half})\right]P(\text{the sample } X_{t1}= -E \text{ occurs in second half})$$

$$= (\tfrac{1}{2})(\tfrac{1}{3})+(1)(\tfrac{2}{3}) = \tfrac{5}{6} \qquad (G-32)$$

Then $R_X(\tau)$, for $\tau = nT_1$, is

$$R_X(nT_1)=\left(\frac{E^2}{4}\right)(\tfrac{1}{2})-\left(\frac{E^2}{4}\right)(\tfrac{1}{2})-\left(\frac{3E^2}{4}\right)(\tfrac{1}{6})+\left(\frac{3E^2}{4}\right)(\tfrac{5}{6}) = \frac{E^2}{2} \qquad (G-33)$$

If τ is some multiple of half the bit period

$$\tau = n\,\frac{T_1}{2} \quad , \ |n|> 1 \qquad (G-34)$$

then the conditional probabilities of equation (G-9) are:

$$P(X_{t2}=E|X_{t1}=E) = 0, \qquad (G-35)$$

since X_{t1} must be a sample of the first half of a bit period, and X_{t2} is therefore a sample of the second half of another bit period.

$$P(X_{t2}= -E|X_{t1}=E) = 1 \qquad (G-36)$$

$P(X_{t2}= E|X_{t1}= -E) =$

$$P\left[(X_{t2}=E|X_{t1}= -E)\,|\,(\text{the sample } X_{t1}= -E \text{ occurs in first half})\right]P(\text{the sample } X_{t1}= -E \text{ occurs in first half})$$

$$+ P\left[(X_{t2}=E|X_{t1}= -E)\,|\,(\text{the sample } X_{t1}= -E \text{ occurs in second half})\right]P(\text{the sample } X_{t1}= -E \text{ occurs in second half})$$

$$= (0)(\tfrac{1}{3})+(\tfrac{1}{2})(\tfrac{2}{3}) = \tfrac{1}{3} \qquad (G-37)$$

$$P(X_{t2} = -E | X_{t1} = -E) =$$

$$P\left[(X_{t2} = -E | X_{t1} = -E) \,\middle|\, \begin{matrix}\text{(the sample } X_{t1} = -E \\ \text{occurs in first half)}\end{matrix}\right] P\begin{pmatrix}\text{the sample } X_{t1} = -E \\ \text{occurs in first half}\end{pmatrix}$$

$$+ P\left[(X_{t2} = -E | X_{t1} = -E) \,\middle|\, \begin{matrix}\text{(the sample } X_{t1} = -E \\ \text{occurs in second half)}\end{matrix}\right] P\begin{pmatrix}\text{the sample } X_{t1} = -E \\ \text{occurs in second half}\end{pmatrix}$$

$$= (1)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{2}{3} \qquad \text{(G-38)}$$

Then $R_X(\tau)$, for $\tau = n \frac{T_1}{2}$, is

$$R_X\left(n \frac{T_1}{2}\right) = \left(\frac{E^2}{4}\right)(0) - \left(\frac{E^2}{4}\right)(1) - \left(\frac{3E^2}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{3E^2}{4}\right)\left(\frac{2}{3}\right) = 0 \qquad \text{(G-39)}$$

Figure G-2 contains a plot of $R_X(\tau)$ for all values of τ.

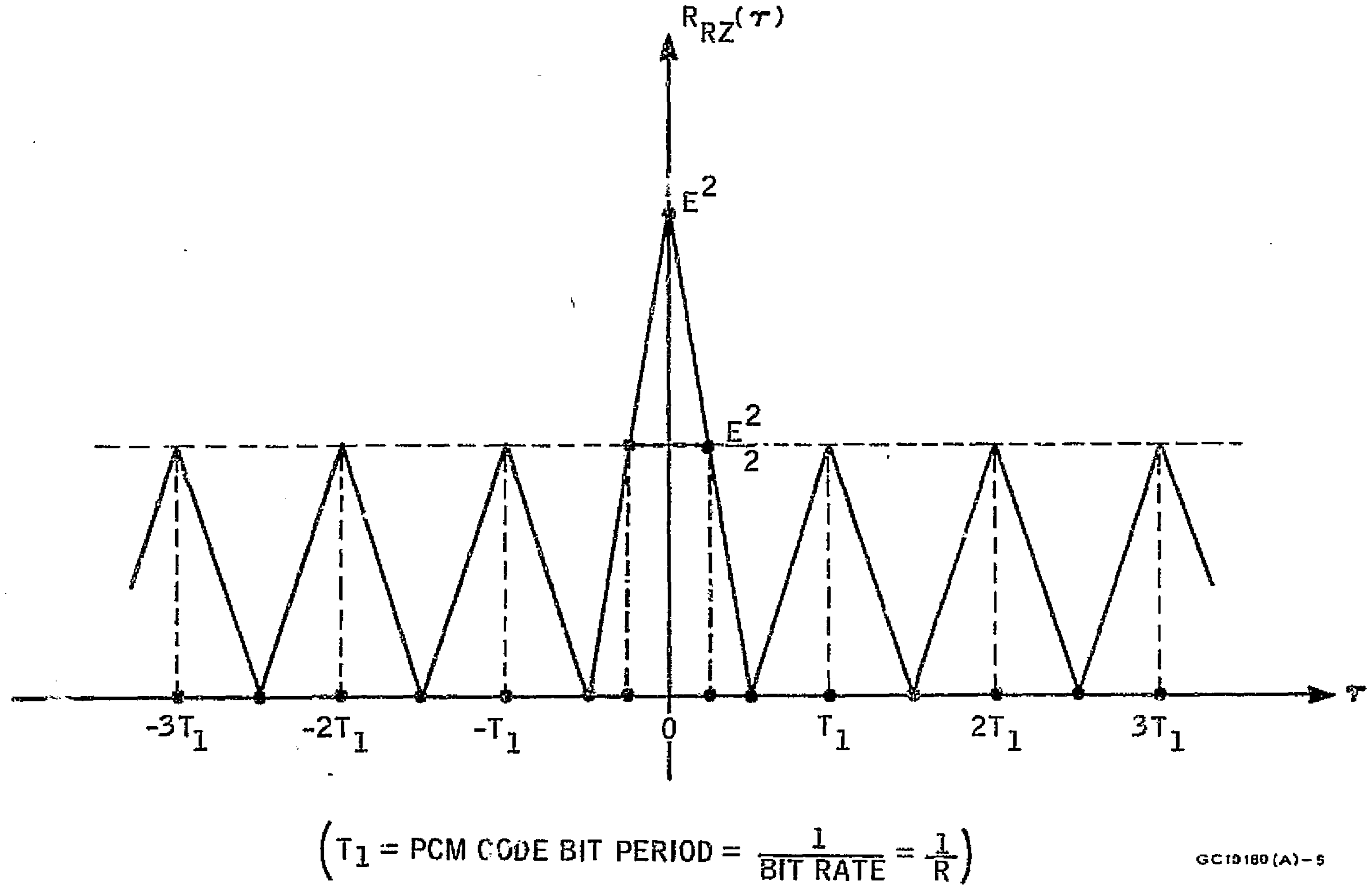


Figure G-2 Ensemble-Average Autocorrelation Function for a Random PCM Code

APPENDIX H

BANDWIDTH REQUIREMENTS FOR RZ CODES

The power density spectrum of an RZ PCM code with a random bit pattern, as determined in Paragraph 5.2, is given by

$$S_{RZ}(\omega) = \frac{E^2 T_1}{8\pi} \left(\frac{\sin \frac{\omega T_1}{4}}{\frac{\omega T_1}{4}} \right)^2 + \sum_{n=-\infty}^{\infty} \frac{E^2}{4} \left(\frac{\sin \frac{n\omega_0 T_1}{4}}{\frac{n\omega_0 T_1}{4}} \right)^2 \delta\,(\omega - n\omega_0) \qquad \text{(H-1)}$$

The amount of power present in a frequency band extending from $-\omega_B$ to $+\omega_B$, as shown in Figure H-1, is given by

$$P_{2\omega_B} = \int_{-\omega_B}^{+\omega_B} S_{RZ}(\omega)\, d\omega = \int_{-\omega_B}^{+\omega_B} \frac{E^2 T_1}{8\pi} \left(\frac{\sin \frac{\omega T_1}{4}}{\frac{\omega T_1}{4}} \right)^2 d\omega$$

$$+ \int_{-\omega_B}^{+\omega_B} \sum_{n=-\infty}^{\infty} \frac{E^2}{4} \left(\frac{\sin \frac{n\omega_0 T_1}{4}}{\frac{n\omega_0 T_1}{4}} \right)^2 \delta\,(\omega - n\omega_0)\, d\omega$$

$$= \frac{E^2 T_1}{8\pi} \int_{-\omega_B}^{+\omega_B} \left(\frac{\sin \frac{\omega T_1}{4}}{\frac{\omega T_1}{4}} \right)^2 d\omega + \frac{E^2}{4} \sum_{n=-\infty}^{\infty} \left(\frac{\sin \frac{n\omega_0 T_1}{4}}{\frac{n\omega_0 T_1}{4}} \right)^2 \int_{-\omega_B}^{+\omega_B} \delta\,(\omega - n\omega_0)\, d\omega \qquad \text{(H-2)}$$

By substituting $x = \frac{\omega T_1}{4}$, the first integral of the above expression

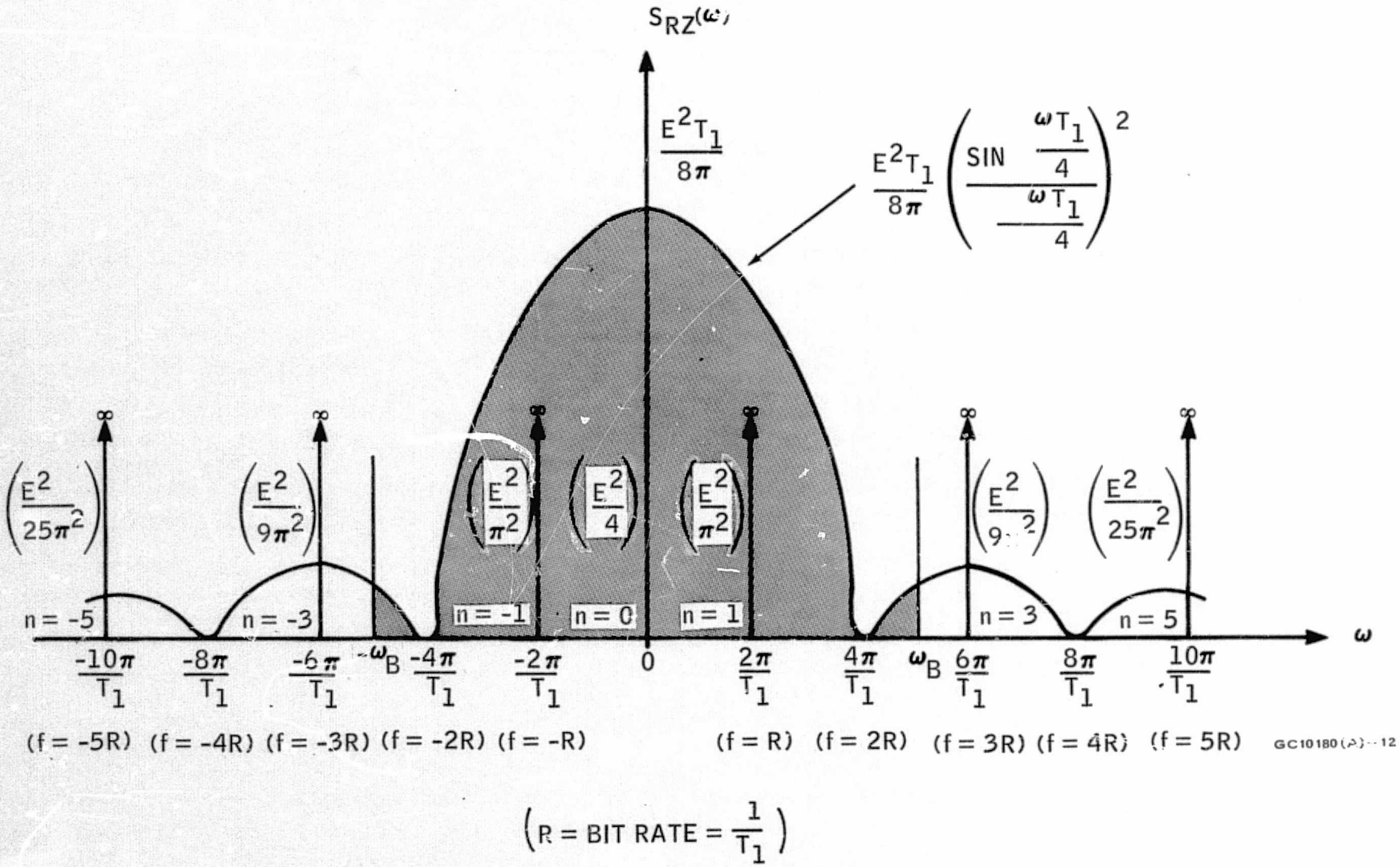


Figure H-1 Power Spectrum of an RZ Code (Random Bit Pattern) reduces to

$$P_{2\omega_B} = \frac{E^2 T_1}{8\pi} \int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \left(\frac{\sin x}{x}\right)^2 \frac{4}{T_1} dx = \frac{E^2}{2\pi} \int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \left(\frac{\sin x}{x}\right)^2 dx$$

$$= \frac{E^2}{4\pi} \int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{1-\cos 2x}{x^2} dx = \frac{E^2}{4\pi} \int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{dx}{x^2} - \frac{E^2}{4\pi} \int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\cos 2x}{x^2} dx$$

$$= \frac{E^2}{4\pi} \left[\frac{-1}{x}\right]_{-\frac{\omega_B T_1}{4}}^{\frac{\omega_B T_1}{4}} - \frac{E^2}{4\pi} \left\{ - \left[\frac{\cos 2x}{x}\right]_{-\frac{\omega_B T_1}{4}}^{\frac{\omega_B T_1}{4}} - 2 \int_{-\frac{\omega_B T_1}{4}}^{+\frac{\omega_B T_1}{4}} \frac{\sin 2x \, dx}{x} \right\}$$

$$= \frac{E^2}{4\pi} \left[-\frac{8}{\omega_B T_1}\right] + \frac{2E^2}{\omega_B T_1 \pi} \cos\left(\frac{\omega_B T_1}{2}\right) + \frac{E^2}{\pi} \int_0^{+\frac{\omega_B T_1}{2}} \frac{\sin \xi \, d\xi}{\xi}$$

$$\doteq - \frac{2E^2}{\omega_B T_1 \pi} + \frac{2E^2}{\omega_B T_1 \pi} \cos\left(\frac{\omega_B T_1}{2}\right) + \frac{E^2}{\pi} \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right)$$

$$= - \frac{4E^2}{\omega_B T_1 \pi} \left\{\frac{1}{2}\left[1 - \cos\left(\frac{\omega_B T_1}{2}\right)\right]\right\} + \frac{E^2}{\pi} \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right)$$

$$= - \frac{4E^2}{\omega_B T_1 \pi} \sin^2\left(\frac{\omega_B T_1}{4}\right) + \frac{E^2}{\pi} \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right) \qquad \text{(H-3)}$$

Equation (H-3) expresses the power (in a two-sided bandwidth of $2\omega_B$) contributed by the continuous component of the RZ power spectrum. The total power contained in the continuous part of the spectrum may be found by letting $_B$ approach infinity. Then, the first term of equation (H-3) becomes zero, and the Si integral approaches $\frac{\pi}{2}$. The total power is given by

$$P_T{}' = \left(\frac{E^2}{\pi}\right)\left(\frac{\pi}{2}\right) = \frac{E^2}{2} \tag{H-4}$$

The second integral of equation (H-2) represents the power contributed by the discrete components of the RZ spectrum. That integral can be simplified by noting that

$$\int_{-\omega_B}^{+\omega_B} \delta(\omega - n\omega_0)\,d\omega = \begin{cases} 1 \text{ when } \omega = n\omega_0 \\ 0 \text{ when } \omega \neq n\omega_0 \end{cases} \tag{H-5}$$

and that the $\pm\omega_B$ limits impose a constraint on the maximum value of ω and, therefore, on the number of impulses which is summed. The total power contribution of all the weighted unit impulses can be found by letting ω_B approach infinity (or, alternately, letting $|n|$ approach infinity). Then,

$$P_T{}'' = \frac{E^2}{4}\sum_{n=-\infty}^{\infty}\left(\frac{\sin\frac{n\omega_0 T_1}{4}}{\frac{n\omega_0 T_1}{4}}\right)^2 = \frac{E^2}{4}\sum_{n=-\infty}^{\infty}\left(\frac{\sin\frac{n\left(\frac{2\pi}{T_1}\right)T_1}{4}}{\frac{n\left(\frac{2\pi}{T_1}\right)T_1}{4}}\right)^2$$

$$= \frac{E^2}{4}\sum_{n=-\infty}^{\infty}\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right)^2 \tag{H-6}$$

but

$$\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right)^2 = 1 \text{ for } n = 0 \tag{H-7}$$

and

$$\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right)^2 = \left(\frac{\sin\frac{-n\pi}{2}}{\frac{-n\pi}{2}}\right)^2 = \begin{cases} \left(\frac{1}{\frac{n\pi}{2}}\right)^2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases} \tag{H-8}$$

so

$$P_T'' = \frac{E^2}{4}\left[1 + 2\sum_{n=0}^{\infty} \frac{4}{n^2\pi^2}\right] = \frac{E^2}{4} + \left(\frac{E^2}{2}\right)\left(\frac{4}{\pi^2}\right)\sum_{n=0}^{\infty}\frac{1}{n^2}$$

$$= \frac{E^2}{4} + \frac{2E^2}{\pi^2}\sum_{r=0}^{\infty}\frac{1}{n^2} \tag{H-9}$$

But

$$\sum_{n=0}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} \tag{H-10}$$

so

$$P_T'' = \frac{E^2}{4} + \frac{E^2}{4} = \frac{E^2}{2} \tag{H-11}$$

The total power contribution of the periodic (clock) component of the RZ code, then, is equal to the total power contribution of the random component. The total power contained in the RZ signal is

$$P_T = P_T' + P_T'' = \frac{E^2}{2} + \frac{E^2}{2} = E^2 \tag{H-12}$$

By equation (H-3), the percentage of total power due to the random component of the RZ code, present in the frequency band between $-\omega_B$ and $+\omega_B$, is

$$\frac{P_{2\omega_B}'}{P_T} \times 100 = -\frac{400}{\omega_B T_1 \pi} \sin^2\left(\frac{\omega_B T_1}{4}\right) + \frac{100}{\pi} \mathrm{Si}\left(\frac{\omega_B T_1}{2}\right) \qquad \text{(H-13)}$$

Similarly, from equation (H-9), the percentage of total power due to the periodic (clock) component of the RZ code, present in the frequency band between $-\omega_B$ and $+\omega_B$, is

$$\frac{P_{2\omega_B}''}{P_T} \times 100 = 25 + \frac{200}{\pi^2} \sum_{\substack{n=0 \\ (n\ \text{odd})}}^{X} \frac{1}{n^2} \qquad \text{(H-14)}$$

where X is an integer equal to, or less than, $\omega_B / \frac{2\pi}{T_1}$.

Equations (H-13) and (H-14) are evaluated for various values of ω_B, and the results are given in Table H-1.

TABLE H-1

PERCENTAGE OF TOTAL POWER OF THE COMPONENTS OF AN RZ PCM CODE

ω_B	TWO-SIDED BANDWIDTH ($2\omega_B$)		$\frac{P_{2\omega_B}}{P_T}$ X 100% (RANDOM COMPONENT)	$\frac{P_{2\omega_B}}{P_T}$ X 100% (PERIODIC COMPONENT)	$\frac{P_{2\omega_B}}{P_T}$ X 100 (PERCENT)
0+	0+		0	25.0	25.0
$\frac{\pi}{2T_1}$	$\frac{\pi}{T_1}$	($\frac{1}{2}$ X Bit Rate)	12.6	25.0	37.6
$\frac{\pi}{T_1}$	$\frac{2\pi}{T_1}$	(Bit Rate)	24.0	25.0	49.0
$\frac{3\pi}{2T_1}$	$\frac{3\pi}{T_1}$	($1\frac{1}{2}$ X Bit Rate)	31.9	25.0	56.9
$\frac{2\pi}{T_1}$ +	$\frac{4\pi}{T_1}$ +	(2 X Bit Rate)	38.7	45.2	83.9
$\frac{5\pi}{2T_1}$	$\frac{5\pi}{T_1}$	($2\frac{1}{2}$ X Bit Rate)	42.7	45.2	87.9
$\frac{3\pi}{T_1}$	$\frac{6\pi}{T_1}$	(3 X Bit Rate)	44.5	45.2	89.7
$\frac{7\pi}{2T_1}$	$\frac{7\pi}{T_1}$	($3\frac{1}{2}$ X Bit Rate)	45.0	45.2	90.2
$\frac{4\pi}{T_1}$	$\frac{8\pi}{T_1}$	(4 X Bit Rate)	45.1	45.2	90.3
$\frac{9\pi}{2T_1}$	$\frac{9\pi}{T_1}$	($4\frac{1}{2}$ X Bit Rate)	45.2	45.2	90.4
$\frac{5\pi}{T_1}$	$\frac{10\pi}{T_1}$	(5 X Bit Rate)	45.4	45.2	90.6
$\frac{11\pi}{2T_1}$	$\frac{11\pi}{T_1}$	($5\frac{1}{2}$ X Bit Rate)	45.9	45.2	91.1
$\frac{6\pi}{T_1}$ +	$\frac{12\pi}{T_1}$ +	(6 X Bit Rate)	46.7	47.4	94.1
$\frac{13\pi}{2T_1}$	$\frac{13\pi}{T_1}$	($6\frac{1}{2}$ X Bit Rate)	46.9	47.4	94.3
$\frac{7\pi}{T_1}$	$\frac{14\pi}{T_1}$	(7 X Bit Rate)	47.4	47.4	94.8

APPENDIX I

CALCULATION OF THE POWER SPECTRAL DENSITY OF A PHASE-SHIFT-KEYED SINUSOID (NONCOHERENT MODULATION)

Equations (46), (47), and (48) show that the autocorrelation function of a sinusoidal carrier which is noncoherently phase-modulated by an NRZ PCM code is given by

$$R_{PSK}(\tau) = R_{NRZ}(\tau) R_{CARRIER}(\tau) \tag{I-1}$$

But

$$R_{CARRIER}(\tau) = \frac{A^2}{2} \cos(\omega_c \tau) \tag{I-2}$$

and inspection of Figure 3-1 reveals that

$$R_{NRZ}(\tau) = \begin{cases} \dfrac{E^2}{T_1}(\tau + T_1) & \text{For } -T_1 \leq \tau \leq 0 \\[2ex] \dfrac{-E^2}{T_1}(\tau - T_1) & \text{For } 0 \leq \tau \leq T_1 \end{cases} \tag{I-3}$$

Then

$$\begin{aligned} R_{PSK}(\tau) &= \frac{E^2}{T_1}(\tau + T_1)\frac{A^2}{2}\cos(\omega_c \tau) \\ &= \frac{A^2E^2}{2T_1}(\tau + T_1)\cos(\omega_c \tau) \qquad \text{For } -T_1 \leq \tau \leq 0 \end{aligned} \tag{I-4}$$

and

$$\begin{aligned} R_{PSK}(\tau) &= -\frac{E^2}{T_1}(\tau - T_1)\frac{A^2}{2}\cos(\omega_c \tau) \\ &= -\frac{A^2E^2}{2T_1}(\tau - T_1)\cos(\omega_c \tau) \qquad \text{For } 0 \leq \tau \leq T_1 \end{aligned} \tag{I-5}$$

The power density spectrum of the noncoherent PSK signal is given by

$$S_{PSK}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{PSK}(\tau)e^{-j\omega\tau}d\tau$$

$$= \frac{1}{2\pi}\int_{-T_1}^{0} \frac{A^2E^2}{2T_1}(\tau+T_1)\cos(\omega_c\tau)e^{-j\omega\tau}d\tau$$

$$-\frac{1}{2\pi}\int_{0}^{T_1} \frac{A^2E^2}{2T_1}(\tau-T_1)\cos(\omega_c\tau)e^{-j\omega\tau}d\tau$$

$$= \frac{A^2E^2}{4\pi T_1}\int_{-T_1}^{0}(\tau+T_1)\left(\frac{e^{j\omega_c\tau}+e^{-j\omega_c\tau}}{2}\right)e^{-j\omega\tau}d\tau$$

$$-\frac{A^2E^2}{4\pi T_1}\int_{0}^{T_1}(\tau-T_1)\left(\frac{e^{j\omega_c\tau}+e^{-j\omega_c\tau}}{2}\right)e^{-j\omega\tau}d\tau$$

$$= \frac{A^2E^2}{8\pi T_1}\int_{-T_1}^{0}(\tau+T_1)\left[e^{j(\omega_c-\omega)\tau}+e^{-j(\omega_c+\omega)\tau}\right]d\tau$$

$$-\frac{A^2E^2}{8\pi T_1}\int_{0}^{T_1}(\tau-T_1)\left[e^{j(\omega_c-\omega)\tau}+e^{-j(\omega_c+\omega)\tau}\right]d\tau \qquad \text{(I-6)}$$

Integration of the above expression yields

$$
\begin{aligned}
S_{PSK}(\omega) = \frac{A^2E^2}{8\pi T_1} \Bigg\{ & \frac{e^{-j(\omega-\omega_c)\tau}}{(\omega-\omega_c)^2}\,[1+j(\omega-\omega_c)\tau] + \frac{e^{-j(\omega+\omega_c)\tau}}{(\omega+\omega_c)^2}\,[1+j(\omega+\omega_c)\tau] \\
& + \frac{T_1 e^{-j(\omega-\omega_c)\tau}}{-j(\omega-\omega_c)} + \frac{T_1 e^{-j(\omega+\omega_c)\tau}}{-j(\omega+\omega_c)} \Bigg\} \Bigg|_{\tau=-T_1}^{\tau=0} \\
+ \frac{A^2E^2}{8\pi T_1} \Bigg\{ & \frac{-e^{-j(\omega-\omega_c)\tau}}{(\omega-\omega_c)^2}\,[1+j(\omega-\omega_c)\tau] - \frac{e^{-j(\omega+\omega_c)\tau}}{(\omega+\omega_c)^2}\,[1+j(\omega+\omega_c)\tau] \\
& + \frac{T_1 e^{-j(\omega-\omega_c)\tau}}{-j(\omega-\omega_c)} + \frac{T_1 e^{-j(\omega+\omega_c)\tau}}{-j(\omega+\omega_c)} \Bigg\} \Bigg|_{\tau=0}^{\tau=T_1} \\
= \frac{A^2E^2}{8\pi T_1} \Bigg\{ & \frac{1}{(\omega-\omega_c)^2} + \frac{1}{(\omega+\omega_c)^2} + \cancel{\frac{j(\omega-\omega_c)T_1}{(\omega-\omega_c)^2}} + \cancel{\frac{j(\omega+\omega_c)T_1}{(\omega+\omega_c)^2}} \\
& - \frac{e^{j(\omega-\omega_c)T_1}\,[1-\cancel{j(\omega-\omega_c)T_1}]}{(\omega-\omega_c)^2} - \frac{e^{j(\omega+\omega_c)T_1}\,[1-\cancel{j(\omega+\omega_c)T_1}]}{(\omega+\omega_c)^2} \\
& - \cancel{\frac{j(\omega-\omega_c)T_1 e^{j(\omega-\omega_c)T_1}}{(\omega-\omega_c)^2}} - \cancel{\frac{j(\omega+\omega_c)T_1 e^{j(\omega+\omega_c)T_1}}{(\omega+\omega_c)^2}} \\
& - \frac{e^{-j(\omega-\omega_c)T_1}\,[1+\cancel{j(\omega-\omega_c)T_1}]}{(\omega-\omega_c)^2} - \frac{e^{-j(\omega+\omega_c)T_1}\,[1+\cancel{j(\omega+\omega_c)T_1}]}{(\omega+\omega_c)^2} \\
& + \cancel{\frac{j(\omega-\omega_c)T_1 e^{-j(\omega-\omega_c)T_1}}{(\omega-\omega_c)^2}} + \cancel{\frac{j(\omega+\omega_c)T_1 e^{-j(\omega+\omega_c)T_1}}{(\omega+\omega_c)^2}} \\
& + \frac{1}{(\omega-\omega_c)^2} + \frac{1}{(\omega+\omega_c)^2} - \cancel{\frac{j(\omega-\omega_c)T_1}{(\omega-\omega_c)^2}} - \cancel{\frac{j(\omega+\omega_c)T_1}{(\omega+\omega_c)^2}} \Bigg\}
\end{aligned}
$$

$$= \frac{A^2E^2}{8\pi T_1}\left\{\frac{2}{(\omega-\omega_c)^2} + \frac{2}{(\omega+\omega_c)^2} - \frac{2}{(\omega-\omega_c)^2}\left[\frac{e^{j(\omega-\omega_c)T_1}+e^{-j(\omega-\omega_c)T_1}}{2}\right]\right.$$

$$\left. - \frac{2}{(\omega+\omega_c)^2}\left[\frac{e^{j(\omega+\omega_c)T_1}+e^{-j(\omega+\omega_c)T_1}}{2}\right]\right\}$$

$$= \frac{A^2E^2}{4\pi T_1}\left\{\frac{1}{(\omega-\omega_c)^2} + \frac{1}{(\omega+\omega_c)^2} - \frac{1}{(\omega-\omega_c)^2}\cos\left[(\omega-\omega_c)T_1\right] - \frac{1}{(\omega+\omega_c)^2}\cos\left[(\omega+\omega_c)T_1\right]\right\}$$

$$= \frac{A^2E^2}{2\pi T_1}\left\{\left[\frac{1}{(\omega-\omega_c)^2}\right]\left[\frac{1}{2}-\frac{1}{2}\cos\left[(\omega-\omega_c)T_1\right]\right] + \left[\frac{1}{(\omega+\omega_c)^2}\right]\left[\frac{1}{2}-\frac{1}{2}\cos\left[(\omega+\omega_c)T_1\right]\right]\right\}$$

$$= \frac{A^2E^2}{2\pi T_1}\left[\frac{1}{(\omega-\omega_c)^2}\sin^2\left(\frac{\omega-\omega_c}{2}T_1\right) + \frac{1}{(\omega+\omega_c)^2}\sin^2\left(\frac{\omega+\omega_c}{2}T_1\right)\right] \quad \text{(I-7)}$$

The above expression may be reduced to

$$S_{PSK}(\omega) = \frac{A^2E^2}{2\pi T_1}\left\{\frac{\sin^2\left(\frac{\omega-\omega_c}{2}T_1\right)}{\frac{4}{T_1^2}\left(\frac{\omega-\omega_c}{2}T_1\right)^2} + \frac{\sin^2\left(\frac{\omega+\omega_c}{2}T_1\right)}{\frac{4}{T_1^2}\left(\frac{\omega+\omega_c}{2}T_1\right)^2}\right\}$$

$$= \frac{A^2E^2T_1}{8\pi}\left\{\frac{\sin^2\left(\frac{\omega-\omega_c}{2}T_1\right)}{\left(\frac{\omega-\omega_c}{2}T_1\right)^2} + \frac{\sin^2\left(\frac{\omega+\omega_c}{2}T_1\right)}{\left(\frac{\omega+\omega_c}{2}T_1\right)^2}\right\} \quad \text{(I-8)}$$

Since the autocorrelation function of the noncoherent PSK signal is equal to the product of the autocorrelation functions of the carrier and the PCM code, the power spectrum of the PSK signal should be equal to the convolution of the power spectra of the carrier and the code (Ref. 8).

$$S_{PSK}(\omega) = S_{NRZ}(\omega) * S_{CARRIER}(\omega)$$

$$= \int_{-\infty}^{\infty} S_{NRZ}(\omega-\xi)\ S_{CARRIER}(\xi)d\xi$$

$$= \int_{-\infty}^{\infty} S_{NRZ}(\xi)\ S_{CARRIER}(\omega-\xi)d\xi \qquad \text{(I-9)}$$

The power spectrum of the carrier consists simply of two impulses located at $\omega = +\omega_c$ and at $\omega = -\omega_c$, or

$$S_{CARRIER}(\omega) = \frac{A^2}{4}\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right] \qquad \text{(I-10)}$$

The power spectrum of the NRZ code, as determined in Paragraph 3.2, is

$$S_{NRZ}(\omega) = \frac{E^2 T_1}{2\pi}\left(\frac{\sin \frac{\omega T_1}{2}}{\frac{\omega T_1}{2}}\right)^2 \qquad \text{(I-11)}$$

Substitution of equations (I-10) and (I-11) into equation (I-9) yields

$$S_{PSK}(\omega) = \int_{-\infty}^{\infty} \frac{E^2 T_1}{2\pi}\left[\frac{\sin \frac{(\omega-\xi)T_1}{2}}{\frac{(\omega-\xi)T_1}{2}}\right]^2 \frac{A^2}{4}\left[\delta(\xi-\omega_c)+\delta(\xi+\omega_c)\right]d\xi$$

$$= \frac{A^2 E^2 T_1}{8\pi} \int_{-\infty}^{\infty}\left[\frac{\sin \frac{(\omega-\xi)T_1}{2}}{\frac{(\omega-\xi)T_1}{2}}\right]^2 \left[\delta(\xi-\omega_c)+\delta(\xi+\omega_c)\right]d\xi \qquad \text{(I-12)}$$

But

$$\int_a^b f(x)\delta(x-x_1)dx = f(x_1) \quad \text{For } a \le x_1 \le b \qquad \text{(I-13)}$$

So

$$S_{PSK}(\omega) = \frac{A^2E^2T_1}{8\pi}\left\{\left[\frac{\sin\frac{(\omega-\omega_c)T_1}{2}}{\frac{(\omega-\omega_c)T_1}{2}}\right]^2 + \left[\frac{\sin\frac{(\omega+\omega_c)T_1}{2}}{\frac{(\omega+\omega_c)T_1}{2}}\right]^2\right\} \qquad \text{(I-14)}$$

The above result agrees exactly with that obtained (equation I-8) by taking the Fourier transform of $R_{PSK}(\tau)$.