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CONVERGENCE ACCELERATION PROCEDURE FOR THE METHOD OF STEEPEST DESCENT

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FOR THE
METHOD OF STEEPEST DESCENT

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

ϕ	scalar performance index
x	n vector of state variables
t	scalar independent variable, time
u	m vector of control variables
f	n vector of constraining functions
ψ	q vector of terminal constraint relations
Ω	scalar stopping condition
dx, $d\phi$, $d\Omega$, $d\psi$	differentials of x, ϕ, Ω, ψ, respectively
δx, δu	variation of x or u, respectively
λ	n vector of adjoint variables
λ_ϕ, λ_ψ, λ_Ω	defined in equation (9)
dS	scalar step size in control space
$\lambda_{\phi\Omega}$, $\lambda_{\psi\Omega}$	defined after equation (15)
ν	q vector of constant Lagrange multipliers
μ	scalar constant Lagrange multiplier
W	$m \times m$ arbitrary weighting matrix
$d\beta$	q vector defined after equation (20)
$I_{\psi\psi}$, $I_{\psi\phi}$, $I_{\phi\phi}$	defined in equations (21), (22) and (23)
E	Weierstrass E-Function defined in equation (27)
H	scalar Generalized Hamiltonian ($H = \lambda^T f$)
C	scalar iteration factor defined in equation (37)
P	scalar penalty function

LIST OF SYMBOLS (Concluded)

K	scalar constant defined in equation (45)
I	$m \times m$ unity matrix

Superscripts

$(\dot{\ })$	$\frac{d(\ )}{dt}$ total time derivative
$(\ )^T$	transpose of ( )
$(\ )^{-1}$	inverse of ( )
$(\ )^*$	optimal value of ( )

Subscripts

$(\ )_x$	$\frac{\partial(\ )}{\partial x}$ partial derivative
$(\ )_{xy}$	$\frac{\partial(\ )^2}{\partial x \partial y}$ second partial derivative
$(\ )_o$	initial value of ( )
$(\ )_f$	terminal value of ( )

CONVERGENCE ACCELERATION PROCEDURE
FOR THE
METHOD OF STEEPEST DESCENT

ABSTRACT

A procedure is proposed which accelerates the convergence rate of the steepest descent or gradient optimization methods. The previously suggested procedures of selecting a preferred gradient step size for each iteration is extended by defining an easily determined, time dependent weighting matrix that approximately extremizes the penalty function or performance index. Numerical results with this modification are obtained and compared with results obtained by applying the conventional technique. A significant acceleration in the shaping of the optimal control program is realized.

INTRODUCTION

In studying procedures for accelerating the convergence rates of the classical gradient methods, it is necessary to first define the optimization problem. The purpose of this investigation is stated, and a brief background sketch is made.

Definition of the Optimization Problem

In one class of optimization problems, in particular the spacecraft trajectory optimization problem, it is desired to determine the history of the control variables in such a manner that certain specified initial and terminal constraints are satisfied while some performance index is extremized. The control variables are unspecified inputs to the system which may be chosen to control the spacecraft state; i.e., the position and velocity. The initial and terminal constraints are simply conditions on the positions and velocities that must be satisfied at the initial and terminal time, respectively. The performance index is usually a scalar function associated with the spacecraft performance and is the quantity to be extremized.

The terminal constraints are handled in either the so-called "hard" or "soft" form. In the "hard" form an effort is made to satisfy the terminal constraints identically while in the "soft" form the constraints are satisfied only approximately. It is in the former case that the performance index approach is taken because this index is extremized separate from the satisfaction of the terminal constraints. It is with the latter case that the penalty function concept emanates; i.e., a certain penalty is accepted because of the less stringent demand of only approximate terminal constraint satisfaction.

Purpose of the Investigation

The ultimate purpose of this investigation is to develop an insight into the convergence characteristics of some of the direct optimization methods. This ultimate purpose is approached by satisfying the following secondary objectives:

(1) Increase the understanding of the currently popular optimization methods so that the deficient convergence characteristics of each method are discovered.

(2) Extend and modify these methods to eliminate the deficiencies.

(3) Formulate and successfully implement a realistic example.

(4) Compare the convergence characteristics of the proposed procedures with those derived from previously proposed schemes.

Background Study of the Gradient Methods

An analytical development of a trajectory optimization theory was published by Kelley[1] in 1960. This method, referred to as the gradient method, is based on an extension of some ideas presented by Courant in 1941. A similar formulation was made, simultaneously and independently, by Bryson, Denham, Carroll and Mikami[2], and Bryson and Denham[3]. Kelley, Kopp, and Moyer[4] presented an analysis of several gradient techniques using inequality constraints on the control variables and a penalty function concept for handling terminal constraints. In an effort to determine the thrust steering program for the optimization of a second stage booster, Pfeiffer[5] developed a method of "critical direction" which is similar to the gradient techniques of Kelley and Bryson.

In 1963, more attention began to center around convergence acceleration for the gradient methods. Wagner and Jazwinski[6] presented a gradient method incorporating both terminal and instantaneous inequality constraints. This investigation also included an interesting method for determining the control step size magnitude that should be taken in the gradient direction to approximately maximize the decrease in the penalty function. A new step size is calculated for each iteration. This scheme involves making three trial forward integrations with different control step sizes, and recording the three resulting penalty function values. A second order polynominal is fitted through these points, and the step size that corresponds to the minimum value of the penalty function is selected for the next iteration. This method, therefore, takes full advantage of each adjoint integration by selecting an optimal step size for that particular iteration.

Rosenbaum[7], also in 1963, developed a method similar to a closed-loop guidance scheme that "provides rapid convergence for a variety of missions." The distinctive feature of this method is that the control step size in the gradient direction is calculated and becomes a time dependent quantity. The significant result is that larger deviations from the nominal trajectory can be tolerated while still satisfying the terminal constraints; thus, it is possible to move more rapidly toward the optimal trajectory. The approach is similar to the Λ-matrix control scheme proposed by Bryson and Denham[8]. Unfortunately, the rates of convergence relative to previously proposed methods are not adequately illustrated.

In 1964, Stancil[9] proposed a slightly different approach to the inherent gradient convergence problem. This approach is similar to Rosenbaum[7] in that a time dependent weighting matrix is calculated. Basically, the formulation follows a suggestion made, but not used, by Bryson, Denham, Carroll and Mikami[2], in which the current control program is averaged with the Eulerian control. The procedure eliminates the guessing of the performance index decrease and other weighting factors. Again, the convergence characteristics of this method are not directly compared with the previously proposed techniques.

Lewallen[10], in 1966, derives a time dependent and easily determined weighting matrix which is applicable to either minimizing a performance index or a penalty function. An analysis and comparison is made using both the proposed weighting matrix and the unity matrix. When the proposed matrix is used a significant convergence acceleration is realized.

FORMULATION

The gradient method is formulated with both hard and soft constraints even though the formulation, perhaps in different form, is presented in available literature. Inclusion of this information in the present report is encouraged (1) to make the report more self contained and (2) to provide a basis for the extensions required for convergence acceleration discussions.

Gradient Method Using Hard Constraints

It is desired to determine the control program $u(t)$, where u is an m vector, which will yield an extreme value of some performance index

$$\phi = \phi(x_f, t_f) \tag{1}$$

subject to the differential equations of motion

$$\dot{x} = f(x,u,t) \tag{2}$$

where x is an n vector and u is an m vector, while satisfying the terminal constraint relations

$$\psi = \psi(x_f,t_f) = 0 \tag{3}$$

in hard form, where ψ is a q vector. One of the terminal constraint relations may be selected as a stopping condition for the integration process,

$$\Omega = \Omega(x_f,t_f) = 0 \ . \tag{4}$$

If the differential equations (2) are linearized about some nominal path, the resulting equations become,

$$\delta\dot{x} = f_x\ \delta x + f_u\ \delta u \tag{5}$$

where f_x and f_u are partial derivatives of f with respect to x and u, respectively, and are evaluated on the nominal trajectory.

The equations adjoint to (5) are

$$\dot{\lambda} = -f_x^T\lambda \tag{6}$$

where λ is an n vector of adjoint variables. This equation may be combined with (5) to yield

$$\frac{d}{dt}(\lambda^T\delta x) = \lambda^T\ f_u\ \delta u \ . \tag{7}$$

Integrating this equation yields

$$(\lambda^T \delta x)_f = \int_{t_o}^{t_f} \lambda^T f_u \, \delta u \, dt + (\lambda^T \delta x)_o \tag{8}$$

which is designated the Fundamental Guidance Equation. The object now is to determine how initial state variations and integrated control variations influence the performance index, stopping condition, and the terminal corstraints. If, on separate trials, the terminal values of the adjoint variables are set equal to

$$\lambda_\phi^T(t_f) = \left[\frac{\partial \phi}{\partial x}\right]_f \qquad \lambda_\psi^T(t_f) = \left[\frac{\partial \psi}{\partial x}\right]_f \qquad \lambda_\Omega^T(t_f) = \left[\frac{\partial \Omega}{\partial x}\right] \tag{9}$$

where λ_ϕ is an n vector, λ_ψ is a $n \times q$ matrix and λ_Ω is an n vector, the desired relations are seen to be

$$d\phi = \int_{t_o}^{t_f} \lambda_\phi^T f_u \, \delta u \, dt + (\lambda_\phi^T \delta x)_o + \dot{\phi} \, dt_f \tag{10}$$

$$d\psi = \int_{t_o}^{t_f} \lambda_\psi^T f_u \, \delta u \, dt + (\lambda_\psi^T \delta x)_o + \dot{\psi} \, dt_f \tag{11}$$

$$d\Omega = \int_{t_o}^{t_f} \lambda_\Omega^T f_u \, \delta u \, dt + (\lambda_\Omega^T \delta x)_o + \dot{\Omega} \, dt_f \tag{12}$$

where $(\dot{\ }) = \left[\frac{\partial()}{\partial x}\dot{x} + \frac{\partial()}{\partial t}\right]_f$ and $d() = [\delta() + (\dot{\ })\,dt]_f$.

This formulation allows the specification of an allowable step size to be taken in control space defined by

$$dS = \int_{t_o}^{t_f} \tfrac{1}{2}\,\delta u^T\, W\, \delta u\, dt \tag{13}$$

where the step is a weighted quadratic function of the control deviation. The weighting matrix W is included to improve the convergence characteristics by giving more weight to regions of low sensitivity. However, it is often chosen unity because of the lack of knowledge concerning the region of sensitivity. The criteria used for determining the best elements for this weighting matrix are not easy to determine and are usually found through trial and error procedures.

The stopping condition (4) is to be identically satisfied so $d\Omega$ in (12) is equated to zero. The terminal time variation dt_f is eliminated from (10) and (11) to yield

$$d\phi = \int_{t_o}^{t_f} \lambda_{\phi\Omega}^T\, f_u\, \delta u\, dt + (\lambda_{\phi\Omega}^T \delta x)_o \tag{14}$$

$$d\psi = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T\, f_u\, \delta u\, dt + (\lambda_{\psi\Omega}^T \delta x) \tag{15}$$

where $\lambda_{\phi\Omega} = \lambda_\phi - \frac{\dot{\phi}}{\dot{\Omega}}\lambda_\Omega \qquad \lambda_{\psi\Omega} = \lambda_\psi - \lambda_\Omega \frac{\dot{\psi}^T}{\dot{\Omega}}$.

The total variation of the performance index may be represented by

$$d\phi = \int_{t_o}^{t_f} \lambda_{\phi\Omega}^T f_u \, \delta u \, dt + (\lambda_{\phi\Omega}^T \delta x)_o$$

$$+ \nu^T \left[d\psi - \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u \, \delta u \, dt - (\lambda_{\psi\Omega}^T \, \delta x)_o \right]$$

$$+ \mu \left[dS - \int_{t_o}^{t_f} \frac{1}{2} \, \delta u^T \, W \, \delta u \, dt \right] \tag{16}$$

where the terminal constraints and the control step are adjoined by the use of the ν^T and μ Lagrange multipliers, respectively. Since it is desired to determine the control variation which corresponds to the maximum change in the performance index, the first variation of (16) must vanish; therefore

$$\delta(d\phi) = \int_{t_o}^{t_f} (\lambda_{\phi\Omega}^T f_u - \nu^T \lambda_{\psi\Omega} f_u - \mu \, \delta u^T W) \, \delta^2 u \, dt \equiv 0 \, . \tag{17}$$

This implies that the desired control variation is

$$\delta u = \frac{1}{\mu} W^{-1} f_u^T (\lambda_{\phi\Omega} - \lambda_{\psi\Omega} \nu) \tag{18}$$

and when this equation is substituted back into (13) and (15), the values of ν and μ are seen to be

$$\nu = -\mu I_{\psi\psi}^{-1} \, d\beta + I_{\psi\psi}^{-1} \, I_{\psi\phi} \tag{19}$$

and

$$\mu = \pm\left[\frac{I_{\phi\phi} - I_{\psi\phi}^T \, I_{\psi\psi}^{-1} \, I_{\psi\phi}}{dS - d\beta^T \, I_{\psi\psi}^{-1} \, d\beta}\right]^{1/2} \tag{20}$$

where $\quad d\beta = d\psi - (\lambda_{\psi\Omega}^T \delta x)_o$

$$I_{\psi\psi} = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T \, f_u \, W^{-1} \, f_u^T \, \lambda_{\psi\Omega} \, dt \tag{21}$$

$$I_{\psi\phi} = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T \, f_u \, W^{-1} \, f_u^T \, \lambda_{\phi\Omega} \, dt \tag{22}$$

$$I_{\phi\phi} = \int_{t_o}^{t_f} \lambda_{\phi\Omega}^T \, f_u \, W^{-1} \, f_u^T \, \lambda_{\phi\Omega} \, dt \tag{23}$$

and $I_{\psi\psi}$ is a $q \times q$ matrix, $I_{\psi\phi}$ is a q vector, and $I_{\phi\phi}$ is a scalar.

Now combining (18) through (23) yields the desired control program

$$\delta u = \pm W^{-1} f_u^T(\lambda_{\phi\Omega} - \lambda_{\psi\Omega} I_{\psi\psi}^{-1} I_{\psi\phi}) \left[\frac{dS - d\beta^T I_{\psi\psi}^{-1} d\beta}{I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi} I_{\psi\phi}}\right]^{1/2}$$

$$+ W^{-1} f_u^T \lambda_{\psi\Omega} I_{\psi\psi}^{-1} d\beta \qquad (24)$$

where the positive (negative) sign is used if ϕ is to be maximized (minimized). The previous control program is now modified by

$$u_{new} = u_{old} + \delta u . \qquad (25)$$

The computational procedure for the Gradient Method using hard constraints may be summarized as follows:

(1) Integrate the n differential equations of motion (2) forward, using an assumed control program and the desired initial conditions. This integration is continued until the stopping condition (4) is satisfied. The state variable values are stored at each integration step.

(2) Integrate the adjoint equations (6) backwards $q+2$ times with the starting conditions (9). The coefficient matrix f_x is formed from the state variables stored during the forward integration.

(3) Integrate the I equations (21) through (23) backwards simultaneously with the adjoint equations using initial conditions of zero to yield values at t_o for $I_{\psi\psi}$, $I_{\psi\phi}$, and $I_{\phi\phi}$.

(4) Select a desired improvement in the terminal dissatisfaction $d\psi$ for the next iteration.

(5) Select a reasonable value for the mean square control deviation from the previous control program by using

$$dS = \frac{1}{2}\,\delta u^2_{ave}\,(t_f - t_o),$$

which provides a value of dS .

(6) Use the selected value of $d\psi$ and dS to calculate the numerator under the radical in (24). If this quantity is negative, determine the $d\psi$ that makes the quantity vanish. If it is positive, use the quantity as it is.

(7) Calculate the δu as given by (24) and alter the assumed control program. The quantity dS must be decreased according to some selected criteria to prevent stepping across the optimal point into a nonoptimal region.

(8) The procedure is continued until the control variations are less than some preselected value.

Convergence Acceleration Using Hard Constraints

A primary objective of the present investigation is to develop an iterative scheme that reduces some of the arbitrariness and increases the convergence rate of the Gradient Method when using hard constraints. Since the weighting matrix W, introduced in (13) is arbitrary, some rational basis for its selection is needed.

The problem is approached by examining an integral form of the Weierstrass E-Function which approximates the change in the performance index. This change is

$$d\phi \cong \int_{t_o}^{t_f} E(x^*,\dot{x}^*,\dot{x},t)\, dt \tag{26}$$

where E is the Weierstrass E-Function as developed by Gelfand and Fomin[11]. The E-Function is defined as

$$E = f(x^*,\dot{x},t) - f(x^*,\dot{x}^*,t) - \frac{\partial f}{\partial \dot{x}^*}(x^*,\dot{x}^*,t)\,(\dot{x} - \dot{x}^*) \tag{27}$$

and for the system to be considered

$$f(x,\dot{x},t) = H(x,u,t) - \lambda^T\dot{x} . \tag{28}$$

The asterisks refer to the optimal path, and the absence of asterisks refer to any nearby path. From the calculus of variations, a necessary condition for the existence of a minimum valued performance index is that E be non-negative during the interval $t_o \leq t \leq t_f$.

It is noted, by examining (2), that a variation in control is accompanied by a variation in $\dot{x}$, and that a state variation will occur only after a finite duration of time. Hence, the expansion of (26) is made by considering that the control deviation is not accompanied by a change in state. The relation (26) is now written

$$d\phi \cong \int_{t_o}^{t_f} (H - H^*)\, dt \,. \tag{29}$$

The first term in the integrand may be expanded in a Taylor's series about the optimal path at each point in time to yield

$$H \cong H^* + H_u^* \,\delta u + \frac{1}{2}\, \delta u^T H_{uu}^* \,\delta u + \cdots\cdots \tag{30}$$

and substituting the above equation into (29) and recalling that $H_u^* = 0$ on the optimal path results in,

$$d\phi \cong \int_{t_o}^{t_f} \frac{1}{2}\, \delta u^T H_{uu}^* \,\delta u \, dt \,. \tag{31}$$

This equation represents the performance index change associated with the deviation of the control program from an optimal control program. It must be stated that H_{uu}^* is not known until the optimal trajectory is converged upon, but the expression (31) becomes increasingly accurate as convergence progresses. It is during this terminal phase of convergence that the Gradient Methods have the greatest need for convergence acceleration.

The convergence acceleration technique proposed in the present investigation uses the expression (31) to approximate the performance index change rather than the one previously mentioned in (14). When this is done and the control step size constraint is not included, $d\phi$ may be written

$$d\phi = \int_{t_o}^{t_f} \tfrac{1}{2}\, \delta u^T H_{uu}^{*} \,\delta u \, dt$$

$$+ \nu^T \left[d\beta - \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u \,\delta u \, dt \right] . \tag{32}$$

Requiring that $\delta(d\phi) \equiv 0$ yields

$$\delta u = (H_{uu}^{*})^{-1} f_u^T \lambda_{\psi\Omega} \nu \tag{33}$$

and when this equation is substituted into (15),

$$\nu = I_{\psi\psi}^{-1} d\beta \tag{34}$$

where

$$I_{\psi\psi} = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u (H_{uu}^{*})^{-1} f_u^T \lambda_{\psi\Omega} \, dt . \tag{35}$$

Therefore, the desired control deviation becomes

$$\delta u = (H_{uu}^{*})^{-1} f_u^T \lambda_{\psi\Omega} I_{\psi\psi}^{-1} d\beta . \tag{36}$$

In comparing equations (36) and (24), it should be noted that (36) is simply the last term of (24) where W has been replaced with H^*_{uu} . Since the step size constraint was eliminated, the control variation can be controlled by requesting only a percentage of the terminal dissatisfaction to be corrected by

$$d\beta = -C \, d\beta \, , \tag{37}$$

where $0 \leq C \leq 1.0$.

It is interesting to note that the control variation law (36) proposed in the present investigation is similar to the one successfully used by Tapley and Fowler[12] in a closed-loop control scheme.

Gradient Method Using Soft Constraints

The theoretical development of the Gradient Method using soft terminal constraints is similar to that used for hard constraints. The primary difference is that the terminal constraints are adjoined to the performance index to form a penalty function

$$P(x_f,t_f) = W_o \, \phi^2(x_f,t_f) + \sum_{i=1}^{9} W_i \, \psi_i^2(x_f,t_f) \tag{38}$$

where the W_i's are weighting constants. If these constants are sufficiently large, minimizing the penalty function is essentially the same as minimizing the performance index while driving the terminal constraints to zero.

To determine how this penalty function is related to initial state variations and the integrated control variations, the Fundamental Guidance Equation (8) is used. Selecting the starting conditions for the adjoint equations (6) to be

$$\lambda_P^T(t_f) = \left[\frac{\partial P}{\partial x}\right]_f \qquad \lambda_\Omega^T(t_f) = \left[\frac{\partial \Omega}{\partial x}\right]_f \tag{39}$$

where λ_P is an n vector and λ_Ω is a scalar, yields

$$dP = \int_{t_o}^{t_f} \lambda_P^T f_u \, \delta u \, dt + (\lambda_P^T \delta x)_o + \dot{P} \, dt_f \tag{40}$$

$$d\Omega = \int_{t_o}^{t_f} \lambda_\Omega^T f_u \, \delta u \, dt + (\lambda_\Omega^T \delta x)_o + \dot{\Omega} \, dt_f \, . \tag{41}$$

If the stopping condition $d\Omega$ is identically satisfied, the penalty function change may be expressed as

$$dP = \int_{t_o}^{t_f} \lambda_{P\Omega}^T f_u \, \delta u \, dt + (\lambda_{P\Omega}^T \delta x)_o \tag{42}$$

where

$$\lambda_{P\Omega}^T = \lambda_P - \frac{\dot{P}}{\dot{\Omega}} \lambda_\Omega \, . \tag{43}$$

Now, it is desired to determine the control variation which maximized the penalty function change dP . Adjoining an unweighted control step constraint to the penalty function change yields

$$dP = \int_{t_o}^{t_f} \lambda_{P\Omega}^T f_u \, \delta u \, dt + \mu \left[dS - \int_{t_o}^{t_f} \tfrac{1}{2} \delta u^T \delta u \, dt \right] + (\lambda_{P\Omega}^T \delta x)_o \, . \tag{44}$$

Requiring that $\delta(dP)$ vanish implies that

$$\delta u = K f_u^T \lambda_{P\Omega} = K H_u^T \tag{45}$$

where K is a constant equal to $1/\mu$, H_u is defined as $\lambda_{P\Omega}^T f_u$. This equation is similar to the one developed by Wagner and Jazwinski(6). The constant K can be interpreted as a control step size in the gradient direction.

The penalty function change is evaluated by substituting (45) into (42) to yield

$$dP = K \int_{t_o}^{t_f} H_u H_u^T \, dt \, . \tag{46}$$

The computational procedure for the Gradient Method using soft constraints may be summarized as follows:

(1) Integrate the n differential equations of motion (2) forward using an assumed control program and the desired initial conditions. This integration is continued until the stopping condition (4) is satisfied. The state variable values are stored at each integration step.

(2) Integrate the n adjoint equations (6) backward one time with the starting condition (43). The coefficient matrix f_x is formed from the state variables stored during the forward integration.

(3) Having obtained the solution $\lambda_{P\Omega}^T(t)$, the term $H_u = \lambda_{P\Omega}^T f_u$ may be formed. The square of H_u may be integrated from t_o to t_f and the step size K may be determined by specifying a desired penalty function change dP .

(4) The control variation may be determined from (45) and applied to the previous control program.

(5) The procedure is continued until the control variations are less than some preselected value.

It must be noted that the specified penalty function change, and hence the step size K , is arbitrary, and the judicious selection of K becomes a key factor in increasing the convergence rate. An automatic procedure for its selection is desired.

Convergence Acceleration Using Soft Constraints

The formulation for determining a convergence acceleration procedure when using soft constraints is similar to that when hard constraints are used. The performance index change or penalty function change in the case of soft terminal constraints is approximated by

$$dP \cong \int_{t_o}^{t_f} E(x^*, \dot{x}^*, \dot{x}, t)\, dt \, . \tag{47}$$

In the same manner as discussed for hard constraints, this relation may be reduced to

$$dP \cong \int_{t_o}^{t_f} (H - H^*)\, dt \tag{48}$$

which states that the penalty function change may be approximated by a time integral of the Hamiltonian deviation from the optimal value.

Now, the second term in the integrand of (48) may be expanded in a Taylor's series about the current (nonoptimal) path at each point in time to yield

$$H^* \cong H + H_u\, \delta u + \frac{1}{2}\, \delta u^T\, H_{uu}\, \delta u + \cdots \, . \tag{49}$$

Substituting this expression into (48) results in

$$dP \cong \int_{t_o}^{t_f} - (H_u \, \delta u + \frac{1}{2} \, \delta u^T \, H_{uu} \, \delta u) \, dt \tag{50}$$

which states the penalty function change in terms of the control deviations.

It is desired to determine a control deviation which will maximize the penalty function change on each iteration, and hence a necessary condition is that $\delta(dP) \equiv 0$ which leads to

$$\delta u = -H_{uu}^{-1} \, H_u^T \tag{51}$$

where H_{uu} and H_u are evaluated on the current trajectory. This equation implies that the optimal control is in the negative gradient direction, weighted by H_{uu}^{-1}. By comparing (51) with (45), it is seen that the constant K, calculated only once for each iteration, is replaced by the time dependent and easily calculated weighting matrix H_{uu}^{-1}.

APPLICATION AND RESULTS

The theoretical developments made in the previous section are applied to a realistic example which is difficult enough to demonstrate the convergence advantages but simple enough to be easily implemented. The example chosen is the two-dimensional minimum time, constant low thrust Earth-Mars transfer trajectory. The transfer is assumed to leave the Earth region with initial conditions corresponding to that of Earth, transfer through heliocentric space to match condition of the Mars orbit. The orbital parameters used are shown in Appendix A, and the differential equations used are shown in Appendix B. An example procedure for setting up the Method of Steepest Descent using hard constraints is shown in Appendix C.

The Method of Steepest Descent using hard constraints is selected to illustrate the convergence acceleration procedure. The value of using the H^*_{uu} matrix in the expression for control deviation (36) rather than using the unity matrix in the conventional formulation (24) is illustrated in Figures 1, 2, and 3. The three figures represent cases where three widely different initially assumed solutions or control programs are used. The plots are of thrust angle above local horizontal in degrees as a function of mission time in days. Each figure also includes the Eulerian or optimal control program solution so that the state of convergence of the other illustrated solutions may be assessed. The two remaining solutions on each figure are the ones that have been developed after 13 iterations. The curves marked by $W = I$ used the conventional technique described by (24) and the ones marked by $W = H^*_{uu}$ used the proposed technique (36).

The significant fact illustrated is that for all three assumed solutions, the control program, after 13 iterations, that uses the proposed acceleration procedure is well ahead of the conventional procedure in shaping the curve. Both procedures ultimately approach the Eulerian program, therefore illustration of the converged solution is not instructive.

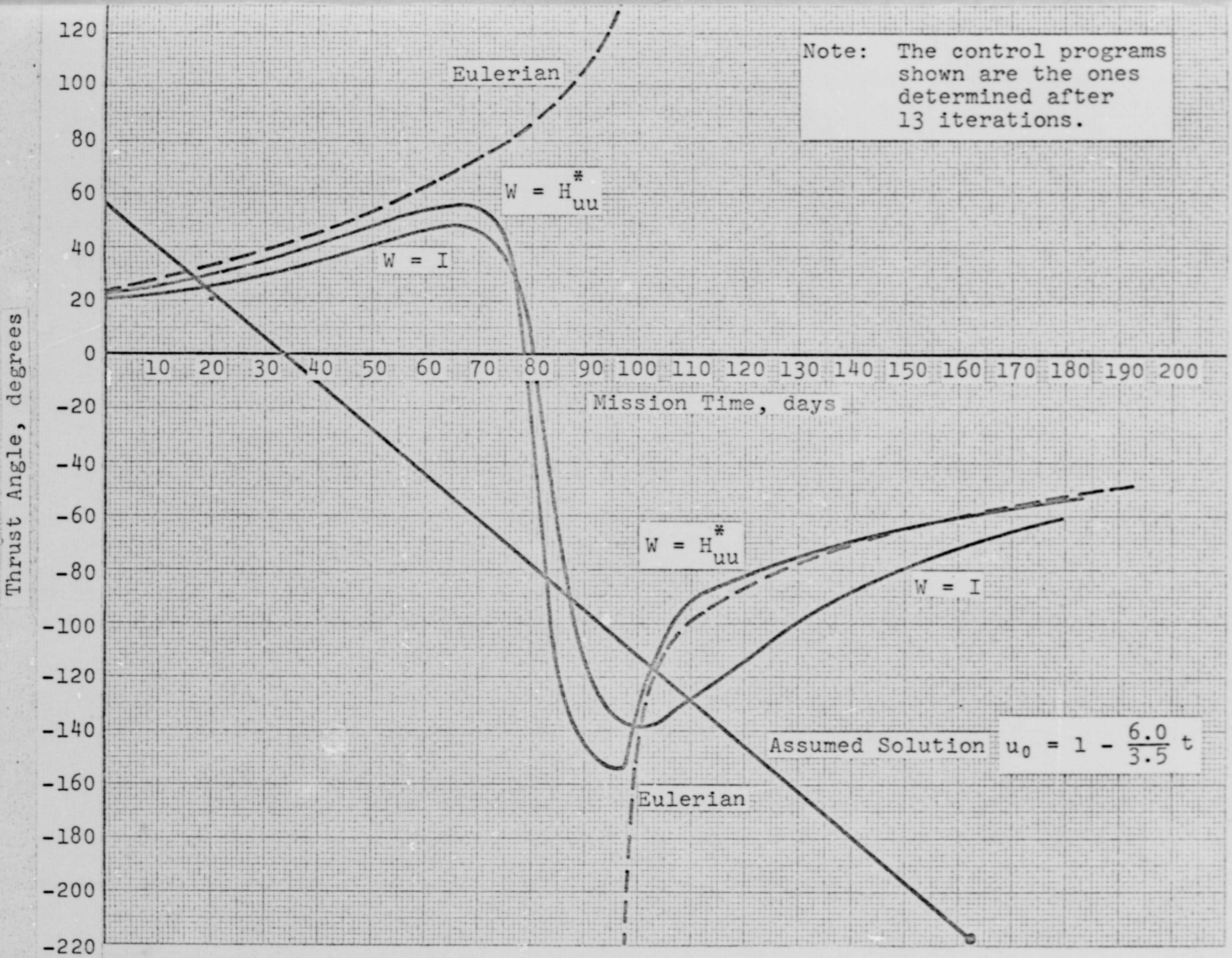

Figure 1. Influence of Weighting Matrix H^*_{uu} on Optimal Control Programs Shaping Using MSD.

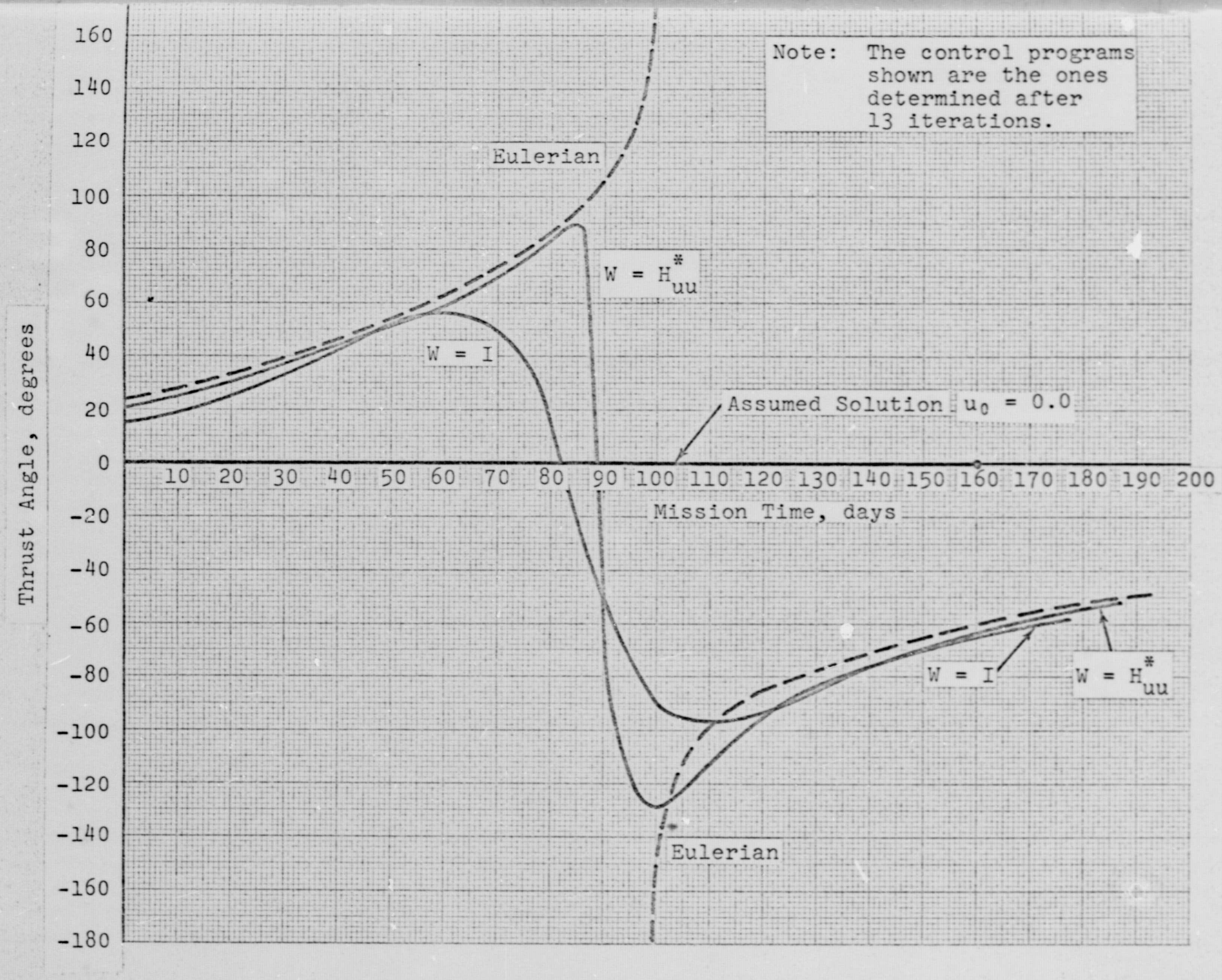

Figure 2. Influence of Weighting Matrix H^*_{uu} on Optimal Control Program Shaping Using MSD.

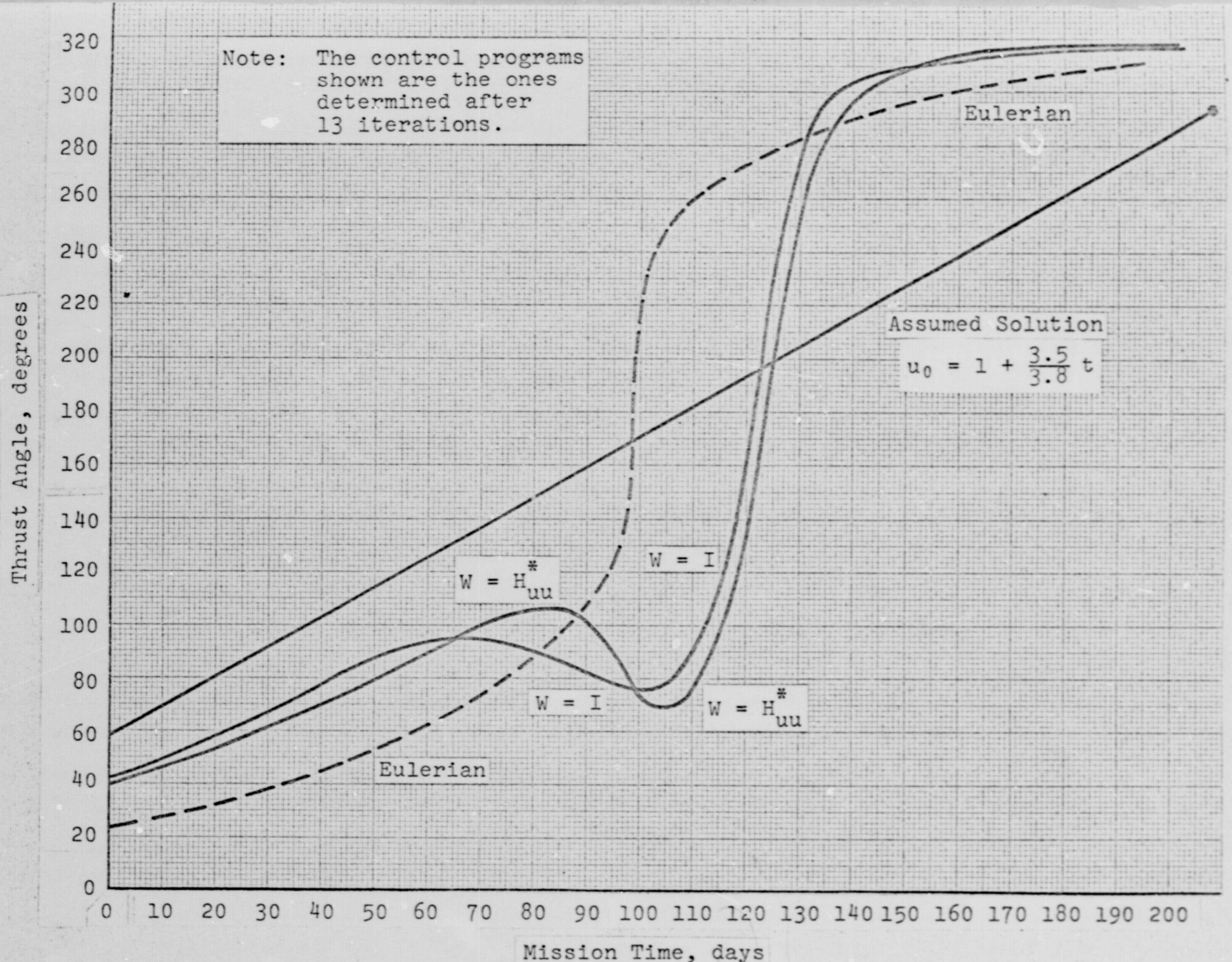

Figure 3. Influence of Weighting Matrix H^*_{uu} on Optimal Control Program Shaping Using MSD.

Although it is not shown, for the $W = I$ case illustrated in Figure 2, approximately 12 more iterations are required to duplicate the shape obtained by the $W = H^*_{uu}$ case in only 13 iterations. Hence, for this particular case, the proposed procedure reduces the computational time required to 50 percent of the time required by the conventional method.

One additional piece of information that can be extracted from the figures is that of how the assumed solution influences the convergence rate for this particular problem. Since the control program state of development is shown for the 13^{th} iteration in each case, a comparison may be made.

CONCLUSIONS AND RECOMMENDATIONS

The conclusions of this investigation are that the proposed procedure, where $W = H^*_{uu}$, produces a significant acceleration in the convergence rate of the Method of Steepest Descent using hard constraints. In one of the cases presented, the computational time was reduced to one-half of that previously required. A byproduct in the investigation is seen in that convergence occurs for three widely different initially assumed control programs--a real contrast to the highly sensitive indirect methods. It is seen, however, that the assumed control program does influence the rate of convergence. This is illustrated in the figures by comparing the control program shape after 13 iterations for each different initially assumed solution.

This theoretical development for the Method of Steepest Descent with soft constraints has not been verified by application to the above example. It is recommended that this be completed, so that a comparison of the two procedures can be made.

Appendix A

ORBITAL PARAMETERS

Appendix A

ORBITAL PARAMETERS

Astronomical Unit, AU	$.14959870 \times 10^{12}$ meters
Orbital Radius of Earth, r_e	$.10000000 \times 10^{1}$ AU
Orbital Radius of Mars, v_m	$.15236790 \times 10^{1}$ AU
Gravitational Constant of Sun, GM	$.13271504 \times 10^{21}$ meters^3/sec^2
Initial Spacecraft Mass, m_c	$.67978852 \times 10^{3}$ kilograms
Spacecraft Thrust, T	$.40312370 \times 10^{1}$ newtons
Spacecraft Mass Rate, $\dot{m}$	$.10123858 \times 10^{-4}$ kilograms/sec

Appendix B
DIFFERENTIAL EQUATIONS

Appendix B
DIFFERENTIAL EQUATIONS

The differential equations used in this investigation are (1) the differential equations of motion

$$z_1 = \dot{u} = \frac{r^2}{r} - \frac{GM}{r^2} + \frac{T \sin \beta}{m} = f_1$$

$$z_2 = \dot{v} = -\frac{uv}{r} + \frac{T \cos \beta}{m} = f_2$$

$$z_3 = \dot{r} = u = f_3$$

$$z_4 = \dot{\theta} = \frac{v}{r} = f_4$$

and (2) the adjoint differential equations

$$\dot{\lambda}_1 = \left(\frac{v}{r}\right) \lambda_2 - \lambda_3$$

$$\dot{\lambda}_2 = -\left(\frac{2v}{r}\right) \lambda_1 + \left(\frac{u}{r}\right) \lambda_2 - \left(\frac{1}{r}\right) \lambda_4$$

$$\dot{\lambda}_3 = \left(\frac{\Upsilon^2}{r^2} - \frac{2GM}{r^3}\right) \lambda_1 - \left(\frac{uv}{r^2}\right) \lambda_2 + \left(\frac{v}{r^2}\right) \lambda_4$$

$$\dot{\lambda}_4 = 0 .$$

Appendix C
EXAMPLE PROCEDURE

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EXAMPLE PROCEDURE

The differential equations of motion are integrated forward from t_o with starting conditions

$$z(t_o) = \begin{bmatrix} u \\ v \\ r \\ \theta \end{bmatrix}_{t_o}$$

and some initially assumed control program $\beta(t)$.

The performance index to be minimized is

$$\phi = t_f$$

and the terminal constraints are

$$\psi_1 = u(t_f) - u_f = 0$$

$$\psi_2 = v(t_f) - v_f = 0$$

$$\psi_3 = r(t_f) - r_f = 0 .$$

The stopping condition used is $\Omega = \theta(t_f) - \theta_f = 0$, and the starting conditions for the backward integration are

$$\lambda_\phi^T(t_f) = \left[\frac{\partial\phi}{\partial x}\right]_f = [0 \quad 0 \quad 0 \quad 0]$$

$$\lambda_4^T(t_f) = \left[\frac{\partial \phi}{\partial x}\right]_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_\Omega^T(t_f) = \left[\frac{\partial \Omega}{\partial x}\right]_f = [0 \quad 0 \quad 0 \quad 1] \ .$$

The time rates of change of performance index, terminal constraints, and stopping condition are

$$\dot{\phi} = \left[\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \dot{x}\right]_f = 1$$

$$\dot{\psi} = \left[\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} \dot{x}\right]_f = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix}_f$$

$$\dot{\Omega} = \left[\frac{\partial \Omega}{\partial t} + \frac{\partial \Omega}{\partial x} \dot{x}\right]_f = \dot{\theta}_f \ .$$

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