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AN INVESTIGATION OF ASYMPTOTIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

MANNED SPACECRAFT CENTER

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LIST OF SYMBOLS

A_n, B_n	Constant square matrices
$A(x), B(x)$	Variable coefficient matrices
$\lvert\lvert E \rvert\rvert$	Norm of matrix E
h, k, ℓ, n, N	Integers
$\underline{p}_n, \underline{q}_n, \underline{r}_n, \underline{y}_n$	Constant vectors
$P(x)$	Variable transformation matrix
$\underline{w}(x)$	Dependent vector solution of a comparison O D E
$W(x), Y(x)$	Fundamental matrix solutions of O D E 's
x	Independent variable
$\underline{y}(x)$	Dependent vector solution of the O D E
$\underline{\tilde{y}}(x)$	Dependent vector solution of the transformed O D E
$\underline{\tilde{y}}_1^n(x)$	*nth* variable vector iterant
$\{\underline{\tilde{y}}_1\}$	One-parameter family of solutions of the transformed O D E

$\{\tilde{\underline{y}}_2\}$	Two-parameter family of solutions of the transformed O D E
$\|\|\underline{y}\|\|$	Norm of the vector $\underline{y}$

AN INVESTIGATION OF ASYMPTOTIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

By Robert M Myers

SUMMARY

This paper presents a brief study of power series, Frobenius and Thome' solutions of ordinary differential equations In addition, an elementary proof of the existence of actual solutions corresponding to the formal Thome' solutions is given Moreover, a numerical method is developed which enables one to continue the accurate values of the Thome' solutions to small values of the independent variable

INTRODUCTION

Consider the following *Nth* order homogenous system of ordinary differential equations (O D E)

$$\frac{d\underline{y}(x)}{dx} = \left\{ x^h \sum_{n=o}^{\infty} A_n x^{-n} \right\} \underline{y}(X) \tag{1}$$

where $\sum_{n=o}^{\infty} A_n x^{-n}$ converges for $|x| \geq a > 0$ and h is an integer

It is quite natural to attempt to determine solutions of the form

$$(2) \qquad \underline{y}(x) = \sum_{n=o}^{\infty} \underline{y}_n x^{-n}$$

If (2) is formally substituted into the system of O D E 's (1) one will obtain the equation

$$(3) \qquad -\sum_{n=2}^{\infty} (n-1)\underline{y}_{n-1} x^{-n} = x^h \sum_{n=o}^{\infty} \left(\sum_{k=o}^{n} A_{n-k} \underline{y}_k \right) X^{-n}$$

Assume for now that A_o is a nonsingular matrix and $h \geq 0$ Then, one obtains by equating coefficients of like powers of x

$$x^h \quad \underline{0} = A_o \underline{y}_o , \quad \underline{y}_o = \underline{0} \text{ since } A_o \text{ is nonsingular}$$

$$(4) \quad x^{h-1}. \quad \underline{0} = A_1 \underline{y}_o + A_o \underline{y}_1 = A_o \underline{y}_1 \rightarrow \underline{y}_1 = \underline{0}$$

$$x^{h-2} \quad \underline{0} = A_2 \underline{y}_o + A_1 \underline{y}_1 + A_o \underline{y}_2 = A_o \underline{y}_2 \rightarrow \underline{y}_2 = \underline{0}$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$x^o \quad \underline{0} = A_h \underline{y}_o + \cdots + A_1 \underline{y}_{h-1} + A_o \underline{y}_h = A_o \underline{y}_h \rightarrow \underline{y}_h = \underline{0}$$

$$x^{-1} \quad \underline{0} = A_{h+1} \underline{y}_o + \cdots + A_1 \underline{y}_h + A_o \underline{y}_{h+1} \rightarrow \underline{y}_{h+1} = \underline{0}$$

$$x^{-2} \quad -\underline{y}_1 = A_{h+2} \underline{y}_o + \cdots + A_1 \underline{y}_{h+1} + A_o \underline{y}_{h+2}$$

$$\text{but} \quad \underline{y}_1 = \underline{0} \rightarrow \underline{y}_{h+2} = \underline{0}$$

It is clear that substitution of a power series solution into the O D E when A_o is nonsingular and $h \geq o$ leads only to the trivial solution, i e , one obtains only the solution $\underline{y}(x) \equiv \underline{0}$ Also it is rather exceptional to obtain nontrivial power series solutions for the case where A_o is singular

Now consider the case $h = -1$ and A_o does not have a nonpositive integer eigenvalue Once again, one can attempt to substitute formally a power series solution into the differential equation

$$\text{(5)} \qquad \frac{x d\underline{y}(x)}{dx} = \left(\sum_{n=o}^{\infty} A_n x^{-n}\right) \underline{y}(x)$$

$$-\sum_{n=1}^{\infty} n \underline{y}_n x^{-n} = \sum_{n=o}^{\infty}\left(\sum_{k=o}^{n} A_{n-k}\underline{y}_k\right) x^{-n}$$

One can attempt to solve for $\underline{y}_n$ by equating like powers of x

(6)

$x^o \qquad \underline{0} = A_o \underline{y}_o + \underline{y}_o = \underline{0}$ since zero is not an eigenvalue of A_o .

$x^{-1} \qquad -\underline{y}_1 = A_1 \underline{y}_o + A_o \underline{y}_1 = A_o \underline{y}_1$ but -1 is not an eigenvalue of $A_o \therefore \underline{y}_1 = \underline{0}$

Similarly, one can conclude $\underline{y}_2, \underline{y}_3, \cdots$ are all zero vectors Hence, the formal procedure leads only to the trivial solution $\underline{y}(x) \equiv \underline{0}$

Now consider the case $h = -2$

$$(7)(a) \quad \sum_{n=2}^{\infty} -(n-1)\underline{y}_{n-1}x^{-n} = \sum_{n=2}^{\infty}\left(\sum_{k=o}^{n} A_{n-2-k}\underline{y}_k\right)x^{-n}$$

x^{-2} $\quad -\underline{y}_1 = A_o\underline{y}_o$

choose an arbitrary value of $\underline{y}_o$ and this determines $\underline{y}_1$

(b) $\quad x^{-3}$ $\quad -2\underline{y}_2 = A_1\underline{y}_o + A_o\underline{y}_1$

clearly $\underline{y}_2$ is determined from $\underline{y}_o$ and $\underline{y}_1$

. . . .

$x^{-(n+1)}$ $\quad -n\underline{y}_n = A_{n-1}\underline{y}_o + A_{n-2}\underline{y}_1 + \cdots + A_o\underline{y}_{n-1}$ thus, $\underline{y}_n$ is determined by

$A_{n-1}, \cdots, A_o, \underline{y}_o, \cdots, \underline{y}_{n-1}$

Notice that the determination of $\underline{y}_n$ proceeds regardless of the exceptional properties of the lead coefficient matrix A_o. Indeed, the power series solution can be used when $h \leq -2$ and h is an integer. So far, the procedure outlined above has been strictly formal, i.e., no justification of the validity of the formal solution has been given. What is required is a justification of the procedure of term-by-term differentiation and equating like powers of x. It is somewhat surprising that this procedure is completely justified provided one establishes convergence of the formal power series (see [1] or [3]). For proof of convergence, see [2] or [4].

Now, return to the case $h = -1$ For simplicity, consider $N = 1$, i e, the order of the system is one (Scalar problem)

$$(8) \qquad xy' = \left(\sum_{n=o}^{\infty} a_n x^{-n}\right) y$$

For this problem, a slightly more general substitution turns out to be successful Consider

$$(9) \qquad y = \left(\sum_{n=o}^{\infty} y_n x^{-n}\right) x^{\alpha} = \sum_{n=o}^{\infty} y_n x^{\alpha-n}$$

Substitute (9) into (8) and equate like powers of x

x^{α} $\quad \alpha y_o = a_o y_o$

(Clearly this equation will be satisfied for $\alpha = a_o$ and arbitrary y_o)

(10)

$x^{\alpha-1}$ $\quad (a_o - 1)y_1 = a_o y_1 + a_1 y_o$

or $-y_1 = a_1 y_o$

(Thus, once a value of y_o is assigned, then y_1 is determined)

. . . .

$x^{\alpha-n}$ $\quad (a_o - n)y_n = a_o y_n + a_1 y_{n-1} + \cdots + a_n y_o$

$-ny_n = a_1 y_{n-1} + \cdots + a_n y_o$

(y_n is determined from $a_1, \cdots, a_n, y_o, \cdots, y_{n-1}$)

The solution (9) is referred to as a Frobenius solution The more general problem of an Nth order system also admits solutions of this form There are certain exceptional cases that must be dealt with separately for $N \geq 2$ Since there are several good references on the so-called regular singularity [2] or singularity of the first kind [4], the case $h = -1$ will not be treated more fully here The formal manipulations can be *a posteriori* justified by merely establishing convergence of the expansion (see [4])

Now, consider the case $h \geq 0$ and $N = 1$

$$(11) \qquad y'(x) = \left(x^h \sum_{n=o}^{\infty} a_n x^{-n}\right) y$$

Attempt to determine a solution of the foım

$$(12) \qquad y(x) = \left(\sum_{n=o}^{\infty} y_n x^{-n}\right) x^{\sigma} e^{p(x)}$$

where

$$p(x) = p_{h+1} x^{h+1} + \cdots + p_1 x$$

If (12) is formally substituted into (11), then one obtains

$$\left\{\sum_{n=o}^{\infty}(\alpha-n)y_n x^{\alpha-(n+1)} + \sum_{n=o}^{\infty} y_n x^{\alpha-n}\left((h+1)p_{h+1}x^h\right.\right.$$

$$(13) \qquad \left.\left. + \cdots + 2p_2 x + p_1\right)\right\} e^{p(x)} = x^h\left\{\sum_{n=o}^{\infty} a_n x^{-n}\right\} \times$$

$$\left\{\sum_{n=o}^{\infty} y_n x^{\alpha-n} e^{p(x)}\right\}$$

Formally solving (13) leads to the relations

$$p_{h+1} = \frac{a_o}{h+1}$$

$$p_h = \frac{a_1}{h}$$

$$\cdots\cdots\cdots$$

$$(14) \qquad p_1 = a_h$$

$$\alpha = a_{h+1}$$

y_o is arbitrary

$$-y_1 = a_{h+2}y_o$$

$$-2y_2 = a_{h+3}y_o + a_{h+2}y_1$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$-ny_n = a_{h+n+1}y_o + a_{h+n}y_1 + \cdots + a_{h+2}y_{n-1}$$

The success of formal solutions (Thome' solutions) of the form (12) have been known for many years Unfortunately, these solutions seldom lead to convergent expansions when $N \geq 2$ Thus, for this case ($h \geq 0$) one can rarely justify *a posteriori* the formal manipulations Around the turn of the century, Poincare' proved that there is a connection between actual solutions of the differential equation and these formal Thome' solutions

The integer $h + 1$ is referred to as the rank of the singularity at $x = \infty$

THEORETICAL INVESTIGATION OF THOME' SOLUTIONS

It is best now to consider a particular problem If the problem is carefully selected, the general theory will be illustrated but perhaps the involved analysis can be simplified In any event, one would wish to develop an intuitive understanding of the general theory and perhaps eventually read one of the excellent references ([4], [6] or [8]) available to more fully appreciate and utilize the general theory

Consider the third-order system with a singularity of rank one

$$(15) \qquad \underline{y}'(x) = \left\{\text{diagonal}(0,1,-1) + \frac{1}{x}\,\text{diagonal}\left(-1,\frac{1}{2},\frac{1}{2}\right) + \frac{1}{x^2}\,\text{diagonal}(2,1,10)\right\}\underline{y}(x)$$

(16) let
$$A_o = \text{diagonal}(0,1,-1)$$
$$A_1 = \text{diagonal}\left(-1,\tfrac{1}{2},\tfrac{1}{2}\right)$$
$$A_2 = \text{diagonal}(2,1,10)$$
$$A_n = 0 \quad \text{for} \quad n \geq 3$$
$$(x) = A_o + \frac{1}{x} A_1 + \frac{1}{x^2} A_2$$

Since the coefficient matrix $A(x)$ is diagonal, one can immediately determine a fundamental solution (nonsingular matrix solution) See [4]

(17)
$$Y(x) = \left[\underline{y}_1(x) \ \underline{y}_2(x), \underline{y}_3(x)\right]$$
$$\underline{y}_1(x) = \begin{bmatrix}1\\0\\0\end{bmatrix} \frac{1}{x} e^{-2/x}$$
$$\underline{y}_2(x) = \begin{bmatrix}0\\1\\0\end{bmatrix} x^{\frac{1}{2}} e^{\left(x-\frac{1}{x}\right)}$$
$$\underline{y}_3(x) = \begin{bmatrix}0\\0\\1\end{bmatrix} x^{\frac{1}{2}} e^{-\left(x+\frac{10}{x}\right)}$$

Now, this problem is much too simple to illustrate any general principles Therefore, the problem is transformed

(18) let $\underline{y}(x) = P(x)\tilde{\underline{y}}(x)$

where $P(x) = I + \frac{1}{x^2} C$

$$I = \text{diagonal}(1,1,1)$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 3 & 1 & 1 \end{bmatrix}$$

It is easily shown that $\tilde{\underline{y}}(x)$ satisfies (19)

(19) $$\tilde{\underline{y}}'(x) = \{P^{-1}(x)(A(x)P(x) - P'(x))\}\tilde{\underline{y}}(x)$$

(20) let $B(x) = P^{-1}(x)(A(x)P(x) - P'(x))$

then $PB = AP - P'$

(21) assume $B(x) = \sum_{n=o}^{\infty} B_n x^{-n}$

The relationships (22) are a result of substituting (21) into (20)

(22)
$$\begin{aligned} B_o &= A_o \\ B_1 &= A_1 \\ B_2 &= A_2 + A_oC - CA_o \\ B_3 &= A_1C - CA_1 + 2C \\ B_4 &= A_2C - CB_2 \\ \text{for } n \geq 5 \quad B_n &= -CB_{n-2} \end{aligned}$$

$$\therefore (23) \qquad \underline{\tilde{y}}'(x) = \left\{\left(B_o + \frac{B_1}{x}\right) + \sum_{n=2}^{\infty} B_n x^{-n}\right\}\underline{\tilde{y}}(x)$$

Since $\sum_{n=2}^{\infty} B_n x^{-n}$ is small for large x, it seems reasonable to compare solutions $\underline{\tilde{y}}(x)$ with solutions of (24).

$$(24) \qquad \underline{w}'(x) = \left\{B_o + \frac{1}{x} B_1\right\}\underline{w}(x)$$

Recall that $B_o = A_o$ and $B_1 = A_1$, i e, B_o and B_1 are diagonal matrices and hence one can immediately obtain a fundamental solution W for (24)

$$(25) \qquad W(x) = [\underline{w}_1(x), \underline{w}_2(x), \underline{w}_3(x)]$$

$$\underline{w}_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{x}$$

$$\underline{w}_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x^{\frac{1}{2}} e^{x}$$

$$\underline{w}_3(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x^{\frac{1}{2}} e^{-x}$$

One might ask what connection, if any, exists between the solutions of (23) and (24) There is a remarkable connection first established by Poincare' and extended by many other researchers (see [8])

THEOREM If $|\arg x| < \frac{\pi}{2}$, there exist actual solutions $\tilde{\underline{y}}_1(x), \tilde{\underline{y}}_2(x), \tilde{\underline{y}}_3(x)$ of (23)

such that

$$\text{i)} \quad ||\tilde{\underline{y}}_1(x) - \underline{w}_1(x)|| = ||\underline{w}_1|| \times 0\left(\frac{1}{x}\right)$$

$$\text{ii)} \quad ||\tilde{\underline{y}}_2(x) - \underline{w}_2(x)|| = ||\underline{w}_2|| \times 0\left(\frac{1}{x}\right)$$

$$\text{iii)} \quad ||\tilde{\underline{y}}_3(x) - \underline{w}_3(x)|| = ||\underline{w}_3|| \times 0\left(\frac{1}{x}\right)$$

where

$$||\underline{y}|| = \left|\left|\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right|\right| = \max_{1 \le i \le 3} |y_i|$$

and $f(x) = 0\left(\frac{1}{x}\right)$ implies (in the context of this problem) that there exists a constant M, such that, $0 < M < \infty$ and $|f(x)| < \frac{M}{x}$ as $x \to \infty$.

Notice that $||\tilde{\underline{y}}_2(x) - \underline{w}_2(x)||$ is not required to tend to zero Since $||\underline{w}_2(x)||$ is exponentially increasing the product, $||\underline{w}_2(x)|| \times 0\left(\frac{1}{x}\right)$ need not tend to zero However, the relative difference becomes small, i e , the number of significant figure agreement between $\tilde{\underline{y}}_2(x)$ and $\underline{w}_2(x)$ increases with increasing x The reader may be familiar with Stirling's formula for $n!$ Precisely, the same behaviour is observed, i e , Stirling's formula yields more significant figure agreement with $n!$ as n increases The difference between $n!$ and the Stirling approximation is unbounded as n tends to infinity

Now, return to the problem of establishing the existence of actual solutions $\tilde{\underline{y}}_i(x)$, such that $||\tilde{\underline{y}}_i(x) - \underline{w}_i(x)|| = ||\underline{w}_i(x)|| \times 0\left(\frac{1}{x}\right)$

(26) let $$E(x) = \sum_{n=2}^{\infty} B_n x^{-n}$$

then (23) becomes

(27) $$\tilde{\underline{y}}'(x) = \left\{B_o + \frac{1}{x} B_1 + E(x)\right\} \tilde{\underline{y}}(x)$$

One can also introduce a norm for a matrix A possible compatible norm is given in (28)

(28) $$||E|| = ||[e^{ij}]|| = \max_{1 \leq i \leq 3} \sum_{j=1}^{3} |e^{ij}|$$

It is easily shown that for this choice of the norm of a matrix, the usual norm properties (29) are satisfied

$$||E\underline{y}|| \leq ||E|| \times ||\underline{y}||$$

$$||E_1 E_2|| \leq ||E_1|| \times ||E_2||$$

(29) $$||E_1 + E_2|| \leq ||E_1|| + ||E_2||$$

$$||E|| = 0 \text{ IFF } E = 0$$

$$||\alpha E|| = |\alpha| \times ||E||$$

Since $E(x)$ (26) is small $\left(\|E(x)\| = O\left(\frac{1}{x^2}\right)\right)$ one might expect $E(x)\tilde{\underline{y}}(x)$ to be small in comparison with $\left(B_o + \frac{1}{x} B_1\right)\tilde{\underline{y}}(x)$ In fact in comparison with $\left(B_o + \frac{1}{x} B_1\right)\tilde{\underline{y}}(x)$, the term $E(x)\tilde{\underline{y}}(x)$ is approximately known, i e , this term is approximately zero Thus, (27) takes the form of an inhomogenous equation where $E(x)\tilde{\underline{y}}(x)$ is treated like an inhomogenous term

One can attempt to use the method of variation of parameters to obtain the general solution of (27) If $E(x)\tilde{\underline{y}}(x)$ is precisely known, then one can obtain the general solution of (27) in the form of an indefinite integral Since $\tilde{\underline{y}}(x)$ is not known, one arrives at an indefinite integral equation for the general solution of (27)

The homogenous portion of (27) is merely (24) Recall that a fundamental solution of (24) is given by (25)

Assume a solution $\tilde{\underline{y}}(x)$ of the form

(30) $$\tilde{\underline{y}}(x) = W(x)\underline{c}(x)$$

where $\underline{c}(x)$ is to be determined Substitute (30) into (27) then one obtains

(31) $$\tilde{\underline{y}}(x) = W(x)\int^{x} W^{-1}(s)E(s)\tilde{\underline{y}}(s)\, ds$$

It is easily shown that any continuous solution of the integral equation (31) is a differentiable solution of (27) This is one big bonus that integral equations possess and it can be expected that the analysis can proceed more easily if one considers the integral equation (31) rather than the differential equation (27)

OBSERVATIONS (See (25))

1 $\underline{w}_1(x)$ dominates $\underline{w}_3(x)$ for x increasing to infinity
$\underline{w}_1(x)$ dominates $\underline{w}_2(x)$ for x starting at large values and decreasing to small values of x. (Recall $|\arg x| < \frac{\pi}{2}$)

2 $\underline{w}_3(x)$ dominates $\underline{w}_1(x)$ and $\underline{w}_2(x)$ for x decreasing from a large x value to a small x value

3 $\underline{w}_2(x)$ dominates $\underline{w}_1(x)$ and $\underline{w}_3(x)$ for x increasing to infinity

Intuitively, what is meant by the dominance of one solution over another is the relative growth of one solution compared to another For example, $\underline{w}_1(x)$ dominates $\underline{w}_2(x)$ for x decreasing For decreasing x, $\underline{w}_1(x)$ is an increasing function and $\underline{w}_2(x)$ decreases exponentially fast Hence, $\underline{w}_1(x)$ dominates $\underline{w}_2(x)$ for decreasing x

One can use the above observations to construct integral equations which force the dominance of a solution which behaves like $\underline{w}_i(x)$ Three different integral equations will result, each corresponding to the three distinct solutions $\underline{w}_i(x)(i = 1,2,3)$

Recall $W(x) = \text{diagonal}\left(\frac{1}{x}, x^{\frac{1}{2}}e^{x}, x^{\frac{1}{2}}e^{-x}\right)$

(32) (a) $W^{-1}(s) = \text{diagonal}\left(s, s^{-\frac{1}{2}}e^{-s}, s^{-\frac{1}{2}}e^{s}\right)$

(b) let $W_1(s) = \text{diagonal}\left(0, 0, s^{-\frac{1}{2}}e^{s}\right)$

(c) let $W_2(s) = \text{diagonal}\left(s, s^{-\frac{1}{2}}e^{-s}, 0\right)$

clearly $W_1(s) + W_2(s) = W^{-1}(s)$

(33) (a) let $K_1(x,s) = W(x)W_1(s)E(s)$

(b) let $K_2(x,s) = W(x)W_2(s)E(s)$

One should note that $W(x)W_1(s)$ includes the effects of only $\underline{w}_3(x)$ which is the solution of (24) which is dominated by $\underline{w}_1(x)$ for increasing x. $W(x)W_2(s)$ includes the effects of $\underline{w}_1(x)$ and $\underline{w}_2(x)$. Recall that $\underline{w}_2(x)$ is dominated by $\underline{w}_1(x)$ for decreasing values of x.

Consider the following integral equation

(34) $$\tilde{\underline{y}}_1(x) = \underline{w}_1(x) + \int_a^x K_1(x,s)\tilde{\underline{y}}_1(s)\,ds + \int_\infty^x K_2(x,s)\tilde{\underline{y}}_1(s)\,ds$$

For simplicity, restrict the remaining portion of the discussion to the case $\arg x = 0$, i e , x is real and positive

If $a \leq x < \infty$, then the integral from a to x in (34) proceeds with $ds > 0$ and the integral from ∞ to x proceeds with $ds < 0$ Due to the above split that has been performed (see (32) and (33)), one has forced the dominance of a solution of (27) which behaves like $\underline{w}_1(x)$ which is a solution of (24)   It is relatively simple to differentiate formally the integral equation (34) to demonstrate that continuous, absolutely integrable solutions of the integral equation are indeed differentiable solutions of (27)

The proof of the theorem now proceeds in two stages   First, it is necessary to establish the existence of a solution $\tilde{\underline{y}}_1(x)$ to (34)   Secondly, it is demonstrated that

$$||\tilde{\underline{y}}_1(x) - \underline{w}_1(x)|| = ||\underline{w}_1(x)|| \times 0\left(\frac{1}{x}\right)$$

In order to establish the existence of a solution to the integral equation (34), consider the Picard iteration

$$\tilde{\underline{y}}_1^{o}(x) \equiv \underline{0}$$

(35)
$$\tilde{\underline{y}}_1^{n+1}(x) = \underline{w}_1(x) + \int_a^x K_1(x,s)\tilde{\underline{y}}_1^{n}(s)\, ds + \int_\infty^x K_2(x,s)\tilde{\underline{y}}^{n}(s)\, ds$$

Since $||E(s)|| = 0\left(\frac{1}{s^2}\right)$ let M be chosen so that

(36) $$||E(s)|| \leq \frac{M}{s^2}$$

LEMMA If $a \geq \max(4, 6M)$ and $a \leq x < \infty$ then

(37) $$||\tilde{\underline{y}}_1^{n+1}(x) - \tilde{\underline{y}}_1^n(x)|| \leq \left(\frac{1}{2}\right)^n ||\underline{w}_1(x)|| = \left(\frac{1}{2}\right)^n \times \left(\frac{1}{x}\right)$$

PROOF $$\tilde{\underline{y}}_1^o(x) = \underline{0}$$

$$\tilde{\underline{y}}_1^1(x) = \underline{w}_1(x)$$

$$\therefore ||\tilde{\underline{y}}_1^1(x) - \tilde{\underline{y}}_1^o(x)|| = ||\underline{w}_1|| = \left(\frac{1}{2}\right)^o ||\underline{w}_1||$$

$\therefore$ for $n = 0$ the lemma is true Assume true for some $n - 1$ and attempt to show true for n

$$||\tilde{\underline{y}}_1^{n+1}(x) - \tilde{\underline{y}}_1^n(x)|| \leq \int_a^x \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{s^2} ||\tilde{\underline{y}}_1^n(s)$$

$$- \tilde{\underline{y}}_1^{n-1}(s)|| \; |ds| + \int_\infty^x \left(\frac{s}{x}\right) \frac{M}{s^2} ||\tilde{\underline{y}}_1^n(s)$$

$$- \tilde{\underline{y}}_1^{n-1}(s)|| \; |ds|$$

From the inductive hypothesis one concludes

$$||\tilde{Y}_1^{n+1}(x) - \tilde{Y}_1^n(x)|| \le \int_a^x \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{s^2} \left(\frac{1}{2}\right)^{n-1} \frac{1}{s} |ds|$$

$$+ \int_\infty^x \left(\frac{s}{x}\right) \frac{M}{s^2} \left(\frac{1}{2}\right)^{n-1} \frac{1}{s} |ds|$$

CASE 1 $\quad a \ge \frac{x}{2}$

$$\therefore \quad \int_a^x \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{s^2} \left(\frac{1}{2}\right)^{n-1} \frac{1}{s} |ds|$$

$$\le \int_{\frac{x}{2}}^x \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{s^2} \left(\frac{1}{2}\right)^{n-1} \frac{1}{s} |ds|$$

$$\left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} < 1 \quad \text{for} \quad a \ge 4$$

$$\cdot \quad \int_a^x \le \left(\frac{1}{2}\right)^n \frac{3M}{x^2}$$

$6M \le a \le x$

$$\therefore \int_a^x < \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)^n \times \left(\frac{1}{x}\right) = \frac{1}{2} \times \left(\frac{1}{2}\right)^n \times ||\underline{w}_1(x)||$$

$$\int_\infty^x \left(\frac{s}{x}\right) \frac{M}{s^2} \left(\frac{1}{2}\right)^{n-1} \frac{1}{s} |ds| = \frac{M\left(\frac{1}{2}\right)^{n-1}}{x^2} \leq \frac{1}{3}$$

$$\times \left(\frac{1}{2}\right)^n \times ||\underline{w}_1(x)||$$

Thus for case 1 $\left(a \geq \frac{x}{2}\right)$

$$||\underline{\tilde{y}}_1^{n+1}(x) - \underline{\tilde{y}}_1^n(x)|| \leq \left(\frac{1}{2} + \frac{1}{3}\right) \times \left(\frac{1}{2}\right)^n \times ||\underline{w}_1(x)|| < \left(\frac{1}{2}\right)^n$$

$$\times ||\underline{w}_1(x)||$$

CASE 2 $\quad a < \frac{x}{2}$

$$\int_a^x = \int_a^{x/2} + \int_{\frac{x}{2}}^x$$

$$\left|\left|\int_a^{x/2}\right|\right| \leq \int_a^{x/2} \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{s^2} \times \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{s}\, ds$$

For $a \geq 4$, it is easily shown that $\left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)}$ is monotonically increasing with s for fixed x on the interval $a \leq s \leq x$ It is also easily shown that

$e^{-x/2} < \frac{1}{x}$ for $4 \leq a \leq x$

Thus,

$$\left|\left|\int_a^{x/2}\right|\right| \leq \int_a^{x/2} \sqrt{2}\, e^{-x/2} \frac{M}{s^3} \left(\frac{1}{2}\right)^{n-1} ds$$

$$\leq \frac{\sqrt{2}}{a} \cdot \frac{M}{a} \times \left(\frac{1}{2}\right)^n \times \frac{1}{x} < \frac{1}{6} \times \left(\frac{1}{2}\right)^n \times ||\underline{w}_1(x)||$$

$$\cdot \left|\left|\int_a^{x/2}\right|\right| < \frac{1}{6} \times \left(\frac{1}{2}\right)^n ||\underline{w}_1(x)||$$

The results of case 1 can be applied to obtain estimates

for $\left|\left|\int_{\frac{x}{2}}^{x}\right|\right|$ and $\left|\left|\int_{\infty}^{x}\right|\right|$

$$\left|\left|\int_a^{x/2}\right|\right| + \left|\left|\int_{\frac{x}{2}}^{x}\right|\right| + \left|\left|\int_{\infty}^{x}\right|\right| \leq \left(\frac{1}{6} + \frac{1}{2} + \frac{1}{3}\right) \times \left(\frac{1}{2}\right)^n$$

$$\times ||\underline{w}_1||$$

or $\quad ||\tilde{\underline{y}}_1^{n+1}(x) - \tilde{\underline{y}}_1^{n}(x)|| \leq \left(\frac{1}{2}\right)^n \times ||\underline{w}_1(x)||$ QED

It is readily verified that

$$(38) \qquad \tilde{\underline{y}}_1^{n+1}(x) = \sum_{k=o}^{n} \left\{ \tilde{\underline{y}}_1^{k+1}(x) - \tilde{\underline{y}}_1^{k}(x) \right\}$$

Therefore, the above lemma establishes the uniform convergence of the above sum as $n \to \infty$ and hence $\tilde{\underline{y}}_1^{n+1}(x)$ converges uniformly. Denote this limit vector by $\tilde{\underline{y}}_1(x)$. $\tilde{\underline{y}}_1(x)$ is clearly a continuous solution of the integral equation. In addition, the above telescoping sum coupled with the triangle inequality yields an estimate of the growth of $||\tilde{\underline{y}}_1^n(x)||$

$$\text{(a)} \qquad ||\tilde{\underline{y}}_1^{n+1}(x)|| \le \sum_{k=o}^{n} \left(\frac{1}{2}\right)^k ||\underline{w}_1(x)||$$

(39)

$$\text{(b)} \qquad \therefore ||\tilde{\underline{y}}_1(x)|| \le 2\ ||\underline{w}_1(x)|| = \frac{2}{x}$$

This estimate of the rate of growth of $\tilde{\underline{y}}_1(x)$ is sufficient to establish that $||\tilde{\underline{y}}_1(x) - \underline{w}_1(x)|| \le ||\underline{w}_1(x)|| \times 0\left(\frac{1}{x}\right)$

Proof $$||\tilde{\underline{y}}_1(x) - \underline{w}_1(x)|| \leq \int_a^{x/2} \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{x^2} \left(\frac{2}{s}\right) ds$$

$$+ \int_{\frac{x}{2}}^{x} \left(\frac{x}{s}\right)^{\frac{1}{2}} e^{-(x-s)} \frac{M}{s^2} \left(\frac{2}{s}\right) ds$$

$$+ \int_{\infty}^{x} \left(\frac{s}{x}\right) \frac{M}{s^2} \left(\frac{2}{s}\right) |ds|$$

(40) $$||\tilde{\underline{y}}_1(x) - \underline{w}_1(x)|| \leq \int_a^{x/2} \frac{2M}{s^3}\, ds \times \sqrt{2}\, e^{-x/2} + \frac{3M}{x^2} + \frac{2M}{x^2}$$

Clearly $e^{-x/2}$ is an exponentially decreasing function and hence, the first term on the right-hand side of (40) is $0\left(\frac{1}{x^2}\right)$ The entire expression is $0\left(\frac{1}{x^2}\right)$ or $||\underline{w}_1(x)|| \times 0\left(\frac{1}{x}\right)$ This establishes the existence of an actual solution of (27) which behaves like $\underline{w}_1(x)$ as $x \to \infty$.

One could construct other integral equations to prove that there exists an actual solution $\tilde{\underline{y}}_2(x)$ of (27) which behaves like $\underline{w}_2(x)$ as $x \to \infty$, and similarly one could demonstrate a connection between a $\tilde{\underline{y}}_3(x)$ solution of (27) and $\underline{w}_3(x)$

This method of proof is not going to be repeated for $\tilde{\underline{y}}_2(x)$ and $\tilde{\underline{y}}_3(x)$ Only the integral equations will be determined

Recall that $\underline{w}_2(x)$ dominates $\underline{w}_1(x)$ and $\underline{w}_2(x)$ for increasing x

Thus, let $$W_1(x) = \text{diagonal}\left(s, 0, s^{-1/2}e^{s}\right)$$

$$W_2'(s) = \text{diagonal}\left(0, s^{-1/2}e^{-s}, 0\right)$$

$$K_1(x,s) = W(x)W_1(s)E(s)$$

$$K_2(x,s) = W(x)W_2(s)E(s)$$

(41) $$\tilde{\underline{y}}_2(x) = \underline{w}_2(x) + \int_a^x K_1(x,s)\tilde{\underline{y}}_2(s)\,ds + \int_\infty^x K_2(x,s)\tilde{\underline{y}}_2(s)\,ds$$

Since $\underline{w}_3(x)$ does not dominate any solution for increasing x , the corresponding $\tilde{\underline{y}}_3(x)$ integral equation is

(42) $$\tilde{\underline{y}}_3(x) = \underline{w}_3(x) + \int_\infty^x W(x)W^{-1}(s)E(s)\tilde{\underline{y}}_3(s)\,ds$$

So far it has been possible to avoid using Thome' solutions in an attempt to establish a connection between (27) and (24)

It turns out that for the particular problem under investigation, one is able to determine (guess) by inspection the first few terms in the Thome' expansion

Now, consider a different method of attack which more closely parallels the approach developed earlier in this paper That is, attempt to determine solutions via formal substitutions Since, the rank of the singularity of (23) or (27) is one, the formal solution takes the form

$$(43) \qquad \tilde{\underline{y}}(x) = \left(\sum_{n=0}^{\infty} \underline{y}_n x^{-n}\right) x^{\alpha} e^{\lambda x}$$

If one formally substitutes (43) into (23), then one obtains

$$(44) \qquad \lambda \underline{y}_o = B_o \underline{y}_o$$

One clearly obtains the values 0, 1 and -1 for λ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ for the corresponding $\underline{y}_o$ vectors Actually any multiple of these vectors is satisfactory

Corresponding to any of the three possibilities for λ, one can formally determine $\alpha, \underline{y}_1, \underline{y}_2, \underline{y}_3, \cdots$ For example, consider $\lambda = 0$ The corresponding formal solution is of the form

$$(45) \qquad \tilde{\underline{y}}(x) = \sum_{n=0}^{\infty} \underline{p}_n x^{\alpha - n}$$

$$\alpha \underline{p}_o = B_o \underline{p}_1 + B_1 \underline{p}_o$$

$$\alpha = -1$$

(46) $$-2\underline{p}_1 = B_o \underline{p}_2 + B_1 \underline{p}_1 + B_2 \underline{p}_o$$

$$-3\underline{p}_2 = B_o \underline{p}_3 + B_1 \underline{p}_2 + B_2 \underline{p}_1 + B_3 \underline{p}_o$$

$$-4\underline{p}_3 = B_o \underline{p}_4 + B_1 \underline{p}_3 + B_2 \underline{p}_2 + B_3 \underline{p}_1 + B_4 \underline{p}_o$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

If one solves the relations (46), then one obtains for

$$\underline{p}_o = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(47) $$\underline{p}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{p}_2 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \underline{p}_3 = \begin{bmatrix} 2/3 \\ 2 \\ 6 \end{bmatrix}$$

Note that $\underline{w}_1(x) = \underline{p}_o \times \left(\frac{1}{x}\right)$ which is merely the result of truncating the formal sum (45) after the first term

One could proceed to recursively determine $\underline{p}_4, \underline{p}_5, \cdots$. It turns out that if one considers the truncated formal solution $\sum_{k=o}^{\ell} \underline{p}_k x^{\alpha-k}$, then there exists an actual solution $\tilde{\underline{y}}_1(x)$ of (24) such that $||\tilde{\underline{y}}_1(x) - \sum_{k=o}^{\ell} p_k x^{\alpha-n}|| = x^{\alpha} \times 0\left(\frac{1}{x^{\ell+1}}\right)$

This result was established in the previous theorem for $\ell = 0$ This more general result will not be established here (see [4], [8]) Poincare' introduced the following definition A function $f(x)$ is said to be asymptotic to $\sum_{n=o}^{\infty} a_n x^{-n}$ as $x \to \infty$ in the sector $\int$ if $|f(x) - \sum_{k=o}^{N} a_k x^{-k}| = 0\left(\frac{1}{x^{N+1}}\right)$ This property is denoted by

$$(48) \qquad f(x) \sim \sum_{n=o}^{\infty} a_n x^{-n}, \quad x \in \int$$

If one evaluates $\sum_{n=o}^{3} p_n x^{-(n+1)}$ at $x = 100$ one obtains

$$(49) \qquad \begin{bmatrix} 009801007 \\ -\ 00000098 \\ -\ 00000294 \end{bmatrix}$$

Since one can obtain exact solutions via the transformation, (18) one can easily show that

$$(50) \qquad ||\tilde{y}_1 \text{EXACT} - \sum_{n=o}^{3} p_n x^{-n}|| \approx 3 \times 10^{-9}$$

Since $||\tilde{y}_1(100)|| \approx 01$, one thus concludes that the formal truncated solution yields six significant figures with a maximum error of three units in the seventh place Notice that in the above discussion, some of the zeros in the second and third components are counted as significant figures

One can similarly compute approximate values of $\tilde{\underline{y}}_2(100)$ and $\tilde{\underline{y}}_3(100)$

$$\tilde{\underline{y}}_2(x) \approx \sum_{n=o}^{3} \underline{q}_n x^{\frac{1}{2}-n} e^x \tag{51}$$

$$\underline{q}_o = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{q}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{q}_2 = \begin{bmatrix} -2 \\ -3\ 5 \\ -1 \end{bmatrix}$$

$$\underline{q}_3 = \begin{bmatrix} 2 \\ 3\ 8333\cdots \\ 1 \end{bmatrix}$$

$$\tilde{\underline{y}}_1(100) \approx \begin{bmatrix} -\ 532247 \times 10^{41} \\ 2660306 \times 10^{45} \\ -\ 266124 \times 10^{41} \end{bmatrix} \tag{52}$$

Comparing this value with the exact value, one can determine the accuracy to be seven significant figures with a three-unit error in the eighth place The actual error is roughly 3×10^{37} which is a very large number This illustrates the situation where two large numbers can have several significant figures of agreement and still differ by a very large number.

Similarly

$$\tilde{\underline{y}}_3(x) \approx \sum_{n=o}^{3} \underline{r}_n x^{-n}\ x^{\frac{1}{2}} e^{-x} \tag{53}$$

$$\underline{r}_o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{r}_1 = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}, \quad \underline{r}_2 = \begin{bmatrix} -3 \\ -9 \\ 49 \end{bmatrix}$$

$$\underline{r}_3 = \begin{bmatrix} 30 \\ 90 \\ -156\tfrac{2}{3} \end{bmatrix}$$

$$\tilde{\underline{y}}_3(100) \approx \begin{bmatrix} -\ 100442 \times 10^{-45} \\ -\ 301326 \times 10^{-45} \\ 3365713 \times 10^{-42} \end{bmatrix} \tag{54}$$

The exact solution yields the following information. The approximate value (54) is accurate to five significant figures with an error less than two units in the sixth place.

For most problems there is of course no known exact (closed-form) solution. Thus, one must attempt to determine the possible error involved by a more detailed investigation of the asymptotic expansions. One technique is to merely truncate the expansion at the smallest term (see [6]). Olver has recently investigated the problem of determining error estimates [9] and it appears that, for certain cases, this procedure is quite bad, i.e., the errors are much larger than one would expect. This behaviour is most pronounced when one attempts to use the asymptotic expansions near a Stokes' line. For the problem considered in this paper, the Stokes' lines correspond to the rays $\arg x = \pm \frac{\pi}{2}$. Since the relative dominance of one solution over another changes when these lines are crossed, one might expect the inaccuracies that arise when one attempts to use the truncated formal expansions.

Note that the discussion so far depends on the relative dominance of one solution over another In fact, it is clearly possible to add any multiple of $\tilde{y}_1(x)$ and $\tilde{y}_3(x)$ to $\tilde{y}_2(x)$ without changing the asymptotic properties of $\tilde{y}_2(x)$ One could also add any multiple of $\tilde{y}_3(x)$ to $\tilde{y}_1(x)$ without changing the asymptotic properties of $\tilde{y}_1(x)$ Of course one must impose the restriction $|\arg x| < \frac{\pi}{2}$

If one is interested in determining $\tilde{y}_1(x)$ and one only demands

$$(55) \qquad \tilde{y}_1(x) \sim \sum_{n=o}^{\infty} p_n x^{-(n+1)} \quad \text{where} \quad p_o = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

then the possible $\tilde{y}_1(x)$ form a one-parameter family of solutions $\{\tilde{y}_1\}$ Similarly, $\tilde{y}_2(x)$ form a two-parameter family $\{\tilde{y}_2\}$ Since no multiple of $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ can be added to $\tilde{y}_3(x)$ and leave its asymptotic properties unchanged, one can conclude that this solution is uniquely defined $\left(r_o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$, i.e , there corresponds precisely one actual solution of (23) asymptotic to the formal solution

NUMERICAL INVESTIGATION OF THOME' SOLUTIONS

The following is a numerical procedure which, to the author's knowledge, has never been published

The only accurate information obtained concerning the solutions $\tilde{y}_1(x)$, $\tilde{y}_2(x)$ and $\tilde{y}_3(x)$ is for large values of x For many practical problems one is interested in these solutions for fairly small values of x In principle, one can use the values $\tilde{y}_1(a)$, $\tilde{y}_2(a)$ and $\tilde{y}_3(a)$ to continue these solutions back to small values of x Because of the different exponential rates of growth, this corresponds to an unstable numerical problem Indeed, if one starts at $x = 100$ and attempts to use $\tilde{y}_1(100)$ to determine $\tilde{y}_1(20)$, one will find that $\tilde{y}_3(x)$ grows so rapidly (for numerical integration with a negative step the $\tilde{y}_3(x)$ solution of (23) dominates) that one soon obtains only a multiple of $\tilde{y}_3(x)$

In order to obtain a vector $\tilde{y}_1(x)$ at $x = 20$ which is a member of the one-parameter family $\{\tilde{y}_1\}$, one must either increase the precision of the numerical calculations or develop some other numerical technique

From the asymptotic relations one might expect

$$(56) \qquad ||\tilde{y}_1(x)|| \approx 05 \quad \text{at} \quad x = 20$$

The growth of $\tilde{y}_3(x)$ is expected to be $\varepsilon \times 10^{34}$ where ε is a sum of the errors in the initial condition $(\tilde{y}_1(100))$, the roundoff error, and truncation error Hence, one would expect to require about 40 places of accuracy to obtain four significant figures in the determination of $\tilde{y}_1(20)$. One would also require an extremely small step size in the calculations to maintain this precision

Fortunately one can avoid this high-precision arithmetic One need only exploit the property that $\tilde{y}_1(x)$ can be considered a member of the one-parameter family $\{\tilde{y}_1\}$

Recall that any multiple of $\tilde{y}_3(x)$ added to $\tilde{y}_1(x)$ yields a solution of (23) or (27) which is asymptotic to $\tilde{y}_1(x)$, i e, the asymptotic properties are unchanged If one proceeds to integrate from $x = 100$ with a negative step, one will soon observe very rapid growth of the third component of the vector solution

The rapid growth of the third component occurs since $\tilde{y}_3(x)$ grows most rapidly in the third component If one CORRECTS $\tilde{y}_1(x)$ by adding a multiple of $\tilde{y}_3(x)$ so as to force the third component to be zero at say $x = 99$, then one has not changed the asymptotic properties of $\tilde{y}_1(x)$ One is merely calculating a different member (after the correction) of the one-parameter family $\{\tilde{y}_1\}$. If one repeats this procedure at $x = 98$, $x = 97$, $\cdots$, $x = 20$ one arrives at a solution of the form

$$\tilde{y}_1(20) = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ 0 \end{bmatrix} \tag{57}$$

Since there is also an exact solution for this particular problem one can determine the accuracy of this procedure (See Tables I and III)

TABLE I

x	$\tilde{Y}_1$ (EXACT)	$\tilde{Y}_1$ (NI)*
100	$\begin{bmatrix} 98010077 \\ -\ 00009771 \\ -\ 00029399 \end{bmatrix} \times 10^{-2}$	$\begin{bmatrix} 9810070 \\ -\ 00009800 \\ -\ 00029400 \end{bmatrix} \times 10^{-2}$
90	$\begin{bmatrix} 10865581 \\ -\ 00001336 \\ -\ 00004024 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 10865560 \\ -\ 00001334 \\ -\ 00005884 \end{bmatrix} \times 10^{-1}$
80	$\begin{bmatrix} .12189473 \\ -\ 00001896 \\ -\ 00005713 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 12207266 \\ 00051641 \\ -\ 38101953 \end{bmatrix} \times 10^{-1}$
70	$\begin{bmatrix} 13880500 \\ -\ 00002815 \\ -\ 00008496 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 48525 \\ 1\ 41486 \\ -770\ 99997 \end{bmatrix}$
60	$\begin{bmatrix} 16115805 \\ -\ 00004438 \\ -.00013425 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 00012768 \\ 00038333 \\ -.15351575 \end{bmatrix} \times 10^{8}$
50	$\begin{bmatrix} 19208139 \\ -\ 00007588 \\ -\ 00023038 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 00035723 \\ 00107282 \\ -.29852279 \end{bmatrix} \times 10^{12}$
40	$\begin{bmatrix} 23765984 \\ -.00014569 \\ -\ 00044524 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} .00104414 \\ .00313765 \\ -.55931436 \end{bmatrix} \times 10^{16}$
30	$\begin{bmatrix} 31149376 \\ -\ 00033425 \\ -\ 00104679 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 00324519 \\ 00976449 \\ -\ 98114926 \end{bmatrix} \times 10^{20}$
20	$\begin{bmatrix} 45132092 \\ -\ 00104197 \\ -\ 00337387 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 0010994 \\ 0033204 \\ -\ 14917207 \end{bmatrix} \times 10^{25}$

(NI) — numerically integrated

For more practical problems there is no known closed-form solution. Indeed, there would be no need to integrate numerically in such a problem. One can attempt to calculate a solution asymptotic to $\tilde{y}_2(x)$ at $x = 20$ by numerically integrating from $x = 100$ to $x = 20$. For this problem, one adds multiples of $\tilde{y}_1(x)$ and $\tilde{y}_3(x)$ to correct $\tilde{y}_2(x)$ at $x = 99$, $x = 98$, $\cdots$, $x = 20$. At each of these values of x, one corrects $y_2(x)$ so that it is of the form

$$\tilde{y}_2(x) = \begin{bmatrix} 0 \\ \beta_2(x) \\ 0 \end{bmatrix} \tag{58}$$

The value of β_2 so obtained at $x = 20$ can then be used to numerically integrate (23) with a positive step. One can expect to maintain good accuracy on $\tilde{y}_2(x)$ for integration with a positive step since $\tilde{y}_2(x)$ dominates the other solutions of (23) in this direction. When one reaches the value $x = 100$, one can compare the value of $\tilde{y}_2(x)$ calculated using $\begin{bmatrix} 0 \\ \beta_2(20) \\ 0 \end{bmatrix}$ as an initial condition at $x = 20$ with the value one started the calculation at $x = 100$. These two values will not compare well unless one corrects the values of both to the same form (58). There is no reason to expect the initial condition $\begin{bmatrix} 0 \\ \beta_2(20) \\ 0 \end{bmatrix}$ to be of the form (58) at $x = 100$ since there were 80 separate corrections and hence 80 different members of the two-parameter family $\{\tilde{y}_2\}$ which have been used in the calculations. The agreement (or lack of it) at $x = 100$ is some indication of the accuracy of all the numerical solutions obtained at $x = 20$ This follows since all the solutions are used in the calculation of $\tilde{y}_2(20)$ One may find that more precision is required, but it appears that one will require less precision than a numerical integration without utilizing corrections (see Tables II, III and IV)

All numerical integrations were carried out using the UNIVAC 1108, single precision (eight significant figures) with a fifth-order Runge-Kutta method [7] and a fixed step size H ($|H| = 1/32$)

TABLE II

x	$\tilde{Y}_3$(EXACT)	$\tilde{Y}_3$(NI)*
100	$\begin{bmatrix} -\ 00010090 \\ -\ 00030278 \\ 33657280 \end{bmatrix} \times 10^{-42}$	$\begin{bmatrix} -\ 0001004 \\ -\ 0003013 \\ 3365713 \end{bmatrix} \times 10^{-22}$
90	$\begin{bmatrix} -\ 00025738 \\ -\ 00077239 \\ 69551972 \end{bmatrix} \times 10^{-38}$	$\begin{bmatrix} -\ 00025738 \\ -\ 00077238 \\ 69551542 \end{bmatrix} \times 10^{-18}$
80	$\begin{bmatrix} -\ 00006670 \\ -\ 00020017 \\ 14244000 \end{bmatrix} \times 10^{-33}$	$\begin{bmatrix} -\ 00006670 \\ -\ 00020017 \\ 14243887 \end{bmatrix} \times 10^{-13}$
70	$\begin{bmatrix} -\ 00017624 \\ -\ 00052901 \\ 28827363 \end{bmatrix} \times 10^{-29}$	$\begin{bmatrix} -\ 00017524 \\ -\ 00052901 \\ 28827086 \end{bmatrix} \times 10^{-9}$
60	$\begin{bmatrix} -\ 00047740 \\ -\ 00143325 \\ 57398994 \end{bmatrix} \times 10^{-25}$	$\begin{bmatrix} -\ 00047739 \\ -\ 00143323 \\ 57398345 \end{bmatrix} \times 10^{-5}$
50	$\begin{bmatrix} -\ 00013357 \\ -\ 00040113 \\ 11161680 \end{bmatrix} \times 10^{-20}$	$\begin{bmatrix} -\ 00013356 \\ -\ 00040112 \\ 11161535 \end{bmatrix}$
40	$\begin{bmatrix} -\ 00039040 \\ -\ 00117316 \\ 20912637 \end{bmatrix} \times 10^{-16}$	$\begin{bmatrix} -\ 00039040 \\ -\ 00117314 \\ 20912327 \end{bmatrix} \times 10^{4}$
30	$\begin{bmatrix} -\ 00121337 \\ -\ 00365093 \\ 36685010 \end{bmatrix} \times 10^{-12}$	$\begin{bmatrix} -\ 00121335 \\ -\ 00365087 \\ 36684401 \end{bmatrix} \times 10^{8}$
20	$\begin{bmatrix} -\ 00411079 \\ -\ 01241501 \\ 55775284 \end{bmatrix} \times 10^{-8}$	$\begin{bmatrix} -\ 00411072 \\ -\ 01241479 \\ 55774266 \end{bmatrix} \times 10^{12}$

*(NI) – numerically integrated

Notice the numerically integrated (NI) solution has been scaled No corrections are required to maintain good accuracy on $\tilde{Y}_3(x)$ The dominance of $\tilde{Y}_3(x)$ for integration with a negative step completely explains this accuracy

TABLE III

x	$\tilde{\Sigma}_1$ (EXACT)	$\tilde{\Sigma}_1$ (NIWC)[a]	$\tilde{\Sigma}_1$ (NIW/OC)[b]
100	$\begin{bmatrix} 98010068 \\ -00009771 \\ 00029399 \end{bmatrix} \times 10^{-2}$	$\begin{bmatrix} 98010070 \\ -00009800 \\ -00029400 \end{bmatrix} \times 10^{-2}$	$\begin{bmatrix} 98010070 \\ -00009800 \\ -00029400 \end{bmatrix} \times 10^{-2}$
90	$\begin{bmatrix} 10865580 \\ -00001341 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 10865559 \\ -00001341 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 10865558 \\ -00001341 \\ 00000000 \end{bmatrix} \times 10^{-1}$
80	$\begin{bmatrix} 12189470 \\ -00001903 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 12189426 \\ -00001903 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 12189426 \\ -00001903 \\ 00000000 \end{bmatrix} \times 10^{-1}$
70	$\begin{bmatrix} 13880495 \\ -00002830 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 13880425 \\ -00002830 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 13880435 \\ -00002786 \\ 00000000 \end{bmatrix} \times 10^{-1}$
60	$\begin{bmatrix} 16115794 \\ -00004472 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 16115688 \\ -00004472 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 16235352 \\ -00488281 \\ 00000000 \end{bmatrix} \times 10^{-1}$
50	$\begin{bmatrix} 19208112 \\ -00007671 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 19207944 \\ -00007671 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$
40	$\begin{bmatrix} 23765900 \\ -00014817 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 23765649 \\ -00014816 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 13107200 \\ 26214400 \\ 00000000 \end{bmatrix} \times 10^{6}$
30	$\begin{bmatrix} 31149033 \\ -00034457 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 31148659 \\ -00034456 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 42949673 \\ 85899346 \\ 00000000 \end{bmatrix} \times 10^{10}$
20	$\begin{bmatrix} 45129605 \\ -00111707 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 45128971 \\ -00111705 \\ 00000000 \end{bmatrix} \times 10^{-1}$	$\begin{bmatrix} 04222125 \\ 11258999 \\ 00000000 \end{bmatrix} \times 10^{16}$

[a](NIWC) — numerically integrated with corrections

[b](NIW/OC) — numerically integrated without corrections

Notice for $x = 90$, $x = 80$, $\cdot$, $x = 20$ all solutions are reduced to the same form

$$\tilde{\underline{y}}_1(x) = \begin{bmatrix} \alpha_1(x) \\ \beta_1(x) \\ 0 \end{bmatrix} \tag{59}$$

In order to facilitate a comparison one must examine the same member of the one-parameter family $\{\tilde{\underline{y}}_1\}$ Thus, all vectors are reduced to the form (59) For the numerical integration with corrections (NIWC), one uses this corrected value in the numerical integration, but the corrected value is ignored in the numerical integration without corrections (NIW/OC) Table III clearly indicates the value of the correction procedure

One may wonder why the comparison of $\tilde{\underline{y}}_1$(EXACT) and $\tilde{\underline{y}}_1$(NI) in Table I seems to be much worse than $\tilde{\underline{y}}_1$(EXACT) and $\tilde{\underline{y}}_1$(NIW/OC) The reason for this is simply that one is obtaining a member of $\{\tilde{\underline{y}}_1\}$ when one computes $\tilde{\underline{y}}_1$(NI) One would need much more precision to more accurately specify a particular member of this family at $x = 100$ in order to achieve agreement between $\tilde{\underline{y}}_1$(EXACT) and $\tilde{\underline{y}}_1$(NI) This emphasizes the numerical instability of integration without corrections.

It appears that numerical integration without correction (NIW/OC) yields a member of the one-parameter family $\{\tilde{\underline{y}}_1\}$ to two significant figures at $x = 60$ This exceeds our expectations The reason is rather simple It is expected that the first component of $\tilde{\underline{y}}_1(x)$ should be of the order 10^{-2} Since the inaccuracies in the numerical calculations are going to be a result of rapid growth in the third component, one may determine how large the third component will have to be in order to affect the first significant figure of $\tilde{y}_3(x)$ From Table II one observes that $\tilde{y}_3(x)$ is approximately of the form

$$\tilde{y}_3(x) \approx \begin{bmatrix} -10^a \\ -10^a \\ 10^{a+3} \end{bmatrix} \tag{60}$$

Now, one expects to lose all significant figures in the determination of $\tilde{y}_1(x)$ when $a = 6$ $(||\tilde{y}_1(x)|| \approx 10^{-2})$ since one maintains only eight significant figures in the calculations Since the error in the initial condition is of the order 10^{-9} and the third component will have to attain the approximate value 10^9, this implies that a growth of 10^{18} times the initial error is necessary to lose all significant figures The $\tilde{y}_3(x)$ solution of (23) attains a growth of 10^{18} times its original value for a decrease of x between 40 and 50 units Thus, at $x = 60$ $\tilde{y}_1$(NIW/OC) is a two significant figure member of $\{\tilde{y}_1\}$ and at $x = 50$ has no significant figures of agreement

TABLE IV

x	$\tilde{\Sigma}_2$(EXACT)	$\tilde{\Sigma}_2$(NIWC)[a]	$\tilde{\Sigma}_2$(NIW/OC)[b]	$\tilde{\Sigma}_2$(NI)[c]
100	$\begin{bmatrix} 00005319 \\ 26603058 \\ 00002658 \end{bmatrix} \times 10^{45}$	$\begin{bmatrix} 00005322 \\ 26603060 \\ -00002661 \end{bmatrix} \times 10^{25}$	$\begin{bmatrix} 00005322 \\ 26603060 \\ 00002661 \end{bmatrix} \times 10^{25}$	$\begin{bmatrix} 00005322 \\ 26603060 \\ -00002661 \end{bmatrix} \times 10^{25}$
90	$\begin{bmatrix} 00000000 \\ 11444181 \\ 00000000 \end{bmatrix} \times 10^{41}$	$\begin{bmatrix} 00000000 \\ 11444123 \\ 00000000 \end{bmatrix} \times 10^{21}$	$\begin{bmatrix} 00000000 \\ 11444162 \\ 00000000 \end{bmatrix} \times 10^{21}$	$\begin{bmatrix} 00017569 \\ 01208596 \\ -57792246 \end{bmatrix} \times 10^{22}$
80	$\begin{bmatrix} 00000000 \\ 48910659 \\ 00000000 \end{bmatrix} \times 10^{36}$	$\begin{bmatrix} 00000000 \\ 48910335 \\ 00000000 \end{bmatrix} \times 10^{16}$	$\begin{bmatrix} 00000000 \\ 11197289 \\ 00000000 \end{bmatrix} \times 10^{17}$	$\begin{bmatrix} 00005542 \\ 00016633 \\ -11835600 \end{bmatrix} \times 10^{27}$
70	$\begin{bmatrix} 00000000 \\ 20730232 \\ 00000000 \end{bmatrix} \times 10^{32}$	$\begin{bmatrix} 00000000 \\ 20730060 \\ 00000000 \end{bmatrix} \times 10^{12}$	$\begin{bmatrix} 00000000 \\ 18447855 \\ 00000000 \end{bmatrix} \times 10^{21}$	$\begin{bmatrix} 00014644 \\ 00043957 \\ -23953141 \end{bmatrix} \times 10^{31}$
60	$\begin{bmatrix} 00000000 \\ 86900848 \\ 00000000 \end{bmatrix} \times 10^{27}$	$\begin{bmatrix} 00000000 \\ 86899977 \\ 00000000 \end{bmatrix} \times 10^{7}$	$\begin{bmatrix} 00000000 \\ 90896679 \\ 00000000 \end{bmatrix} \times 10^{22}$	$\begin{bmatrix} 00039668 \\ 00119091 \\ -47693709 \end{bmatrix} \times 10^{35}$
50	$\begin{bmatrix} 00000000 \\ 35878055 \\ 00000000 \end{bmatrix} \times 10^{23}$	$\begin{bmatrix} 00000000 \\ 35877634 \\ 00000000 \end{bmatrix} \times 10^{3}$		NUMBERS TOO LARGE EXCEED 10^{38}
40	$\begin{bmatrix} 00000000 \\ 14483300 \\ 00000000 \end{bmatrix} \times 10^{19}$	$\begin{bmatrix} 00000000 \\ 14483106 \\ 00000000 \end{bmatrix} \times 10^{-1}$		
30	$\begin{bmatrix} 00000000 \\ 56362816 \\ 00000000 \end{bmatrix} \times 10^{14}$	$\begin{bmatrix} 00000000 \\ 56361958 \\ 00000000 \end{bmatrix} \times 10^{-6}$		
20	$\begin{bmatrix} 00000000 \\ 20434715 \\ 00000000 \end{bmatrix} \times 10^{10}$	$\begin{bmatrix} 00000000 \\ 20434361 \\ 00000000 \end{bmatrix} \times 10^{-10}$		

[a](NIWC) — numerically integrat d with corrections

[b](NIW/OC) — numerically integrated without corrections

[c](NI) — numerically integrated

From Table IV one concludes that $\tilde{y}_2(\mathrm{NIW/OC})$ is much more accurate than $\tilde{y}_2(\mathrm{NI})$ Since $\tilde{y}_2(\mathrm{NI})$ is used to compute $\tilde{y}_2(\mathrm{NIW/OC})$ one may wonder why there is such a discrepancy Before performing the comparison, one must put the solutions in the corrected form (58) If this is not done, one will be comparing different members of the two-parameter family $\{\tilde{y}_2\}$

One can predict the inaccuracies of $\tilde{y}_2(\mathrm{NIW/OC})$ Since $\tilde{y}_2(x)$ is decreasing exponentially fast for integration with a negative step, one expects a breakdown in the numerical solution, i e , no significant figure agreement with a member of $\{\tilde{y}_2\}$ for x near the value $x = 80$

It is easily seen from the tables that for numerical integration with corrections one obtains $\tilde{y}_1(x)$ to five significant figures with a seven unit error in the sixth significant figure, $\tilde{y}_2(x)$ to five significant figures with a four unit error in the sixth significant figure and $\tilde{y}_3(x)$ to four significant figures with a one unit error in the fifth significant figure

If one uses the value of $\tilde{y}_2(\mathrm{NIWC})$ at $x = 20$ as an initial condition and integrates with a positive step to $x = 100$ then one obtains $\begin{bmatrix} \alpha_2(100) \\ \beta_2(100) \\ \gamma_2(100) \end{bmatrix}$ If this value is reduced to the form (58) and if the initial vector is also reduced to this form one can compare these values The results of such a comparison yield four significant figure agreement with a one unit difference in the fifth significant figure This difference is seen to be of the same order as the error in $\tilde{y}_2(x)$ at $x = 20$ Since $\tilde{y}_1(x)$ and $\tilde{y}_3(x)$ are used in determining this value of $\tilde{y}_2(x)$, one might expect this difference to also be an indication of the errors present in these solutions

It is rather surprising to obtain a better value of $\tilde{y}_2(x)$ at $x = 20$ than was obtained for $\tilde{y}_3(x)$ Apparently this is due to the relative inaccuracy of $\tilde{y}_3(100)$ as compared to $\tilde{y}_2(100)$ The inaccuracies in $\tilde{y}_3(x)$ and $\tilde{y}_1(x)$ slightly change the multiples of these solutions required for the correction process but do not seem to affect greatly the accuracy of $\tilde{y}_2(x)$ This is a very useful property since for larger systems, i e , for higher order systems one can hope to handle all the solutions in the above manner If the accuracy decreased, one would require increased precision for higher order systems

OBSERVATIONS AND FURTHER RESEARCH

It is interesting to note that if one introduces the inner product

(61) $$(\underline{y},\underline{v}) = y_1v_1 + y_2v_2 + y_3v_3$$

where $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then one can introduce the angle θ,

(62) $$\theta = \text{arc cos}\left(\frac{(\underline{y},\underline{v})}{\sqrt{(\underline{y},\underline{y}) \times (\underline{v},\underline{v})}}\right)$$

The corrected solutions $\tilde{y}_1(x)$, $\tilde{y}_2(x)$, $\tilde{y}_3(x)$ are practically mutually orthogonal The minimum angle occurring for any pair of solutions $(100 \leq x \leq 20)$ is nearly 89° The correction process is roughly equivalent to a Gram-Schmidt process Since a method of this type has already been investigated by Conte [5] it would be interesting to see the method in this paper and Conte's method compared In this connection, it is even more interesting to note that this author has more nearly satisfied the conditions that Conte wanted to satisfy, i e , Conte was willing to tolerate 1° values of θ

One should exploit, if possible, any freedom that a problem allows For example, one can define an inner product so that an arbitrary set of n linearly independent vectors in an n-dimensional space form an orthonormal basis One should choose this basis after one has simplified the first few coefficient matrices [8].

Since Conte's method and the method in this paper in their present forms seem to be intended for use in different problems one would first need to extend both methods before attempting a comparison

If one attempts to utilize the correction procedure described in this paper one must be careful to integrate in a direction so that the correction procedure is justified For example if one attempted to integrate with a positive step and obtain $\tilde{y}_1(x)$ one would anticipate the need to correct with a multiple of $\tilde{y}_2(x)$ Since $\tilde{y}_2(x)$ dominates $\tilde{y}_1(x)$ asymptotically there is no justification for adding multiples of $\tilde{y}_2(x)$ to $\tilde{y}_1(x)$, i e , one does not maintain the same asymptotic properties

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