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PROJECT APOLLO

APPLICATION OF THE MATCHED CONIC MODEL IN THE STUDY OF CIRCUMLUNAR TRAJECTORIES

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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Prepared by: Thomas F. Gibson

Thomas F. Gibson
AST, Flight Mechanics

Authorized for Distribution:

Maxime A. Faget

Maxime A. Faget
Assistant Director

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APPLICATION OF THE MATCHED CONIC MODEL IN THE STUDY OF CIRCUMLUNAR TRAJECTORIES

ABSTRACT

Matched conic and integrated circumlunar trajectories with the same constraints are compared at injection, pericynthion and perigee. Results indicate that the matched conic model may be used for design purposes and for obtaining trends of circumlunar trajectories. In addition, an iteration scheme is presented to obtain any desired integrated trajectory. The scheme utilizes the matched conic model in conjunction with the integrated n-body model. Convergence is rapid.

INTRODUCTION

Two problems encountered in the study of circumlunar trajectories are: (1) obtaining a reliable analytic model of the earth-moon system and (2) obtaining a fast converging iteration scheme for the determination of initial conditions for precision trajectories. These problems arise because there is no known analytical solution to the general n-body problem where there are more than two bodies involved. It is therefore desirable to obtain an approximate earth-moon model which has an analytical solution and yields results which are good approximations to precision (integrated) results. This model may then be used for parametric studies, as an initial condition generator, and to form a basis for an iteration scheme to converge upon initial conditions for a more accurate model. The purpose of this paper is to show that the matched conic is a model which closely approximates the results of precision trajectories and to show how the matched conic model may be linked with a precision model to give a fast converging, reliable iteration scheme.

These applications divide the discussion into two sections, both of which refer to the matched conic and 7-body trajectory models. A brief discussion of each is given to familiarize the reader with the models. For more detailed information concerning the models, see references 1 and 2.

SYMBOLS

N	translunar or transearth trajectory above the moon orbital plane, deg
R	distance of closest approach (r in example)
S	translunar or transearth trajectory below the moon orbital plane, deg
T	time of flight, hr
V	velocity, ft/sec
h	injection altitude, ft
i	inclination
r	radius
Ω	angle of ascending node, deg
γ	flight, path angle, deg
ϕ	lead angle, deg
Ψ	argument of radius, deg
ω	argument of periapsis, deg

Prefixes

Δ	correction term for matched conic initial conditions
∇	error in integrated end conditions

Subscripts

c	matched conic
d	desired
e	earth

i	integrated
m (e.c.)	moon
r	return
tl	translunar

Superscripts:

*	constrained element

DISCUSSION

Matched Conic

The particular matched conic model used in this discussion was initially developed by the Martin Company for NASA in connection with the Apollo study contract (ref. 1). The model simplifies the complicated earth-moon system to just two point masses which represent the earth and moon. The motion of a third body (particle) in this system is governed completely by the gravitational force of either the earth or moon. To determine which force is to be used, a fictitious sphere, called the moon's sphere of influence, is constructed such that it encloses the moon and moves with it. The radius of the sphere is constant and was chosen empirically by comparison with integrated trajectories. If the third body is inside the sphere of influence, the moon's gravitational force is used; otherwise, the earth's gravity is used. Since the force acting on the third body is always inverse square, the orbits are always solutions to the Kepler problem or conic sections. Thus, a lunar trajectory in this system is a series of conics which are matched at the sphere of influence. Specifically for circumlunar trajectories, the particle would travel out to the sphere of influence on an earth-focused ellipse determined by the initial conditions. At the sphere, the velocity and position vectors are transformed to the moving moon system which results in a hyperbolic orbit about the moon. When the particle again reaches the sphere of influence its orbit is transformed back to the earth reference. (See fig. 1.)

All the orbital parameters are computed by the usual conic formulas with the exception of the velocity at pericynthion which is computed by Jacobi's Integral to the Restricted Three-Body Problem in order to give a better approximation to integrated results.

Although analytic expression exists for a particle's trajectory in this model, the expressions relating initial and final conditions are transcendental and require iteration for solution. However, because of the simplicity of the equations, the iteration can be performed very quickly on a digital computer. A solution where three end conditions are satisfied requires about 10 seconds on the IBM 7090.

7-Body

The 7-body model used for comparison is the NASA Interplanetary program (ref. 2). Briefly this model computes a particle trajectory by Encke's Method subject to the gravitational forces of the following bodies: Earth, Moon, Sun, Venus, Mars, and Jupiter. Perturbations due to the non-spherical gravitational fields of the earth and moon are also taken into account. Initial conditions are given to start the program and a trajectory results upon numerical integration. The computational time for a single trajectory of 150 hours requires about 1.5 minutes on the IBM 7090.

Results of Trajectory Comparisons

A comparison was made with circumlunar trajectories which were computed on the matched conic and 7-body models that satisfied the same constraints. The trajectories are the figure eight type as shown in figure 2. The constraints were imposed at three points of particular interest - at the injection, at pericynthion, and at the return perigee. At injection, there are three constant constraints imposed: radius (r), flight-path angle (γ), and translunar inclination (i_{tl}). At pericynthion, one constraint, a constraint, a constant pericynthion radius (r_m), is imposed. At perigee, two constant constraints, perigee (r_e) and return inclination (i_{re}), are imposed. The geometry and definition of the constraints at injection, pericynthion, and perigee are given in figures 3 to 5, respectively. Specifying the six constraints and the initial time determines a circumlunar trajectory uniquely on both models. A total of 18 elements are defined at the three points, six at each point. Six of the 18 elements have the same constant values on both models because of the constraints; however, the remaining 12 elements will generally have different values on the two models. The 12 unconstrained elements are called free elements. At injection, the free elements are velocity (V), lead angle (ϕ), and argument of injection (Ψ). At pericynthion, the five free elements are time from injection (T_m), velocity (V_m), inclination (i_m), angle of ascending node (Ω_m), and argument of pericynthion (ω_m). The four free elements at perigee are time from injection (T_m), velocity (V_e), angle of ascending node (Ω_e), and argument of perigee (ω_e). The deviations of the matched conic elements from the 7-body elements will give a good indication of the accuracy of the matched conic model. The definition of the free elements at the three points is defined and denoted on figures 3 to 5. A summary of the free and constrained elements is given in table I.

A sample matched conic and 7-body comparison is given in table II. The injection time for all the trajectories presented is Greenwich midnight January 15, 1967. The moon is near apogee at the time of arrival at pericynthion. The two trajectories have a pericynthion altitude of 100 nautical miles and a perigee altitude of 20 nautical miles. The inclination at perigee is 5° south. The differences are computed by subtracting the matched conic element from the corresponding 7-body element. At injection, the differences in the constrained elements are 0 because they are direct inputs. Small difference in the constraints exists at pericynthion and perigee because of the small convergence tolerances in the iteration scheme.

To give further evidence that the differences are small, trajectories were run on the two models where one of the six constraints was varied. The return inclination (i_{re}) was varied between 60°S and 60°N. The trajectories were injected at an altitude of 600,000 ft, 0° flight-path angle (γ), and 1°N translunar inclination (i_{tl}). The pericynthion altitude was 100 nautical miles and the perigee altitude 20 nautical miles. Plots of the free elements as functions of i_{re} for both models at injection, pericynthion and perigee are given in figures 6 to 8, respectively. It is seen from the plots that the shapes of the matched conic curves are very near the same shapes as the corresponding precision curves. In addition, the matched conic curves are displaced from the integrated curves by small increments which are nearly constant. Hence a small constant correction term may be added to the matched conic value to give even a better approximation. These facts are the foundation of the iteration scheme presented in the following section.

In addition to the few comparisons presented here, the author has made many spot checks with variations in the six constraints. In all cases, the matched conic performed as well as presented here.

On this basis, it seems justified to infer that the differences remain small for all figure eight circumlunar trajectories.

Iteration Scheme

The problem of calculating a trajectory which satisfies certain constraints along its path is called a boundary-value problem. The solution to the circumlunar boundary-value problem requires a numerical solution because of the absence of an analytic solution to the n-body problem. Generally, in obtaining a numerical solution, a first guess is made to the initial conditions and a trajectory is computed by numerical integration. The values of the end conditions are compared with the desired ones. If any of the end conditions are not within the allowable error, another guess of the initial conditions is made. The procedure of guessing (iteration) and integrating continues until the desired end conditions are met or until the results indicate that the desired solution does not exist. Obviously the rate of convergence to the proper initial conditions depends upon the particular iteration scheme. Since the computational time of a single 7-body integrated trajectory is about $1\frac{1}{2}$ minutes, an iteration scheme is desired so that only a few integrated trajectories need be computed for convergence.

Ordinarily, an iterative scheme would rely on a near-linear relationship existing between the initial conditions and end conditions. However, for circumlunar trajectories it is well known that the relationship is very nonlinear.

The Δ iteration scheme presented here relies on the near constant differences existing between matched conic and 7-body trajectories which were presented in the previous section of this paper. This means that a set of constant correction terms may be used to update the matched conic initial conditions. Also, changes in the initial conditions will affect the end condition in the same manner for both models.

The simple theory of the Δ iteration scheme can best be shown by figure 9 which exhibits a single variable iteration. The figure shows two curves of injection velocity as a function of return inclination. The correction term ΔV is assumed to be known from a previous trajectory comparison and is used throughout the iteration. For the first guess, the matched conic velocity is computed. This matched conic velocity corresponds to point 1_c on the matched conic curve. The term ΔV is added to the matched conic velocity, and this new velocity is used in the integrating program. Upon integration, the trajectory arrives at the point 1_i on the integrated curve. An error ∇_1 now exists in the return inclination. Since trends are nearly the same, a change of ∇_1 in the return inclination of the matched conic will produce nearly the same change in the return inclination of the integrated. Therefore, the inclination is moved to the right by ∇_1 on the matched conic curve, which gives point 2_c. Again ΔV is added and this gives point 2_I on the integrated curve. A much smaller error ∇_2 now exists. The matched conic is changed by ∇_2 and the procedure continues as before until ∇ is within the allowable tolerance. (Note that the error in the integrated end condition is always added algebraically to the matched conic end condition.) If the ΔV used was the exact one for the desired inclination, no iteration would be necessary. But, in general, the ΔV used will not be the exact one. It is immediately evident that the function of ΔV is to make the iteration occur between closer curves. For multiple convergence, the procedure is the same; correction terms are added to the free initial conditions, and the errors are added to the end conditions of the matched conic. In general, the correction terms are necessary because uncorrected initial conditions give rise to integrated trajectories which are usually grossly in error.

A step-by-step procedure for a multiple convergence follows. For clarity, the method will obtain initial conditions for a circumlunar trajectory which has desired value of pericynthion, perigee, and return inclination (see fig. 2). Injection is from a fixed translunar inclination, altitude, and flight-path angle. The injection geometry is shown in figure 3. A set of correction Δ's is known from a previous trajectory comparison and is used throughout the iteration. In case a set of Δ's is not available or a closer set is desired, they may be obtained by the following steps:

<u>Step 1</u>: Run a matched conic trajectory with the desired constraints.

<u>Step 2</u>: Run integrated trajectory using matched conic initial conditions.

<u>Step 3</u>: Vary one of the integrated free initial conditions (V, Ψ, or ϕ) until a trajectory has the proper return perigee (no more than five integrated trajectories are required in using linear scheme).

Step 4: Run a matched conic which has the same pericynthion and return inclination as the integrated trajectory with the proper perigee.

Step 5: Compute the difference in the free initial conditions:

$$\left.\begin{aligned} \Delta V &= V_i - V_c \\ \Delta \Psi &= \Psi_i - \Psi_c \\ \Delta \phi &= \phi_i - \phi_c \end{aligned}\right\} \Delta\text{'s}$$

The same set of Δ's has been adequate for all the free return trajectories computed by the author to date. However, for faster convergence, the Δ's shown in figure 10 should be used. With a set of Δ's available, the steps in the Δ iteration scheme are outlined below and summarized in block diagram form in figure 11.

Step 1: Run matched conic with desired end conditions. The output is a set of initial conditions.

Step 2: Correct initial conditions by Δ's.

Step 3: Compute integrated trajectory by using corrected initial conditions.

Step 4: Compute the error differences (Δ's) in the end conditions (desired value minus integrated value).

Step 5: If errors are small enough, iteration is terminated. If not, errors are added to previous matched conic end conditions.

Step 6: Repeat the process, but in Step 1 use new matched conic end conditions.

In general, the number of iterations necessary depends upon the allowable end condition tolerances and accuracy of the Δ's. Ordinarily 6 or 7 iterations are required (10 minutes computer time).

To illustrate, a sample iteration is shown in figure 12. The constraints are at the top of the figure together with the correction Δ's. The diagram is largely self-explanatory, however, the following explanation is given for the convergence to return perigee radius. The other parameters, pericynthion radius and return inclination, are iterated simultaneously with the perigee radius and in the same fashion. For the first iteration, the desired value of 3,464 nautical miles was used in the matched conic. The resulting integrated value was 3,164 nautical miles or 300 nautical miles too low. Hence, for the next iteration, perigee on the matched conic was increased by 300 nautical miles to equal 3,764 nautical miles. Instead of increasing by 300 nautical miles, the integrated value only increased 274 nautical miles or, still an error of 26 nautical miles. On the next iteration the matched conic was further increased by 26 nautical miles to a perigee of 3,790 nautical miles. The use of a corrected matched conic initial condition for this trajectory resulted in an integrated trajectory with an error in perigee of only 1.1 nautical miles. In each successive iteration the errors were reduced. The magnitude of the errors shows that perigee radius is the most sensitive. The exact Δ's for the trajectory are obtained by subtracting the first matched conic from the last corrected matched conic. These Δ's are shown at the bottom of the figure.

CONCLUDING REMARKS

The matched conic elements approximate precision elements adequately for many types of parametric studies. An even better approximation may be obtained by adding constant correction terms to the conic element. The Δ iteration scheme has been tested and in all cases proved to be superior to linear perturbation type iteration schemes in both dependability and computational time.

REFERENCES

1. Anon: Apollo Final Report - Trajectory Analysis. Martin Rep. No. ER 12003, June 1961.

2. Pines, Samuel, and Wolf, Henry: Interplanetary Trajectory by the Encke Method Programmed for the IBM 704 and 7090. Republic Aviation Corporation Rep. No. RAC-656-451, Dec. 1960.

TABLE I.- ORBITAL ELEMENTS

Point	Elements	
	Fixed constrained	Free
Injection	r, γ, i_{tl}	V, ψ, ϕ
Pericynthion	r_m	T_m, i_m, V_m, Ω_m, ω_m
Return perigee	r_e, i_{re}	T_e, V_e, Ω_e, ω_e

TABLE II.- TRAJECTORY COMPARISON

Point	Element	Matched conic	7-Body	Difference
Injection	V, ft/sec	35,956.901	35,959.242	2.341
	ψ, deg	7.75578	7.330060	-.42572
	ϕ, deg	38.14462	38.97043	.82581
	i_{t1}*, deg	30 N	30 N	0
	γ*, deg	0	0	0
	r*, ft	21,525,244	21,525,244	0

TABLE II.- TRAJECTORY COMPARISON - Continued

Point	Element	Matched conic	7-Body	Difference
Pericynthion	r_m*, ft	6,288,518	6,288,626	108
	V_m, ft/sec	8,193.7	8,200.4	-6.7
	T_m, hr	75.045	75,339	.294
	Ω_m, deg	34,586	35.293	.707
	i_m, deg	4,599	4.638	.039
	ω_m, deg	-144.502	-143.798	.704

TABLE II.- TRAJECTORY COMPARISON - Concluded

Point	Element	Matched conic	7-Body	Difference
Perigee	r_e*, ft	21,133,527	21,133,948	421
	i_{re}*, deg	5.186 S.	5.184 S.	.003
	T_e, hr	148.83	147.98	-.85
	Ω_e, deg	177.38	176.87	-.51
	ω_e, deg	-11.78	-10.98	-.79
	V_e, ft/sec	36,626	36,631	5

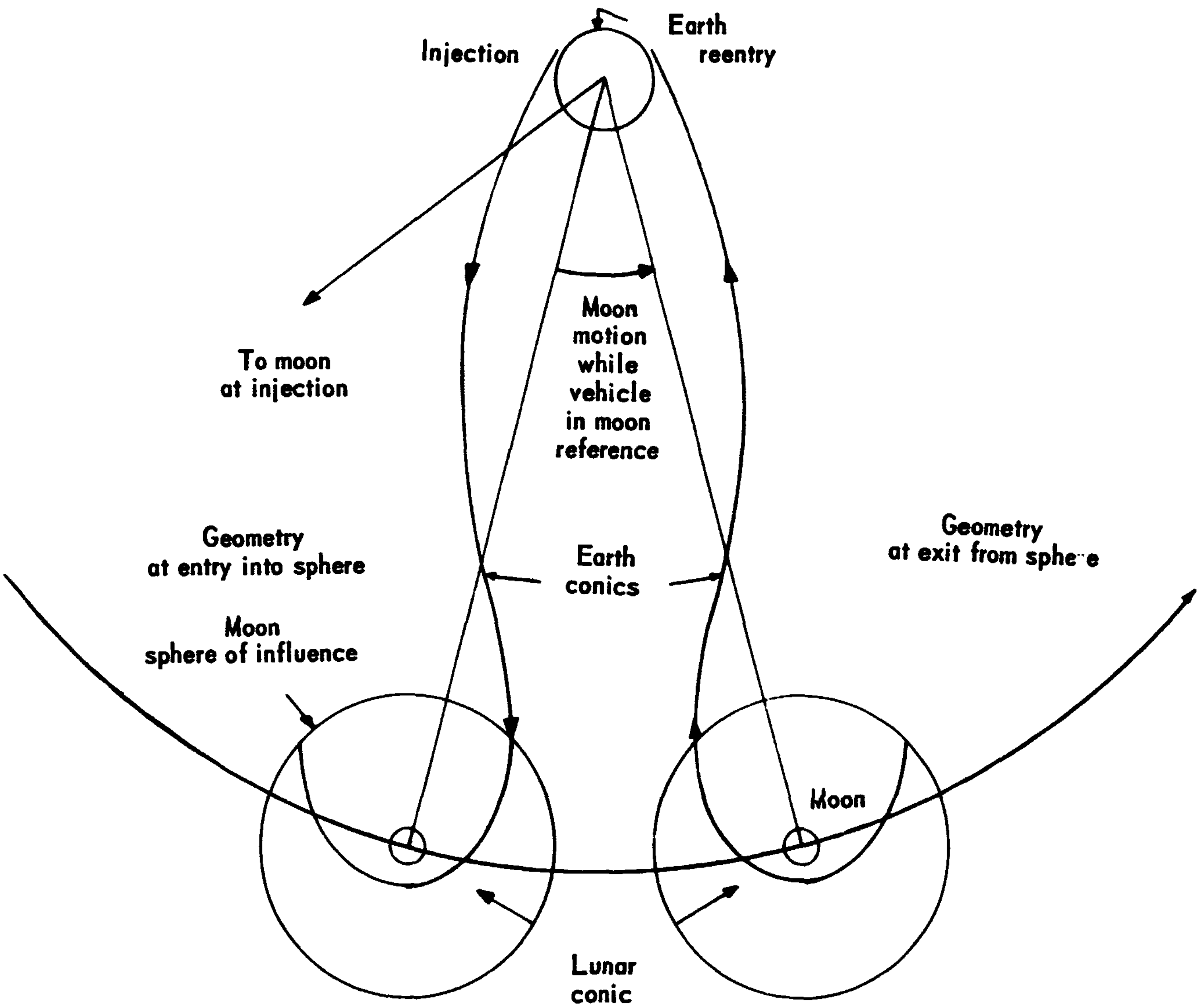


Figure 1. — Matched conic model

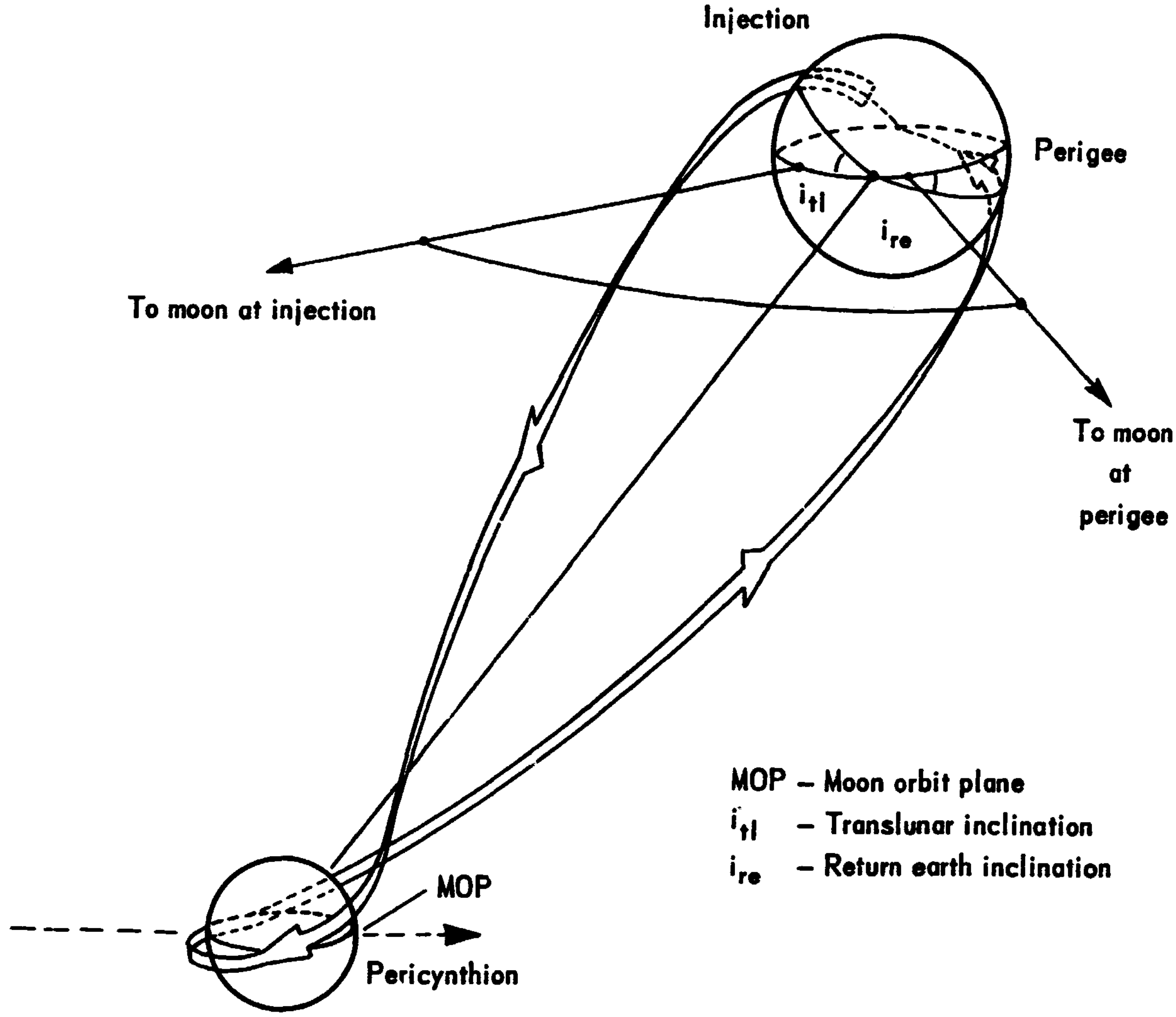


Figure 2.– Free return circumlunar trajectory

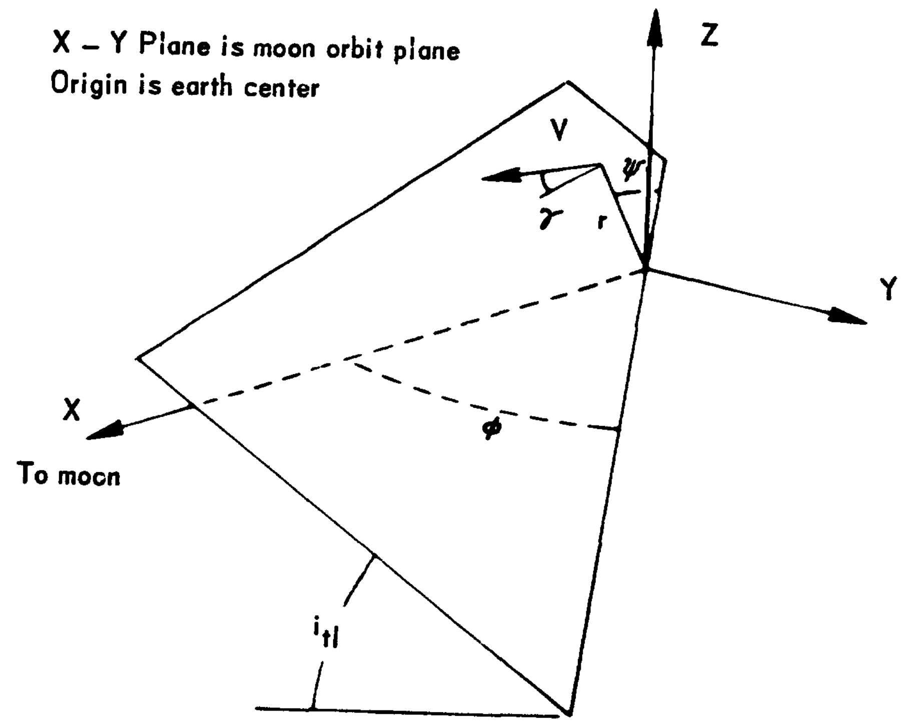


Figure 3.— Initial condition geometry

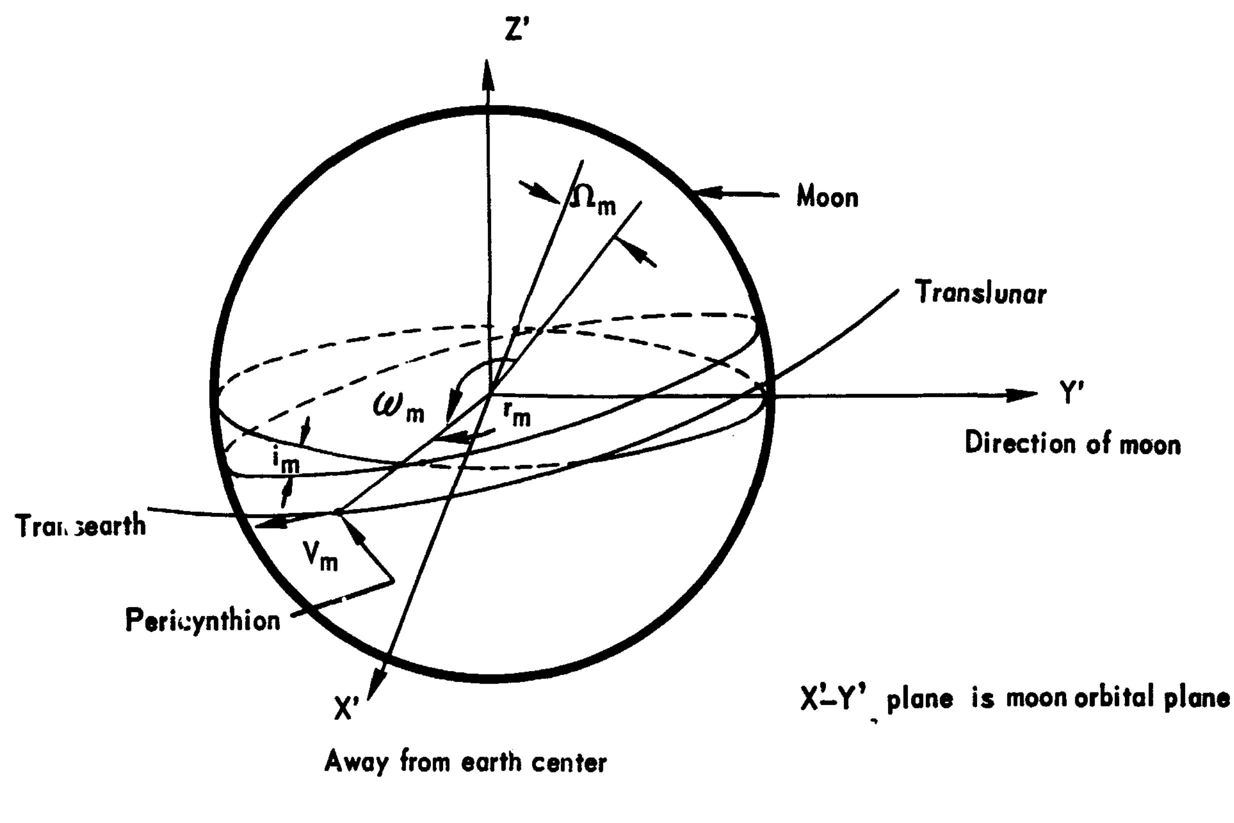


Figure 4. – Pericynthion parameters

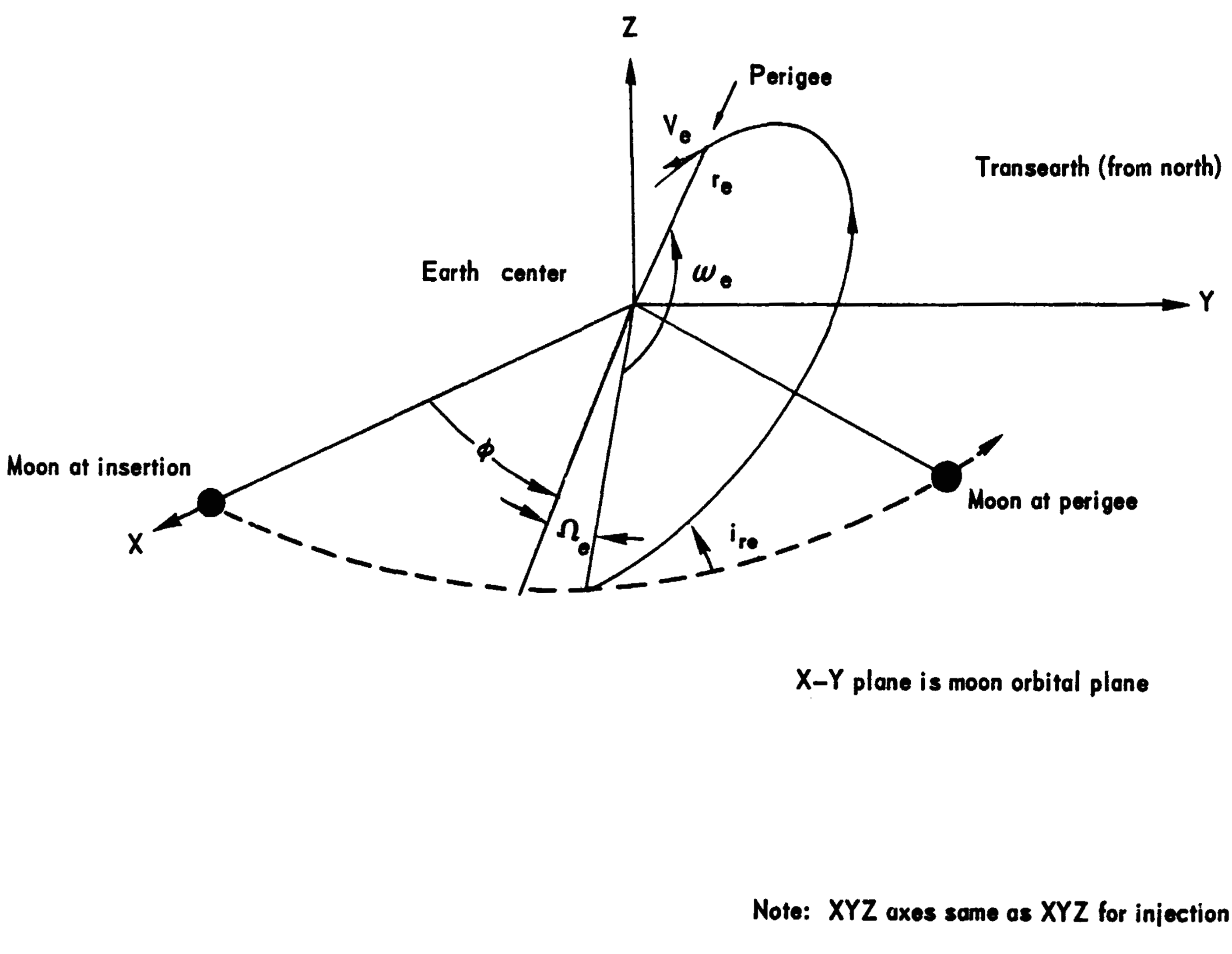
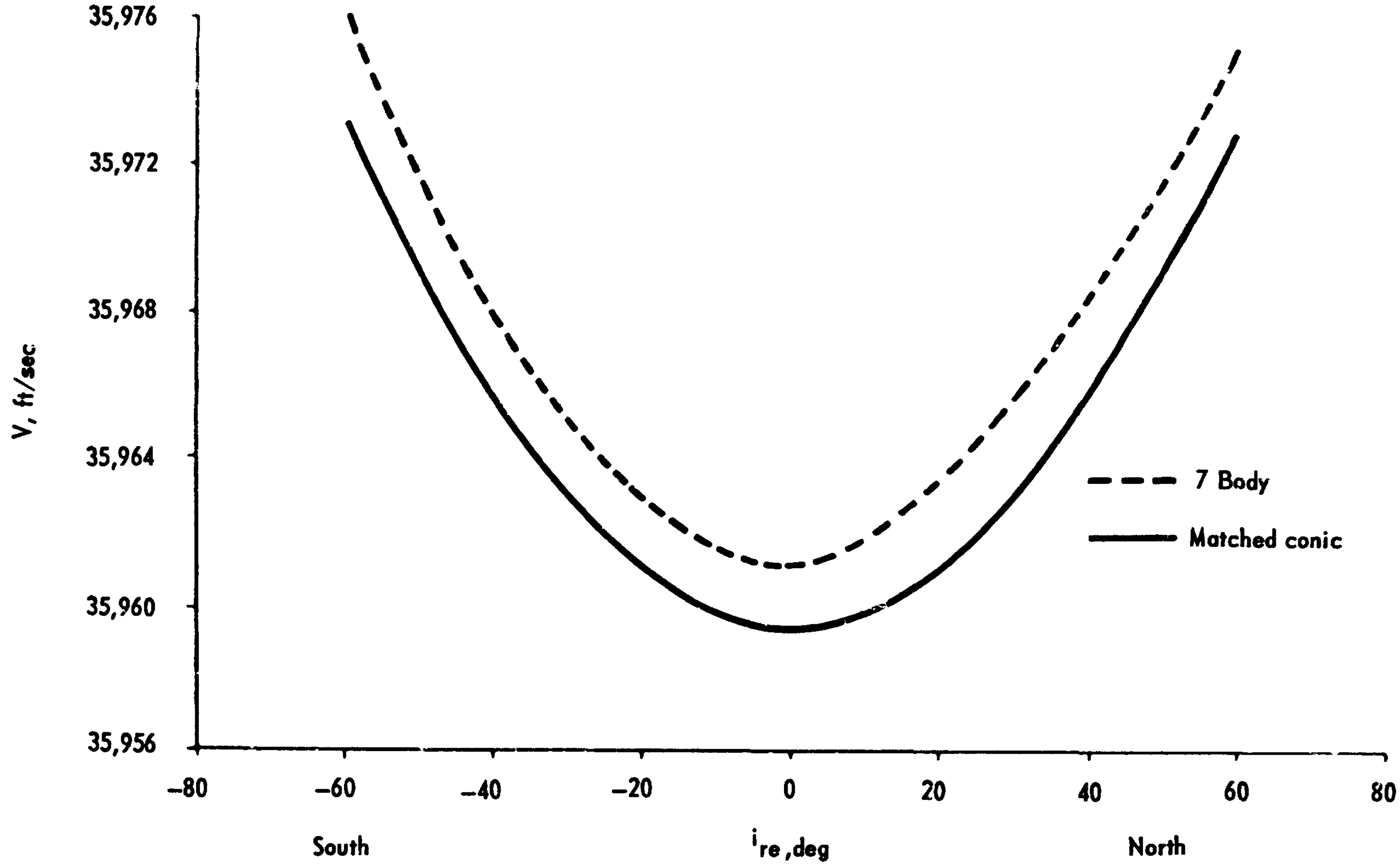
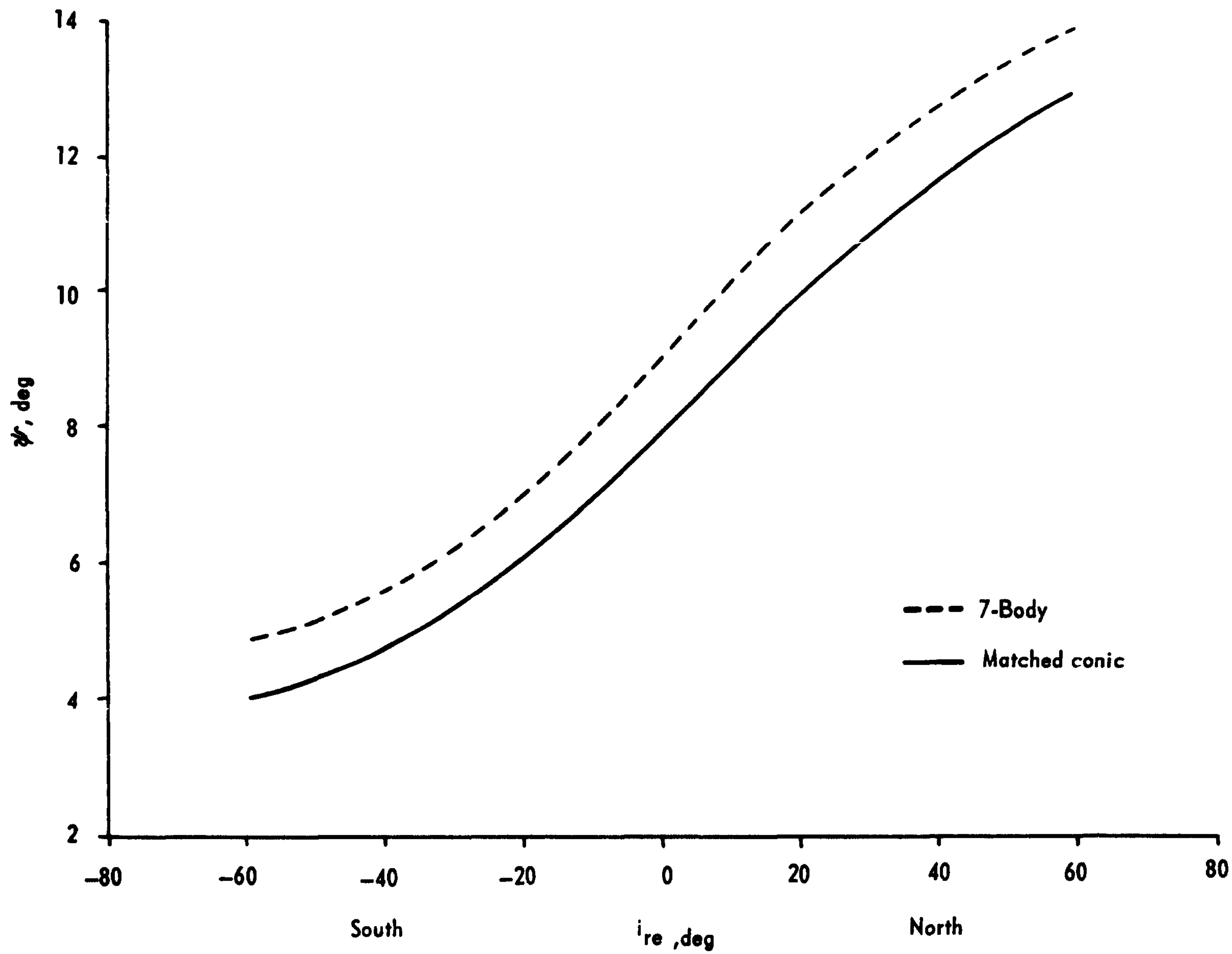


Figure 5. – Return parameters at perigee



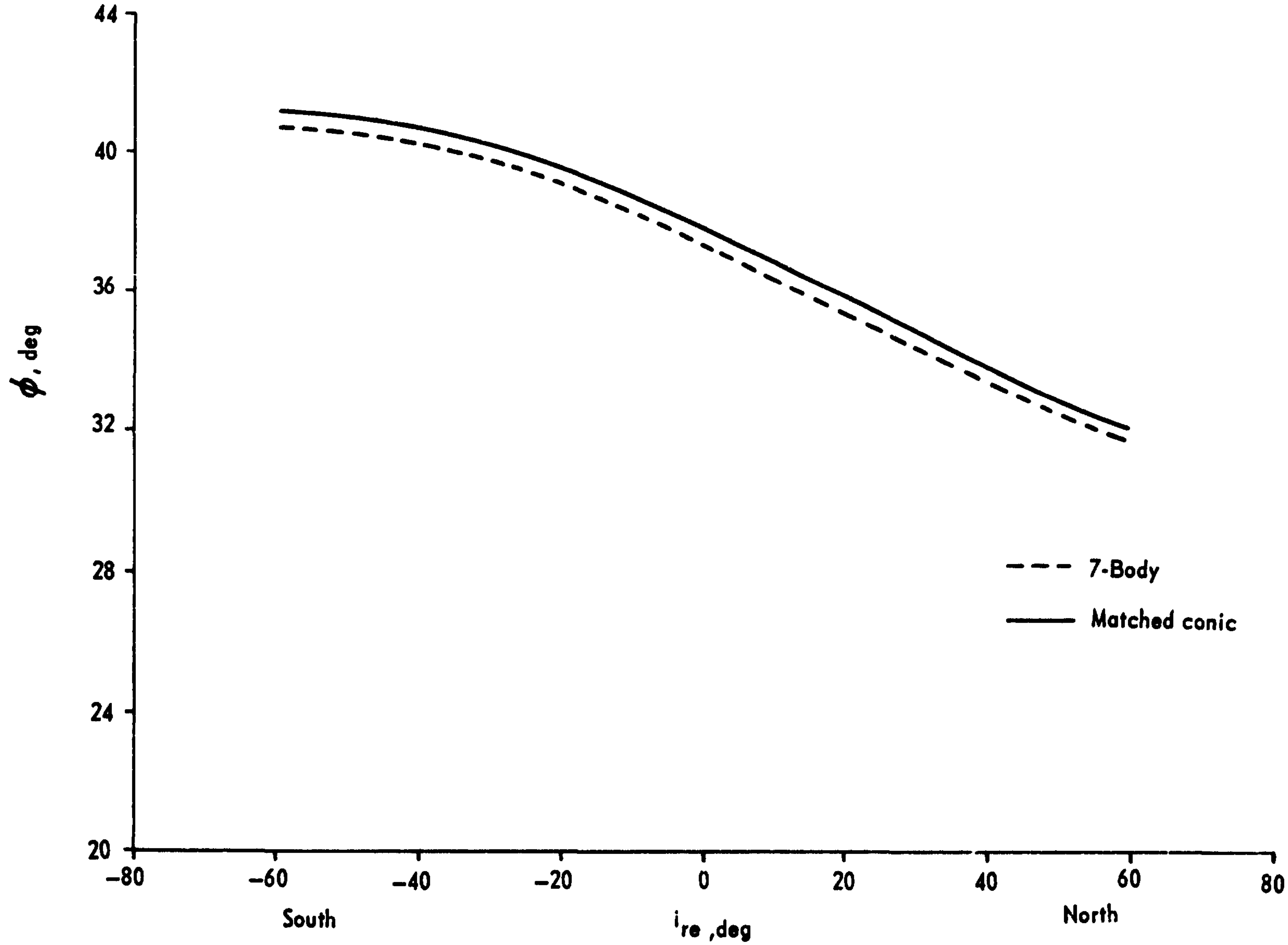
(a) Velocity

Figure 6.— Free orbital elements at injection as functions of the return inclination



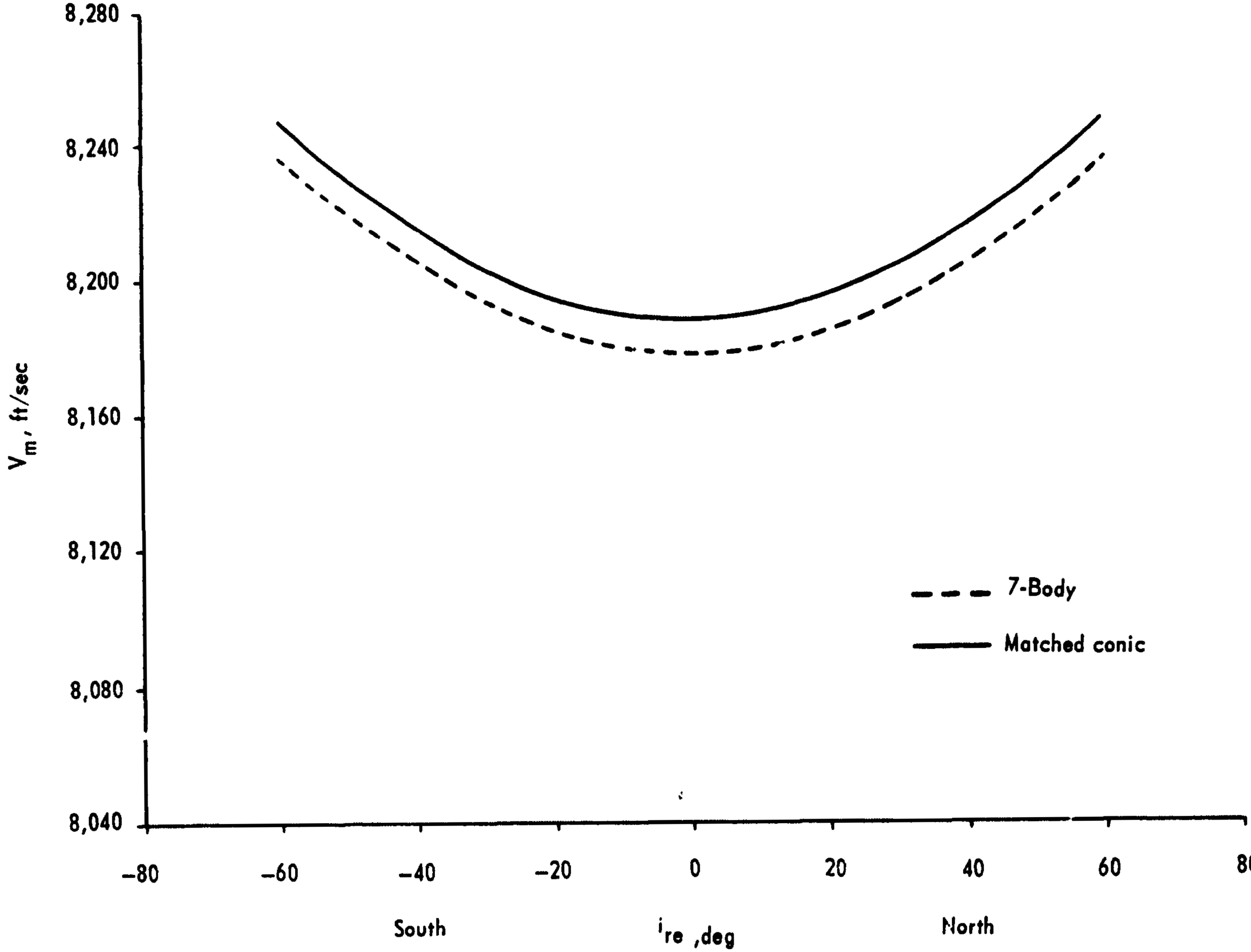
(b) ψ

Figure 6.– Continued



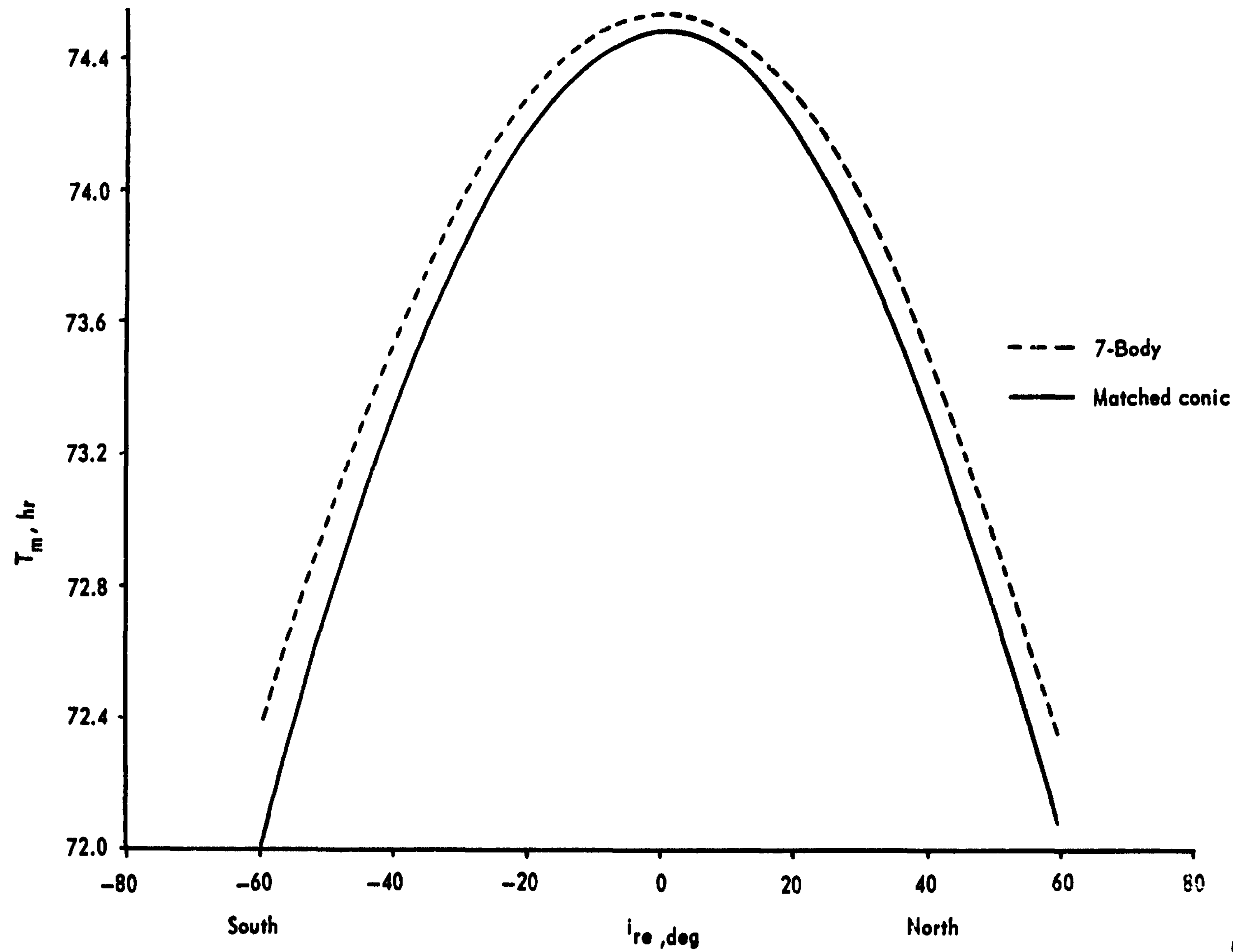
(c) ϕ

Figure 6.– Continued



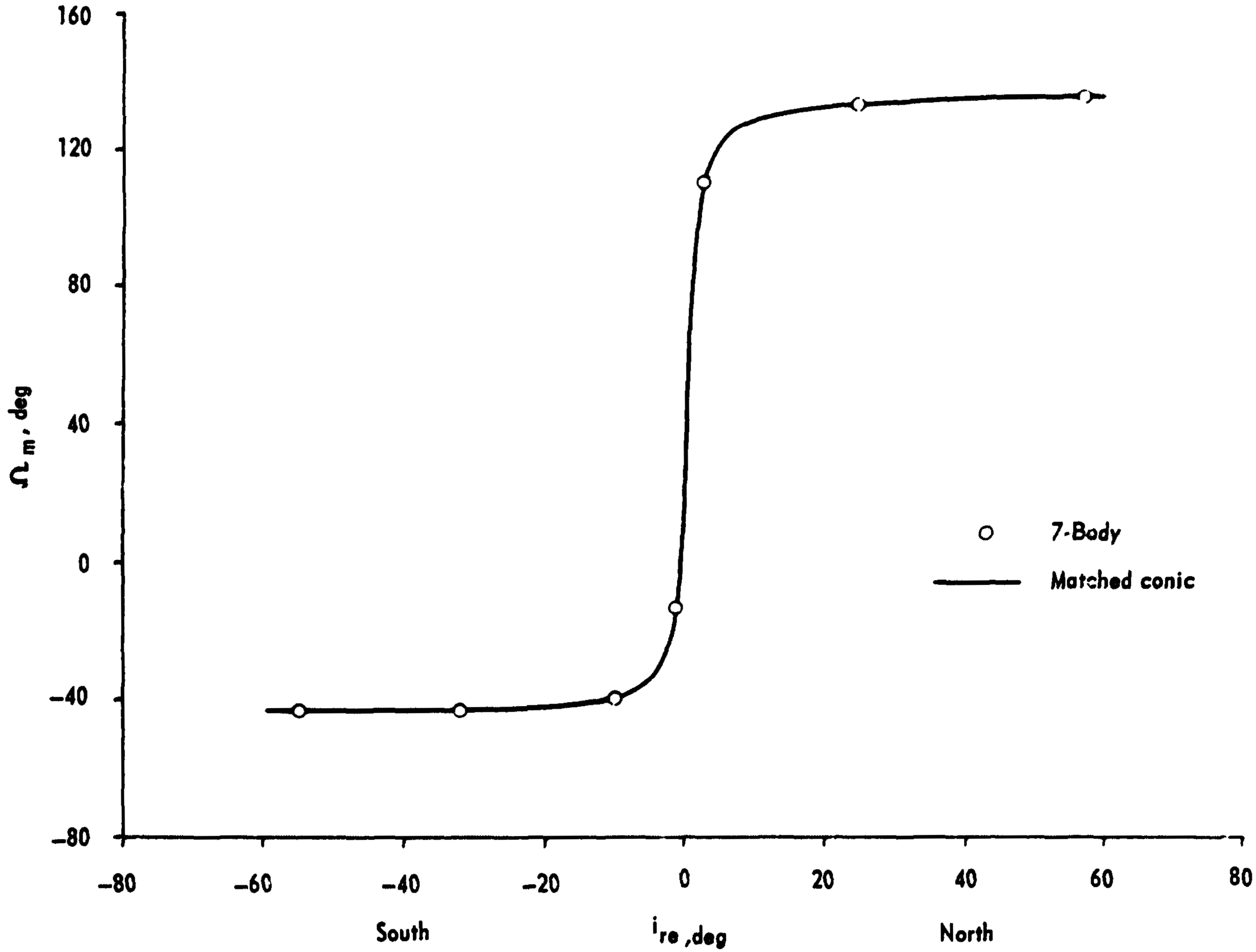
(a) V_m

Figure 7. —Free orbital elements at pericynthion as functions of return inclination



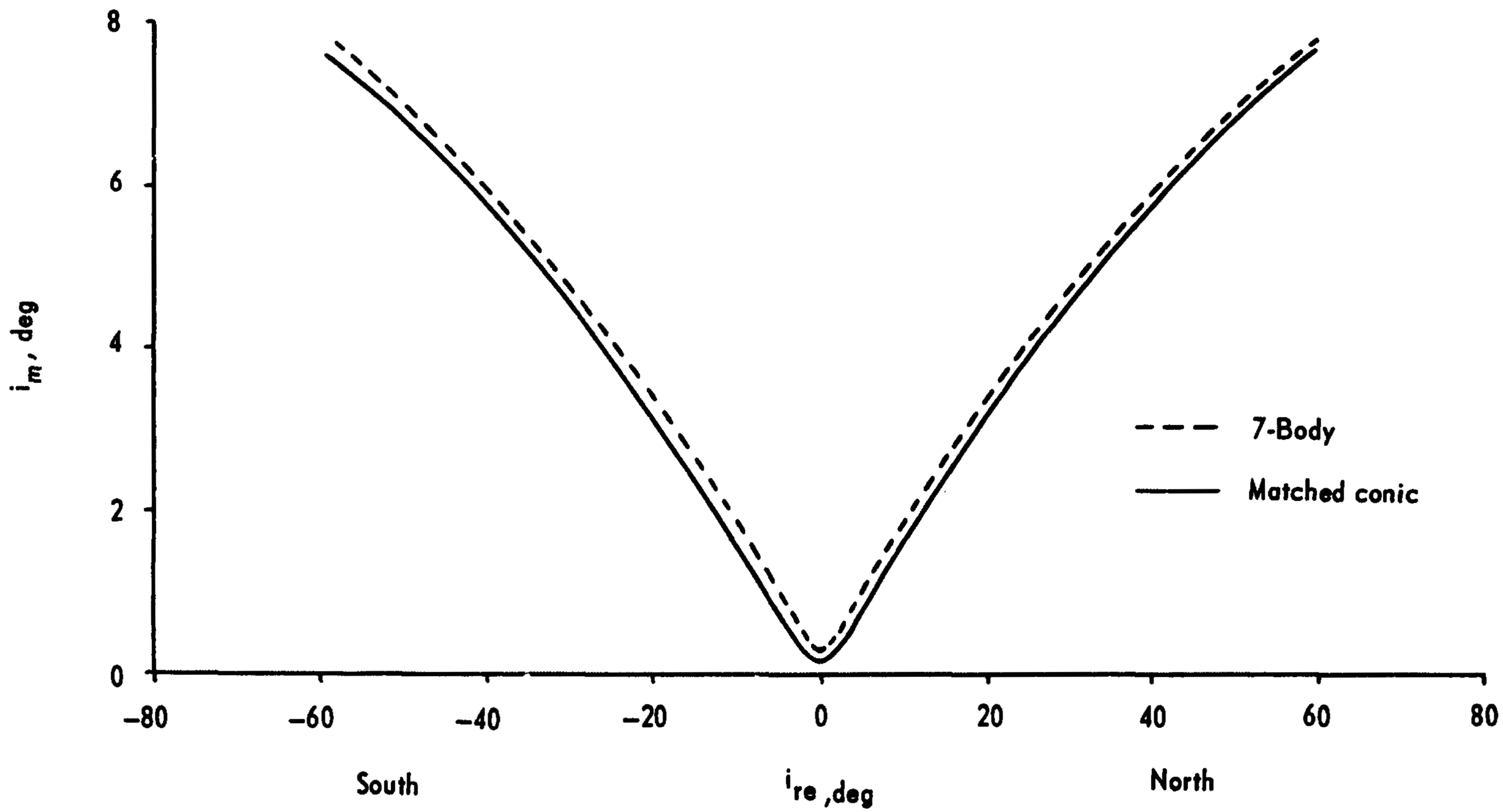
(b) T_m

Figure 7. – Continued



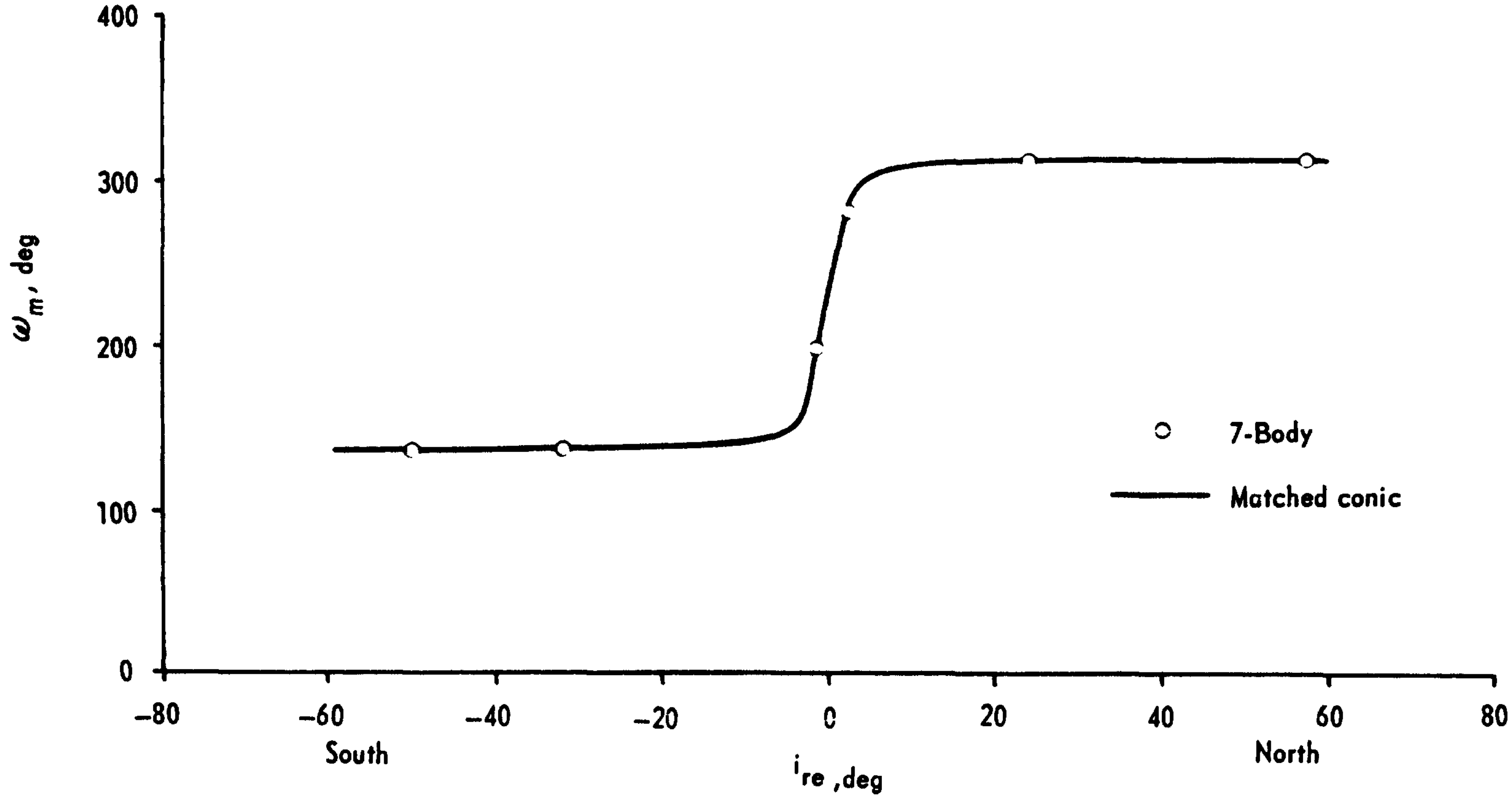
(c) Ω_m

Figure 7. – Continued



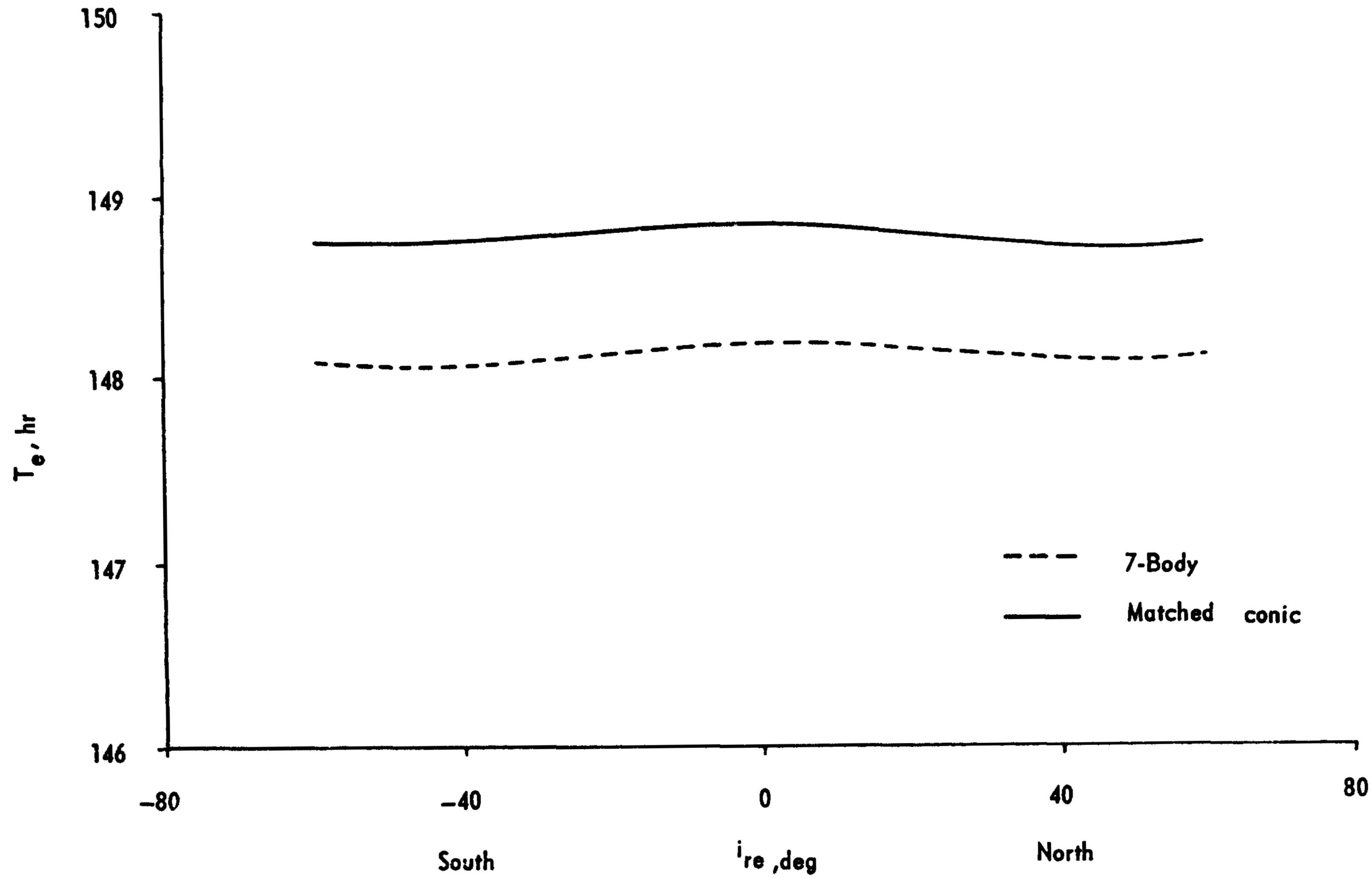
(d) i_m

Figure 7. – Continued



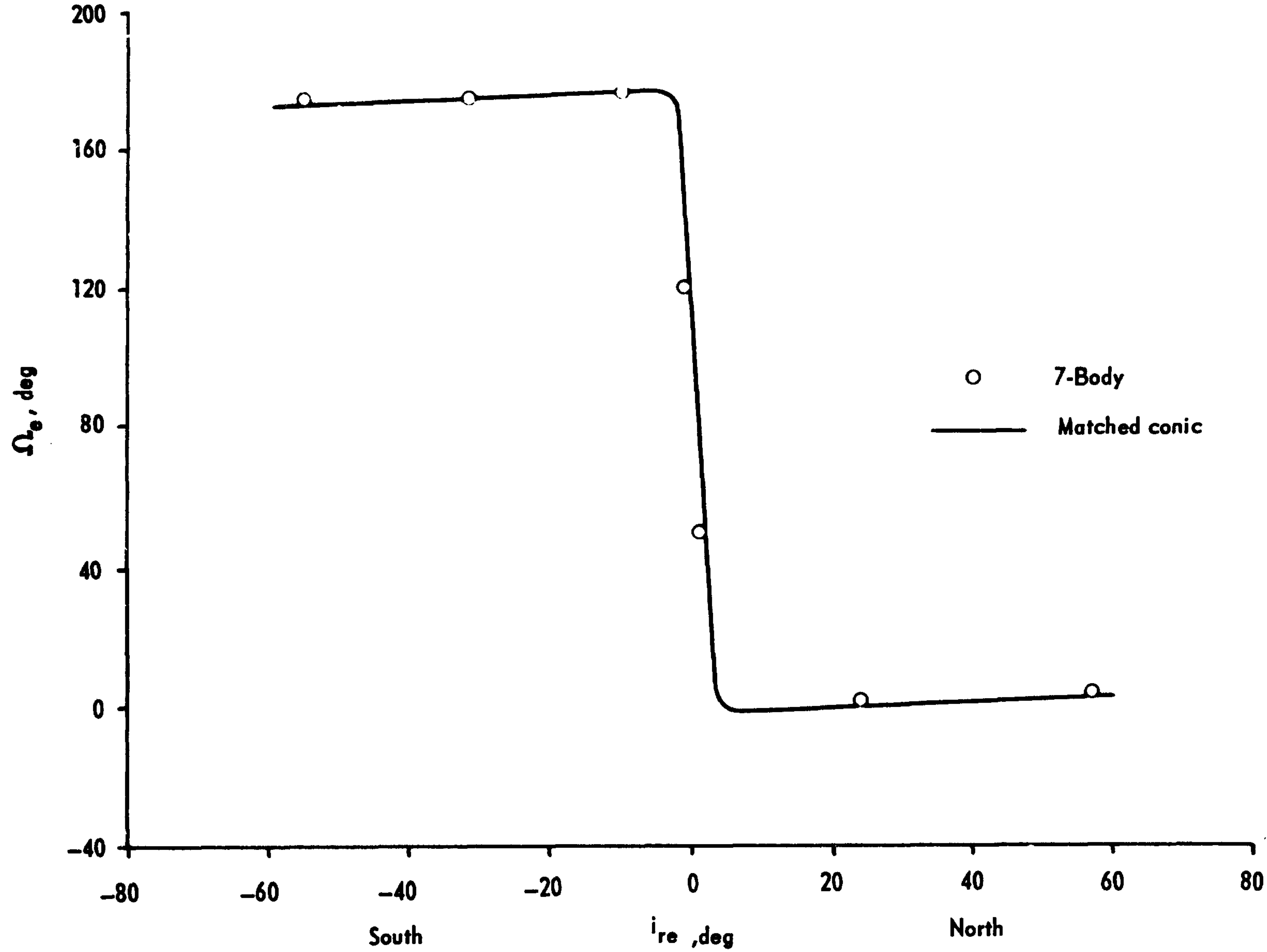
(e) ω_m

Figure 7. – Continued



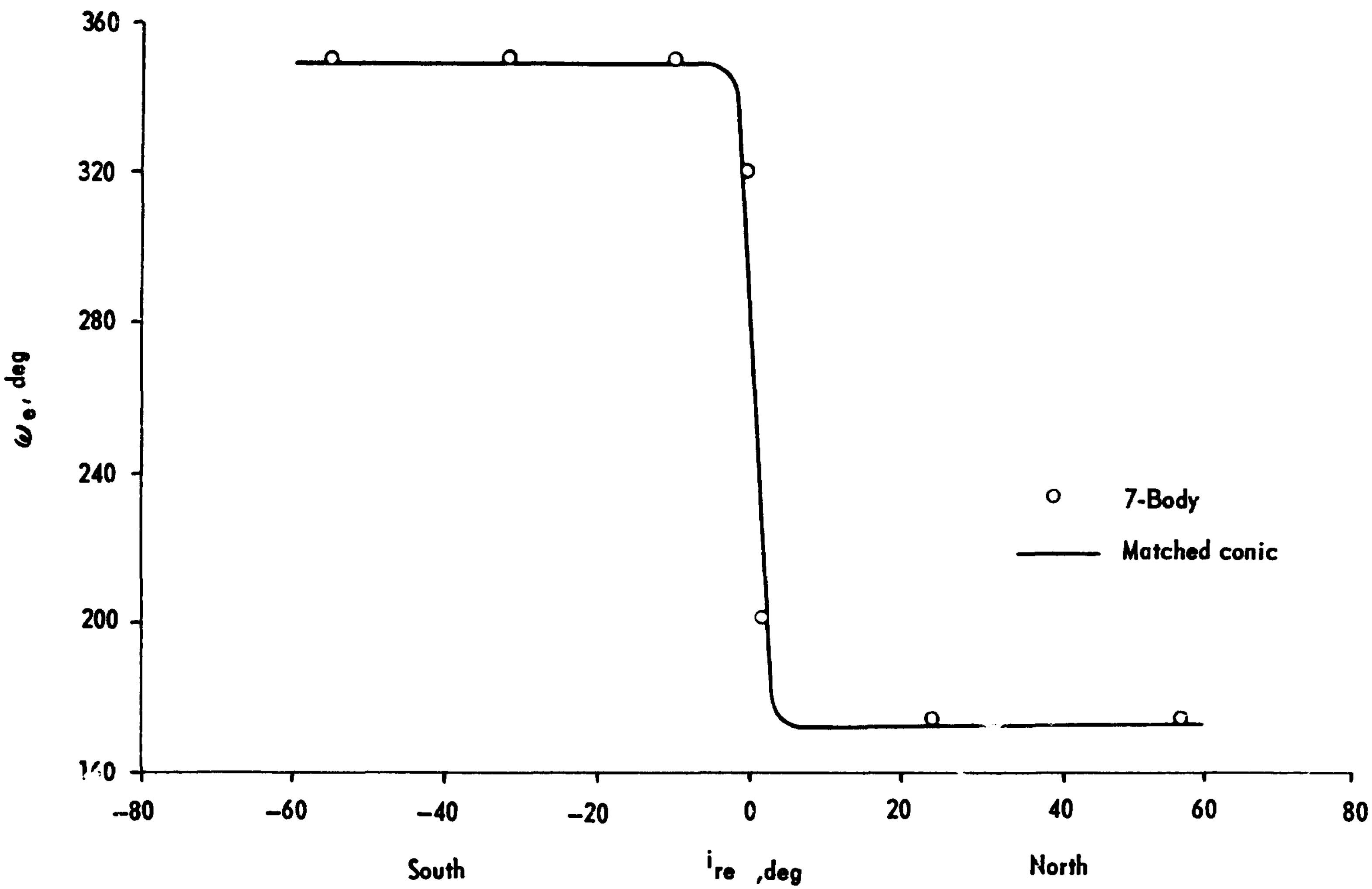
(a) T_e

Figure 8.– Free orbital elements at perigee as functions of return inclination



(b) Ω_e

Figure 8. – Continued



(c) ω_e

Figure 8.– Continued

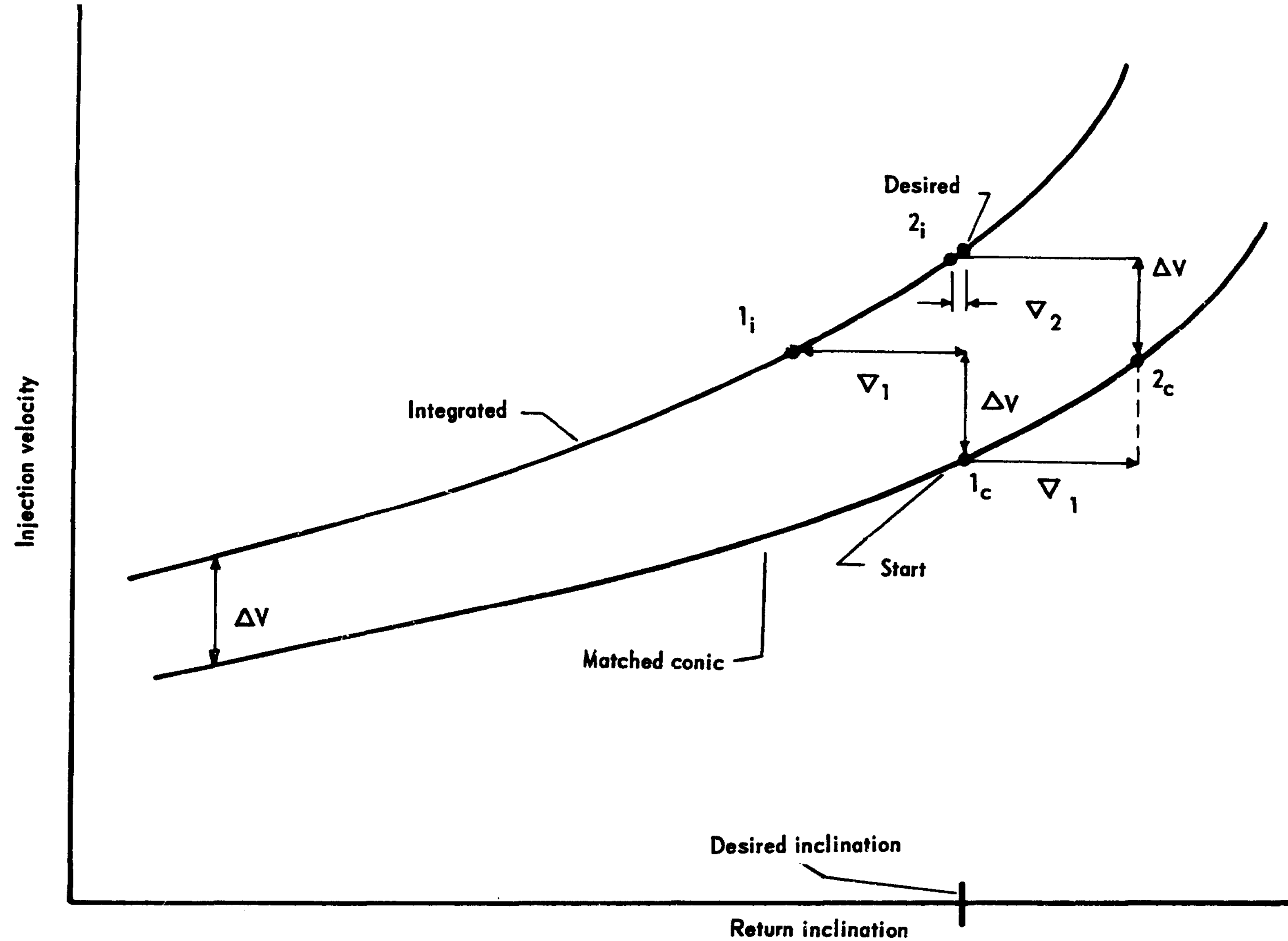


Figure 9. – Single variable Δ iteration

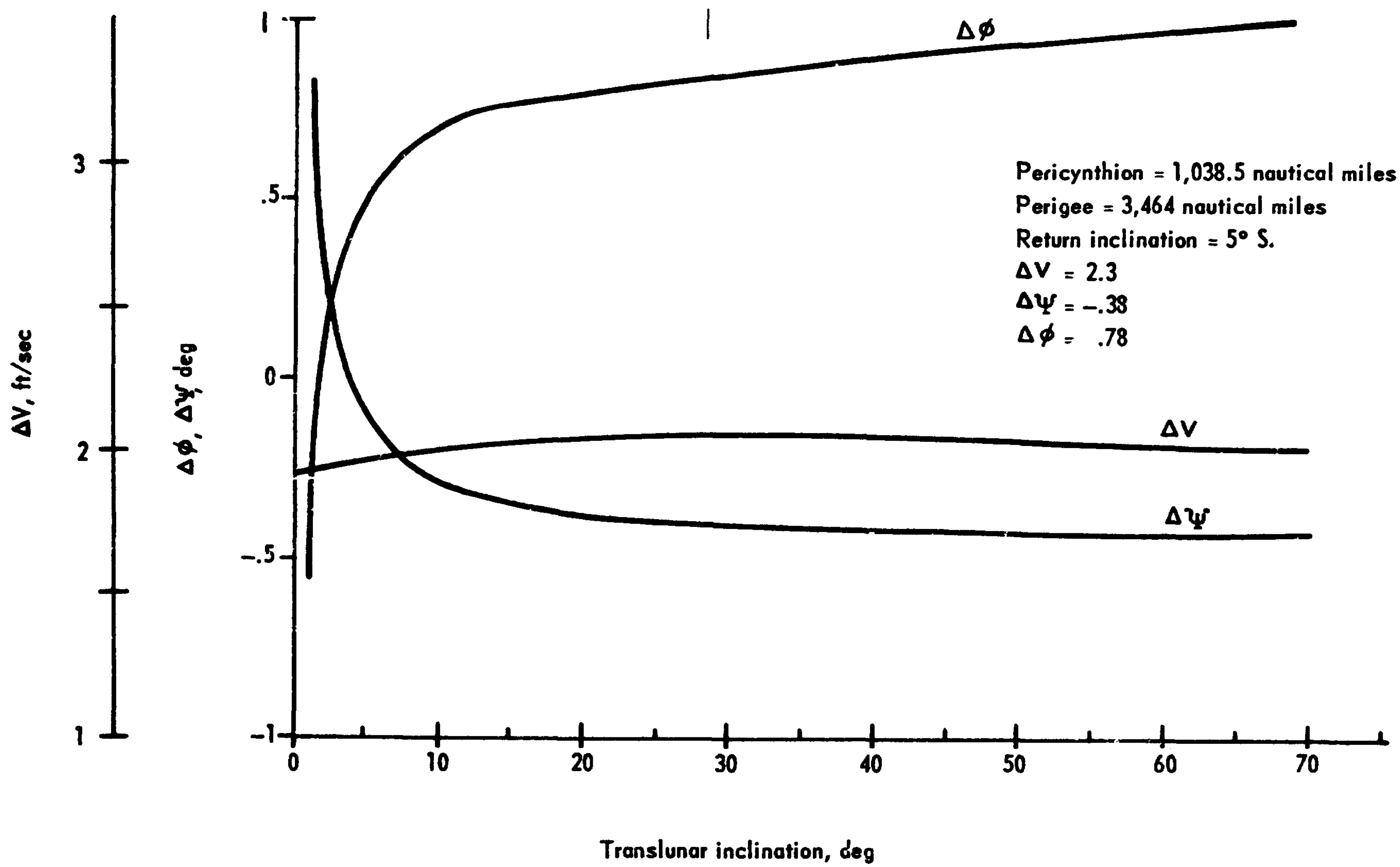


Figure 10. – Correction Δ's

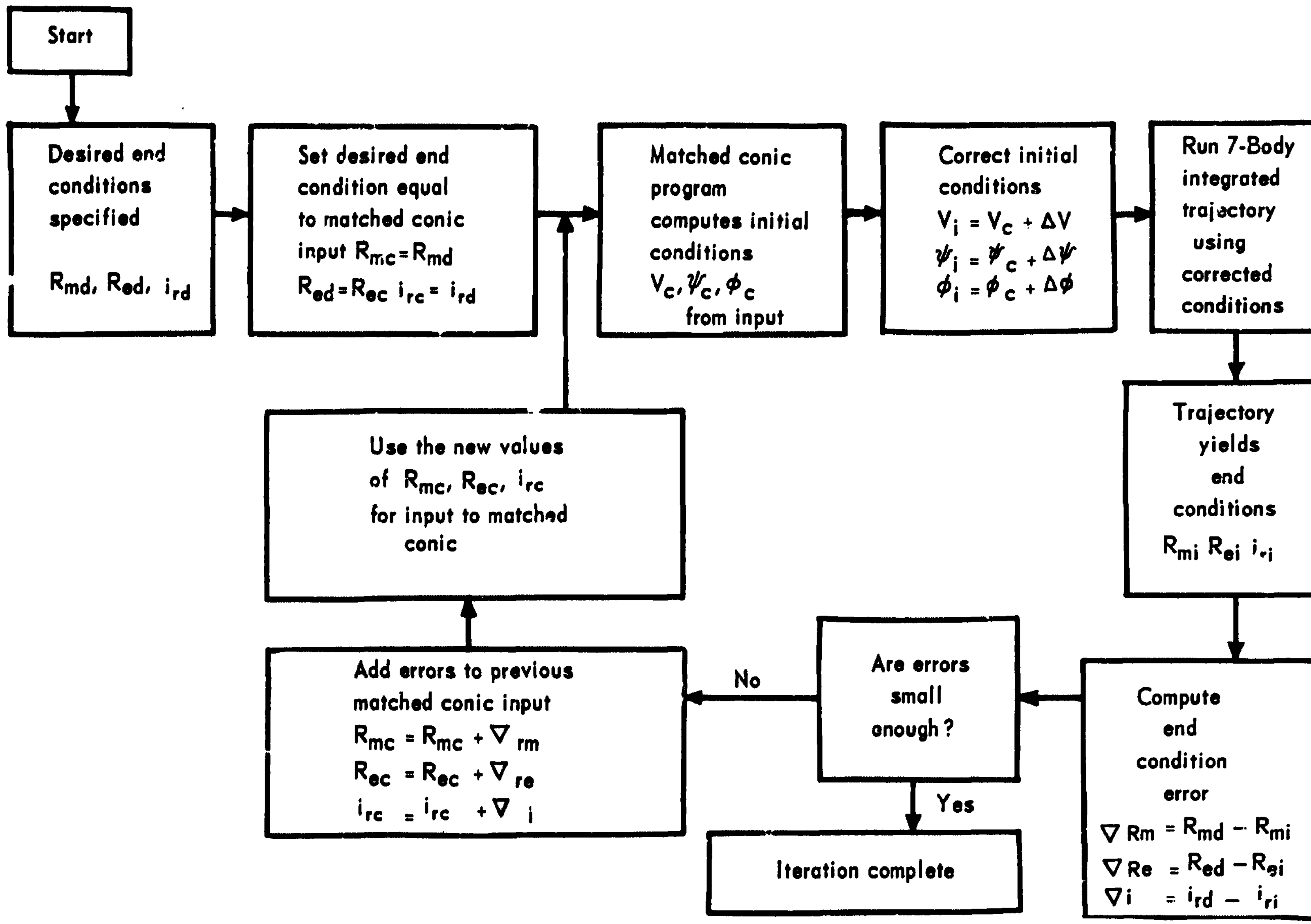


Figure 11.– Flow diagram of Δ interation scheme

Injection time, Jan. 15, 1967
h = Injection altitude = 600,000 ft
Translunar inclination 20° N
$\gamma = 0$

$\Delta V = 2.30$ ft/sec
$\Delta \psi = -\ 0.38°$
$\Delta \phi = 0.78°$

Desired pericynthion (r_m) = 1,038.5 nautical miles (100 nautical miles altitude)

Desired perigee (r_e) = 3,464.0 nautical miles (20 nautical miles altitude)

Desired return inclination = 5° S

First iteration

	Matched conic		Integrated	
Input	r_m	1,038.5	V	35,960.56
Input	r_e	3,464	ψ	7.434
Input	i_r	5° S	ø	38.800
Output	V	35,958.26	r_m	1,054.5
Output	ψ	7.814°	r_e	3,164.0
Output	ø	38.020°	i_r	3.758 S

Add deltas

Errors

∇ r_m −16 mi
∇ r_e 300 mi
∇ i_r 1.2°

Second iteration

	Matched conic		Integrated	
Input	r_m	1,022.5	V	35,960.53
Input	r_e	3,764	ψ	7.428
Input	i_r	6.2° S	ø	38.799
Output	V	35,958.23	r_m	1,036.5
Output	ψ	7.808	r_e	3,438.5
Output	ø	38.019	i_r	4.378° S

Add deltas

Errors

∇ r_m 2mi
∇ r_e 26mi
∇ i_r 0.6°

Third iteration

	Matched conic		Integrated	
Input	r_m	1,024.5	V	35,960.52
Input	r_e	3,790	ψ	7.421
Input	i_r	6.8° S	ø	38.806
Output	V	35,958.22	r_m	1,038.9
Output	ψ	7.801	r_e	3,462.9
Output	ø	38.026	i_r	5.06

Add deltas

Errors

∇ r_m −0.4
∇ r_e 1.1mi
∇ i_r −.06°

Actual Δ's: $\Delta V = 2.257$ $\Delta \psi = 0.3925$ $\Delta \phi = 0.7859$

Figure 12.– Sample iteration