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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MSC 00122

INTERNAL NOTE

AN ITERATIVE METHOD DERIVED FROM EXISTENCE AND UNIQUENESS THEOREMS FOR SYSTEMS OF SECOND-ORDER, NONLINEAR, TWO-POINT-BOUNDARY-VALUE DIFFERENTIAL EQUATIONS

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CONTENTS

AN ITERATIVE METHOD DERIVED FROM EXISTENCE AND UNIQUENESS THEOREMS FOR SYSTEMS OF SECOND-ORDER, NONLINEAR, TWO-POINT-BOUNDARY-VALUE DIFFERENTIAL EQUATIONS

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SUMMARY

This paper develops some applications of the contraction mapping principle. It shows sufficient conditions to guarantee existence of a unique solution of a two-point-boundary-value system of differential equations. Moreover, it exposes an iterative method that must converge from any initial guess to the unique solution, provided certain criteria are satisfied. An example is given to clarify the procedure.

INTRODUCTION

A boundary-value problem in ordinary differential equations differs from an initial-value problem in that it must satisfy conditions at two or more points. An *nth* order differential equation can have n conditions at one point or one condition at n points. In fluid mechanics and trajectory mechanics, systems of second-order equations with conditions specified at two points are often encountered. This paper will consider only systems of this form.

SYSTEM WITH IMPOSED CONDITIONS

Consider

$$
(\mathrm{I})\quad \begin{cases}
y_1''(t) + f_1(t, y_1(t), y_2(t), \cdots, y_n(t), y_1'(t), y_2'(t), \cdots, y_n'(t)) = 0 \\
y_2''(t) + f_2(t, y_1(t), y_2(t), \cdots, y_n(t), y_1'(t), y_2'(t), \cdots, y_n'(t)) = 0 \\
\vdots \qquad \vdots \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \vdots \\
y_n''(t) + f_n(t, y_1(t), y_2(t), \cdots, y_n(t), y_1'(t), y_2'(t), \cdots, y_n'(t)) = 0 \\
y_i(a) = A_i \ , \quad y_i(b) = B_i
\end{cases}
$$

or in vector notation

$$\vec{y}'' + \vec{f}(t, \vec{y}, \vec{y}') = 0 \quad \text{with} \quad \vec{y} = (y_1, y_2, \cdots, y_n)^T$$

$$\vec{y}(a) = \vec{A} \ , \quad \vec{y}(b) = \vec{B}$$

When $\vec{f}$ satisfies a *Lipschitz condition* †, it means that each component satisfies the following:

$$|f_i(t, y_1, y_2, \cdots, y_i, \cdots, y_n, y_1', \cdots, y_i', \cdots, y_n')$$

$$- f_i(t, y_1, y_2, \cdots, x_i, \cdots, y_n, y_1', \cdots, x_i', \cdots, y_n')|$$

$$\leq K_i |y_i - x_i| + L_i |y_i' - x_i'| \quad \text{with} \quad K_i, L_i \geq 0$$

and $(t, y_1, y_2, \cdots, y_i, \cdots, y_n, y_1', \cdots, y_i', \cdots, y_n')$, $(t, y_1, y_2, \cdots, x_i, \cdots, y_n, y_1', \cdots, x_i', \cdots, y_n')$ elements of the domain of $f_i = [a,b] \times R^{2n}$, $1 \leq i \leq n$. Define the minimum K_i, L_i that will preserve the inequality as the *Lipschitz constants* of f_i. Notice that this condition does not presuppose the existence of a solution $\vec{y}(t)$ of system (I).

GREEN'S FUNCTION FOR THE SYSTEM

Now, in order to construct an iterative procedure, it is sufficient to use a Green's function to transform each equation of a homogeneous boundary-value system into an *equivalent* integral equation system. (For the definition and properties of Green's functions, see No. 2 in Bibliography.) Such a Green's function is:

$$G(t,s) = \begin{cases} \dfrac{(b - t)(s - a)}{b - a} & a \leq s \leq t \leq b \\[2ex] \dfrac{(b - s)(t - a)}{b - a} & a \leq t \leq s \leq b \end{cases}$$

(This Green's function is for the operator $-y''$.) Since zero-boundary conditions are rarely encountered, it is usually necessary to transform the system into one with zero-boundary conditions and satisfy the original *Lipschitz condition*. Such a transformation is

$$\vec{\tau}(t) = \frac{a\vec{B} - b\vec{A} + (\vec{A} - \vec{B})t}{a - b}$$

Clearly, $\tau_i(a) = A_i$ and $\tau_i(b) = B_i$. Therefore, if $\vec{y}(t)$ is a solution of problem (I), then $\vec{z}(t) = \vec{y}(t) - \vec{\tau}(t)$ is a solution of

$$\text{(II)} \qquad \begin{cases} \vec{z}\,'' + \vec{F}(t, \vec{z}, \vec{z}\,') = 0 \\ \vec{z}(a) = 0 = \vec{z}(b) \end{cases}$$

where

$$\vec{F}(t, \vec{z}, \vec{z}\,') = \vec{f}(t, \vec{z} + \vec{\tau}(t), \vec{z}\,' + \vec{\tau}\,'(t))$$

Since

$$\left|F_i(t, \cdots, z_i, \cdots, z_n, \cdots, z_i', \cdots, z_n') - F_i(t, \cdots, q_i, \cdots, z_n, \cdots, q_i', \cdots, z_n')\right|$$

$$\equiv \Big|f_i(t, \cdots, z_i + \tau_i(t), \cdots, z_n + \tau_n(t), \cdots, z_i' + \tau_i'(t), \cdots, z_n' + \tau_n(t)$$

$$- f_i(t, \cdots, q_i + \tau_i(t), \cdots, z_n + \tau_n(t), \cdots, q_i' + \tau_i'(t), \cdots, z_n' + \tau_n(t)\Big|$$

$$\leq K_i\left|(z_i + \tau_i(t)) - (q_i + \tau_i(t))\right| + L_i\left|(z_i' + \tau_i'(t)) - (q_i' + \tau_i(t))\right|$$

$$= K_i\left|z_i - q_i\right| + L_i\left|z_i' - q_i'\right|$$

this is exactly the same *Lipschitz condition* that each f_i , $1 \leq i \leq n$, must satisfy.

Theorem:

Problem (II) is equivalent to

$$\vec{z}(t) = \int_a^b G(t, s)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds \qquad (1)$$

(Two problems are equivalent if and only if they have exactly the same solutions.)

Proof:

$$\vec{z}(t) = \int_a^b G(t, s)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds$$

$$= \frac{b - t}{b - a} \int_a^t (s - a)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds$$

$$+ \frac{t - a}{b - a} \int_t^b (b - s)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds$$

Clearly, $\vec{z}(a) = 0 = \vec{z}(b)$.

By the product rule for differentiation, it may be found that

$$\vec{z}'(t) = \frac{-1}{b - a} \int_a^t (s - a)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds$$

$$+ \frac{b - t}{b - a} (t - a)\vec{F}(t, z(t), z'(t))$$

$$+ \frac{1}{b - a} \int_t^b (b - s)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds$$

$$+ \frac{a - t}{b - a} (b - t)\vec{F}(t, \vec{z}(t), \vec{z}'(t))$$

$$
\begin{aligned}
\vec{z}''(t) &= \frac{-1}{b-a}(t-a)\vec{F}(t, \vec{z}(t), \vec{z}'(t)) \\
&+ (-2t + a + b)\vec{F}(t, \vec{z}(t), \vec{z}'(t)) \\
&+ \frac{(b-t)}{b-a}(t-a)\vec{F}'(t, \vec{z}(t), \vec{z}'(t)) \\
&+ \frac{-(b-t)}{b-a}\vec{F}(t, \vec{z}(t), \vec{z}'(t)) \\
&+ (2t - a - b)\vec{F}(t, \vec{z}(t), \vec{z}'(t)) \\
&+ \frac{(a-t)(b-t)}{b-a}\vec{F}'(t, \vec{z}(t), \vec{z}'(t)) \\
&= \frac{(a-t) + (t-b)}{b-a}\vec{F}(t, \vec{z}(t), \vec{z}(t)) \\
&= -\vec{F}(t, \vec{z}(t), \vec{z}'(t))
\end{aligned}
$$

or

$$\vec{z}''(t) = -\vec{F}(t, \vec{z}(t), \vec{z}'(t))$$

The fundamental theorem of contraction mappings will be applied to this class of systems. (A function $T : S \xrightarrow{\text{into}} S$ where S is a normed linear space is said to be a contraction mapping if and only if there is a real number $\alpha, 0 < \alpha < 1$, such that for all $x, y \in S$ $||Tx - Ty|| \leq \alpha||x - y||$.)

CONTRACTION MAPPING THEOREM

For every contraction mapping defined on a complete normed linear space S, there exists a unique fixed point $x \varepsilon S$ such that $Tx = x$. Moreover, for any $x_0 \varepsilon S$, the iterates

$$T^n x_0 = T^{n-1} x_1 = T^{n-2} x_2 = \cdots = Tx_{n-1} = x_n$$

converge to the fixed point x, and

$$||x_n - x|| \leq \frac{\alpha^n}{1 - \alpha} ||x_1 - x_0||$$

(See Bibliography No. 3, pp. 213-216.)

THEOREMS FOR UNIQUE EXISTENCE

First consider the system $\vec{y}'' + \vec{f}(t,\vec{y}) = 0$, $\vec{y}(a) = \vec{A}$, and $\vec{y}(b) = \vec{B}$. Let S be the space of continuous functions on [a,b]. This linear space is complete with norm

$$||v|| = \max_{1 \leq i \leq n} \left\{ \max_{[a,b]} |v_i(t)| \right\}$$

In the following theorems it will be necessary to obtain upper bounds for $\int_a^b G(t,s)\, ds$ and $\int_a^b \left|\frac{\partial G(t,s)}{\partial t}\right| ds$.

<u>Lemma 1</u>:

$$\int_a^b G(t,s)\, ds \leq \frac{(b - a)^2}{8} \quad \text{and} \quad \int_a^b \left|\frac{\partial G(t,s)}{\partial t}\right| ds \leq \frac{b - a}{2}$$

Proof:

$$\int_a^b G(t,s)\,ds = \frac{b-t}{b-a}\int_a^t (s-a)\,ds + \frac{t-a}{b-a}\int_t^b (b-s)\,ds$$

$$= \frac{(b-t)(t-a)^2}{2(b-a)} + \frac{(t-a)(b-t)^2}{2(b-a)}$$

$$= \frac{(b-t)(t-a)}{2} \leq \frac{(b-a)^2}{8}$$

and

$$\int_a^b \left|\frac{\partial G(t,s)}{\partial t}\right| = \frac{1}{b-a}\int_a^t (s-a)\,ds + \frac{1}{b-a}\int_t^b (b-s)\,ds$$

$$= \frac{(t-a)^2 + (b-t)^2}{2(b-a)} \leq \frac{b-a}{2}$$

Theorem 1:

If the system $\overline{y}'' + \overline{f}(t,\overline{y}) = 0$, $\overline{y}(a) = \overline{A}$, and $\overline{y}(b) = \overline{B}$ satisfies the *Lipschitz condition* † and $b - a < (8/K)^{1/2}$ where $K = \max\limits_{1\leq i\leq n} K_i$, then a unique solution exists.

Proof:

First, the system must be transformed into a homogeneous system; it then becomes $\overline{z}'' + \overline{F}(t,\overline{z}) = 0$, $\overline{z}(a) = 0 = \overline{z}(b)$. With the right side of equation (1) as the operator in the following equation, an attempt is made to find sufficient conditions when it is contracting. Notice that the operator T defined by

$T(\vec{F}) = \int_a^b G(t,s)\vec{F}(s, \vec{z}(s), \vec{z}'(s))\, ds$ is a linear mapping from S into S .

$$Tu_i(t) - Tv_i(t) = \int_a^b G(t,s)[F_i(s, u_1(s), u_2(s), \cdots, u_i(s), \cdots, u_n(s)) - F_i(s, \cdots, v_i(s), \cdots, v_n(s)]\, ds$$

Now, the *Lipschitz condition* is used to find

$$|Tu_i(t) - Tv_i(t)| \leq \int_a^b G(t,s)K_i|u_i(s) - v_i(s)|\, ds$$

$$|Tu_i(t) - Tv_i(t)| \leq K_i||\bar{u} - \bar{v}|| \int_a^b G(t,s)\, ds$$

Since by Lemma 1

$$\int_a^b G(t,s)\, ds \leq \frac{(b - a)^2}{8}$$

therefore

$$||T\bar{u} - T\bar{v}|| \leq K||\bar{u} - \bar{v}|| \frac{(b - a)^2}{8}, \quad K = \max_{1\leq i\leq n} k_i$$

The mapping is contracting when

$$K \frac{(b - a)^2}{8} = \alpha < 1$$

or if

$$b - a < (8/K)^{1/2}$$

a unique solution exists, namely: $\vec{y}(t) = \vec{z}(t) + \vec{\tau}(t)$.

Now, consider the system $\vec{y}'' + \vec{f}(t, \vec{y}, \vec{y}') = 0$, $\vec{y}(a) = \vec{A}$, and $\vec{y}(b) = \vec{B}$. This time, let S be the space of continuously differentiable functions. The space is complete with norm $||v|| = \max\limits_{1 \le i \le n} \left\{ \max\limits_{[a,b]} \left(K_i |v_i(t)| + L_i |v_i'(t)| \right) \right\}$

<u>Theorem 2</u>:

If the system $\vec{y}'' + \vec{f}(t, \vec{y}, \vec{y}') = 0$, $\vec{y}(a) = \vec{A}$, and $\vec{y}(b) = \vec{B}$ satisfies the *Lipschitz condition* †, and

$$\left[\frac{K(b - a)^2}{8} + \frac{L(b - a)}{2} \right] < 1 , \qquad \text{with } K = \max_{1 \le i \le n} K_i \quad \text{and } L = \max_{1 \le i \le n} L_i$$

then a unique solution exists.

<u>Proof</u>:

The same transformation as Theorem 1 shall be used; again using $\vec{\tau}(t)$, the system is transformed into a new one with homogeneous boundary conditions:

$$\vec{x}'' + \vec{F}(t, \vec{x}, \vec{x}') = 0$$

$$\vec{x}(a) = 0 = \vec{x}(b)$$

Now form,

$$|Tu_i(t) - Tv_i(t)| \le \int_a^b G(t,s)|f_i(s, u_1(s), \cdots, u_i(s), \cdots, u_n(s), u_1'(s), \cdots, u_n'(s))$$

$$- f_i(s, v_1(s), \cdots, v_i(s), \cdots, v(n), v_1'(s), \cdots, v_n'(s))| \, ds$$

$$\le \int_a^b G(t,s)(K_i|u_i(s) - v_i(s)| + L_i|u_i'(s) - v_i'(s)|) \, ds$$

$$\le ||\vec{u} - \vec{v}|| \int_a^b G(t,s) \, ds \le \frac{(b - a)^2}{8} ||\vec{u} - \vec{v}||$$

and

$$\left|\frac{d}{dt} Tu_i(t) - \frac{d}{dt} Tv_i(t)\right| \le \int_a^b \left|\frac{\partial G(t,s)}{\partial t}\right| \Big|f_i(s, \cdots, u_n'(s))$$

$$- f_i(s, \cdots, v_n'(s))\Big| \, ds$$

$$\le ||\vec{u} - \vec{v}|| \int_a^b \left|\frac{\partial G(t,s)}{\partial t}\right| ds$$

Hence by Lemma 1

$$\le \frac{b - a}{2} ||\vec{u} - \vec{v}||$$

which together imply

$$||T\vec{u} - T\vec{v}|| \le \left[\frac{K(b - a)^2}{8} + \frac{L(b - a)}{2}\right] ||\vec{u} - \vec{v}||$$

where $K = \max\limits_{1 \le i \le n} K_i$, $L = \max\limits_{1 \le i \le n} L_i$

Therefore, if

$$\left[\frac{K(b - a)^2}{8} + \frac{L(b - a)}{2}\right] < 1$$

the mapping T is contracting. Hence, a unique solution exists, namely: $\vec{y}(t) = \vec{x}(t) + \vec{\tau}(t)$.

ITERATIVE PROCEDURE

If the system is of the form $\vec{z}'' + \vec{F}(t,\vec{z}) = 0$, $\vec{z}(a) = 0 = \vec{z}(b)$, the second part of the contraction mapping theorem states that for any starting value $\overset{(0)}{\vec{z}}(t) \in S$, the iterates of the *ith* component converge to $z_i(t)$, that is

$$\overset{(n)}{z_i}(t) = \int_a^b G(t,s)F_i\left(s, \overset{(n-1)}{z_1}(s), \cdots, \overset{(n-1)}{z_i}(s), \cdots, \overset{(n-1)}{z_n}(s)\right) ds$$

converges to $z_i(t)$ for any $\overset{(0)}{\vec{z}}(t) = \left(\overset{(0)}{z_1}(t), \overset{(0)}{z_2}(t), \cdots, \overset{(0)}{z_n}(t)\right)^T$ or

$$\overset{(n)}{\vec{z}} + \vec{F}\left(t, \overset{(n-1)}{\vec{z}}\right) = 0$$

$$\overset{(n)}{\vec{z}}(a) = 0 = \overset{(n)}{\vec{z}}(b)$$

converges to a solution.

Moreover, convergence to the solution is uniform, and a bound for error on each iterate is given by

$$\left\|\overset{(n)}{\vec{z}} - \vec{z}\right\| \leq \frac{\alpha^n}{1 - \alpha}\left\|\overset{(1)}{\vec{z}} - \overset{(0)}{\vec{z}}\right\| .$$

The method is as follows:

(1) Select any appropriate $\overset{(0)}{\vec{z}}(t)$.

(2) Calculate $\overset{(1)}{\vec{z}}(t)$; select this $\overset{(1)}{\vec{z}}(t)$ as a new $\overset{(0)}{\vec{z}}(t)$; proceed like this using each iterate as a new $\overset{(0)}{\vec{z}}(t)$.

EXAMPLE

Consider the system

(1) $y_1''(t) + y_1(t) + y_2(t) = 0$

(2) $y_2''(t) + y_1(t) + \sin y_2(t) = 0$

$$y_1(0) = 1 \qquad y_1(1) = 1$$

$$y_2(0) = 0 \qquad y_2(1) = 0$$

so that

$$|f_i(t, y_1, y_2, \cdots, y_i, \cdots, y_n, y_1, \cdots, y_n')$$

$$- f_i(t, y_1, y_2, \cdots, x_i, \cdots, y_n, y_i, \cdots, y_n')|$$

$$\leq K_i|y_i - x_i|$$

is for (1)

$$|y_i + y_2 - x_i - y_2| \leq (1)|y_1 - x_1|$$

and for (2)

$$|y_1 + \sin y_2 - y_1 - \sin x_2| \leq (1)|y_2 - x_2|$$

$$\alpha = \max K_i \frac{(b - a)^2}{8} = 1/8 .$$

$$\tau_1(t) = \frac{0 \cdot 1 - 1 \cdot 1 + (1 - 1)t}{0 - 1} = 1$$

$$\tau_2(t) = 0 , \text{ so } \bar{\tau}(t) = (1,0)^T$$

The system is transformed into the homogeneous system

$$z_1''(t) + z_1(t) + z_2(t) + 1 = 0$$

$$z_2''(t) + z_1(t) + 1 + \sin z_2(t) = 0$$

since $\bar{F}(t, \bar{z}, \bar{z}') \equiv \bar{f}(t, \bar{z} + \bar{\tau}(t), z' + \bar{\tau}(t))$

Let $\overset{(0)}{\underset{\sim}{z}} = (1, \pi/2)^T$. Now, using the Green's function for this system, it is found that

$$\overset{(1)}{z_1}(t) = \int_0^1 G(t,s)(2 + \pi/2)\, ds$$

$$= (1 - t) \int_0^t s(2 + \pi/2)\, ds + (-t) \int_t^1 (s - 1)(2 + \pi/2)\, ds$$

$$= \frac{t^2(1 - t)(4 + \pi)}{4} - \frac{t(s - 1)^2}{2} \left(\frac{4 + \pi}{2}\right)\Bigg|_t^1$$

$$= \frac{t^2(1 - t)}{2} \left(\frac{4 + \pi}{2}\right) + \frac{t(t - 1)^2}{2} \left(\frac{4 + \pi}{2}\right)$$

$$= \frac{4 + \pi}{4} [t^2 - t^3 + t^3 - 2t^2 + t] = \frac{4 + \pi}{4} t(1 - t) .$$

Similarly,

$$\overset{(1)}{z_2}(t) = \frac{3t(1 - t)}{2}$$

so

$$\overset{(1)}{\underset{\sim}{z}} = \left(\left(\frac{4 + \pi}{4}\right)t(1 - t), \frac{3t(1 - t)}{2}\right)^T$$

By the contraction mapping theorem,

$$||\bar{z} - \overset{(1)}{\bar{z}}|| \leq \frac{1/8}{1 - 1/8} \left|\left|(1, \pi/2)^T - \left(\frac{4 + \pi}{4} t(1 - t), \frac{3t(1 - t)}{2}\right)^T\right|\right|$$

$$= \frac{1}{7} \max_{[0,1]} \left\{\max \overset{(0)}{z_i} - \overset{(1)}{z_i}\right\} \qquad \text{where} \quad i = 1,2$$

$$\cong .385 \quad .$$

Since this must converge uniformly, the next iterate obtained from using $\overset{(1)}{\bar{z}}$ as a new $\overset{(0)}{\bar{z}}$ must be even "closer," (which is made precise by the definition of the norm).

CONCLUSION

This investigation has shown how an iterative method to solve two-point-boundary-value systems of differential equations is obtained from the principle of contraction mappings. Moreover, it states some conditions that will guarantee this solution to be unique, and that the method will uniformly converge to this unique solution for any starting value. Possible logical extensions of these results could be found by assuming a variable but bounded terminal value b . In cases where the length b-a is too large to guarantee unique existence, find conditions that can determine specific intervals on the real axis for which unique solutions exist.

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