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AN APPLICATION OF COLLISIONLESS ION ACOUSTIC WAVE PHENOMENA TO SATELLITE EXPERIMENTS

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AN APPLICATION OF COLLISIONLESS ION ACOUSTIC WAVE PHENOMENA TO SATELLITE EXPERIMENTS

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I. INTRODUCTION

Many experimental techniques have been employed to investigate ionospheric properties; however, little attention appears to have been given to the use of ion acoustic-wave phenomena. Although naturally occurring ion waves have been detected in the ionosphere [1], the author is unaware of any attempts to purposely create and exploit ion-wave phenomena. This dearth of interest in artificially generated ion waves is undoubtedly due to the difficulty encountered in laboratory attempts to generate and detect ion waves. However, sufficient success in the generation and detection of ion waves in the laboratory has now been attained [2, 3] to warrant the consideration of satellite-borne ion-wave experiments. The purpose of this note is to discuss some of the problems and possibilities pertaining to the use of ion acoustic-wave phenomena for satellite experimentation in the ionosphere.

II. POTENTIAL APPLICATIONS

Ion-wave measurements have several potential applications. An accurate measurement of ion-wave velocities can yield direct information about electron temperatures. Furthermore, measurements of ion waves can provide information about the direction of magnetic field lines and information about the relative motion between a satellite and the ionospheric plasma. A knowledge of electron temperatures is important not only in itself but for the determination of electron number densities.

A frequently used technique for measuring electron number densities is the electrostatic plasma probe. The current, I_e, drawn by an electrostatic probe is given by the relation:

$$I_e = j_e A_s F \tag{1}$$

where j_e is the average electron current density (amp/cm^2) in the plasma, A_s is the area of the plasma sheath, and F is the fraction of the current crossing the sheath which intercepts the probe. The area of the plasma sheath can be determined by self-consistent analyses of Poisson's equation, while F can be calculated by assuming that the plasma sheath edge and plasma probe form a diode [4]. The term j_e is given by the relationship:

$$j_e = n_e \left(\frac{kT_e}{4\pi m_e} \right)^{\frac{1}{2}} = 2.5 \times 10^{-5}\, n_e (T_e)^{\frac{1}{2}} \tag{2}$$

where n_e is the electron number density and T_e is the electron temperature. If T_e can be determined independently and j_e has been measured with the probe (equation 1), then equation (2) can be employed to calculate the electron number density directly.

The speed of propagation of an ion acoustic wave through a plasma may be specified as:

$$v_i \approx \left(\frac{\gamma_e k T_e}{m_i} \right)^{\frac{1}{2}} \tag{3}$$

where v_i is magnitude of the ion-wave velocity, γ_e is the ratio of specific heats, k is Boltzmann's constant, and m_i is the ion mass. The atomic constituents of the ionosphere are well known and are displayed in Figure 1. From Figure 1 it is seen that atomic oxygen is the predominant element above the altitude of 400 km. Consequently, the magnitude of the ion mass can be determined and, if the ion-wave velocity can be measured accurately, the electron temperature can be calculated.

The application of ion-wave velocity measurements to the determination of the relative motion between a satellite and the ionospheric plasma requires that the ion-wave velocity $\vec{v}_i$ be of the same order of magnitude as the satellite velocity $\vec{v}_s$. Moreover, in order that the direction of the velocity vector may be determined, it is necessary to make simultaneous measurements of ion acoustic velocity both in the direction of the satellite's orbital path and perpendicular to the satellite's orbital path. A quadrupole array of two perpendicular sets of transmitting-receiving plates which would permit orthogonal measurement of ion-wave velocities is shown in Figure 2. To measure the satellite velocity $\vec{v}_s$ by the method illustrated, the array must be rotated until the planes of plates a and a' are parallel with the satellite's velocity vector. If $\vec{v}_p$ is defined as the ion-wave velocity parallel to the orbital path, and $\vec{v}_n$ as the ion-wave normal to the orbital path, then the satellite velocity is defined as:

$$\vec{v}_s = \vec{v}_p - \vec{v}_n \; . \tag{4}$$

An experimental arrangement of equipment patterned after the quadrupole array shown in Figure 2 is discussed in greater detail later (page 9). However, it should be noted at this point that it will normally be necessary to rotate these electrodes in order to determine the direction of the orbital velocity vector.

The formula for the magnitude of the ion-wave velocity given in equation (3) is derived under the assumption that the electron temperature is much greater than the ion temperature. Because of the importance of this formula, a brief derivation is given in the section following.

III. ION ACOUSTIC WAVE VELOCITY

The correct expression for the ion acoustic-wave velocity has been the subject of some controversy, with most of the uncertainty centered upon the correct value of γ. A simple derivation of an ion-wave equation is given by Spitzer [5]; it begins with the equations of motion for ions and electrons:

$$\rho_i \frac{\partial v_i}{\partial t} = -\nabla p_i \, , \quad \text{and} \quad \rho_e \frac{\partial v_e}{\partial t} = -\nabla p_e . \tag{5}$$

If we assume an adiabatic pressure variation, the electron pressure may be expressed as:

$$p_e = \gamma_e k T_e n_e \quad , \tag{6}$$

and there is a corresponding relation for the ion pressure. A correspondence between number density and velocity may be established through the equation of continuity:

$$\frac{\partial n_e}{\partial t} = -\frac{Z}{m_i} \nabla \cdot (\rho_e \vec{v}_e) = -\frac{Z}{m_i} (\rho_e \nabla \vec{v}_e + \vec{v}_e \cdot \nabla \rho_e) ,$$

where Z represents the degree of ionization of an atom. For small oscillations, the quantity $\vec{v}_e \cdot \nabla \rho_e$ may be neglected since it is the product of two first order terms. Hence, the equation of continuity for electrons reduces to:

$$\frac{\partial n_e}{\partial t} = -\frac{Z}{m_i} \rho \nabla \vec{v}_e , \tag{7}$$

and there is a corresponding relation for ions. If we differentiate equation (6) and combine the result with equation (7), we obtain the general result for electrons or ions:

$$\frac{\partial p}{\partial t} = -\frac{(1 + Z)\gamma k T}{m_i} \vec{\rho} \nabla \cdot \vec{v} ,$$

where it is assumed that the ion and electron kinetic temperatures are equal. Finally, if we combine this expression with equations (5) and derive a more accurate average over electrons and ions we obtain:

$$\frac{\partial^2 v_x}{\partial t^2} = \left(\frac{Z \gamma_e k T_e + \gamma_i k T_i}{m_i} \right) \frac{\partial v_x}{\partial x^2} . \tag{8}$$

Equation (8) has the well known form of wave equation satisfied by a wave traveling in the x direction with a velocity obtained from the equation:

$$v^2 = \frac{Z\gamma_e k T_e + \gamma_i k T_i}{m_i} \quad .$$

It is shown below that unless $T_e \gg T_i$, ion waves are strongly damped; accordingly, the above expression for the ion wave velocity reduces to that given by equation (3).

The magnitude of γ has not been well defined since it is difficult to ascertain whether ion vibrations are isothermal or adiabatic. Essentially, γ depends upon the number of degrees of freedom which the plasma particles may have, and is given by the relation: $\gamma = (2 + m)/m$. Spitzer [5] argues that m = 1 in a fully ionized, compressionless plasma, and thus γ should be 3. Stix [6] contends that ion oscillations are adiabatic (m = 3), and thus γ should be 5/3. Experimental investigations [3] appear to indicate that γ should actually be closer to unity; hence, γ will be assumed to have unity value for the calculations which follow.

IV. LANDAU DAMPING OF ION ACOUSTIC WAVES

The probability of detecting ion waves depends to a large extent upon the amount of damping present. Unfortunately, theoretical consideration of collisionless damping is quite complicated since it involves the analysis of dispersion relations in the complex frequency plane. Because of its importance, and for completeness, a brief discussion of the theory of collisionless damping of ion waves is given here.

In 1945, Landau [7], through an analysis of the linearized, collisionless Boltzmann equation, discovered that under certain conditions a "collisionless" damping of plasma oscillations could occur. Subsequent to Landau's work, a number of investigators studied the problem of "collisionless" damping. One of the best treatments of ion acoustic waves is contained in a paper by Fried and Gould [8]. Fried and Gould solve the collisionless Boltzmann equation along with the Poisson equation to obtain a dispersion relation of the form:

$$\lambda - \tfrac{1}{2}\left[Z'(\xi) + \theta Z'(\xi\sqrt{\theta/\delta})\right] = 0, \qquad (9)$$

where

$\xi = \frac{\omega k}{a} = \frac{k}{a}\ [\mathrm{Re}(\omega) \pm i\,\mathrm{Im}(\omega)] = x + iy$; $\theta = T_e/T_i$; $\underline{\omega}$ is the complex frequency; $\underline{k}$ is the wave number; $\underline{a}$ is the mean thermal velocity; T_e and T_i are the electron and ion temperatures, respectively; and $Z'(\xi)$ is the derivative of the "plasma dispersion relation." A discussion of the derivation of equation (9) is given in the appendix. Equation (9) can be numerically solved for ξ by an iterative process in which values of $\underline{x}$ and $\underline{y}$ are chosen such that equation (9) is satisfied. Figure 3 exhibits solutions of equation (9) for θ = 1, 4, and 25. It can be seen that as θ increases, the ratio $\mathrm{Re}(\omega)/\mathrm{Im}(\omega)$ decreases, implying that for values of θ = 3, 4, or greater, ion waves should be observable.

In the laboratory experiments performed by Alexeff and Jones [3], electron temperatures as high as 10^5 °K were obtained, hence the requirement for $T_e \gg T_i$ was satisfied. Unfortunately, in the ionosphere T_e is only about twice T_i; hence, ion waves probably would be heavily damped. However, experimental results with cesium plasmas [2] in which T_e and T_i are approximately equal indicate that even for small ratios of θ, ion acoustic waves still can be detected. In this case, however, it is not certain how applicable equation (3) may be.

Fried and Gould show that, if the ions and electrons are drifting relative to each other, ion acoustic oscillations may be enhanced. To an observer on a satellite, the ions have a undirectional drift velocity with respect to the satellite, while the electrons are essentially isotropic. Consequently, the motion of a satellite through the ionosphere may enhance the detection of ion waves.

V. SATELLITE ION WAVE EXPERIMENTS

Satellite-borne ion-wave experiments will have to be planned carefully since there may be complicating factors. The ionospheric plasma differs considerably from a quiescent laboratory plasma in that the Earth's magnetic field and perturbations caused by the passage of the satellite will affect the behavior of the ion waves. Furthermore, when ion waves are generated, perhaps by a voltage pulse, it is possible to excite electromagnetic waves which may tend to confuse experimental data.

A. Theoretical Considerations

The probable ion-wave velocity spectrum of the ionosphere can be established after the electron temperature range has been determined. Figures 4 and 5 show the approximate electron temperatures and electron concentrations in the ionosphere for altitudes 100 km to above 700 km [9]. The acoustic velocities for waves in the ionosphere may be determined by substituting the appropriate values obtained from Figure 4 into equation (3). For proton ion waves (m_i = 1 amu), the velocity is expressed as:

$$v_p = 91\sqrt{T_e} \quad \text{meter/sec,} \tag{10}$$

while for any other atom the velocity is expressed as: $v = v_p\sqrt{A}$, where A is the atomic number. Shown in Figure 6 is a range of acoustic velocities for protons, oxygen, and nitrogen. From Figures 4 and 6 we can deduce that at an altitude of approximately 500 km, where $T_e \sim 2 \times 10^3$ °K, an oxygen acoustic wave would have a velocity of 1.0 km/sec. Typical satellite velocities are in the range of 7 to 8 km/sec; consequently, the ion wave velocities have a velocity sufficiently comparable to the satellite velocity to be detectable.

For situations in which dispersion effects can be ignored, the angular frequency of an ion wave is given by the equation:

$$\omega_i^2 = \frac{Zm_e}{m_i}\,\omega_p^2 , \tag{11}$$

in which

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e}$$, and ω_p is the electron plasma frequency.

In general, the ions encountered in the ionosphere are singly ionized; that is, $Z = 1$. The numerical value of $f_p = \omega_p/2\pi$ is $8.97 \times 10^3 \left(n_e^{\frac{1}{2}}\right)$ [5] ; therefore:

$$f_i = 209(n_e/A)^{\frac{1}{2}} \text{ hertz (Hz).} \tag{12}$$

Thus, the ion wave frequency for protons is: $f = 209 \left(n_e^{\frac{1}{2}}\right)$ Hz, for nitrogen it is: $f = 56 \left(n_e^{\frac{1}{2}}\right)$ Hz, and for oxygen it is: $f = 52.2 \left(n_e^{\frac{1}{2}}\right)$ Hz. Ion wave frequencies corresponding to the electron densities depicted in Figure 5 are shown in Figure 7.

The Earth's magnetic field exerts an influence upon the ion waves and, therefore, may affect their detectability. The parameters which determine the magnitude of the influence are the ion cyclotron frequency:

$$\omega_c = \frac{ZeB}{m_i} \tag{13}$$

and the ion gyro radius:

$$r = \frac{m_i \omega_\perp}{Be} , \tag{14}$$

in which $\underline{B}$ is the magnetic field intensity in gauss and $\omega_\perp$ is the component of $\underline{\omega}$ perpendicular to $\underline{B}$. The cyclotron frequency $f_c = \omega_c/2\pi$ has the numerical form $f_c = (1.54 \times 10^3 \, ZB)/A$. The Earth's magnetic field at satellite altitudes is approximately 0.5 gauss [11] ; thus, for singly ionized atoms the cyclotron frequency is:

$$f_c = 0.77A^{-\frac{1}{2}} \text{ kilohertz (kHz).} \tag{15}$$

Hence, $f_c = 0.77$ kHz for protons, 0.206 kHz for nitrogen ions, and 0.193 kHz for oxygen ions. As indicated by Figure 7, the acoustic frequency will be much higher than the ion cyclotron frequency; therefore, there should be little interference between the cyclotron motions and the acoustic waves from a frequency standpoint.

The cyclotron radius has the numerical form:

$$r = 1.04 \times 10^{-2} \frac{A\omega_{\perp}}{B} \quad \text{centimeters.} \tag{16}$$

In a 0.5 gauss magnetic field, r = 16 cm for protons, 60 cm for nitrogen, and 64 cm for oxygen. In order to assess the influence of the ion cyclotron radius, we must ascertain the wavelength of the ion acoustic waves, $\lambda = v/f$. The wavelength may be obtained from equations (10) and (11) in the form:

$$\lambda = 43.5(T_e/n_e)^{\frac{1}{2}} \quad \text{centimeters} \tag{17}$$

for which λ is independent of A. A graph of wavelength versus altitude, in which data from Figures 4 and 5 are used, is given in Figure 8. From Figure 8 we see that at 500 km the ion acoustic wavelength is 4.35 cm.

From this discussion of wavelengths we conclude two things. First, the acoustic wavelength is of a magnitude commensurate with reasonable spacecraft dimensions and, second, the acoustic wavelength is smaller than the ion cyclotron radius. Consequently, the ion waves will appear as a perturbation on the ion cyclotron motion and the cyclotron motion should not limit the detectability of the ion waves.

B. Experimental Apparatus

The experimental apparatus for creating and detecting ion waves can be patterned after the equipment employed by Alexeff and Jones [3]. Shown in Figure 9 is a possible experimental arrangement of the quadrupole transmitter-receiver mentioned on page 3 to create and detect waves perpendicular to each other. The apparatus should be constructed to permit variable plate separation, and the entire receiver should be able to rotate with respect to the satellite. The latter capability is important when measurements are being made of the satellite velocity vector relative to the plasma. Variable plate separation is important because the damping factor is uncertain. Wong et al [2] show that when T_e and T_i are approximately equal, an acoustic wave is effectively damped in approximately 1/4 wavelength (1/e); however, for best wave resolution, plate separation should be at least one wavelength.

An ion wave having a velocity of 1 km/sec traverses 0.1 cm per μsec. Thus for a plate separation of one wavelength ($\sim$5 cm), it takes an ion wave approximately 50 μsec to pass between the plates. A conception of a time-of-flight circuit capable of measuring ion waves with this magnitude of transit time is illustrated in Figure 9. Basic timing is provided by a crystal-controlled oscillator. The output of the oscillator is differentiated to provide excitation pulses at the transmitting plates and for the "one-shot" multivibrators OS-1 and OS-2. Both "one-shots" have varible pulse widths, which permit the direct measurement of the transit time of a wave between plates. The pulse from OS-2 acts as a timing gate. If the received pulse and the OS-2 pulse are coincident, the "and" circuit will respond and the received pulse will be detected. Varying the pulse width of OS-1 will give the OS-2 pulse the proper position, in time, to be coincident with the received pulse. Initially, the OS-2 pulse should be fairly wide; then, when the received pulse is centered within the OS-2 pulse, the OS-2 pulse can be narrowed to refine the measurement of the acoustic-wave transit time. Resolution on the order of 1 μsec should be attainable over a range of approximately 1 msec. Thus, a velocity increment resolution of about 1 cm/sec should be possible.

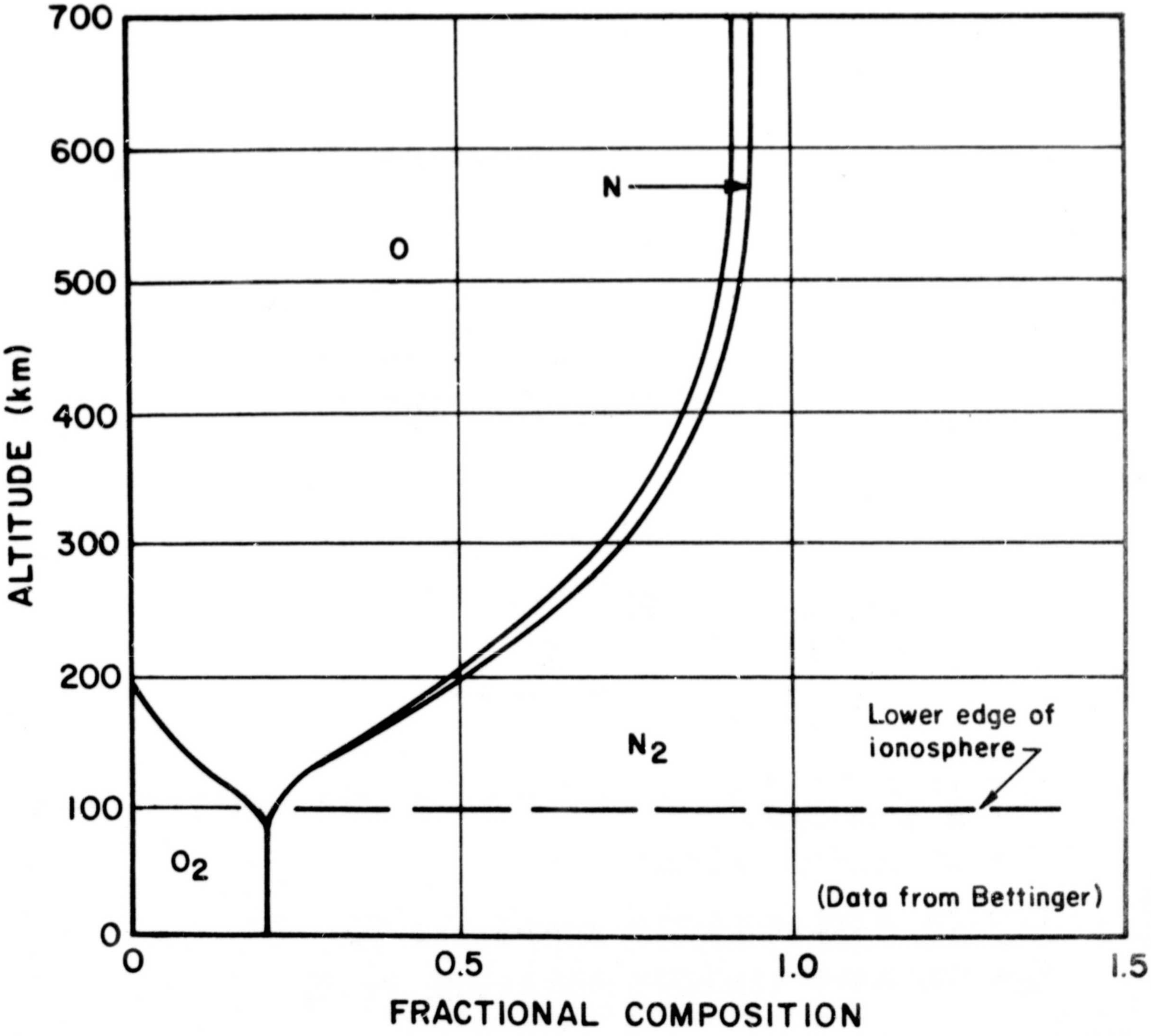
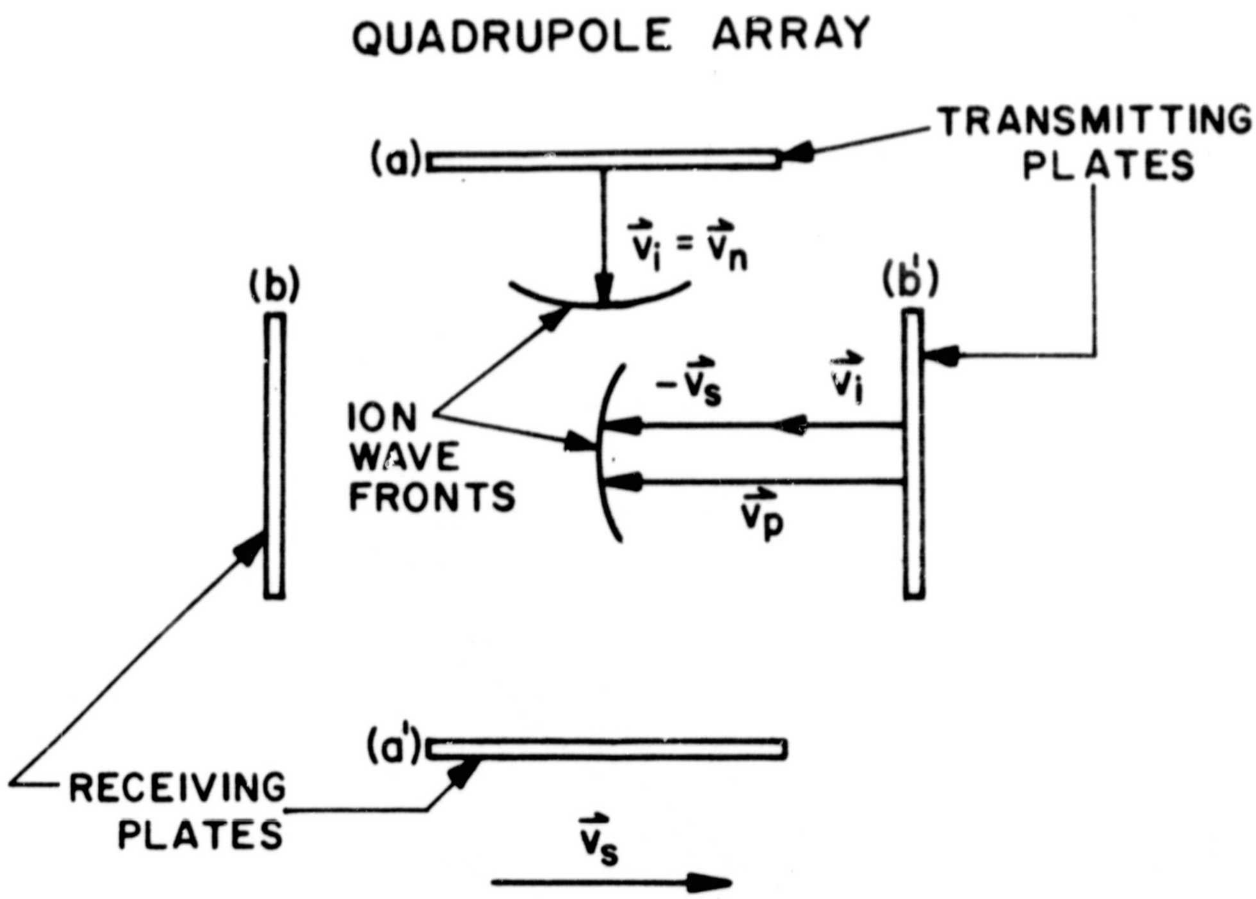


FIGURE 1. APPROXIMATE FRACTIONAL COMPOSITION OF THE UPPER ATMOSPHERE



$\vec{v}_i$ ≡ VELOCITY OF PROPAGATION OF AN ION ACOUSTIC WAVE IN A PLASMA

$\vec{v}_s$ ≡ SATELLITE VELOCITY

$\vec{v}_n$ ≡ ION WAVE VELOCITY NORMAL TO $\vec{v}_s$

$\vec{v}_p$ ≡ ION WAVE VELOCITY PARALLEL TO $\vec{v}_s$

FIGURE 2. RELATIONSHIP BETWEEN SATELLITE VELOCITY AND ION ACOUSTIC-WAVE VELOCITY

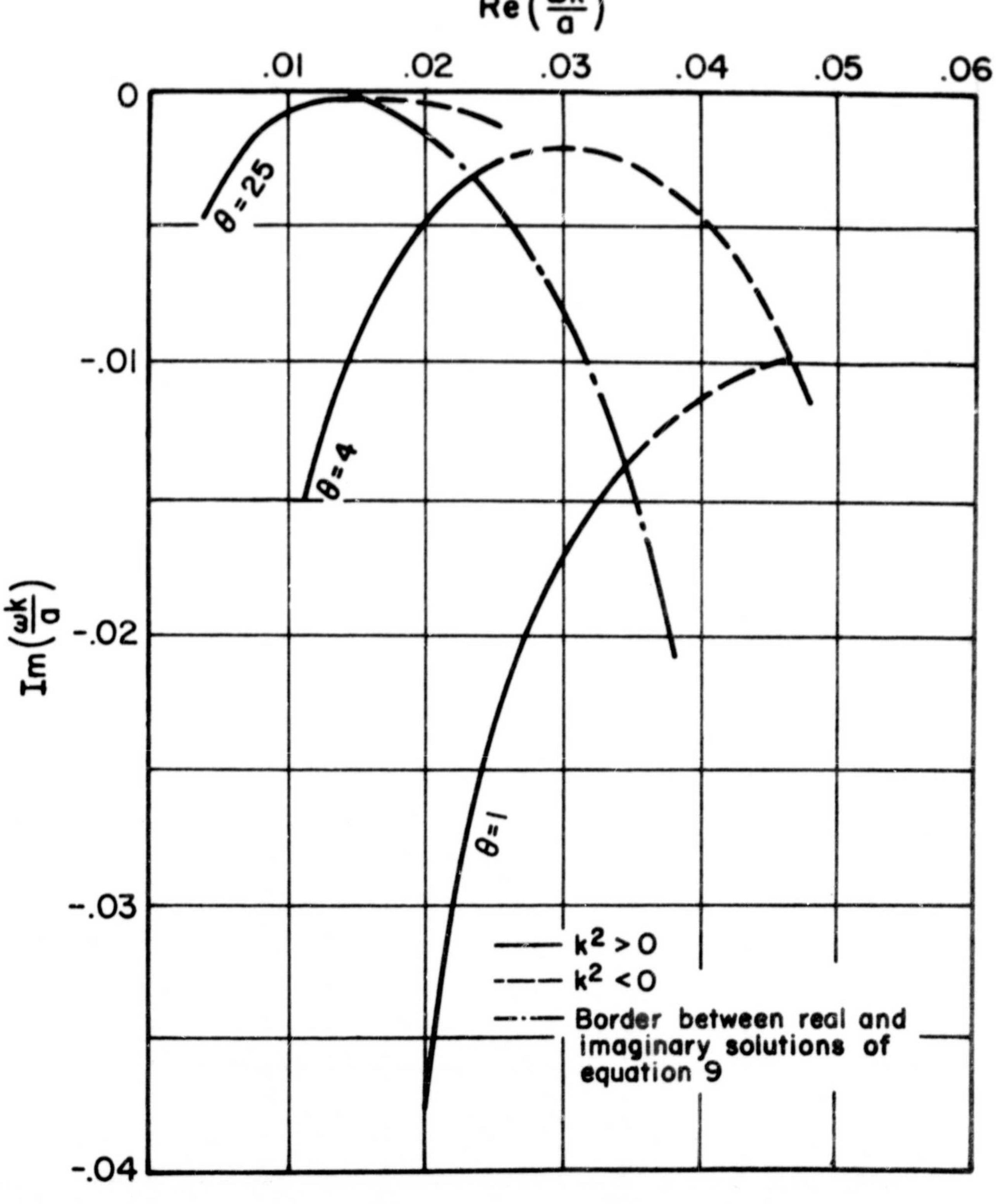


FIGURE 3. SOLUTIONS OF EQUATION 9 ILLUSTRATING THE EFFECT OF $\theta = T_e/T_i$ UPON ION ACOUSTIC-WAVE DAMPING

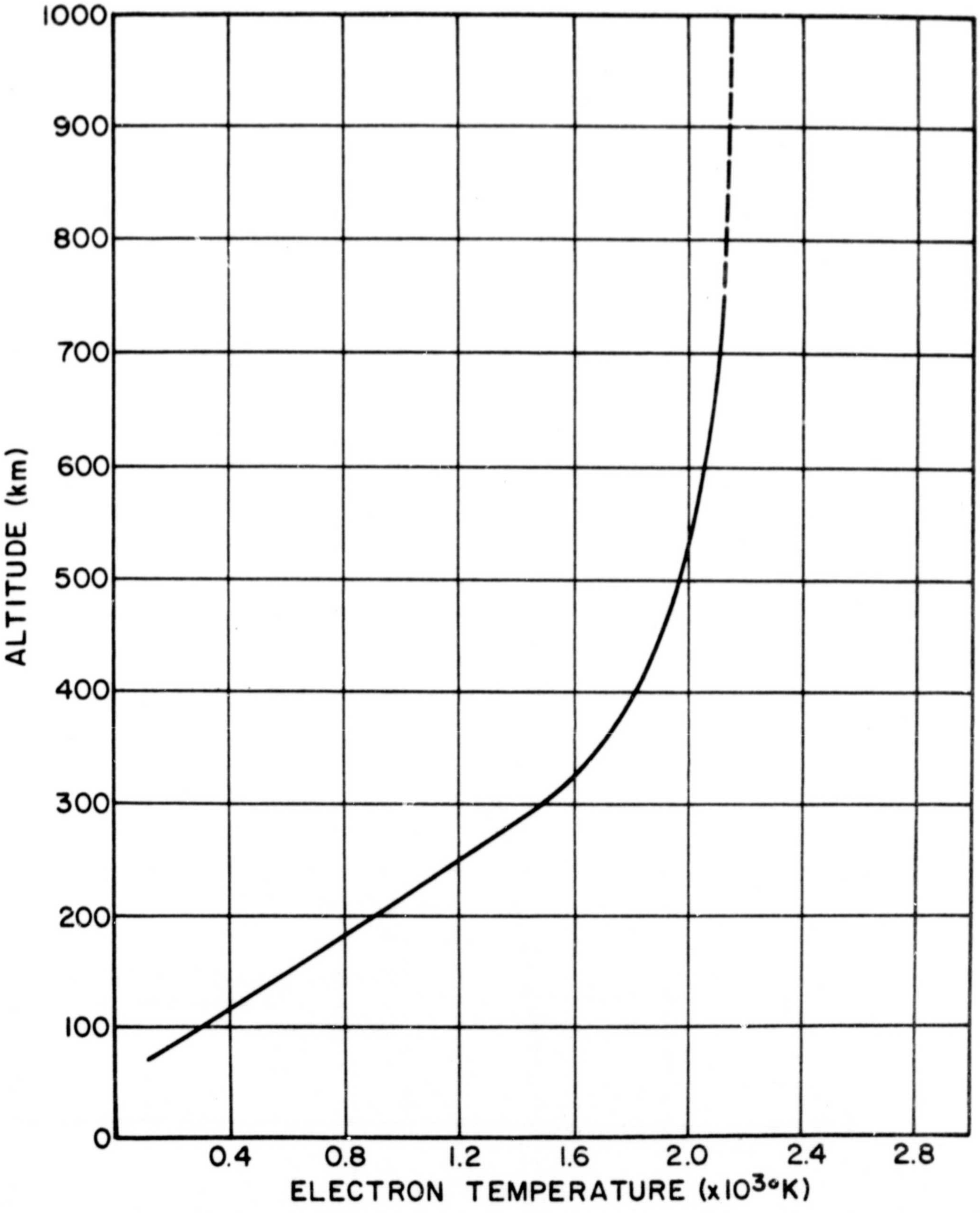


FIGURE 4. APPROXIMATE IONOSPHERIC ELECTRON TEMPERATURES

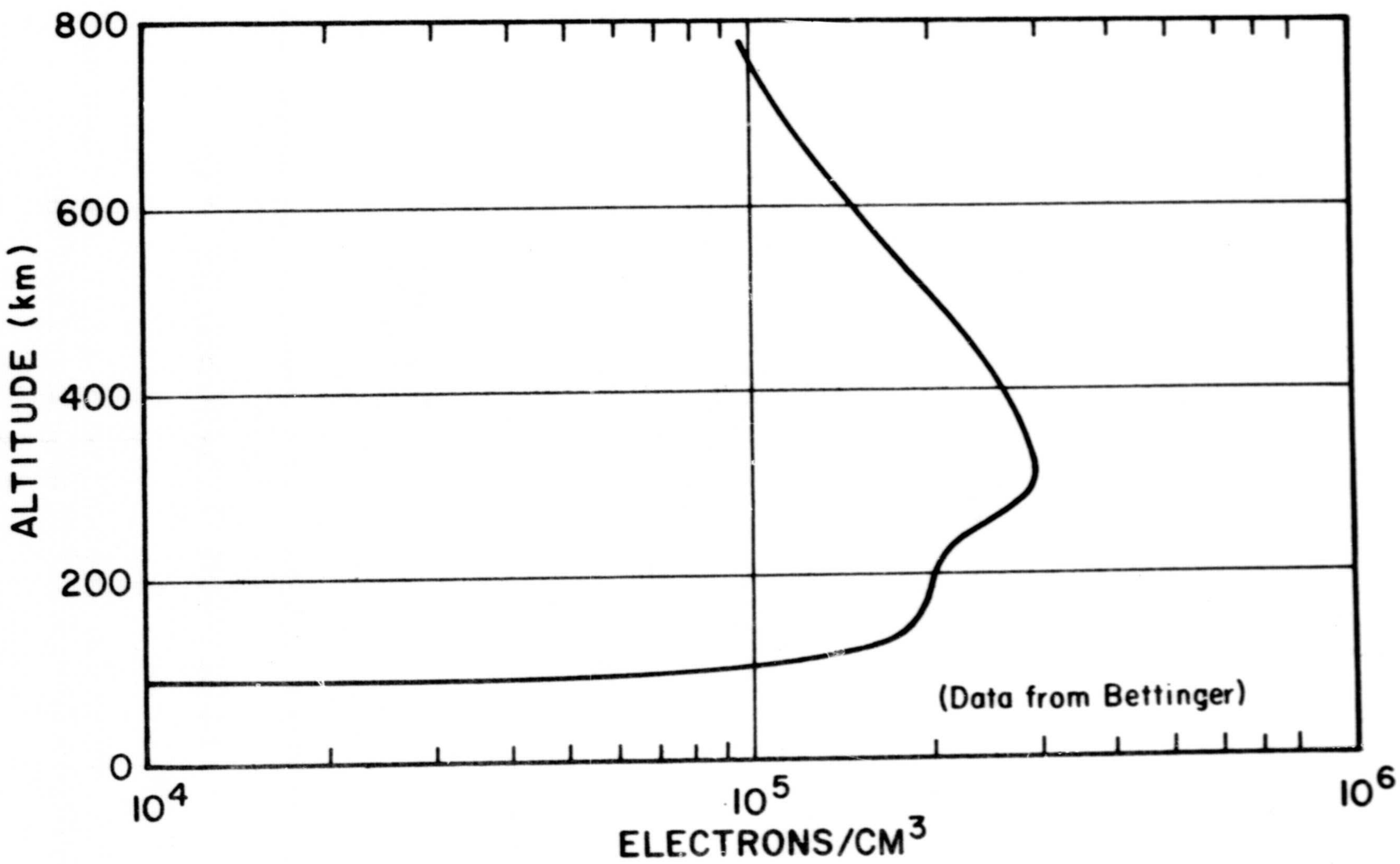


FIGURE 5. DAYTIME IONOSPHERIC ELECTRON CONCENTRATIONS

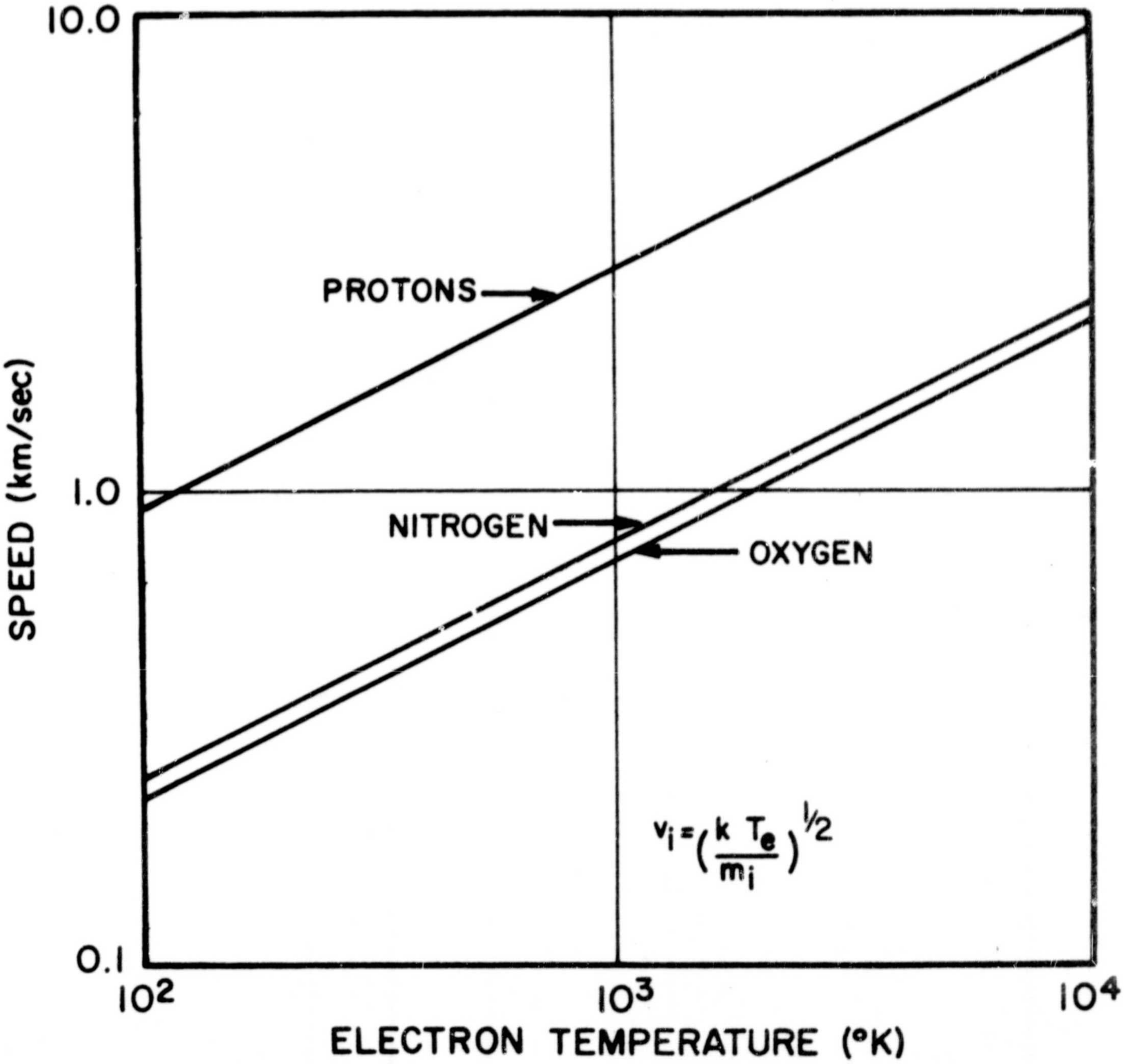


FIGURE 6. ION ACOUSTIC-WAVE SPEEDS FOR TYPICAL IONOSPHERIC ELECTRON TEMPERATURES

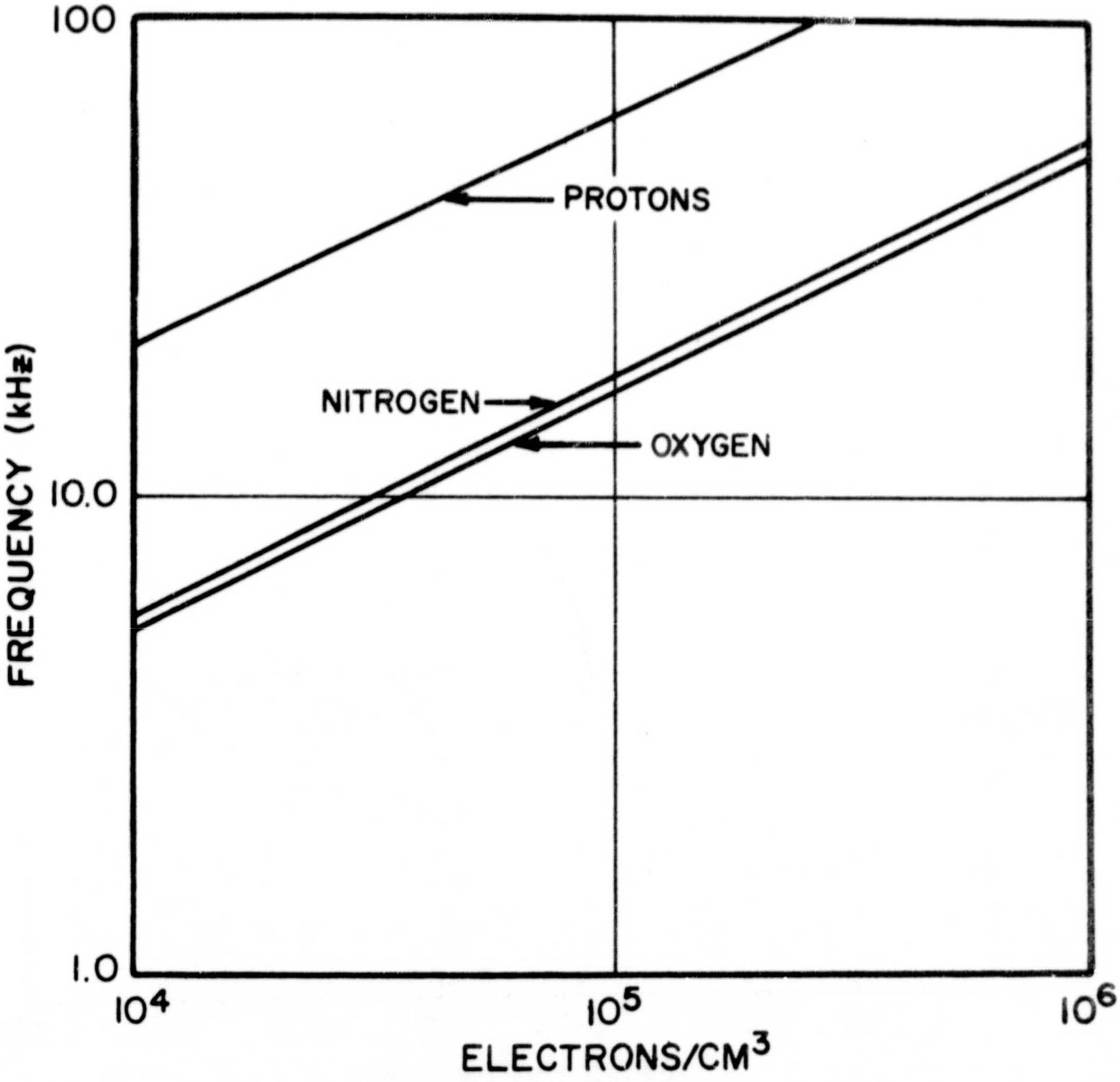


FIGURE 7. ION ACOUSTIC-WAVE FREQUENCIES FOR TYPICAL IONOSPHERIC ELECTRON CONCENTRATIONS

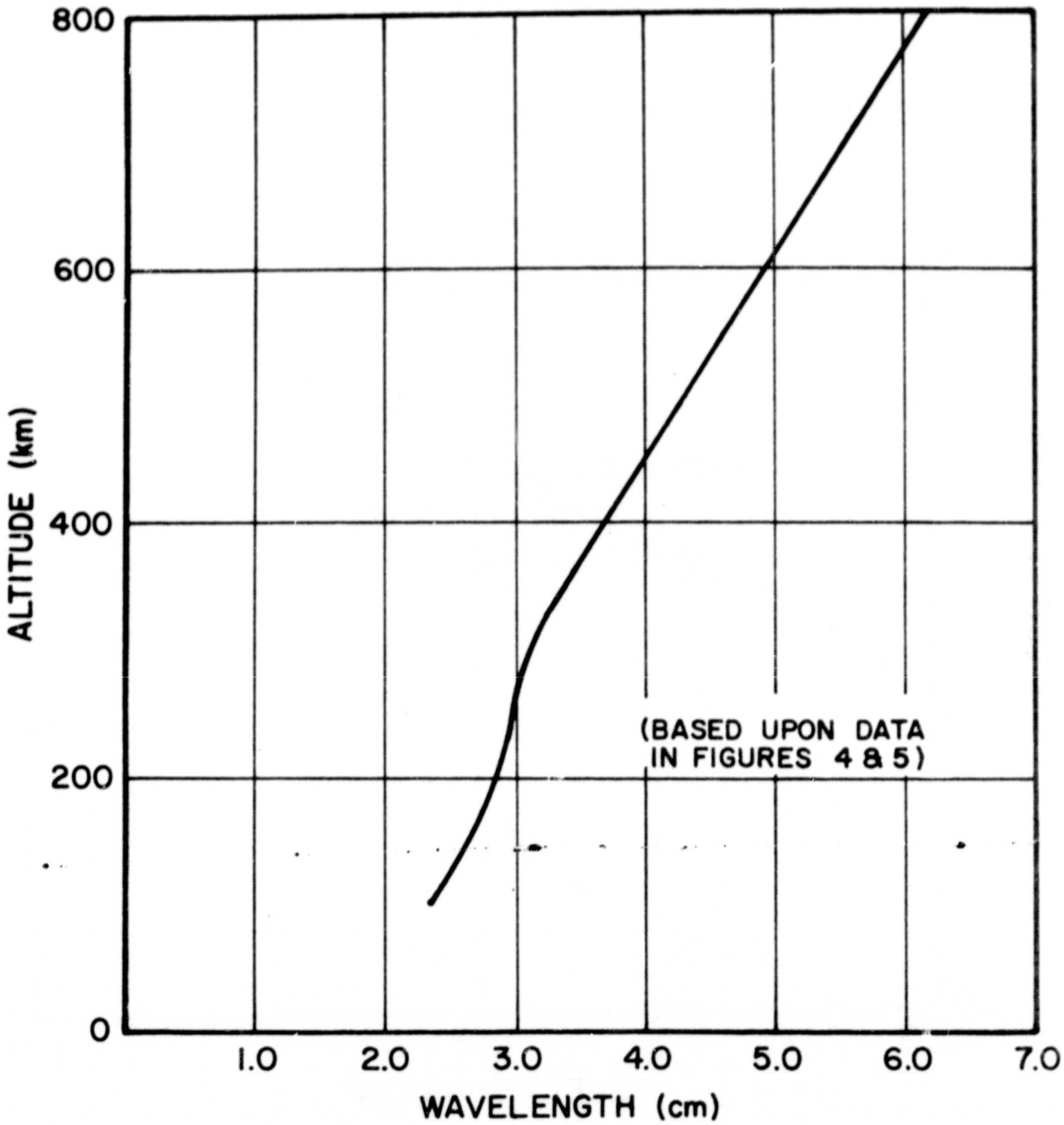


FIGURE 8. ION ACOUSTIC WAVELENGTH VERSUS ALTITUDE

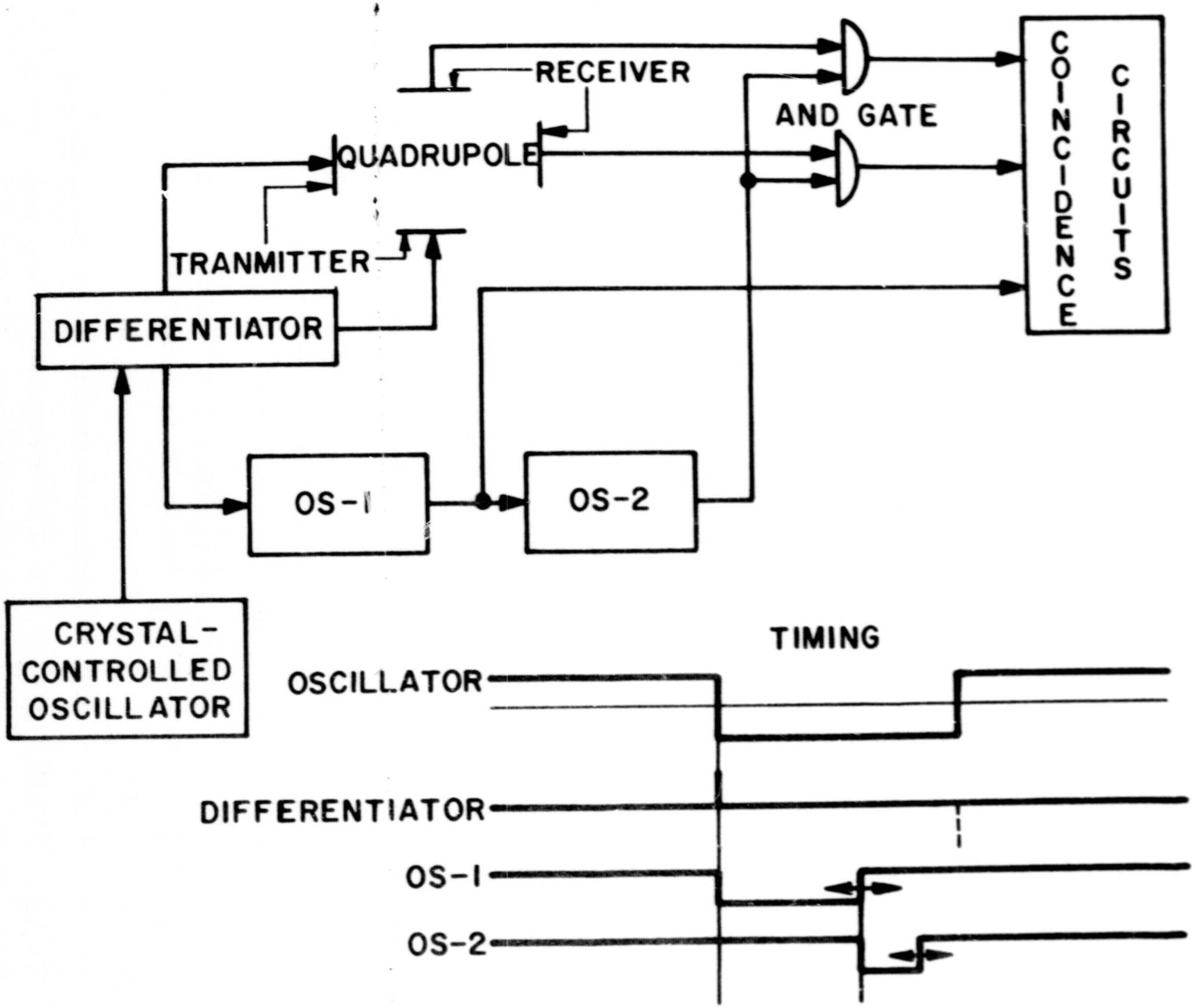


FIGURE 9. SCHEMATIC REPRESENTATION OF A POSSIBLE SATELLITE ION ACOUSTIC-WAVE APPARATUS

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APPENDIX

DERIVATION OF LONGITUDINAL OSCILLATION DAMPING FACTOR

One of the more rigorous approaches to the analysis of plasma oscillation phenomena involves the use of the plasma dispersion relation applicable to the plasma situation under investigation. Plasma dispersion relations have been investigated by many authors; the following analysis is a recapitulation of the work of Fried and Gould [8] and Jackson [10].

We begin with the collisionless Boltzmann equation for electrons:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} f - \frac{e}{m_e} (\vec{E} + \vec{v} \times \vec{B}) \nabla_{\vec{v}} f = 0 \qquad (1a)$$

where $f(\vec{r}, \vec{v}, t)$ is the electron distribution function, $\vec{r}$ and $\vec{v}$ are the displacement vectors in configuration and velocity space, e/m_e is the ratio of electron charge to mass, ∇ is the gradient operator, and $\vec{E}$ and $\vec{B}$ are the electric field and magnetic field intensities. Since we are interested in longitudinal oscillations, the cross-product of $\vec{v}$ and $\vec{B}$ is assumed equal to zero. We shall further assume that the oscillations are one-dimensional and that the velocity space distribution function is spherically symmetric; therefore, equation (1a) has the form:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial t} - \frac{e}{m_e} E \frac{\partial f}{\partial v} = 0 \,. \qquad (2a)$$

Next let us assume that f and E have the form:

$$f(x, v, t) = f_0(v) + f_1(x, v, t)$$

$$E(x, v, t) = E_0(v) + E_1(x, v, t)$$

with $f_1 << f_0$ and $E_1 << E_0$, that is, f_1 and E_1 are small disturbances to the equilibrium quantities f_0 and E_0. Substituting into equation (2a) we obtain:

$$\frac{\partial f}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m_e}(E_0 + E_1)\left(\frac{\partial f_0}{\partial v} + \frac{\partial f_1}{\partial v}\right) = 0$$

since by definition $\partial f_0/\partial t = \partial f_0/\partial x = 0$. The electric field E_0 can be omitted with no loss of generality since E_0 represents an unchanging field and we are interested only in the behavior of the perturbed field E. In addition, we shall neglect second order (nonlinear) effects, that is, $E_1\ \partial f_1\ \partial v$. Thus we arrive at the linearized Boltzmann equation for electrons:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial t} - \frac{e}{m_e} E_1 \frac{\partial f_0}{\partial t} = 0. \qquad (3a)$$

A similiar equation can be obtained for ions, with F replacing f and $+eZ/m_i$ replacing $-e/m_e$. We shall assume that Z = 1.0 in the following analysis.

The coupling between the electric field and the distribution function arises through the Poisson equation, which for one dimension has the form:

$$\frac{\partial E_1}{\partial x} = 4\pi\rho = -e4\pi(n_i - n_e) = 4\pi e \int_{-\infty}^{\infty} dv(f_1 - F_1), \qquad (4a)$$

where ρ is the electric charge density.

Equations (3a) and (4a) are conveniently simplified by employing the Fourier and Laplace transforms defined by:

$$F\,[f(t)] \equiv \int_{-\infty}^{\infty} \exp(i\omega t)f(t)dt \equiv h(\omega)$$

and (5a)

$$L\,[f(t)] \equiv \int_{-\infty}^{\infty} \exp - (pt)f(t)dt \equiv g(p)$$

with the inversion relations

$$L^{-1}[g(p)] \equiv \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \exp(pt)g(p)dp$$

and (6a)

$$F^{-1}[h(\omega)] \equiv \int_{-\infty}^{\infty} \exp - (i\omega t)h(\omega)d\omega \quad .$$

Applying the Fourier transform to x (integrating by parts and observing that $f_1(\pm\infty = 0)$), we obtain:

$$\int_{-\infty}^{\infty} v \frac{\partial f_1}{\partial x} \exp(ikx)dx$$

$$= v_1 f_1 \exp(ikx) \Big|_{-\infty}^{\infty} - ivk \int_{-\infty}^{\infty} f_1 \exp(ikx)dx$$

$$= -ivkf_1(k, v, t) \quad .$$

Applying the Laplace transform to t (setting $p = +i\omega$), we obtain:

$$\int_0^{\infty} \frac{\partial f_1}{\partial t} \exp -(i\omega t)dt = i\omega f_1(k, v, \omega)$$

equation (3a) then becomes:

$$i\omega f_1(k, v, \omega) - ivk \cdot f_1(k, v, \omega) - \frac{e}{m_e} E_1 \frac{\partial f_0}{\partial v} = 0$$

from which we obtain

$$f_1 = \frac{e}{m_e} E_1 \frac{\partial f_0}{\partial v} (i\omega - ivk)^{-1} \quad . \tag{7a}$$

In an analogous manner we obtain F_1 :

$$F_1 = \frac{e}{m_i} E_1 \frac{\partial f_0}{\partial v} (i\omega - ivk)^{-1} \quad . \tag{8a}$$

Finally, we Fourier transform Poisson's equation to obtain:

$$-ikE_1 = 4\pi e \int_{-\infty}^{\infty} dv(F_1 - f_1) \quad . \tag{9a}$$

Substituting f_1 and F_1 into equation (9a), we obtain:

$$kE_1 = \frac{4\pi e^2}{km_e} \int_{-\infty}^{\infty} dv E_1 \left(\frac{\partial f_0}{\partial v} + \frac{m_e}{m_i} \frac{\partial F_0}{\partial v} \right) (v - \frac{\omega}{k})^{-1} ,$$

which is rearranged to:

$$E_1 \left[k^2 - \frac{4\pi e^2 n_0}{m_e} \int_{-\infty}^{\infty} \left(\frac{1}{n_0} \frac{\partial f_0}{\partial v} + \frac{m_e}{m_i n_0} \frac{\partial F_0}{\partial v} \right) (v - \frac{\omega}{k})^{-1} \right] = 0,$$

where n is the particle number density. Some simplification is possible by noting that the "plasma" frequency relation is:

$$4\pi n e^2 / m_e = \omega_p^2$$

and by introducing:

$$\delta \equiv m_e / m_i ,$$

whereupon we have the dispersion relation:

$$H(k, \frac{\omega}{k}) \equiv \frac{k^2}{\omega_p^2} - \int_{-\infty}^{\infty} \left(\frac{\partial f_0}{\partial v} + \delta \frac{\partial F_0}{\partial v} \right) (v - \frac{\omega}{k})^{-1} = 0 . \qquad (10a)$$

In principle, if f_0 and F_0 are known, equation (10a) can be solved for $\underline{\omega}$, and it is seen that $\underline{\omega}$ is a function of velocity, that is, the frequencies are dispersed in velocity space. In general, $\underline{\omega}$ is a complex number and it is well known that if $Im(\omega) < 0$, a wave of the form exp - ($i\omega t$) will be damped, while if $Im(\omega) > 0$, a wave will grow and unstable oscillations can occur. The question of collisionless damping, usually known as Landau damping, has been debated considerably. A principal difficulty arises because, as Jackson demonstrates [10], the dispersion relation of equation (10a) is valid only for $Im(\omega) > 0$, and to obtain a relation for $Im(\omega) < 0$, equation (10a) must be analytically continued across the real axis.

In the case of Maxwellian distributions, analytic continuation is possible; therefore, we shall consider these types of distributions. Accordingly, we assume f_0 and F_0 have the form:

$$f_0(v) = \frac{\exp-(v^2/a^2)}{a\pi^{\frac{1}{2}}}$$

and (11a)

$$F_0(v) = \frac{\exp-(v^2/A^2)}{A\,\pi^{\frac{1}{2}}} \quad ,$$

where the relation $a^2 = 2kT_e/m$ is the mean thermal speed squared. Substituting equation (11a) into equation (10a), we have

$$0 = \frac{k^2}{\omega_p^2} + \int_{-\infty}^{\infty} dv \left[\frac{2v}{a} \frac{1}{a^2\pi^{\frac{1}{2}}} \exp-\left(\frac{v^2}{a^2}\right) + \delta \frac{2v}{A} \frac{1}{A^2\pi^{\frac{1}{2}}} \exp-\left(\frac{v^2}{A^2}\right)\right] \left(v - \frac{\omega}{k}\right)^{-1}$$

This expression may be reduced if we employ:

$$\theta \equiv \frac{m_e}{m_i} \frac{a^2}{A^2} = \frac{T_e}{T_i} \quad , \text{ and } x = v/a \; .$$

Therefore:

$$0 = \frac{k^2}{\omega_p^2} + \frac{2}{a\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx \left[x \exp-(x^2) + \theta\, x \frac{a}{A} \exp-(x^2\theta/\delta)\right] \left(xa - \frac{\omega}{k}\right)^{-1}$$

The term $\underline{\omega}$ is a well defined analytic function and may be extended into the complex plane when the following definition is used:

$$\xi = x + iy = \omega/ka \, .$$

Thus the dispersion relation is reduced to:

$$0 = \frac{k^2}{\omega_p^2} + \frac{2}{a^2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx \; [x \exp - (x^2)$$

$$+ \theta x \sqrt{\theta/\delta} \;\; \exp - (x^2\theta/\delta)][x - \xi]^{-1}. \qquad (12a)$$

The Debye wave number is given as follows:

$$k_D^2 = 2\omega_p^2/a^2$$

and if we define

$$\lambda \equiv k^2/k_D^2$$

we obtain

$$0 = \lambda + \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx\, x\, [\exp - (x^2)$$

$$+ \theta\sqrt{\theta/\delta} \;\exp - (x^2\,\theta/\delta)]\; [x - \xi]^{-1}. \qquad (13a)$$

It is now convenient to consider the plasma dispersion function

$$Z(z) \equiv \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} du \exp - (u^2)\,(u - z)^{-1}. \qquad (14a)$$

Equation (13a) is in the form of a Cauchy integral $\int dz f(z)/(z - z_0)$, and from the theory of complex variables the derivative of this integral can be written as:

$$\frac{d}{dz}\int \frac{f(z)dz}{z - z_0} = \int \frac{f'(z)}{z - z_0}\, dz \qquad (15a)$$

(prime denotes differentiation). Thus the relation $dZ(z)/dz = Z'(z)$ becomes:

$$Z'(z) = -2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} du\; u \exp - (u^2)\,(u - z)^{-1}.$$

When equation (15a) is compared with equation (13a), it is seen that equation (13a) reduces to:

$$\lambda - \tfrac{1}{2}[Z'(\xi) + \theta\, Z'(\xi\sqrt{\theta/\delta})] = 0. \qquad (16a)$$

Although equation (16a) may be solved for ξ, the process is laborious at best. Fortunately the function $Z(\xi)$ closely resembles the error function for complex argument defined by:

$$W(z) \equiv \frac{1}{\pi i} \int_{-\infty}^{\infty} du \exp-(u^2)(u-z)^{-1} .$$

From a comparison of the definition of W(z) with equation (14a), we obtain:

$$Z(\xi) = \sqrt{\pi}\; W(\xi)/i .$$

The derivative of W(z) is given by the relation:

$$W'(z) = -2zW(z) + 2i\pi^{-\frac{1}{2}} .$$

Since the dispersion relation is symmetric in x, we can obtain values for $\xi = x - iy$ by employing the relation:

$$W(-z) = 2\exp-(z^2) - W(z) .$$

Employing these relationships, we find:

$$Z'(\xi) = \frac{2\sqrt{\pi}}{i} \left[-\xi W(z) + \frac{i}{\sqrt{\pi}}\right]$$

$$= 2\left[1 + i\sqrt{\pi}\;\xi W(\xi)\right]$$

and

$$Z'(-\xi) = 2\left[1 + i\sqrt{\pi}\;\xi\left(2\exp-(\xi^2) - W(\xi)\right)\right] . \qquad (17a)$$

If equation (17a) is substituted in equation (16a) and expanded, numerical results can be obtained for $\xi = \omega k/a$. The procedure, while tedious, is straightforward. One first determines those values of x and y which reduce the imaginary part of equation (16a) to zero. Then the real part of equation (16a) determines λ. This procedure has been carried out by Fried and Gould, and the resultant curves in the ξ plane are exhibited in Figure 3. It should be noted that as θ increases, Im(ω) becomes quite small, indicating that $\theta \geq 2$ or 3, there is a reasonable probability of observing longitudinal plasma waves.