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GRAVITY GRADIENT MEASUREMENTS WITH A LASER ABSOLUTE GRAVIMETER

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16. ABSTRACT

A technique for measuring the gravity gradient is derived from extensions of Dr. Hudson's original theory employed in the laser interferometry method for measuring absolute gravity. The performance of the proposed experiment for measuring gradient is predicted through analyzing the effects from expected sources of measurement error.

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GRAVITY GRADIENT MEASUREMENTS WITH A LASER ABSOLUTE GRAVIMETER

SUMMARY

This study presents the derivation of a technique for measuring gravity gradient, as well as absolute gravity, by modification of an existing laser absolute gravimeter. The technique has the additional advantage of identifying a point in space corresponding to the obtained value of g.

The method of error analysis used to predict the effect on performance resulting from known sources of measurement error is outlined. This analysis, as employed by the authors, indicates that expected accuracy is equal to that obtainable with the present state of the art.

Brief consideration is given to the modifications which would be necessary to make the existing gravimeter compatible with the proposed technique.

INTRODUCTION

This study proposes an experiment designed to measure the rate of change of the acceleration caused by gravity (hereafter called "gradient" or "K") through the employment of a laser interferometry technique. Such a technique is presently being utilized in a device for the measurement of the absolute acceleration caused by gravity. In essence, extensions of the theory employed in such a gravimeter are put forth, and modifications to its configuration which would render it amenable to gradient measurement are suggested.

BACKGROUND MATERIAL

Subsequent discussion relies heavily upon an article concerning laser absolute gravimetry which appeared in Laser Focus [1].

> A laser system for measuring absolute gravity is shown in Figure 1. The instrument is in the form of a Michelson interferometer in which mirror M1 is fixed and mirror M2 is a freely-falling body in the gravitational field g. Interference fringes are detected by the photodetector. Let x be the distance fallen along the gravitational field vector by the mirror M2 . . . ; then, at the photodetector, interference maxima are observed for:
>
> $$2x = 0, \lambda, 2\lambda, 3\lambda, \ldots, N\lambda$$

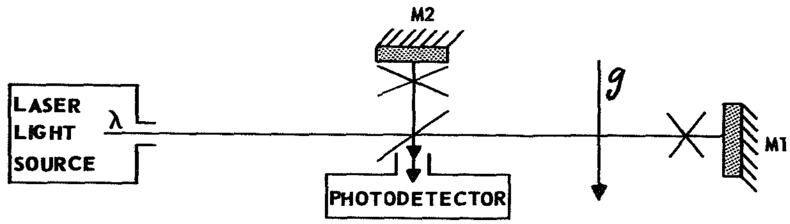

Figure 1. Basic relationship of primary components in laser interferometry system for measuring absolute gravity.

The number of maxima, N, observed in a distance x is:

$$2x = N\lambda$$

$$N = \frac{2}{\lambda} x$$

Lambda is the wavelength of the illumination in a hard vacuum. If the light falling on M2 happens to make a small angle θ with the gravity vector (remember that M2 always falls parallel with the gravity vector), then the component of displacement parallel to g is $x \cos\theta$, and the expression becomes:

$$N = \frac{2}{\lambda} x \cos\theta$$

Differentiating this expression twice with respect to time, we obtain a relation for g:

$$-g \equiv \ddot{x} = \frac{\lambda}{2\cos\theta} \ddot{N}$$

[Note here that displacements downward along the gravity vector are considered positive and that the quantity g is inherently negative.] Subject to the initial conditions: $N = 0, \dot{N} = \dot{N}_0$ at $t = 0$ and integrating, one finds:

$$N = \frac{-g\cos\theta}{\lambda} t^2 + \dot{N}_0 t$$

In the author's experiment, time is determined by counting the number of oscillations of a standard oscillator of frequency f (hereafter called "the clock"). If n is the clock pulse number, then time is determined from:

$$n = \int_0^t f dt$$

$$t = \frac{n}{f} \tag{1}$$

Two electronic counters are used, one to record waves from the standard frequency oscillator and the other to record interference maxima from the photodetector. In this article, only integral values of n are considered. It should be noted that the higher the clock frequency f, the less time is associated with the integer n (see Equation 1). In an actual experiment, n is not really an integer, so the less time an integer represents, the less error it can introduce. The equation may now be written:

$$N = -\left(\frac{g \cos \theta}{f^2 \lambda}\right) n^2 + \left(\frac{\dot{N}_0}{f}\right) n \tag{2}$$

In Equation 2, the unknowns which must be experimentally computed are g and $\dot{N}_0$. To compute g, the quantity of interest, two equations are required to permit a simultaneous solution. Hence, two measurements must be made for each experiment. Let us designate these measurements as A and B. Then, eliminating $\dot{N}_0$ we find:

$$g = \left(\frac{f^2 \lambda}{\cos \theta}\right)\left(\frac{N_A n_B - N_B n_A}{n_A n_B (n_A + n_B)}\right) \tag{3}$$

DISCUSSION OF INITIAL PROBLEM OF INTEREST

This study was initiated by the problem of determining the point along the path of M2 at which the computed g value is applicable. This question arises from the fact that g is in reality not constant over the interval of the drop path. Its value is continuously increasing as M2 approaches the end of the drop. This can be seen from the Universal Law of Gravitation:

$$F = -G \frac{mm'}{r^2}$$

$$m'g = -G \frac{mm'}{r^2}$$

$$g = -G \frac{m}{r^2}$$

where m is the mass of the earth; m′ is the mass of M2; G is the gravitational constant; r is the distance from earth's center of mass; and g is the acceleration due to the earth. Thus, as M2 falls, it decreases the value of r which in turn causes an increase in the value of g. Differentiating the last expression above with respect to distance yields:

$$\frac{dg}{dr} = +2G\frac{m}{r^2} \equiv K \qquad (1)$$

This quantity $\frac{dg}{dr}$ is the gravity gradient, K. It constitutes the basis for the free-air correction utilized in geodesy to reduce gravity measurements to the surface of the accepted reference ellipsoid or spheroid. The universally accepted normal value of K utilized for this purpose is 3.08×10^{-6} gal/cm.[1]

It can be seen that g is assumed to be constant throughout the drop path of M2 [1]. From the preceding discussion, it is evident that the value obtained is merely representative of those values existing along the path. Then, to what point along the path does this representative value correspond?

The effective length of the drop path utilized by the device discussed in Reference 1 is taken to be 16.593492 cm. Therefore, the difference between the g values at the beginning and the end of the path is of the order of 5×10^{-5} gal. Thus, if g measurements are made to seven figures or less, the value obtained is essentially representative of the entire path since the gradient effects do not enter until the eighth figure.

Because advancing the state of the art will necessitate extending the accuracy of g measurements beyond the seventh figure, the problem of gradient effects is definitely of interest. So the problem of determining to what point along the path the measured g value corresponds becomes real.

DERIVING A NEW EQUATION OF MOTION

Since the whole problem centers around the assumption that g is constant over intervals of vertical displacement, the derivation of a new equation of motion in which g varies with distance is in order. In this derivation the gravity gradient, K, will be the constant of motion. First, however, it seems advisable to determine whether K can be considered constant over sufficiently large intervals that the problem incurred by considering g to be constant does not occur again. If we refer to equation (1) and differentiate once with respect to distance, we find

$$\frac{dK}{dr} = -6G\frac{m}{r^4} = -1.445 \times 10^{-14} \text{ gal/cm}^2$$

---

1. 1 gal = 1 cm/sec^2.

where

$$Gm = 3.986032 \times 10^{20} \ \text{cm}^3/\text{sec}^2$$

and

$$r = 6.378165 \times 10^{8} \ \text{cm}$$

Thus, over a 1-m interval, K is seen to change only on the order of about 0.5 ppm.

Taking K to be constant and displacement downward along the gravity vector to be negative, we begin our derivation.

By definition

$$g(x) = \frac{d^2x}{dt^2}$$

Then

$$\frac{dg(x)}{dx} = \frac{d}{dx}\left(\frac{d^2x}{dt^2}\right)$$

$$= \frac{d}{dt}\left(\frac{d^2x}{dt^2}\right)\frac{dt}{dx}$$

$$= \left(\frac{d^3x}{dt^3}\right)\frac{dt}{dx}$$

$$= \dddot{x}\ \frac{1}{\dot{x}}$$

$$= \frac{\dddot{x}}{\dot{x}}$$

We have previously defined $\frac{dg(x)}{dx}$ as K; thus

$$K = \frac{\dddot{x}}{\dot{x}} \tag{2}$$

The solution proposed for equation (2) is

$$x = A\,e^{\sqrt{K}t} + B\,e^{-\sqrt{K}t} + C$$

Differentiating this expression to obtain its first three derivatives yields:

$$\dot{x} = A\sqrt{K}\,e^{\sqrt{K}t} - B\sqrt{K}\,e^{-\sqrt{K}t}$$

$$\ddot{x} = A\,K\,e^{\sqrt{K}t} + B\,K\,e^{-\sqrt{K}t}$$

$$\dddot{x} = A\,K^{3/2}\,e^{\sqrt{K}t} - B\,K^{3/2}\,e^{-\sqrt{K}t}$$

Substituting the first and third derivatives into equation (2) shows that the solution satisfies the differential equation. Subjecting the solution and its derivatives to the initial conditions $x_0 = 0$, $K\dot{x}_0 = \dddot{x}_0$, and $\ddot{x}_0 = g_0$ at $t = 0$, the resulting system can be solved for the constants A, B, and C. This results in the following equation of motion:

$$x = \frac{\sqrt{K}\dot{x}_0}{K}\left[\sinh\sqrt{K}(t)\right] + \frac{g_0}{K}\left[\cosh\sqrt{K}(t) - 1\right] \tag{3}$$

A similar result was derived by A. H. Cook [2].

EXPERIMENTAL SOLUTION FOR QUANTITIES OF INTEREST

In equation (3), the unknowns to be computed from experimental data are $\dot{x}_0$, g_0, and K. Three equations are needed to afford simultaneous solution. Thus, three measurements must be made for each experiment instead of two as employed in the experiment discussed in Reference 1. Designating these measurements as A, B, and C, one then arrives at the following system of equations:

$$x_A = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A)\right] + \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A) - 1\right]$$

$$x_B = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)\right] + \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)\right]$$

$$x_C = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$+ \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right] \qquad (4)$$

(The complete mathematical derivation of equations (4) may be found in Appendix A.) As the nature of the system makes any explicit solution for the unknowns impossible, one is forced to employ iterative techniques for solution. Solving the first equation for $\dot{x}_{0A}$ and substituting this value into the second and third equations results in:

$$x_B = \left[\frac{x_A - \frac{g_{0A}\cosh\sqrt{K}(t_A)}{K} + \frac{g_{0A}}{K}}{\sinh\sqrt{K}(t_A)}\right]\left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)\right]$$

$$+ \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)\right]$$

$$x_C = \left[\frac{x_A - \frac{g_{0A}\cosh\sqrt{K}(t_A)}{K} + \frac{g_{0A}}{K}}{\sinh\sqrt{K}(t_A)}\right]\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$+ \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right]$$

Solving both expressions for g_{0A} and subtracting one from the other, one obtains

$$0 = K\,\frac{x_B - \dfrac{x_A}{\sinh\sqrt{K}(t_A)}A'}{B' - A''\tanh\sqrt{K}\left(\frac{1}{2}t_A\right)} - K\,\frac{x_C - \dfrac{x_A}{\sinh\sqrt{K}(t_A)}C'}{D - C'\tanh\sqrt{K}\left(\frac{1}{2}t_A\right)}$$

where

$$A' = \sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)$$

$$B' = \cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)$$

$$C' = \sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)$$

$$D = \cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)$$

Thus, there results a function $\phi(K, t_A, t_B, t_C, x_A, x_B, x_C) = 0$. This immediately lends itself to an iterative solution for K when values for the six measured parameters are substituted. The value of g_{0A} is readily calculated from the results of that computation.

Thus, one has obtained the values for two fundamental quantities, the absolute acceleration caused by gravity and the gravity gradient. In addition to this multiplicity of function, the value of g_{0A} is assigned to a particular point in space, x_{0A}, the starting point of the first measured interval of distance.

PERFORMANCE ANALYSIS

To predict the performance of this experiment, and thus determine its feasibility, one must estimate the effect of measurement errors. This analysis must be performed on equations (4). The technique employed, the detailed explanation of which is found in Appendix B, was suggested by Charles Dalton of Aero-Astrodynamics Laboratory, Marshall Space Flight Center. Equations (4) are first put in the following form:

$$F = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A)\right] + \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A) - 1\right] - x_A = 0$$

$$G = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)\right] + \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)\right] - x_B = 0$$

$$H = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$+ \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right] - x_C = 0$$

where $F, G,$ and H are all functions of $K, \dot{x}_{0A}, g_{0A}, x_A, x_B, x_C, t_A, t_B,$ and t_C. One then obtains the partial derivatives of the three dependent variables $K, \dot{x}_{0A},$ and g_{0A} with respect to any particular independent variable by setting up a system of linear equations:

Letting $u \epsilon \left\{x_A, x_B, x_C, t_A, t_B, t_C\right\}$,

$$-\frac{\partial F}{\partial u} = \frac{\partial F}{\partial K}\frac{\partial K}{\partial u} + \frac{\partial F}{\partial \dot{x}_{0A}}\frac{\partial \dot{x}_{0A}}{\partial u} + \frac{\partial F}{\partial g_{0A}}\frac{\partial g_{0A}}{\partial u}$$

$$-\frac{\partial G}{\partial u} = \frac{\partial G}{\partial K}\frac{\partial K}{\partial u} + \frac{\partial G}{\partial \dot{x}_{0A}}\frac{\partial \dot{x}_{0A}}{\partial u} + \frac{\partial G}{\partial g_{0A}}\frac{\partial g_{0A}}{\partial u}$$

$$-\frac{\partial H}{\partial u} = \frac{\partial H}{\partial K}\frac{\partial K}{\partial u} + \frac{\partial H}{\partial \dot{x}_{0A}}\frac{\partial \dot{x}_{0A}}{\partial u} + \frac{\partial H}{\partial g_{0A}}\frac{\partial g_{0A}}{\partial u}$$

(Expressions for the coefficients and constant terms involved may be found in Appendix C.) Thus, in matrix notation one has the following equation:

$$\begin{bmatrix} \frac{\partial F}{\partial K} & \frac{\partial F}{\partial \dot{x}_{0A}} & \frac{\partial F}{\partial g_{0A}} \\ \frac{\partial G}{\partial K} & \frac{\partial G}{\partial \dot{x}_{0A}} & \frac{\partial G}{\partial g_{0A}} \\ \frac{\partial H}{\partial K} & \frac{\partial H}{\partial \dot{x}_{0A}} & \frac{\partial H}{\partial g_{0A}} \end{bmatrix} \begin{bmatrix} \frac{\partial K}{\partial u} \\ \frac{\partial \dot{x}_{0A}}{\partial u} \\ \frac{\partial g_{0A}}{\partial u} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F}{\partial u} \\ \frac{\partial G}{\partial u} \\ \frac{\partial H}{\partial u} \end{bmatrix}$$

Solving one such system for each of the independent variables will yield values for the partials of each dependent variable with respect to each independent variable. To obtain the probable error (P.E.) in any one of the three dependent variables, one needs only to substitute the values of the partials into the following equation:

Letting $\nu \epsilon \left\{ K, \dot{x}_{0A}, g_{0A} \right\}$

$$P.E._{\nu} = \sqrt{\left(\frac{\partial \nu}{\partial x_A}\right)^2 \left(P.E._{x_A}\right)^2 + \left(\frac{\partial \nu}{\partial t_A}\right)^2 \left(P.E._{t_A}\right)^2 + \left(\frac{\partial \nu}{\partial x_B}\right)^2 \left(P.E._{x_B}\right)^2 + \left(\frac{\partial \nu}{\partial t_B}\right)^2 \left(P.E._{t_B}\right)^2 + \left(\frac{\partial \nu}{\partial x_C}\right)^2 \left(P.E._{x_C}\right)^2 + \left(\frac{\partial \nu}{\partial t_C}\right)^2 \left(P.E._{t_C}\right)^2} \quad (5)$$

The above technique may be used to predict performance, provided one has the values for all nine variables involved. As it is not possible to obtain the time values associated with each drop segment without constructing an apparatus to do so, solution of equations (4) for the dependent variables K, $\dot{x}_{0A}$, and g_{0A} was out of the question for generating values to be used in the error analysis. Therefore, the values for x_A, X_B, x_C, K, x_{0A}, and g_{0A} were substituted into equations (4) and solved for times t_A, t_B, and t_C. The displacements x_A, x_B, and x_C were each taken to be -10 cm (roughly the length of segments used in the apparatus discussed in Reference 1). The value of g_{0A} was taken as -979.6395 gal, the value on a base plate in a nearby laboratory. The normal value 3.086×10^{-6} gal/cm was used for K; and for purposes of simplification, x_{0A} was taken to be zero. Substituting these values into equations (4) and solving for the time values, one then has values for all nine variables suitable for purposes of error analysis.

For the probable errors associated with the measured parameters x_A, x_B, x_C, t_A, t_B, and t_C, we suppose the errors imposed by the limitations of the measuring devices employed in the apparatus discussed in Reference 1. For the distance measurements, one is, therefore, dependent upon the output of a gas laser of wavelength 6329.9147 Å, which has a long-term stability of 10^{-8}, and upon a fringe counter which is accurate to one-hundredth of a fringe. The time measurements will be derived from electronic counting of wave numbers (accurate to within one count) from a 100-Mc oscillator having a stability of 10^{-8}.

With values for all nine variables and the necessary information concerning probable errors in the measured parameters, one has the data for computing the partials and P.E.'s contained on the right side of equation (5). The accuracy of these computations is admittedly limited by that of the extended precision mode of the IBM 1130 computer that was employed. The data obtained from the analysis indicate that the performance will be well within required limits.

RESULTS OF PERFORMANCE ANALYSIS

It appears that a six-figure repeatability can be expected in the g_{0A} values obtained, which is the state of the art as attained by H. Preston-Thomas [3] and D. R. Tate [4]. In addition, five-figure repeatability in gradient values can be expected. (Sources known to the writers indicate that this would constitute a one-figure advance over present state of the art.)

To check the effect of varying values of K, g_{0A}, and x ($x = x_A = x_A = x_C$) in generating time values and the resulting effect on the performance analysis, trials were made for every possible combination of the following terms:

$K = 2.8 \times 10^{-6}$ gal/cm , 2.9×10^{-6} gal/cm , 3.0×10^{-6} gal/cm , 3.1×10^{-6} gal/cm

g_{0A} = -977 gal , -978 gal , -979 gal , -980 gal

x = -5 cm, -10 cm, -15 cm, -20 cm

The results for the probable error inherent to g_{0A} values remained the same as in the first analysis. However, it was found that in some worst case conditions, the probable error inherent to K reduced the number of repeatable figures in gradient by one, to four repeatable figures.

CONCLUSIONS

This study indicates that the apparatus discussed in Reference 1 can be modified to make both gravity and gravity gradient measurements to state-of-the-art accuracies. The modification entailed would consist of little more than the incorporation of an additional segment of drop path and an additional counter. It would, however, also be of concern to determine the point at which the effective drop path begins, as the g_{0A} value measured would correspond to that point in space. Location of this point within a millimeter seems to be possible by mechanical means.

APPENDIX A

DERIVATION OF EQUATIONS (4)

By definition

$$g(x) \equiv \frac{d^2x}{dt^2}$$

then

$$\frac{dg(x)}{dx} = \frac{d}{dx}\left(\frac{d^2x}{dt^2}\right)$$

$$= \frac{d}{dt}\left(\frac{d^2x}{dt^2}\right)\frac{dt}{dx}$$

$$= \left(\frac{d^3x}{dt^3}\right)\frac{dt}{dx}$$

$$= \dddot{x}\,\frac{1}{\dot{x}}$$

$$= \frac{\dddot{x}}{\dot{x}} \equiv K$$

The proposed solution of the above differential equation is

$$x = A\,e^{\sqrt{K}t} + B\,e^{-\sqrt{K}t} + C \qquad \text{(A-1)}$$

Differentiating,

$$\dot{x} = A\,K^{1/2}\,e^{\sqrt{K}t} - B\,K^{1/2}\,e^{-\sqrt{K}t}$$

$$\ddot{x} = A\,K\,e^{\sqrt{K}t} + B\,K\,e^{-\sqrt{K}t}$$

$$\dddot{x} = A\,K^{3/2}\,e^{\sqrt{K}t} - B\,K^{3/2}\,e^{-\sqrt{K}t} \qquad \text{(A-2)}$$

Initial conditions at $t = 0$ are $x_0 = 0, K\dot{x}_0 = \dddot{x}_0$, and $\ddot{x}_0 = g_0$. Imposing them on equations (A-1) and (A-2), one has

$$0 = A + B + C$$

$$K\dot{x}_0 = A\,K^{3/2} - B\,K^{3/2}$$

$$g_0 = A\,K + B\,K$$

Setting up the augmented matrix for this system and reducing, one finds:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ K^{3/2} & -K^{3/2} & 0 & K\dot{x}_0 \\ K & K & 0 & g_0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2K^{3/2} & 0 & K\dot{x}_0 - K^{1/2}\,g_0 \\ K & K & 0 & g_0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\dfrac{K^{1/2}\dot{x}_0 - g_0}{2K} \\ 1 & 1 & 0 & \dfrac{g_0}{K} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\dfrac{K^{1/2}\dot{x}_0 - g_0}{2K} \\ 0 & 0 & -1 & \dfrac{g_0}{K} \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\dfrac{K^{1/2}\dot{x}_0 - g_0}{2K} \\ 0 & 0 & 1 & -\dfrac{g_0}{K} \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \dfrac{K^{1/2}\dot{x}_0 + g_0}{2K} \\ 0 & 1 & 0 & -\dfrac{K^{1/2}\dot{x}_0 - g_0}{2K} \\ 0 & 0 & 1 & -\dfrac{g_0}{K} \end{array}\right],$$

which yields:

$$A = \frac{\sqrt{K}\dot{x}_0 + g_0}{2K}$$

$$B = -\frac{\sqrt{K}\dot{x}_0 - g_0}{2K}$$

$$C = -\frac{g_0}{K}$$

The original system of equations thus becomes

$$x = \left(\frac{\sqrt{K}\dot{x}_0 + g_0}{2K}\right) e^{\sqrt{K}t} - \left(\frac{\sqrt{K}\dot{x}_0 - g_0}{2K}\right) e^{-\sqrt{K}t} - \left(\frac{g_0}{K}\right)$$

$$\dot{x} = \left(\frac{K\dot{x}_0 + \sqrt{K}g_0}{2K}\right) e^{\sqrt{K}t} + \left(\frac{K\dot{x}_0 - \sqrt{K}g_0}{2K}\right) e^{-\sqrt{K}t}$$

$$\ddot{x} = \left(\frac{\sqrt{K}\dot{x}_0 + g_0}{2}\right) e^{\sqrt{K}t} - \left(\frac{\sqrt{K}\dot{x}_0 - g_0}{2}\right) e^{-\sqrt{K}t}$$

$$\dddot{x} = \left(\frac{K\dot{x}_0 + \sqrt{K}g_0}{2}\right) e^{\sqrt{K}t} + \left(\frac{K\dot{x}_0 - \sqrt{K}g_0}{2}\right) e^{-\sqrt{K}t}$$

They may be simplified to the following equations:

$$x = \frac{\sqrt{K}\dot{x}_0}{2K}\left(e^{\sqrt{K}t} - e^{-\sqrt{K}t}\right) + \frac{g_0}{2K}\left(e^{\sqrt{K}t} + e^{-\sqrt{K}t}\right) - \frac{g_0}{K}$$

$$= \frac{\sqrt{K}\dot{x}_0}{K} \sinh(\sqrt{K}t) + \frac{g_0}{K} \cosh(\sqrt{K}t) - \frac{g_0}{K}$$

$$\dot{x} = \frac{K\dot{x}_0}{2K}\left(e^{\sqrt{K}t} + e^{-\sqrt{K}t}\right) + \frac{\sqrt{K}g_0}{2K}\left(e^{\sqrt{K}t} - e^{-\sqrt{K}t}\right)$$

$$= \dot{x}_0 \cosh(\sqrt{K}t) + \frac{\sqrt{K}g_0}{K} \sinh(\sqrt{K}t)$$

$$\ddot{x} = \frac{\sqrt{K}\dot{x}_0}{2}\left(e^{\sqrt{K}t} - e^{-\sqrt{K}t}\right) + \frac{g_0}{2}\left(e^{\sqrt{K}t} + e^{-\sqrt{K}t}\right)$$

$$= \sqrt{K}\dot{x}_0 \sinh(\sqrt{K}t) + g_0 \cosh(\sqrt{K}t)$$

$$\dddot{x} = \frac{K\dot{x}_0}{2}\left(e^{\sqrt{K}t} + e^{-\sqrt{K}t}\right) + \frac{\sqrt{K}g_0}{2}\left(e^{\sqrt{K}t} - e^{-\sqrt{K}t}\right)$$

$$= K\dot{x}_0 \cosh(\sqrt{K}t) + \sqrt{K}g_0 \sinh(\sqrt{K}t)$$

The equation of motion of an object during the first of three contiguous intervals would thus be

$$x_A = \frac{\sqrt{K}\dot{x}_{0A}}{K} \sinh(\sqrt{K}t_A) + \frac{g_{0A}}{K} \cosh(\sqrt{K}t_A) - \frac{g_{0A}}{K}$$

The equation of motion during the second of the three contiguous intervals is

$$x_B = \frac{\sqrt{K}\dot{x}_{0B}}{K} \sinh(\sqrt{K}t_B) + \frac{g_{0B}}{K} \cosh(\sqrt{K}t_B) - \frac{g_{0B}}{K}$$

where

$$\dot{x}_{0B} = \dot{x}_A = \dot{x}_{0A} \cosh(\sqrt{K}t_A) + \frac{\sqrt{K}g_{0A}}{K} \sinh(\sqrt{K}t_A)$$

$$g_{0B} = \ddot{x}_A = \sqrt{K}\dot{x}_{0A} \sinh(\sqrt{K}t_A) + g_{0A} \cosh(\sqrt{K}t_A)$$

Thus, the equation for x_B is

$$x_B = \frac{\sqrt{K}}{K} \sinh(\sqrt{K}t_B)\left[\dot{x}_{0A}\cosh(\sqrt{K}t_A) + \frac{\sqrt{K}g_{0A}}{K}\sinh(\sqrt{K}t_A)\right]$$

$$+ \frac{1}{K}\cosh(\sqrt{K}t_B)\left[\sqrt{K}\dot{x}_{0A}\sinh(\sqrt{K}t_A) + g_{0A}\cosh(\sqrt{K}t_A)\right]$$

$$- \frac{1}{K}\left[\sqrt{K}\dot{x}_{0A}\sinh(\sqrt{K}t_A) + g_{0A}\cosh(\sqrt{K}t_A)\right]$$

$$= \frac{\sqrt{K}\dot{x}_{0A}}{K}\sinh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A) + \frac{g_{0A}}{K}\sinh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A)$$

$$+ \frac{\sqrt{K}\dot{x}_{0A}}{K}\cosh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A) + \frac{g_{0A}}{K}\cosh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A)$$

$$- \frac{\sqrt{K}\dot{x}_{0A}}{K}\sinh(\sqrt{K}t_A) - \frac{g_{0A}}{K}\cosh(\sqrt{K}t_A)$$

$$= \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A) + \cosh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A) - \sinh(\sqrt{K}t_A)\right]$$

$$+ \frac{g_{0A}}{K}\left[\sinh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A) + \cosh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A) - \cosh(\sqrt{K}t_A)\right]$$

This results in the following equation:

$$x_B = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh(\sqrt{K}t_A + \sqrt{K}t_B) - \sinh(\sqrt{K}t_A)\right]$$

$$+ \frac{g_{0A}}{K}\left[\cosh(\sqrt{K}t_A + \sqrt{K}t_B) - \cosh(\sqrt{K}t_A)\right]$$

The equation of motion during the last of the three intervals is

$$x_C = \frac{\sqrt{K}\dot{x}_{0C}}{K}\sinh(\sqrt{K}t_C) + \frac{g_{0C}}{K}\cosh(\sqrt{K}t_C) - \frac{g_{0C}}{K}$$

where

$$\dot{x}_{0C} = \dot{x}_B = \dot{x}_{0B} \cosh(\sqrt{K}t_B) + \frac{\sqrt{K}g_{0B}}{K} \sinh(\sqrt{K}t_B)$$

$$g_{0C} = \ddot{x}_B = \sqrt{K}\dot{x}_{0B} \sinh(\sqrt{K}t_B) + g_{0B} \cosh(\sqrt{K}t_B)$$

However,

$$\dot{x}_{0B} = \dot{x}_A = \dot{x}_{0A} \cosh(\sqrt{K}t_A) + \frac{\sqrt{K}g_{0A}}{K} \sinh(\sqrt{K}t_A)$$

$$g_{0B} = \ddot{x}_A = \sqrt{K}\dot{x}_{0A} \sinh(\sqrt{K}t_A) + g_{0A} \cosh(\sqrt{K}t_A)$$

Thus,

$$\dot{x}_{0C} = \left[\dot{x}_{0A} \cosh(\sqrt{K}t_A) + \frac{\sqrt{K}g_{0A}}{K} \sinh(\sqrt{K}t_A) \right] \cosh(\sqrt{K}t_B)$$

$$+ \frac{\sqrt{K}}{K} \sinh(\sqrt{K}t_B) \left[\sqrt{K}\dot{x}_{0A} \sinh(\sqrt{K}t_A) + g_{0A} \cosh(\sqrt{K}t_A) \right]$$

and

$$g_{0C} = \sqrt{K} \sinh(\sqrt{K}t_B) \left[\dot{x}_{0A} \cosh(\sqrt{K}t_A) + \frac{\sqrt{K}g_{0A}}{K} \sinh(\sqrt{K}t_A) \right]$$

$$+ \cosh(\sqrt{K}t_B) \left[\sqrt{K}\dot{x}_{0A} \sinh(\sqrt{K}t_A) + g_{0A} \cosh(\sqrt{K}t_A) \right]$$

Reducing, one finds that

$$\dot{x}_{0C} = \dot{x}_{0A} \cosh(\sqrt{K}t_A) \cosh(\sqrt{K}t_B) + \frac{\sqrt{K}g_{0A}}{K} \sinh(\sqrt{K}t_A) \cosh(\sqrt{K}t_B)$$

$$+ \dot{x}_{0A} \sinh(\sqrt{K}t_A) \sinh(\sqrt{K}t_B) + \frac{\sqrt{K}g_{0A}}{K} \sinh(\sqrt{K}t_B) \cosh(\sqrt{K}t_A)$$

$$\dot{x}_{0C} = \dot{x}_{0A}\left[\cosh(\sqrt{K}t_A)\cosh(\sqrt{K}t_B) + \sinh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A)\right]$$

$$+ \frac{\sqrt{K}g_{0A}}{K}\left[\sinh(\sqrt{K}t_A)\cosh(\sqrt{K}t_B) + \sinh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A)\right]$$

$$= \dot{x}_{0A}\cosh(\sqrt{K}t_A + \sqrt{K}t_B) + \frac{\sqrt{K}g_{0A}}{K}\sinh(\sqrt{K}t_A + \sqrt{K}t_B)$$

and

$$g_{0C} = \sqrt{K}\dot{x}_{0A}\sinh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A) + g_{0A}\sinh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A)$$

$$+ \sqrt{K}\dot{x}_{0A}\cosh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A) + g_{0A}\cosh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A)$$

$$= \sqrt{K}\dot{x}_{0A}\left[\sinh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A) + \cosh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A)\right]$$

$$+ g_{0A}\left[\sinh(\sqrt{K}t_B)\sinh(\sqrt{K}t_A) + \cosh(\sqrt{K}t_B)\cosh(\sqrt{K}t_A)\right]$$

$$= \sqrt{K}\dot{x}_{0A}\sinh(\sqrt{K}t_A + \sqrt{K}t_B) + g_{0A}\cosh(\sqrt{K}t_A + \sqrt{K}t_B)$$

Substituting these values in the original equation for x_C, one finds that

$$x_C = \frac{\sqrt{K}}{K}\sinh(\sqrt{K}t_C)\left[\dot{x}_{0A}\cosh(\sqrt{K}t_A + \sqrt{K}t_B) + \frac{\sqrt{K}g_{0A}}{K}\sinh(\sqrt{K}t_A + \sqrt{K}t_B)\right]$$

$$+ \frac{1}{K}\cosh(\sqrt{K}t_C)\left[\sqrt{K}\dot{x}_{0A}\sinh(\sqrt{K}t_A + \sqrt{K}t_B) + g_{0A}\cosh(\sqrt{K}t_A + \sqrt{K}t_B)\right]$$

$$- \frac{1}{K}\left[\sqrt{K}\dot{x}_{0A}\sinh(\sqrt{K}t_A + \sqrt{K}t_B) + g_{0A}\cosh(\sqrt{K}t_A + \sqrt{K}t_B)\right]$$

$$x_C = \frac{\sqrt{K}\dot{x}_{0A}}{K} \sinh(\sqrt{K}t_C) \cosh(\sqrt{K}t_A + \sqrt{K}t_B) + \frac{g_{0A}}{K} \sinh(\sqrt{K}t_C) \sinh(\sqrt{K}t_A + \sqrt{K}t_B)$$

$$+ \frac{\sqrt{K}\dot{x}_{0A}}{K} \cosh(\sqrt{K}t_C) \sinh(\sqrt{K}t_A + \sqrt{K}t_B) + \frac{g_{0A}}{K} \cosh(\sqrt{K}t_C) \cosh(\sqrt{K}t_A + \sqrt{K}t_B)$$

$$- \frac{\sqrt{K}\dot{x}_{0A}}{K} \sinh(\sqrt{K}t_A + \sqrt{K}t_B) - \frac{g_{0A}}{K} \cosh(\sqrt{K}t_A + \sqrt{K}t_B)$$

Further simplifying, one finds that

$$x_C = \frac{\sqrt{K}\dot{x}_{0A}}{K} \left[\sinh(\sqrt{K}t_C) \cosh(\sqrt{K}t_A + \sqrt{K}t_B) + \cosh(\sqrt{K}t_C) \sinh(\sqrt{K}t_A + \sqrt{K}t_B) - \sinh(\sqrt{K}t_A + \sqrt{K}t_B) \right]$$

$$+ \frac{g_{0A}}{K} \left[\sinh(\sqrt{K}t_C) \sinh(\sqrt{K}t_A + \sqrt{K}t_B) + \cosh(\sqrt{K}t_C) \cosh(\sqrt{K}t_A + \sqrt{K}t_B) - \cosh(\sqrt{K}t_A + \sqrt{K}t_B) \right]$$

$$= \frac{\sqrt{K}\dot{x}_{0A}}{K} \left[\sinh(\sqrt{K}t_A + \sqrt{K}t_B + \sqrt{K}t_C) - \sinh(\sqrt{K}t_A + \sqrt{K}t_B) \right]$$

$$+ \frac{g_{0A}}{K} \left[\cosh(\sqrt{K}t_A + \sqrt{K}t_B + \sqrt{K}t_C) - \cosh(\sqrt{K}t_A + \sqrt{K}t_B) \right].$$

Thus, the resulting system is

$$x_A = \frac{\sqrt{K}\dot{x}_{0A}}{K} \left[\sinh\sqrt{K}(t_A) \right] + \frac{g_{0A}}{K} \left[\cosh\sqrt{K}(t_A) - 1 \right]$$

$$x_B = \frac{\sqrt{K}\dot{x}_{0A}}{K} \left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A) \right] + \frac{g_{0A}}{K} \left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A) \right]$$

$$x_C = \frac{\sqrt{K}\dot{x}_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$+ \frac{g_{0A}}{K}\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right]$$

APPENDIX B

MEMORANDUM CONCERNING ERROR ANALYSIS TECHNIQUE

The following is the text of a memorandum by Charles C. Dalton on the subject "Error Propagation in Gravimetry Experiments." This information was provided in response to a specific request by the authors of this paper for technical assistance.

The mathematical aspects of the problem, as it now is, is that three parameters to be calculated, say x, y, and z for initial velocity, gravity value at $t = 0$ and gravity gradient, are related to six measured parameters, say a, b, c, d, e, and f by three simultaneous equations of some considerable complication, say

$$F(x, y, z, a, b, c, d, e, f) = 0 \tag{1}$$

$$G(x, y, z, a, b, c, d, e, f) = 0 \tag{2}$$

$$H(x, y, z, a, b, c, d, e, f) = 0 \tag{3}$$

You want to solve the equations for x, y, and z and to compute the random error of each propagated from the measured parameters.

An explicit solution for x, y, and z might be complicated and difficult to find or even not possible. But the given implicit equations can be programmed for a numerical solution to whatever accuracy is required; e.g., geometrically each of the equations can be considered a surface in the three dimensions x, y, and z. The solution, which for physical reasons is known to exist, must be one of the possibly several points of intersection of the three surfaces. Calculating the propagated error does not require the solution to be explicit.

The six measured parameters are three measurements of time and three measurements of distance in terms of wavelengths of the laser light. So far as I can tell. . . the random errors of the six measured parameters can be considered effectively both statistically independent and all normally distributed. . . . [Some further thought should be given to this point.]

The formulation for the propagation of error parallels that of the total differential; i.e.,

$$dx = \frac{\partial x}{\partial a} da + \frac{\partial x}{\partial b} db + \frac{\partial x}{\partial c} dc + \frac{\partial x}{\partial d} dd + \ldots$$

where an acceptable approximation is to replace each of the differentials by corresponding small finite "delta" increments. But this presupposes that both the (arithmetic) magnitude and the algebraic sign of the increments of the independent parameters are known, whereas the random errors in specific measurements are unknown in both aspects. Therefore, with statistical independence, the different terms of the total differential are added like orthogonal vector components by squaring each component; and the finite "delta" increments can each be expressed as the same small fraction Δ of the corresponding standard deviation σ, say $\Delta\sigma_a$, $\Delta\sigma_b$, etc. Then, by multiplying both sides of the equation by Δ^{-2},

$$\sigma_x{}^2 = (\partial x/\partial a)^2 \sigma_a{}^2 + (\partial x/\partial b)^2 \sigma_b{}^2 + (\partial x/\partial c)^2 \sigma_c{}^2 + \ldots \qquad (4)$$

and similarly for y and z. But without statistical independence there would also be other terms involving correlation factors. The result, rigorous only for linear functions, is an approximation which omits, usually, smaller terms with higher derivatives.

In principle, the six partial derivatives in equation (4) could be evaluated by giving small increments alternately to the six independent measured parameters and noting the change to the solution for x, y, and z by the given simultaneous equations (1) through (3). Undoubtedly this would require special double precision programming and more computation than would be required by solving for each of the six measured parameters a, b, . . . a set of three linear equations relating the partial derivatives of the three dependent variables x, y, and z with respect to the particular dependent variable, e.g., a; i.e., by partial differentiation of equation (1),

. . . .

$$\frac{\partial x}{\partial a} = \frac{\dfrac{\partial F}{\partial a} + (\partial F/\partial y)\dfrac{\partial y}{\partial a} + (\partial F/\partial z)\dfrac{\partial z}{\partial a}}{-\dfrac{\partial F}{\partial x}}$$

If the correctness of this manner of differentiation of an implicit function is not obvious, it may be helpful to rewrite the same result in symmetric form; i.e.,

$$(\partial F/\partial x)\frac{\partial x}{\partial a} + (\partial F/\partial y)\frac{\partial y}{\partial a} + (\partial F/\partial z)\frac{\partial z}{\partial a} + \frac{\partial F}{\partial a} = 0 \quad .$$

By differentiating equations (2) and (3) similarly and switching to subscripts to denote partial differentiation, say

$$F_x \text{ for } \frac{\partial F}{\partial x} \quad \text{and} \quad x_a \text{ for } \frac{\partial x}{\partial a}$$

one gets three linear equations for the partial derivatives of the three variables x, y, and z with respect to the measured parameter a; i.e., in matrix notation

$$\begin{bmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = - \begin{bmatrix} F_a \\ G_a \\ H_a \end{bmatrix} \tag{5}$$

and similarly for each of the other five measured parameters b, c, etc., by replacing a. Note that the coefficient matrix is the same for all of the measured parameters a, b, etc.

APPENDIX C

EXPRESSIONS FOR TERMS INVOLVED IN ERROR ANALYSIS

The terms involved in error analysis are as follows.

$$\frac{\partial F}{\partial K} = \frac{\sqrt{K}}{2K^2}\left[g_{0A}\,t_A - \dot{x}_{0A}\right]\left[\sinh\sqrt{K}(t_A)\right] + \left[\frac{\dot{x}_{0A}}{2K}\,t_A - \frac{g_{0A}}{K^2}\right]\left[\cosh\sqrt{K}(t_A)\right] + \frac{g_{0A}}{K^2}$$

$$\frac{\partial G}{\partial K} = \frac{\sqrt{K}}{2K^2}\left[g_{0A}\,t_A - \dot{x}_{0A}\right]\left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)\right]$$

$$+ \left[\frac{\dot{x}_{0A}}{2K}\,t_A - \frac{g_{0A}}{K^2}\right]\left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)\right]$$

$$+ \frac{t_B}{2K}\left[\dot{x}_{0A}\cosh\sqrt{K}(t_A + t_B) + \frac{\sqrt{K}g_{0A}}{K}\sinh\sqrt{K}(t_A + t_B)\right]$$

$$\frac{\partial H}{\partial K} = \frac{\sqrt{K}}{2K^2}\left[g_{0A}(t_A + t_B) - \dot{x}_{0A}\right]\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$+ \left[\frac{\dot{x}_{0A}}{2K}(t_A + t_B) - \frac{g_{0A}}{K^2}\right]\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right]$$

$$+ \frac{t_C}{2K}\left[\dot{x}_{0A}\cosh\sqrt{K}(t_A + t_B + t_C) + \frac{\sqrt{K}g_{0A}}{K}\sinh\sqrt{K}(t_A + t_B + t_C)\right]$$

$$\frac{\partial F}{\partial \dot{x}_{0A}} = \frac{\sqrt{K}}{K}\left[\sinh\sqrt{K}(t_A)\right]$$

$$\frac{\partial G}{\partial \dot{x}_{0A}} = \frac{\sqrt{K}}{K}\left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)\right]$$

$$\frac{\partial H}{\partial \dot{x}_{0A}} = \frac{\sqrt{K}}{K}\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$\frac{\partial F}{\partial g_{0A}} = \frac{1}{K}\left[\cosh\sqrt{K}(t_A) - 1\right]$$

$$\frac{\partial G}{\partial g_{0A}} = \frac{1}{K}\left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)\right]$$

$$\frac{\partial H}{\partial g_{0A}} = \frac{1}{K}\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right]$$

$$\frac{\partial F}{\partial x_A} = -1 \qquad \frac{\partial F}{\partial x_B} = 0 \qquad \frac{\partial F}{\partial x_C} = 0$$

$$\frac{\partial G}{\partial x_A} = 0 \qquad \frac{\partial G}{\partial x_B} = -1 \qquad \frac{\partial G}{\partial x_C} = 0$$

$$\frac{\partial H}{\partial x_A} = 0 \qquad \frac{\partial H}{\partial x_B} = 0 \qquad \frac{\partial H}{\partial x_C} = -1$$

$$\frac{\partial F}{\partial t_A} = \dot{x}_{0A}\left[\cosh\sqrt{K}(t_A)\right] + \frac{\sqrt{K}g_{0A}}{K}\left[\sinh\sqrt{K}(t_A)\right]$$

$$\frac{\partial G}{\partial t_A} = \dot{x}_{0A}\left[\cosh\sqrt{K}(t_A + t_B) - \cosh\sqrt{K}(t_A)\right] + \frac{\sqrt{K}g_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B) - \sinh\sqrt{K}(t_A)\right]$$

$$\frac{\partial H}{\partial t_A} = \dot{x}_{0A}\left[\cosh\sqrt{K}(t_A + t_B + t_C) - \cosh\sqrt{K}(t_A + t_B)\right] + \frac{\sqrt{K}g_{0A}}{K}\left[\sinh\sqrt{K}(t_A + t_B + t_C) - \sinh\sqrt{K}(t_A + t_B)\right]$$

$$\frac{\partial F}{\partial t_B} = 0$$

$$\frac{\partial G}{\partial t_B} = \dot{x}_{0A} \left[\cosh \sqrt{K}(t_A + t_B)\right] + \frac{\sqrt{K} g_{0A}}{K} \left[\sinh \sqrt{K}(t_A + t_B)\right]$$

$$\frac{\partial H}{\partial t_B} = \dot{x}_{0A} \left[\cosh \sqrt{K}(t_A + t_B + t_C) - \cosh \sqrt{K}(t_A + t_B)\right]$$

$$+ \frac{\sqrt{K} g_{0A}}{K} \left[\sinh \sqrt{K}(t_A + t_B + t_C) - \sinh \sqrt{K}(t_A + t_B)\right]$$

$$\frac{\partial F}{\partial t_C} = 0$$

$$\frac{\partial G}{\partial t_C} = 0$$

$$\frac{\partial H}{\partial t_C} = \dot{x}_{0A} \left[\cosh \sqrt{K}(t_A + t_B + t_C)\right] + \frac{\sqrt{K} g_{0A}}{K} \left[\sinh \sqrt{K}(t_A + t_B + t_C)\right]$$

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GRAVITY GRADIENT MEASUREMENTS WITH A LASER ABSOLUTE GRAVIMETER

By R. C. Borden and O. K. Hudson

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