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PREDICTING PLL THRESHOLD BEHAVIOR WITH SINUSOIDAL AND GAUSSIAN MODULATION USING THE RICE - RIDGEWAY CRITERIA

by

Frank J. Loch

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The equation derived by S. O. Rice in his classic paper relating $(S/N)_o$ to $(S/N)_{in}$ is as follows:

$$S_o/N_o = \frac{3\rho\, B_{IF}{}^3 (2f_a)^{-3}}{\rho\sqrt{3}\,(1 - \text{erf}\sqrt{\rho})\left(\frac{B_{IF}}{f_a}\right)^2 + 1} \qquad (1)$$

This equation is valid for the case where $\frac{\Delta f}{f_a} > 5$, and the predetection bandwidth B_{IF} is given by:

$$B_{IF} = 2\Delta f$$

However, for the cases of interest $\frac{\Delta f}{f_a} < 5$ and the contribution of the peak modulating frequency f_a must be considered. Therefore, $B_{IF} = 2[\Delta f + f_a]$ and Equation (1) becomes:

$$S_o/N_o = \frac{\frac{3}{2}\rho\,\frac{B_{IF}}{f_a}\left(\frac{\Delta f}{f_a}\right)^2}{\rho\sqrt{3}\left(\frac{B_{IF}}{f_a}\right)^2 (1 - \text{erf}\sqrt{\rho}) + 1} \qquad (2)$$

Equations (1) and (2) were derived on the assumption that the input signal + noise is given by:

$$\text{Input} = Q \cos\left(\omega_c t - \frac{A}{\omega_s}\cos \omega_s t\right) + I_N \qquad (3)$$

and

$$I_N = I_C \cos \omega_c t - I_S \sin \omega_c t \qquad (4)$$

where I_C and I_S are the in-phase and quadrature phase components of the noise I_N with respect to the carrier frequency $\frac{\omega_c}{2\pi}$.

Also, the assumption that the signal amplitude A = 0 was made so that Equation (3) reduces to:

$$\text{Input} = Q \cos \omega_c t + I_N \tag{5}$$

The generalized version of Equation (2) can be written as:

$$S_o/N_o = \frac{\frac{3}{2} \rho \frac{B_{IF}}{f_a} \left(\frac{\Delta f}{f_a}\right)^2}{\rho \left(\frac{B_{IF}}{f_a}\right)^2 K \left(\frac{N^+}{B_{IF}}\right) + C} \tag{6}$$

where N^+ is the number of positive clicks per second, and K is a constant. By referring to Equation (2) we can see that for the unmodulated case, where the signal amplitude is 0, K = 12, and

$$\frac{N^+}{B_{IF}} = (1 - \operatorname{erf}\sqrt{\rho})\left(\frac{1}{4\sqrt{3}}\right) \tag{7}$$

This expression for $\frac{N^+}{B_{IF}}$ is obtained from Rice's derivation of N^+;

$$N^+ = \frac{r}{2} (1 - \operatorname{erf}\sqrt{\rho}) \tag{8}$$

where r is defined as being the radius of gyration of the power spectrum w(f) about its axis of symmetry, $f = f_c$.

$$r = \frac{1}{2\pi} \left(\frac{b_2}{b_o}\right)^{1/2} \tag{9}$$

$$b_o = \overline{I_N^2} = \overline{I_C^2} = \overline{I_S^2} = \int_0^\infty w(f)df = \int_0^\infty 2w(f_c + f)df \tag{10}$$

$$b_2 = (2\pi)^2 \int_0^\infty (f - f_c)^2 \, w(f)df \tag{11}$$

Assuming a rectangular filter of bandwidth B_{IF} cps centered on f_c, then it follows that:

$$w(f) = w_o \quad \text{for} \quad f_c - B_{IF}/2 < f < f_c + B_{IF}/2$$

and

$$w(f) = 0 \text{ elsewhere.}$$

$$b_o = \int_{f_c - B_{IF}/2}^{f_c + B_{IF}/2} w_o \, df = w_o[f_c + B_{IF}/2 - f_c + B_{IF}/2] = w_o B_{IF} \tag{12}$$

$$b_2 = 4\pi^2 \int_{f_c - B_{IF}/2}^{f_c + B_{IF}/2} w_o(f - f_c)^2 \, df = 4\pi^2 \, w_o \left. \frac{(f - f_c)^3}{3} \right|_{f_c - B_{IF}/2}^{f_c + B_{IF}/2}$$

$$b_2 = \frac{4\pi^2}{3} w_o[(f_c + B_{IF}/2 - f_c)^3 - (f_c - B_{IF}/2 - f_c)^3]$$

$$= \frac{4}{3} \pi^2 w_o[(B_{IF}/2)^3 - (-\frac{B_{IF}}{2})^3] = \frac{4}{3} \pi^2 w_o(\frac{1}{2})^3 [2B_{IF}{}^3]$$

$$= \frac{4\pi^2 w_o(2)}{(3)(8)} B_{IF}{}^3 = \frac{\pi^2 w_o B_{IF}{}^3}{3} \tag{13}$$

$$\therefore r = \frac{1}{2\pi}\left(\frac{b2}{b_o}\right)^{1/2} \tag{14}$$

$$= \frac{1}{2\pi}\left(\frac{\dfrac{\pi^2 w_o B_{IF}{}^3}{3}}{(w_o B_{IF})^{1/2}}\right)^{1/2} = \frac{1}{2\pi}\pi\left(\frac{w_o}{w_o}\right)^{1/2} B_{IF}\frac{1}{\sqrt{3}}$$

$$r = \frac{B_{IF}}{2\sqrt{3}} \tag{15}$$

$$\therefore N^+ = \frac{r}{2}(1 - \mathrm{erf}\sqrt{\rho}) = \frac{B_{IF}}{4\sqrt{3}}(1 - \mathrm{erf}\sqrt{\rho}) \tag{16}$$

and

$$\frac{N^+}{B_{IF}} = \frac{1}{4\sqrt{3}}(1 - \mathrm{erf}\sqrt{\rho}) \tag{17}$$

For the case of sinusoidal modulation the expression for the number of positive clicks per second, as derived by Rice, is as follows:

$$N^+ \approx \frac{A}{2\pi}\frac{e^{-\rho}}{\pi} + \frac{r\, e^{-\rho}}{(4\pi\ \rho)^{1/2}}\, e^{-a\rho}\, I_O(a\rho) \tag{18}$$

where

$$a = \frac{(A/2\pi)^2}{2r^2} = \left[\frac{\mathrm{rms}(\phi'/2\pi)}{r}\right]^2 \tag{19}$$

and

$$\phi' = A \sin \omega_s t \tag{20}$$

A maximum frequency deviation is desired which will make the carrier fill up the entire rectangular input bandwidth. This is accomplished by letting

$$A/2\pi = B_{IF}/2 \tag{21}$$

Therefore,

$$a = \frac{(B_{IF}/2)^2}{2r^2} \tag{22}$$

and

$$N^+ \sim \frac{B_{IF}}{2}\,\frac{e^{-\rho}}{\pi} + \frac{r\,e^{-\rho}}{(4\pi\rho)^{1/2}}\quad e^{-a\rho}\, I_0(a\rho) \tag{23}$$

Since $r = \frac{B_{IF}}{2\sqrt{3}}$, we can write:

$$N^+ \sim \frac{B_{IF}\,e^{-\rho}}{2\pi} + \frac{B_{IF}\,e^{-\rho}}{(48\pi\rho)^{1/2}}\quad e^{-a\rho}\, I_0\,(a\rho) \tag{24}$$

$$\therefore\ \frac{N^+}{B_{IF}} \sim \frac{e^{-\rho}}{2\pi} + \frac{e^{-\ (1+a)}}{(48\pi\rho)1/2}\ I_0(a\rho) \tag{25}$$

Further simplification can be made by noting that:

$$a = \frac{B_{IF}^2/4}{2r^2} = \frac{B_{IF}^2/4}{2B_{IF}^2/12} = \quad = \frac{3}{2} \tag{26}$$

Finally we have,

$$\frac{N^+}{B_{IF}} \approx \frac{e^{-\rho}}{2\pi}\ \left[1 + \left(\frac{\pi}{12\rho}\right)^{1/2} e^{-3\rho/2}\, I_0\,\left(\frac{3\rho}{2}\right)\right] \tag{27}$$

The exact expression for $\frac{N^+}{B_{IF}}$ is plotted in Rice's classic paper on noise in FM receivers. Since the graph of the expression is extremely linear down to a carrier to noise ratio $\rho = .5$, a simplified but accurate equation can be derived which can replace Equation (27) for use in a computer program, etc.

Referring to Figure IV, page 406 of Rice's paper Noise in FM Receivers, let:

$$y = K X + b \tag{28}$$

$$y = \log \frac{N^+}{B_{IF}} \tag{29}$$

$$\log \frac{N^+}{B_{IF}} = K X + b \tag{30}$$

$$X = \rho \tag{31}$$

$$(y_1 \; x_1) = (.003, 4)$$

$$(y_{21} \; x_2) = (.06, 1)$$

$$\log (.003) = 4K + b$$

$$\underline{-\log (.06) = -K - b}$$

$$\log (.003) - \log (.06) = 3K$$

$$3K = \log \left(\frac{.003}{.06}\right) = \log .05 = \log \frac{1}{20} = -1.3$$

$$\therefore \underline{K = -.434}$$

$$\log\left(\frac{N^+}{B_{IF}}\right) = -.434\ \rho + b$$

$$\log\ (.06) = -.434 + b$$

$$-1.22 = -.434 + b$$

$$\underline{b = -.788}$$

$$\therefore \log\left(\frac{N^+}{B_{IF}}\right) = -(.434\ \rho + .788) \tag{32}$$

or

$$\frac{N^+}{B_{IF}} = +\log^{-1}-(.434\ \rho + .788) \tag{33}$$

Referring to Equation (6), for the case of sinusoidal modulation, we can write:

$$S_o/N_o = \frac{3/2\ \rho\ \frac{B_{IF}}{f_a}\left(\frac{\Delta f}{f_a}\right)^2}{+12\ \rho\left(\frac{B_{IF}}{f_a}\right)^2 \log^{-1}-(.434\ \rho + .788) + c} \tag{34}$$

where

$$c = [J_o{}^2 + J_5{}^2 + 2(J_1{}^2 + J_2{}^2 + J_3{}^2 + J_4{}^2\ )] \tag{35}$$

For the case of a signal ϕ' having Gaussian statistics with zero average and variance $\overline{\phi'^2}$ the expression for N^+ as derived by Rice is as follows:

$$N^+ = N^- \sim r\ e^{-\rho}\left(\frac{1 + 2\ a\rho}{4\pi\rho}\right)^{1/2} \tag{36}$$

where

$$a = \frac{\overline{\phi'^2}}{(2\pi\mu)^2} \tag{37}$$

$$\frac{N^+}{B_{IF}} = e^{-\rho} \left[\frac{1 + 2\ a\rho}{48\pi\rho}\right]^{1/2} \tag{38}$$

Predictions of threshold performance using the Rice-Ridgeway criteria have shown that there is a noticeable improvement in the comparison between the theoretical and experimental curves when the click-noise due to modulation is taken into account. Figure 1 shows the results of this comparison. To summarize, the equation used to predict S/N behavior for the case of sinusoidal modulation and using the Rice-Ridgeway criteria is:

$$S_o/N_o = \frac{3/2\ \rho\ \frac{B_{IF}}{f_a} \left[\frac{\Delta}{f_a}\right]^2}{-12\rho \left[\frac{B_{IF}\ B_D}{f_a^{\ 2}}\right] \log^{-1}(.434\ \rho + .788) + c} \tag{39}$$

For the case of a Gaussian noise type signal the equivalent expression is:

$$S_o/N_o = \frac{3/2\ \rho\ \frac{B_{IF}}{f_a} \left[\frac{\Delta f}{f_a}\right]^2}{12\ \rho \left[\frac{B_{IF}\ B_D}{fa^2}\right] e^{-\rho} \left[\frac{1 + 2\ a\rho}{48\ \pi\rho}\right]^{1/2} + c} \tag{40}$$

When sinusoidal modulation is used, the increase in the number of clicks/second can be shown by the ratio:

$$\frac{N+ \text{ for } A = 0}{N+ \text{ for } A \neq 0} \approx 2\sqrt{3} \left(\frac{\rho}{\pi}\right)^{1/2} + e^{-a\rho}\ I_0(a\rho) \tag{41}$$

It is concluded that the contribution of modulation to click noise is significant below the "knee" of the $(S/N)_o$ versus (Signal $Power_{in}$) curve. From Figure 1 it can be seen that the predicted curve for the case where $A \neq 0$ (Equation 39) is considerably closer to the measured data than the curve for the unmodulated case. Figure 2 shows that the predicted curve for no modulation gives a lower value for threshold than the measured data. The theoretical curve for sinusoidal modulation gives a very accurate prediction of threshold and is also closer to the measured data below threshold.