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A HYBRID METHOD FOR THE OPTIMAL LINEAR CONTROL OF NONLINEAR SYSTEMS

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SUMMARY

The relative advantages and disadvantages of indirect and direct optimization methods have been known for some time. This development illustrates the compatibility between a perturbation method (indirect) and a gradient method (direct). Moreover, the investigation reveals exactly how estimates of the initial Lagrange multipliers, needed for the indirect method, may be obtained from the direct method. After several iterations with the direct method, the initial value of the Lagrange multipliers may be improved to the point that convergence can be achieved with the indirect method.

INTRODUCTION

In recent years, considerable interest has been generated in methods for solving the nonlinear two-point boundary value problem. These methods are quite naturally categorized as either direct or indirect. The indirect methods seek to satisfy the conditions of mathematical optimality, that is, the necessary conditions resulting from the first variation of the functional to be extremized. The classical optimality conditions are satisfied identically, and iterations are continued until the desired terminal constraints are satisfied. The indirect methods that have been successfully implemented are those proposed by Breakwell et al[1], Jazwinski[2], Kenneth and McGill[3], and Lewallen[4].

On the other hand, the direct methods seek to improve an assumed control program, with the aid of influence function concepts, by changing the control program in such a manner that some index of performance is extremized and the desired terminal constraints are approached. The classical optimality conditions are not satisfied identically, and therefore the optimal trajectory is only approximated. The direct methods that have been successfully implemented are those proposed by Kelley[5], Bryson and Denham[6], 1616y et al[7][8], McReynolds[9], and Gottlieb[10].

The major objection to the indirect methods is that the initially assumed Lagrange multipliers must be iterated upon, and often the first guess for their values is so poor that convergence is never achieved. However, if convergence occurs, it does so quadratically and the optimal or Eulerian control program may be easily evaluated. The major objection to the direct methods is that the Eulerian control is only approximated. Experience has revealed that even though the terminal constraints are adequately satisfied and the performance index is extremized, the approximated control program is often significantly different from the Eulerian. However, the convergence process will usually begin from almost any reasonable control program assumption.

Clearly, a hybrid approach, combining the best features of the indirect and direct methods, would be of significant value. In the discussion that follows, the perturbation method proposed in Reference 1 is shown to be compatible with the gradient method proposed in Reference 6. In addition, a hybrid method is proposed which illustrates how initial values of Lagrange multipliers may be obtained for the indirect method by using the direct method.

NECESSARY CONDITIONS FOR AN OPTIMAL PROCESS

In general, the optimal control process can be stated as follows: Determine the m-vector of control variables $u(t)$ in the interval $t_o \leq t \leq t_f$ such that a scalar performance index of the form

$$I = \phi(x_f, t_f) \tag{1}$$

is minimized, while the n-vector of initial conditions

$$x(t_o) = x_o \tag{2}$$

at a known t_o, and the q-vector of terminal conditions

$$M(x_f, t_f) = 0 \tag{3}$$

at an unknown t_f, and the n first-order, nonlinear, differential equations

$$\dot{x} = f(x,u,t) \tag{4}$$

are satisfied.

The necessary conditions required for the accomplishment of the above stated objective are discussed by Tapley and Lewallen[11]. These conditions may be summarized as follows. In the interval of interest,

$$\left.\begin{aligned} \dot{x} &= H_\lambda^T(x,u,\lambda,t) \\ \dot{\lambda} &= -H_x^T(x,u,\lambda,t) \\ 0 &= H_u^T(x,u,\lambda,t) \end{aligned}\right\} \tag{5}$$

At the unknown terminal time,

$$\left.\begin{aligned} M(x_f,\ t_f) &= 0 \\ N(x_f,\ \lambda_f,\ \eta,\ t_f) = \left(P_x - \lambda^T\right)_f &= 0 \\ R(x_f,\ \lambda_f,\ \eta,\ t_f) = (P_t + H)_f &= 0 \end{aligned}\right\} \tag{6}$$

The scalar functions P and H are defined as

$$P = \phi(x_f,\ t_f) + \eta^T M(x_f,\ t_f) \tag{7}$$

$$H = \lambda^T f(x,u,t) \tag{8}$$

where H is referred to as the generalized Hamiltonian, and η is a q-vector of Lagrange multipliers.

With the indirect methods, it is usually assumed that a well defined minimum of $H(x,u,\lambda,t)$ exists so that $H_u = 0$ and H_{uu} is positive definite. With these assumptions, the condition $H_u = 0$ yields m algebraic equations which can be used to eliminate the m control variables in Equations (5-a) and (5-b). The results can be expressed as

$$\dot{x} = H_\lambda^T(x,\lambda,t) \qquad \dot{\lambda} = -H_x^T(x,\lambda,t) \tag{9}$$

where $H = H[x,u(x,\lambda,t),\lambda,t]$. Equations (2), (6) and (9) lead to a conventional two-point boundary-value problem. If the 2n-vectors z and $F(z,t)$ are defined,

$$z^T = \left[x^T \mid \lambda^T\right] \qquad F^T = \left[H_\lambda \mid -H_x\right] \tag{10}$$

then Equations (9) can be expressed as the 2n-vector

$$\dot{z} = F(z,t) \tag{11}$$

Furthermore, Equations (6) define the terminal boundary conditions for Equation (11) to be

$$h(z_f, t_f) = 0 \tag{12}$$

where h is an $n+q+1$ vector. These $n+q+1$ conditions may be used to determine the n values of $\lambda(t_o)$, the q values of η , and the one value of t_f .

INDIRECT OPTIMIZATION METHOD

The indirect optimization methods seek to satisfy the above stated necessary conditions required for optimality, that is, Equations (2), (5), and (6). In implementing the perturbation methods, as proposed in References 1 and 2, Equations (2) and (5) are satisfied identically, and iterations are continued until Equations (6) are satisfied. If convergence occurs, the satisfaction of these terminal constraints is usually approached quadratically. In implementing the quasilinearization methods, as proposed in References 3 and 4, Equations (2) and (6) may be satisfied identically, and iterations are continued on a linearized version of Equations (5) until the Equations (5) themselves are satisfied. In the following investigation, the only indirect method considered will be the perturbation method.

The perturbation methods require a reference solution from which to begin. The equations that describe this reference trajectory are given in Equation (9). Since the initial state is given in Equation (2), a solution may be generated by assuming n initial Lagrange multipliers, $\lambda(t_o)$, and integrating Equations (9) forward. This integration is continued until some assumed terminal time is reached. An estimate of this terminal time may be made by determining the time when one of the specified constraints is satisfied identically. After the terminal time is reached and before the correction for the next iteration is made, q values of η must be determined.

In addition to the above assumptions, some consideration must be given to the behavior of trajectories near the reference path. The study of these nearby trajectories require the perturbation of the equations that define a reference solution, that is, Equations (5) must be perturbed. This operation leads to

$$\delta\dot{x} = H_{\lambda x}\,\delta x + H_{\lambda u}\,\delta u \tag{13}$$

$$\delta\dot{\lambda} = -H_{xx}\,\delta x - H_{xu}\,\delta u - H_{x\lambda}\,\delta\lambda \tag{14}$$

$$0 = H_{ux}\,\delta x + H_{uu}\,\delta u + H_{u\lambda}\,\delta\lambda \tag{15}$$

where the coefficients must be evaluated on the reference path. These relations are simply a set of linearized equations which describe possible perturbations to the reference trajectory.

Equation (15) may be solved for the control variation

$$\delta u = -H_{uu}^{-1}\,(H_{ux}\,\delta x + H_{u\lambda}\,\delta\lambda) \tag{16}$$

provided H_{uu} is nonsingular. This variation is eliminated from Equations (13) and (14) to provide

$$\begin{bmatrix} \delta\dot{x} \\ \hline \delta\dot{\lambda} \end{bmatrix} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & -A_1^T \end{array}\right] \begin{bmatrix} \delta x \\ \hline \delta\lambda \end{bmatrix} \tag{17}$$

where

$$A_1 = H_{\lambda x} - H_{\lambda u}\,H_{uu}^{-1}\,H_{ux}$$

$$A_2 = -H_{\lambda u}\,H_{uu}^{-1}\,H_{u\lambda}$$

$$A_3 = -H_{xx} + H_{xu}\,H_{uu}^{-1}\,H_{ux}\ .$$

These $2n$ first-order differential equations may be used to describe the variations in the x and λ histories due to control variations. In order to relate δx and $\delta\lambda$ at t_f to the corrections $\delta\lambda_o$ at dt_o, the properties of a linear system of ordinary differential equations are used. Let $\Phi(t)$ be a $2n \times n$ matrix whose columns are solutions of Equation (17), hence

$$\begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} = \Phi(t) = \begin{bmatrix} \Phi_x(t) \\ \Phi_\lambda(t) \end{bmatrix} \delta\lambda(t_o) \tag{18}$$

where $\Phi^T(t_o) = [0_{n\times n} \vdots I_{n\times n}]$. The matrix $\Phi(t)$ simply represents solutions that result when a unit perturbation is made in the unknown initial conditions.

Since these perturbed trajectories are considered, allowances must be made for the perturbed path to miss the desired terminal constraints given in Equations (6). Hence, the reference terminal constraints are perturbed to yield

$$dM = M_x\,\delta x + \mathring{M}\,dt\Big|_{t_f} = 0 \tag{19}$$

$$dN = \delta\lambda - P_{xx}\,\delta x - M_x^T\,d\eta - \left(H_x^T - P_{xx}f - P_{xt}\right) dt\Big|_{t_f} = 0 \tag{20}$$

$$dR = r^T\delta\lambda + (H_x + P_{xt})\,\delta x + M_t^T\,d\eta + (H_t + P_{tx}f + P_{tt})\,dt\Big|_{t_f} = 0 \tag{21}$$

where the first-order approximation $d(\ ) = \delta(\ ) + (\dot{\ })\,dt$ has been used to relate total changes in a quantity $(\ )$ to variations in $(\ )$. During the initial iterations, the Equations (19), (20) and (21) are not necessarily satisfied; the associated dissatisfaction or error is denoted by evaluating M, N and R, respectively. Before N and R may be evaluated, however, a nominal value of η must be determined. This is accomplished by solving the first q of the n equations represented by N for the q values of η.

To determine the $n+q+1$ corrections for $\lambda(t_o)$, η, and t_f, Equations (18), (19), (20) and (21) may be combined to yield,

$$\begin{bmatrix} dM \\ \hline dN \\ \hline dR \end{bmatrix} = \left[\begin{array}{c:c:c} M_x\,\Phi_x & 0 & \dot{M} \\ \hline \Phi_\lambda - P_{xx}\,\Phi_x & -M_x^T & \dot{P}_x - H_x^T \\ \hline f^T\Phi_\lambda + (H_x + P_{xt})\Phi_x & M_t^T & \dot{P}_t + H_t \end{array}\right]_f \begin{bmatrix} \delta\lambda_o \\ \hline d\eta \\ \hline dt_f \end{bmatrix} \qquad (22)$$

where $dN^T = [00\cdots0 \;\; dN_{q+1}\cdots dN_n]$. The first q elements of this vector are set equal to zero because the first q equations of N have been satisfied identically to determine the nominal values of η.

In problems where the constraints are relatively simple, the Lagrange multipliers η are eliminated from the start, that is, the terminal constraint M is not included in Equation (7). This requires the elimination of the dependent variations in the transversality condition

$$\left(P_x - \lambda^T\right)_f dx_f + \left(P_t + H\right)_f dt_f = 0 \tag{23}$$

that result from the first variation. This is accomplished by perturbing the desired terminal constraint relation, Equation (3), to produce

$$dM = \left(M_x\right)_f dx_f + \left(M_t\right)_f dt_f = 0 \tag{24}$$

Now, q of the dx_f and dt_f in Equation (24) are solved for in terms of the remaining $n-q$ variations. These q variations are eliminated from the $n+1$ variations in Equation (23) leaving only $n+1-q$ independent variations. The coefficients of the $n+1-q$ variations are equated to zero, and along with the q relations in Equation (3), provide $n+1$ conditions for the $n+1$ values of λ_o and t_f. If the above-discussed $n+1$ terminal conditions become the elements of an $n+1$ vector h, the equation analogous to Equation (22) is seen to be

$$[dh] = \left[h_x \; \Phi_x \;\middle|\; h_\lambda \; \Phi_\lambda\right] \begin{bmatrix} \delta\lambda_o \\ \hline dt_f \end{bmatrix} \tag{25}$$

The computational procedure for the indirect optimization method with the Lagrange multipliers eliminated from the start may be summarized as follows:

(1) Integrate the $2n$ nonlinear differential equations of motion and the Euler-Lagrange equations, Equation (11), forward from t_o to an assumed t_f with starting

conditions consisting of the n known initial conditions satisfying Equation (2) and n assumed values for the unknown Lagrange multipliers.

(2) Simultaneously with the above integration, integrate the $2n$ perturbation equation, Equation (17), with starting conditions described after Equation (18). The perturbation coefficients are formed from the variables that describe the reference trajectory.

(3) Solve the $n+1$ linear algebraic equations, Equation (25), for a linear approximation of the corrections that must be applied to the assumed initial values of the Lagrange multipliers and the terminal time.

(4) Apply these corrections and repeat the process until the corrections on the terminal norm become smaller than some preselected value.

DIRECT OPTIMIZATION METHOD

The objectives of the optimization problem are the same regardless of the method of solution. Hence, the direct optimization formulation involves the previously described equations, Equations (1)(2)(3) and (4). It is found convenient to partition the q-vector of terminal contraints $M(x_f, t_f) = 0$ to

$$M(x_f, t_f) = \begin{bmatrix} \psi(x_f, t_f) \\ \hdashline \Omega(x_f, t_f) \end{bmatrix} \tag{26}$$

where ψ is a q-1 vector and Ω is a scalar stopping condition to be satisfied identically. Although use of this stopping condition is a convenient way to determine an approximate terminal time, one must use judgment to insure the selected terminal constraint will in fact be satisfied during initial iterations.

If the differential equation, Equation (4), is linearized about some nominal path, the resulting equation becomes

$$\delta\dot{x} = f_x \,\delta x + f_u \,\delta u \tag{27}$$

where the partial derivatives f_x and f_u are evaluated on the nominal path. The equation adjoint to Equation (27) is

$$\dot{\lambda} = - f_x^T \lambda \tag{28}$$

where λ is an n-vector of the adjoint variables. It should be noted that Equation (28) and Equation (5-b) are identical. Equation (28) may be combined with Equation (27) to yield

$$\frac{d}{dt}(\lambda^T \delta x) = \lambda^T f_u \,\delta u \tag{29}$$

Integrating this equation and considering Equation (2) yields

$$(\lambda^T \delta x)_f = \int_{t_o}^{t_f} \lambda^T f_u \,\delta u \, dt \tag{30}$$

which is designated the fundamental guidance equation. The object now is to determine how initial state variations and integrated control variations influence the performance index, stopping condition, and the terminal constraints.

If, on separate trials, the terminal values of the adjoint variables are set equal to

$$\left.\begin{aligned} \lambda_{\phi}^{T}(t_f) &= \left[\frac{\partial\phi}{\partial x}\right]_f \\ \lambda_{\psi}^{T}(t_f) &= \left[\frac{\partial\psi}{\partial x}\right]_f \\ \lambda_{\Omega}^{T}(t_f) &= \left[\frac{\partial\Omega}{\partial x}\right] \end{aligned}\right\} \qquad (31)$$

where λ_{ϕ} is an n vector, λ_{ψ} is a $n \times q\text{-}1$ matrix and λ_{Ω} is an n vector. The desired relations are seen to be

$$d\phi = \int_{t_o}^{t_f} \lambda_{\phi}^{T} f_u \, \delta u \, dt + \dot{\phi} \, dt_f \qquad (32)$$

$$d\psi = \int_{t_o}^{t_f} \lambda_{\psi}^{T} f_u \, \delta u \, dt + \dot{\psi} \, dt_f \qquad (33)$$

$$d\Omega = \int_{t_o}^{t_f} \lambda_{\Omega}^{T} f_u \, \delta u \, dt + \dot{\Omega} \, dt_f \qquad (34)$$

where $(\dot{\ }) = \left[\frac{\partial()}{\partial x} \dot{x} + \frac{\partial()}{\partial t}\right]_f$ and $d() = [\delta() + (\dot{\ }) \, dt]_f$.

This formulation allows the specification of an allowable step size to be taken in control space defined by

$$dS = \int_{t_o}^{t_f} \frac{1}{2} \, \delta u^T \, W \, \delta u \, dt \qquad (35)$$

where the step is a weighted quadratic function of the control deviation. The weighting matrix W is included to improve the convergence characteristics by giving more weight to regions of low sensitivity. However, unity is often chosen because of the lack of knowledge concerning this region. The criteria used for determining the best elements for this weighting matrix are not easy to determine and are usually found through trial and error procedures.

The stopping condition, $\Omega = 0$ in Equation (26), is to be identically satisfied so $d\Omega$ in Equation (34) is equated to zero. The terminal time variation dt_f is eliminated from Equations (32) and (33) to yield

$$d\phi = \int_{t_o}^{t_f} \lambda_{\phi\Omega}^T f_u \, \delta u \, dt \tag{36}$$

$$d\psi = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u \, \delta u \, dt \tag{37}$$

where $\lambda_{\phi\Omega} = \lambda_\phi - \dfrac{\dot{\phi}}{\dot{\Omega}} \lambda_\Omega \qquad \lambda_{\psi\Omega} = \lambda_\psi - \lambda_\Omega \dfrac{\dot{\psi}^T}{\dot{\Omega}}$.

The total variation of the performance index may be represented by

$$d\phi = \int_{t_o}^{t_f} \lambda_{\phi\Omega}^T f_u \, \delta u \, dt + \nu^T \left[d\psi - \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u \, \delta u \, dt \right]$$

$$+ \mu \left[dS - \int_{t_o}^{t_f} \frac{1}{2} \, \delta u^T \, W \, \delta u \, dt \right] \tag{38}$$

where the terminal constraints and the control step are adjoined by the use of the ν^T and μ Lagrange multipliers, respectively. It should be noted that the terminal constraints are adjoined in the same manner as in Equation (7) for the indirect method, and that the elements of ν are just the first q-1 elements of the η vector of Equation (7). Since it is desired to determine the control variation which corresponds to the maximum change in the performance index, the first variation of Equation (38) must vanish; therefore

$$\delta\ d\phi = \int_{t_o}^{t_f} \left(\lambda_{\phi\Omega}^T\ f_u - \nu^T\ \lambda_{\psi\Omega}\ f_u - \mu\ \delta u^T\ W \right) \delta^2 u\ dt \equiv 0 \tag{39}$$

This implies that the desired control variation is

$$\delta u = \frac{1}{\mu} W^{-1}\ f_u^T\ (\lambda_{\phi\Omega} - \lambda_{\psi\Omega}\ \nu) \tag{40}$$

and when this equation is substituted back into Equations (35) and (37), the values of ν and μ are seen to be

$$\nu = -\mu I_{\psi\psi}^{-1}\ d\psi + I_{\psi\psi}^{-1}\ I_{\psi\phi} \tag{41}$$

and

$$\mu = \pm \left[\frac{I_{\phi\phi} - I_{\psi\phi}^T\ I_{\psi\psi}^{-1}\ I_{\psi\phi}}{dS - d\psi^T\ I_{\psi\psi}^{-1}\ d\psi} \right]^{1/2} \tag{42}$$

where

$$I_{\psi\psi} = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u W^{-1} f_u^T \lambda_{\psi\Omega} \, dt \tag{43}$$

$$I_{\psi\phi} = \int_{t_o}^{t_f} \lambda_{\psi\Omega}^T f_u W^{-1} f_u^T \lambda_{\phi\Omega} \, dt \tag{44}$$

$$I_{\phi\phi} = \int_{t_o}^{t_f} \lambda_{\phi\Omega}^T f_u W^{-1} f_u^T \lambda_{\phi\Omega} \, dt \tag{45}$$

and $I_{\psi\psi}$ is a q-1 × q-1 matrix, $I_{\psi\phi}$ is a q-1 vector, and $I_{\phi\phi}$ is a scalar.

Now combining Equations (40) through (45) yields the desired control program

$$\delta u = \pm W^{-1} f_u^T(\lambda_{\phi\Omega} - \lambda_{\psi\Omega} I_{\psi\psi}^{-1} I_{\psi\phi}) \left[\frac{dS - d\psi^T I_{\psi\psi}^{-1} d\psi}{I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi} I_{\psi\phi}}\right]^{1/2}$$

$$+ W^{-1} f_u^T \lambda_{\psi\Omega} I_{\psi\psi}^{-1} d\psi \tag{46}$$

where the positive (negative) sign is used if ϕ is to be maximized (minimized). The previous control program is now modified by

$$u_{new} = u_{old} + \delta u . \tag{47}$$

The computational procedure for the direct optimization method may be summarized as follows:

(1) Integrate the n differential equations of motion, Equation (1) forward, using an assumed control program and the desired initial conditions, Equation (2). This integration is continued until the stopping condition, $\Omega = 0$ in Equation (26), is satisfied. The state variable values are stored at each integration step.

(2) Integrate the adjoint equation, Equation (28), backward $q+1$ times with the starting conditions, Equation (31). The coefficient matrix $-f_x^T$ is formed from the state variables stored during the forward integration.

(3) Integrate the I equations, Equations (43) through (45) backward simultaneously with the adjoint equations using initial conditions of zero to yield values at t_o for $I_{\psi\psi}$, $I_{\psi\phi}$, and $I_{\phi\phi}$.

(4) Select a desired improvement in the terminal dissatisfaction $d\psi = -\psi$ for the next iteration.

(5) Select a reasonable value for the mean square allowable control deviation, and from

$$dS = \frac{1}{2}\, \delta u_{ave}^2 \, (t_f - t_o)$$

determine an initial value of dS.

(6) Use the selected value of $d\psi$ and dS to calculate the numerator under the radical in Equation (46). If this quantity is negative, determine the $d\psi$ that makes the quantity vanish. If it is positive, use the quantity as it is.

(7) Calculate the δu as given by Equation (46) and alter the assumed control program. The quantity dS must be decreased according to some selected criteria to prevent stepping across the optimal point into a nonoptimal region.

(8) The procedure is continued until the control variations are less than some preselected value.

HYBRID OPTIMIZATION METHOD

One objective of developing a hybrid method is to combine the best characteristics of the two existing methods. The two methods in this case have been developed above. The indirect method shown has excellent convergence characteristics, but determination of initial Lagrange multipliers is so critical that often the convergence process is never started. On the other hand, the direct method discussed will begin to converge from almost any initial guess on the control program. However, inherent with this method, convergence is never achieved and the classical optimality condition $H_u = 0$ is never satisfied. The proposed hybrid method will show how good initial estimates of the Lagrange multipliers may be obtained for the indirect method from several iterations of the direct method.

It is helpful at this point to review the necessary conditions for mathematical optimality and see how these conditions relate to the methods discussed. In summary, the necessary conditions are:

I.	$\dot{x} = H_\lambda^T$	Equation (5-a)
II.	$\dot{\lambda} = -H_x^T$	Equation (5-b)
III.	$0 = H_u^T$	Equation (5-c)
IV.	$x(t_o) = x_o$	Equation (2)
V.	$M(x_f, t_f) = 0$	Equation (6-a)
	a. $\psi(x_f, t_f) = 0$	Equation (26)
	b. $\Omega(x_f, t_f) = 0$	
VI.	$\lambda^T(t_f) = \phi_x + \nu^T\psi_x + \zeta\Omega_x \Big|_{t_f}$	Equation (6-b)
VII.	$\lambda^T f + \phi_t + \nu^T \psi_t + \zeta\Omega_t \Big|_{t_f} = 0$	Equation (6-c)

where $\eta^T = [\nu^T \mid \zeta]$.

The following table will show which necessary conditions are forced to satisfaction and which conditions are used to iterate upon until adequate results are obtained for several different optimization methods.

	Indirect Methods				Direct Methods	
	MPF	MAF	GNR	MQM	MSD	MHS
Forced to satisfaction	I,II,III,IV		III,IV,V,VI, VII		I,II,IV,Vb	
Iterate on until adequate results are obtained.	V,VI,VII		I,II		III,Va	

where

MPF - Method of Perturbation Functions (see Reference 1 and section titled Indirect Optimization Method)

MAF - Method of Adjoint Functions (see Reference 2)

GNR - Generalized Newton-Raphson (see Reference 3)

MQM - Modified Quasilinearization Method (see Reference 4)

MSD - Method of Steepest Descent (see References 5 and 6 and section titled Direct Optimization Method)

MHS - Min-H Strategy (see Reference 10).

Now, the two transversality conditions VI and VII may be combined to yield

$$\left(\phi_x + \nu^T\psi_x + \zeta\Omega_x\right)f + \phi_t + \nu^T \psi_t + \zeta\Omega_t\Big|_{t_f} = 0 \qquad (48)$$

which may be rearranged to become

$$\phi_x f + \phi_t + \nu^T \psi_x f + \nu^T \psi_t + \zeta\Omega_x f + \zeta\Omega_t \Big|_{t_f} = 0 \tag{49}$$

Recalling that $\dot{\phi} = \phi_x f + \phi_t \Big|_{t_f}$, $\dot{\psi} = \psi_x f + \psi_t \Big|_{t_f}$

and $\dot{\Omega} = \Omega_x f + \Omega_t \Big|_{t_f}$, Equation (49) may be written

$$\dot{\phi} + \nu^T \dot{\psi} + \zeta\dot{\Omega} = 0 \tag{50}$$

Solving for the Lagrange multiplier associated with the stopping condition yields

$$\zeta = -\frac{1}{\dot{\Omega}} \left[\dot{\phi} + \nu^T \dot{\psi}\right] \tag{51}$$

This relation may be substituted back into VI. to provide

$$\lambda^T(t_f) = \left[\phi_x - \frac{\dot{\phi}}{\dot{\Omega}} \Omega_x\right]_{t_f} + \nu^T \left[\psi_x - \frac{\dot{\psi}}{\dot{\Omega}} \Omega_x\right]_{t_f} \tag{52}$$

or by using the relations following Equation (37)

$$\lambda(t_f) = \lambda_{\phi\Omega}(t_f) + \lambda_{\psi\Omega}(t_f)\nu \tag{53}$$

where ν may be calculated from Equation (41).

The computational procedure for the hybrid optimization method may be summarized as follows:

(1) The direct method is applied for an iteration.

(2) The value of ν is determined from Equation (41) and Equation (53) if used to evaluate $\lambda(t_f)$.

(3) Equation (28) is integrated backward from t_f to t_o providing $\lambda(t_o)$.

(4) The iterations of the direct method are continued for some specified length of time.

(5) The $\lambda(t_o)$ obtained from the direct method is used as a starting condition in the indirect method.

(6) If, after two iterations of the indirect method, the terminal norm is increased, continue iterations with the direct method to improve the estimates of $\lambda(t_o)$.

(7) If, after two iterations of the indirect method, the terminal norm is decreased, continue iterations with the indirect method.

CONCLUSIONS

This investigation has revealed how the Method of Perturbation Functions and the Method of Steepest Descent may be combined into a hybrid method. The resulting method has the advantage of having the best merits of both methods while some of the undesirable characteristics of both have been eliminated. Moreover, the hybrid is such that either method may be used individually, if desired.

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