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Electrostatic Cyroscope Drift Caused by Rotor Asphericity
Yu. G. Martynenko
(Reilly Translations/JPL 70-27)

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(Ukhody elektrostaticheskogo giroscopa vyzyvayemyye nespherichnost'yu rotora)

by

Yu. G. Martynenko
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# ELECTROSTATIC GYROSCOPE DRIFT CAUSED BY ROTOR ASPHERICITY

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Expressions are derived for the moments acting on an electrostatic gyroscope rotor, the axisymmetric surface of which is symmetrical with respect to the equatorial plane. With certain assumptions the drift of the gyroscope is determined, the spherical (up to torsion) rotor of which has a finite rigidity.

We note that one of the reasons of the drift of the electrostatic gyroscope is the noncoincidence of the rotor figure [shape] caused by errors in manufacture, by deformations during rotation, or thermal expansion.

Ordinarily, [ref.#1-3], the electrostatic gyroscope rotor is enclosed in a spherical housing which has a vacuum. There is a system of electrodes on the inside surface $S$ of the housing. The position of the rotor with respect to the housing is measured by special sensors as a function of the readings of which a servo suspension system changes the potentials of the electrodes and assures electrostatic suspension of the rotor [ref. #4].

Six suspension electrodes represent spherical segments or bands $S_k$ ($k$ = 1, ..., 5) of the surface of sphere $S$ located at the apexes of a right octahedron [ref. #2].

In order to determine the moments acting on the rotor, it is necessary to find the electrostatic field created by the electrodes $S_k$. In the suspension devices of the electrostatic gyroscope, the distance between the rotor and the electrodes is small in comparison to the linear dimensions, therefore, the field and the gap of the $k$-th electrode-rotor can be considered coincident with the corresponding part of the field in the space between the rotor surface $S_0$ and the conducting sphere $S$ the potential of which is equal to the potential of the $k$-th electrode [ref.#4].

1. Furthermore, it is necessary to determine the potential of the field $U$ which satisfies the basic equation of electrostatics between $S_0$ and $S$—the Laplace equation

$$\nabla^2 U = 0 \tag{1.1}$$

and the following boundary conditions on $S_0$ and $S$:

$$U|_{S_0} = U_0, \qquad U|_S = U_k \tag{1.2}$$

Here $U_0$ is the potential of the rotor, $U_k$ is the potential of the $k$-th electrode. We shall write the equation of the surface $S_0$ and $S$. Let us introduce the following system of coordinates:

$O\xi_1\xi_2\xi_3$ - with the origin at the center $O$ of sphere $S$ and the axes directed along the axes of symmetry of the electrodes;

$O_1\eta_1\eta_2\eta_3$ - with the origin at the geometric center $O_1$ of rotor $S_0$ and the axes parallel to the axes $O\xi_1\xi_2\xi_3$ ($O_1$ is the point of intersection of the axis of symmetry of the rotor and its equatorial plane);

$O_1x_1x_2x_3$ - with axis $O_1x_3$ directed along the axis of symmetry of the rotor;

$O_1y_1y_2y_3$ - with axis $O_1y_3$ directed along $O_1O$ and plane $O_1y_1y_3$ containing $O_1x_3$

In addition, the systems of the spherical coordinates are: $r$, $\vartheta$, $\varphi$, and $r_1$, $\vartheta_1$, $\varphi_1$ with the common origin at point $O_1$ and the polar axes respectively $O_1x_3$ and $O_1y_3$, $0 \leqslant \vartheta, \vartheta_1 \leqslant \pi$ and $0 \leqslant \varphi, \varphi_1 < 2\pi$.

Let the surface of rotor $S_0$ be the surface of rotation symmetric with respect to the equatorial plane $O_1x_1x_2$. In the spherical coordinates $\vartheta$, $\varphi$ the equation of $S_0$ has the form

$$r = r(\vartheta) = h(z) \qquad (z = \cos\vartheta) \tag{1.3}$$

Let $h(z)$ be the continuous function on [-1, +1]. Expanding $h(z)$ into a series by Legendre polynomials $P_a(z)$ and using the properties of parity $P_n(z)$, we obtain

$$r = a\left[1 + \sum_{l=1}^{N} \varepsilon_{2l} P_{2l}(z)\right] \tag{1.4}$$

Here $N$ is the number of terms in the series in the expansion of $h(z)$ by the Legendre polynomials which assure that the necessary accuracy at a given rotor shape

$$\varepsilon_{2l} = \frac{4l+1}{2a}\int_{-1}^{1} h(z)P_{2l}(z)\,dz$$

The value $a$ is naturally called the "radius" of the rotor since the surface of an electrostatic gyroscope rotor is nearly spherical and $\varepsilon_{2l} \ll 1$ are small parameters.

Let us assume that the geometric center of rotor $O_1$ is near $O$, i.e.,

$$\frac{\rho}{b} = \delta \ll 1, \qquad \frac{\rho}{b-a} \ll 1 \qquad (\rho = OO_1,\ b \text{ is the radius of sphere } S$$

The equation of the surface $S$ in coordinates $r_1$, $\vartheta_1$, $\varphi_1$ is found by means of the formula of the counter [reverse] distance [ref. #5]

$$r = b[1 + \delta P_1(\cos\vartheta_1)]$$

In this equation and below we shall disregard terms of higher than the first order of smallness for the small parameters $\delta$, $\varepsilon_2$, ... $\varepsilon_{2N}$.

If $\chi$ is the angle between the axes $O_1x_3$ and $O_1y_3$, then the bond [coupling] between the coordinates $\vartheta$, $\varphi$ and $\vartheta_1$ and $\varphi_1$ have the following form:

$$\sin\vartheta_1\cos\varphi_1 = \cos\chi\sin\vartheta\cos\varphi - \sin\chi\cos\vartheta, \quad \sin\vartheta_1\sin\varphi_1 = \sin\vartheta\sin\varphi$$
$$\cos\vartheta_1 = \cos\chi\cos\vartheta + \sin\chi\sin\vartheta\cos\varphi \tag{1.5}$$

Using the summation theorem for Legendre polynomials [ref. #6], we obtain an equation of the surface of sphere $S$ in the coordinates $r$, $\vartheta$, $\varphi$

$$r = b[1 + \delta(\cos\chi P_1(z) + \sin\chi P_1^{(1)}(z)\cos\varphi)] \qquad (1.6)$$

Here $P_1^{(1)}(z)$ is an associate Legendre function of the first kind.

Considering the structure of expressions (1.4) and (1.6) for the surfaces $S_0$ and $S$ where the boundary conditions of (1.2) are given, we shall seek a solution of equations (1.1) in the form

$$U = (A_0 + B_0 r^{-1}) + (A_1 r + B_1 r^{-2})[P_1(z) + P_1^{(1)}(z)(C_1\cos\varphi + D_1\sin\varphi)] +$$
$$+ \sum_{l=1}^{N} (A_{2l} r^{2l} + B_{2l} r^{-2l-1}) P_{2l}(z) \qquad (1.7)$$

($A_n$, $B_n$, $C_1$, $D_1$ are arbitrary constants)

Let us substitute in (1.7) the formulas (1.4) and (1.6), the setting the coefficients at identical Legendre polynomials by taking into consideration (1.2) in the obtained equations, we shall find

$$U = A_0 + abE_k \frac{1}{r} + \left(\frac{ab^2}{b^3 - a^3} r - \frac{a^4 b^2}{(b^3 - a^3) r^2}\right) \times$$
$$\times \{\cos\chi P_1(z) + \sin\chi P_1^{(1)}(z)\cos\varphi\} \delta E_k +$$
$$+ \sum_{l=1}^{N} \left[\frac{-a^{2l+1} b}{b^{4l+1} - a^{4l+1}} r^{2l} + \frac{a^{2l+1} b^{4l+2}}{(b^{4l+1} - a^{4l+1}) r^{2l+1}}\right] \varepsilon_{2l} E_k P_{2l}(z) \qquad (1.8)$$
$$E_k = (U_0 - U_k)/(b - a)$$

The intensity of the electrostatic field **E** = - grad $U$.

Differentiating (1.8) and using (1.4), we count the components of the field intensity on the surface of the rotor $S_0$

$$-\frac{\partial U}{\partial r}\Big|_{S_0} = \frac{b}{a} E_k - \frac{3ab^2}{b^3 - a^3} \delta E_k (\cos\chi P_1(z) + \sin\chi\cos\varphi P_1^{(1)}(z)) +$$
$$+ \sum_{l=1}^{N} \frac{b[(2l-1) b^{4l+1} + (2l+2) a^{4l+1}]}{a(b^{4l+1} - a^{4l+1})} \varepsilon_{2l} E_k P_{2l}(z)$$

$$-\frac{1}{r}\frac{\partial U}{\partial \vartheta}\Big|_{S_0} = \sum_{l=1}^{N} \frac{b}{a} \varepsilon_{2l} E_k P_{2l}'(\cos\vartheta)\sin\vartheta, \qquad -\frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}\Big|_{S_0} = 0$$

The density of the pondermotive forces on the rotor surface [ref. #7] is

$$f = \frac{1}{8\pi} E^2|_{S_0} = \qquad (1.9)$$
$$= \frac{1}{8\pi}\left\{\frac{b^2}{a^2} E_k^2 - \frac{6b^3\delta}{b^3 - a^3} E_k^2 (\cos\chi P_1(z) + \sin\chi\cos\varphi P_1^{(1)}(z)) + \right.$$

$$+2\sum_{l=1}^{N}\frac{b^2[(2l-1)b^{2l+1}+(2l+2)a^{2l+1}]}{a^2(b^{2l+1}-a^{2l+1})}v_{2l}P'_{2l}(z)E_k^2\Bigg\}$$

The pondermotive forces are directed along external normals $\mathbf{n}$ to the rotor surface $S_0$

$$\mathbf{f}=f\mathbf{n}$$

The components of the unit vector of the normal to the surface (1.3) are:

$$n_{x_1}=\frac{-r'(\vartheta)\cos\vartheta+r(\vartheta)\sin\vartheta}{\sqrt{r^2(\vartheta)+[r'(\vartheta)]^2}}\cos\varphi$$

$$n_{x_2}=\frac{-r'(\vartheta)\cos\vartheta+r(\vartheta)\sin\vartheta}{\sqrt{r^2(\vartheta)+[r'(\vartheta)]^2}}\sin\varphi$$

$$n_{x_3}=\frac{r'(\vartheta)\sin\vartheta+r(\vartheta)\cos\vartheta}{\sqrt{r^2(\vartheta)+[r'(\vartheta)]^2}} \tag{1.10}$$

The density of the moment of the pondermotive forces with respect to point $O_1$ is

$$\mathbf{m}=\mathbf{r}\times\mathbf{f}=f[\mathbf{r}\times\mathbf{n}]$$

$$m_{x_1}=f\frac{r(\vartheta)r'(\vartheta)\sin\varphi}{\sqrt{r^2(\vartheta)+[r'(\vartheta)]^2}},\quad m_{x_2}=-f\frac{r(\vartheta)r'(\vartheta)\cos\varphi}{\sqrt{r^2(\vartheta)+[r'(\vartheta)]^2}},\quad m_{x_3}=0 \tag{1.11}$$

Differentiating (1.4) and substituting $r'(\vartheta)$ in (1.11), we find

$$m_{x_1}=-af\sum_{l=1}^{N}v_{2l}P_{2l}'(\cos\vartheta)\sin\vartheta\sin\varphi$$

$$m_{x_2}=af\sum_{l=1}^{N}v_{2l}P_{2l}'(\cos\vartheta)\sin\vartheta\cos\varphi \tag{1.12}$$

The moment $\mathbf{M}_k$ with which the $k$-th electrode acts on the rotor is

$$\mathbf{M}_k=\iint_{S_{0k}}\mathbf{m}\,dS \tag{1.13}$$

Here $S_{0k}$ is a part of the surface $S_0$ located below $S_k$.

Let $S_k$ be a spherical band of sphere $S$ formed by circular cones with apexes at point $O$ and common axes $\xi_i$. We shall designate the angles between $\xi_i$ and the cones formed respectively by $\psi_1$ and $\psi_2$ $(0 \leqslant \psi_1 < \psi_2 < 1/4\pi)$. (If $\psi_1 = 0$, then $S_k$ is a spherical segment of sphere $S$.)

Initially we shall examine the case when $O_1$ lies on the axis $\xi_1$ which is the axis of symmetry of the $k$-th electrode. Equation $S_k$ in this case in coordinates $r_1$, $\vartheta_1$, $\varphi_1$ has the form

$$\psi_1 + \psi_{01} \leqslant \vartheta_1 \leqslant \psi_2 + \psi_{02}, \quad 0 \leqslant \varphi_1 < 2\pi, \quad r = b[1 + \delta P_1(\cos\vartheta_1)]$$

$$\psi_{0j} = \arcsin \frac{\sin\psi_j}{\sqrt{1 + 2\delta\cos\psi_j + \delta^2}} - \psi_j = -\delta\sin\psi_j \quad (j = 1, 2)$$

Passing, in agreement with (1.5), to the coordinates $\vartheta_1$, $\varphi_1$ in the formulas of (1.9), 1.12) and substituting the obtained expressions in (1.13), we find

$$M_{ky_i} = -\frac{b^2E_k^2}{8\pi a} \int_{\psi_1}^{\psi_2} \int_0^{2\pi} \sum_{l=1}^{N} \varepsilon_{2l} P_{2l}'(\cos\chi\cos\vartheta_1 + \sin\chi\sin\vartheta_1\cos\varphi_1) \times$$
$$\times \sin\vartheta_1 \sin\varphi_1 r^2 \sin\vartheta_1 \, d\varphi_1 \, d\vartheta_1$$

$$M_{ky_1} = \frac{b^2E_k^2}{8\pi a} \int_{\psi_1}^{\psi_2} \int_0^{2\pi} \sum_{l=1}^{N} \varepsilon_{2l} P_{2l}'(\cos\chi\cos\vartheta_1 + \sin\chi\sin\vartheta_1\cos\varphi_1) \times \tag{1.14}$$
$$\times (\cos\chi\sin\vartheta_1\cos\varphi_1 - \sin\chi\cos\vartheta_1) r^2 \sin\vartheta_1 \, d\varphi_1 \, d\vartheta_1$$

Since $P_{2l}'(z)$ is a polynomial of degree $2l - 1$ of $z$, then it is easy to see from (1.14) that $M_{kyi} = 0$ at any $N$, and

$$M_{ky_1} = \frac{1}{8\pi} ab^2E_k^2 \sum_{l=1}^{N} \varepsilon_{2l} I_{2l}$$

$$I_{2l} = \int_{\psi_1}^{\psi_2} \int_0^{2\pi} (\cos\chi\sin\vartheta_1\cos\varphi_1 - \sin\chi\cos\vartheta_1) \times \tag{1.15}$$
$$\times P_{2l}'(\cos\chi\cos\vartheta_1 + \sin\chi\sin\vartheta_1\cos\varphi_1) \, d\varphi_1 \, d\cos\vartheta_1$$

If $N = 1$, then from (1.15)

$$M_{ky_1} = {}^1\!/_{16} \Delta b^2 E_k^2 \sin 2\chi (\sin\psi_2 \sin 2\psi_2 - \sin\psi_1 \sin 2\psi_1) \tag{1.16}$$

Here $\Delta = -3/2a\varepsilon_2$ is the "asphericity" of the rotor—the difference between the largest and smallest radii of the rotor.

Using the recurrent formulas for Legendre polynomials [ref. #6], the summation theorem, and the property of orthogonality cos $n$?$_1$ on [0, $2\pi$], we find

$$I_{2l} = I_{2l-2} + J_l$$

$$J_l = \frac{\pi(4l-1)}{l} \sin\chi \left[ P_{2l}'(\cos\chi) \int_{\cos\psi_1}^{\cos\psi_2} z P_{2l-1}(z) \, dz - \right.$$
$$\left. - \cos\chi P_{2l-1}'(\cos\chi) \int_{\cos\psi_1}^{\cos\psi_2} P_{2l-2}(z) \, dz \right]$$

With tangible vlues of $\varepsilon_{2l}$ this formula makes it possible to estimate the effect of "higher harmonics" in the form [shape] of the rotor on the value of the moment $M_k$.

It is possible to show that in the general case, when the disposition of

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$O_1$ is arbitrary in a fairly small vicinity of $O$, the expression for $\mathbf{M}_k$ will differ from (1.14) in values of the second order of smallness for the small parameters $\delta$, $\varepsilon_{2l}$.

2. If the spherical rotor of the electrostatic gyroscope has a finite rigidity, then under the action of centrifugal forces that arise during the rotation of the rotor its surface will be deformed. According to [ref. #8], the equation of the surface of a rotating elastic sphere has the form of (1.4), in which $N = 1$. Let us consider the moment acting on the deformed rotor from the side of the suspension electrodes. Let the intensity on the $2i - 1$)-th, $2i$-th electrodes having a common axis of symmetry $\xi_i$ be regulated by the $i$-th channel of the servo suspension system ($i = 1, 2, 3$). From (1.16) it follows that the vector of the moment $\mathbf{M}_{\xi i}$ is created by electrodes of the $i$-th channel

$$\mathbf{M}_{\xi_i} = M_0 \sin 2\chi_i (E_{2i-1}^2 + E_{2i}^2) \frac{\xi_i \times x_3}{|\xi_i \times x_3|} \tag{2.1}$$
$$M_0 = {}^{1}/_{1}\varepsilon \Delta b^2 (\sin \psi_2 \sin 2\varphi_2 - \sin \psi_1 \sin 2\varphi_1)$$

Here $\xi_i$, $x_3$ are unit vectors directed along the axis $\xi_i$ and $x_3$; $\chi_i$ is the angle between $\xi_i$ and $x_3$.

The position of the system of coordinates $O_1x_1x_2x_3$ with respect to $O_1\eta_1\eta_2\eta_3$ is defined by the angles $\alpha$, $\beta$. (The system of coordinates $x_1x_2x_3$ is obtained from $\zeta_1\zeta_2\zeta_3$ by rotation through angle $\beta$ about axis $\zeta_2$, the system of coordinates $\zeta_1\zeta_2\zeta_3$ is obtained from $\eta_1\eta_2\eta_3$ by rotation through angle $\alpha$ about axis $\eta_1$.) Projecting (2.1) on the axis $\xi_1\xi_2\xi_3$, we find

$$\begin{aligned} M_{\xi_1} &= 2M_0(E_{\xi_2}^2 - E_{\xi_3}^2)\sin\alpha\cos\alpha\cos^2\beta \\ M_{\xi_2} &= -2M_0(E_{\xi_3}^2 - E_{\xi_1}^2)\cos\alpha\sin\beta\cos\beta \\ M_{\xi_3} &= 2M_0(E_{\xi_1}^2 - E_{\xi_2}^2)\sin\alpha\sin\beta\cos\beta \\ \mathbf{M}_{\Sigma} &= \mathbf{M}_{\xi_1} + \mathbf{M}_{\xi_2} + \mathbf{M}_{\xi_3}, \qquad E_{\xi_i}^2 = E_{2i-1}^2 + E_{2i}^2 \end{aligned} \tag{2.2}$$

Let us examine the case when the base of the electrostatic gyrcscope moves with a constant acceleration. The resulting vector of the forces of gravitation and inertia of the transfer motion applied to the center of mass of the rotor we shall designate $mwe$ ($m$ is the mass of the rotor, $\mathbf{e}$ is the unit vector having the projection $e_i$ on axis $\xi_i$). Let us assume that the geometric center of the rotor coincides with the center of mass, that the servo suspension system linearly changes in field intensity below the electrodes as a function of the shift of the geometric center of the rotor and assures a fairly rapid deformation of the oscillations of the rotor. Here it is possible to consider than in the motion of a base with constant acceleration, the center of mass of the rotor is immovable with respect to the housing and

$$E_{2i-1} = E_0(1 - k_0\rho_{\xi_i}), \quad E_{2i} = E_0(1 + k_0\rho_{\xi_i}) \quad (i = 1, 2, 3) \tag{2.3}$$

and a projection of the shift of the center of mass $\rho_{\xi i}$ under the action of the $mwe$ forces are equal to

$$\rho_{\xi_i} = mwe_i / K \tag{2.4}$$

Here $k_0$ is the coefficient of amplification of the servo suspension system, $E_0$ is the field intensity below the electrodes with the absence of rotor shift

$K$ is the rigidity of the suspension. Substituting (2.3) and (2.4) into (2.2), we obtain

$$\begin{aligned} M_{\Sigma\xi_1} &= 2L(e_3^2 - e_2^2)\sin\alpha\cos\alpha\cos^2\beta \\ M_{\Sigma\xi_2} &= -2L(e_1^2 - e_3^2)\cos\alpha\sin\beta\cos\beta \\ M_{\Sigma\xi_3} &= 2L(e_2^2 - e_1^2)\sin\alpha\sin\beta\cos\beta \\ L &= 2M_0(E_0 k_0 m w / K)^2 \end{aligned} \qquad (2.5)$$

From (2.5) it follows that the suspension will have no perturbing effect on the motion of the rotor about its center of mass if

$$|e_1| = |e_2| = |e_3| = {}^1\!/_3\sqrt{3} \qquad (2.6)$$

i.e., if the vector $mwe$ is colinear to one of the four different directions determined by (2.6).

The moment $M_\Sigma$ also returns to zero in the cases when

$$(\alpha = 0,\ \beta = 0),\ (\alpha = \pi,\ \beta = 0),\ (\alpha = \pm{}^1\!/_2\pi,\ \beta = 0),\ (\beta = \pm{}^1\!/_2\pi) \qquad (2.7)$$

i.e., when the axis of rotation of rotor $x_3$ is colinear to one of the axes of symmetry of electrode $\xi_i$. (Below it will be shown that each of the positions of (2.7) may be both stable and unstable.)

Let us examine the common case when the condition of (2.6) is not fulfilled. We shall write the motion equations of the rotor with respect to the center of the mass $O_1$ in projections on axis $x_1 x_2 x_3$ in the Euler-Bulgakov form

$$\begin{aligned} A\frac{d}{dt}\left[\cos\beta\frac{d\alpha}{dt}\right] + (C - A)\sin\beta\frac{d\alpha}{dt}\frac{d\beta}{dt} + \frac{d\beta}{dt}C\frac{d\gamma}{dt} &= M_{\Sigma x_1} \\ A\frac{d^2\beta}{dt^2} + (A - C)\cos\beta\sin\beta\left(\frac{d\alpha}{dt}\right)^2 - \cos\beta\frac{d\alpha}{dt}C\frac{d\gamma}{dt} &= M_{\Sigma x_2} \\ C\frac{d}{dt}\left[\sin\beta\frac{d\alpha}{dt} + \frac{d\gamma}{dt}\right] &= M_{\Sigma x_3} \end{aligned} \qquad (2.8)$$

Here $A$ is the equatorial, $C$ the polar moment of inertia of the rotor, $\gamma$ is the angle of inherent rotation, $M_{\Sigma xi}$ is the projection of vector $\mathbf{M}_\Sigma$ on axis $x_i$. By virtue of the symmetry of the shape of the rotor $M_{\Sigma x_3} = 0$, therefore, from the last equation (2.8) the first integral follows

$$C\,d\gamma / dt + C\sin\beta\,d\alpha / dt = H \qquad (2.9)$$

Excluding $\gamma$ from (2.8) by means of (2.9) and substituting on the right sides of (2.8) the pojection values of $\mathbf{M}_\Sigma$, we obtain

$$\begin{aligned} A\cos\beta\frac{d^2\alpha}{dt^2} - 2A\sin\beta\frac{d\alpha}{dt}\frac{d\beta}{dt} + H\frac{d\beta}{dt} &= 2L(e_3^2 - e_2^2)\sin\alpha\cos\alpha\cos\beta \\ A\frac{d^2\beta}{dt^2} + A\cos\beta\sin\beta\left(\frac{d\alpha}{dt}\right)^2 - H\cos\beta\frac{d\alpha}{dt} &= \\ = 2L(e_3^2 - e_2^2)(\varkappa + \cos^2\alpha)\sin\beta\cos\beta \end{aligned} \qquad (2.10)$$

$$\varkappa = \frac{e_2^2 - e_1^2}{e_3^2 - e_2^2}$$

Following [ref. #9], we introduce into equations (2.10) a small parameter. We do this to change the time $\tau = \lambda t$ and we select $\lambda$ so that $\lambda H/L \approx 1$. Passing in (2.10) to a dimensionless time $\tau$ and considering that the moments of the first parts of (2.10) have a force function*, we find

$$\begin{gathered}
\mu \cos\beta \frac{d^2\alpha}{d\tau^2} - 2\mu \sin\beta \frac{d\alpha}{d\tau}\frac{d\beta}{d\tau} + \frac{d\beta}{d\tau} + \frac{1}{\cos\beta}\frac{\partial V}{\partial \alpha} = 0 \\
\mu \frac{d^2\beta}{d\tau^2} + \mu \cos\beta \sin\beta \left(\frac{d\alpha}{d\tau}\right)^2 - \cos\beta \frac{d\alpha}{d\tau} + \frac{\partial V}{\partial \beta} = 0 \qquad (2.11) \\
\mu = \frac{A\lambda^2}{L} \approx \frac{AL}{H^2}, \quad V(\alpha, \beta) = (e_3^2 - e_2^2)(\varkappa + \cos^2\alpha)\cos^2\beta
\end{gathered}$$

Multiplying the first equation of (2.11) by $\cos\beta\, d\alpha/d\tau$, the second by $d\beta/d\tau$ and adding, we obtain still another first integral—the integral of energy, the existence of which is obvious by virtue of the potentiality of the moments and independence of the connection on time.

$$\mu\left[\frac{1}{2}\cos^2\beta\left(\frac{d\alpha}{d\tau}\right)^2 + \frac{1}{2}\left(\frac{d\beta}{d\tau}\right)^2\right] + V(\alpha, \beta) = \text{const} \qquad (2.12)$$

With the fast acting gyroscope $\mu \ll 1$ and in the projectile approximation by $\mu$, the motion equation of the axis of symmetry of the rotor along the sphere of unit radius with the center at point $O_1$ will have the form

$$(\varkappa + \cos^2\alpha)\cos^2\beta = (\varkappa + \cos^2\alpha_0)\cos^2\beta_0 \qquad (2.13)$$

Here $\alpha_0$, $\beta_0$ are the initial values of the angles $\alpha$, $\beta$.

As a function of the value of parameter $\varkappa$, the trajectories of (2.13) (figure) represent the intersection of the sphere and

(1) ($\varkappa = \pm\infty$) a plane parallel to the plane of $O_1\eta_2\eta_3$;
(2) ($-\infty < \varkappa < -1$) an elliptical cylinder oblated along axis $\eta_1$;
(3) ($\varkappa = -1$) a circular cylinder;
(4) ($-1 < \varkappa < 0$) an elliptical cylinder extended along axis $\eta_1$;
(5) ($\varkappa = 0$) a plane parallel to plane $O_1\eta_1\eta_2$;
(6) ($0 < 2\varkappa < \infty$) a hyperbolic cylinder.

All cylinders have generatrices parallel to the axis $\eta_2$.

The stability of the position of (2.7) on the figure is obvious.

We shall limit ourselves to the investigation of the first two cases since all other easily lead to them by an appropriate change in the enumeration of the axes of the coordinate system $O\xi_1\xi_2\xi_3$.

In case (1) when $\partial V/\partial\alpha = 0$, from the first equation of (2.11) the integral follows associated with the cylindrical coordinate $\alpha$

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*This case was pointed out to the author by A.I. Kobrin

$$\mu \cos^2 \beta \, d\alpha / d\tau + \sin \beta = \text{const} \qquad (2.14)$$

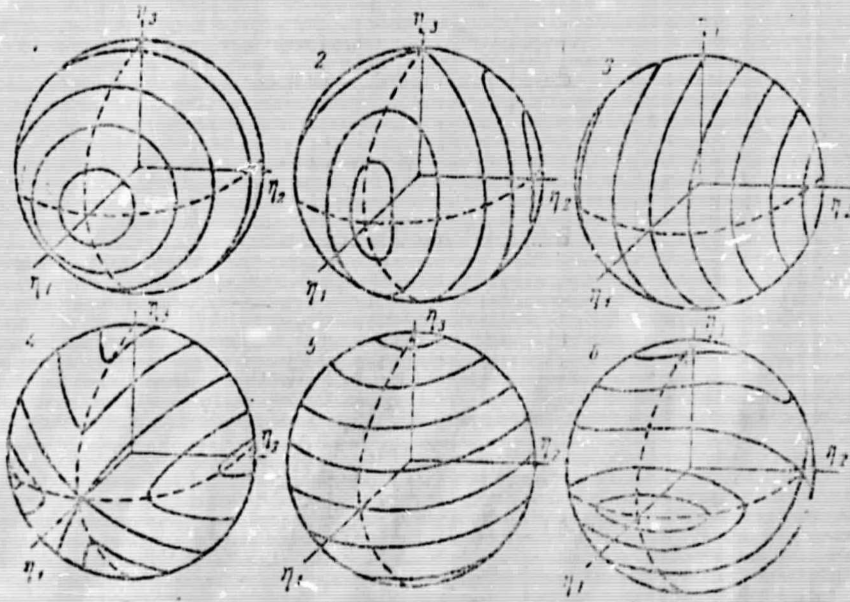
Excluding from (2.12) by means of (2.14) $d\alpha/d\tau$, we obtain the system

$$\mu^2 \left(\frac{du}{d\tau}\right)^2 = \mu(1 - u^2)(h_1 + e_0 u^2) - (h_2 - u)^2, \quad \mu \frac{d\alpha}{d\tau} = \frac{h_2 - u}{1 - u^2}$$
$$u = \sin \beta, \quad h_1 = \mu \cos^2 \beta_0 (\alpha_0')^2 + \mu(\beta_0')^2 - e_0 u_0^2,$$
$$h_2 = \mu \cos^2 \beta_0 \alpha_0' + u_0, \quad u_0 = \sin \beta_0, \quad e_0 = 2(c_2^2 - c_1^2) \qquad (2.15)$$

Here $\alpha_0'$, $\beta_0'$ are values of the velocities $d\alpha/d\tau$, $d\beta/d\tau$ and the initial moment of time. From the first equation of (2.15) we find

$$\tau - \tau_0 = \int_{u_0}^{u} \frac{\mu \, du}{\sqrt{\mu(1 - u^2)(h_1 + e_0 u^2) - (h_2 - u)^2}} \qquad (2.16)$$

After reversing the elliptical integral of (2.16), the calculation of $\alpha$ leads to the quadrature.




From the condition of the non-negativeness of the radicand expression in (2.16), we obtain

$$|u - u_0| < \mu \cos^2 \beta_0 |\alpha_0'| + \sqrt{\mu |e_0| + \mu^2 \cos^2 \beta_0 (\alpha_0')^2 + \mu^2 (\beta_0')^2}$$

i.e., at a fairly small $\mu$ the motion of the axis of symmetry of the rotor will take place in just as small a vicinity of the plotted, previously approximated trajectory (2.13) as desired.

One may expect that in the case (2) the exact solution of equation (2.11) is found near (2.13), i.e., taking into consideration in (2.12) the terms with small parameters leads to a superposition of the precessional motion of (2.13) and a certain continuous nutational motion with fairly small amplitude and frequency $\mu^{-1}$ (in time $\tau$).

The time of the precessional motion along trajectory (2.13) in case (2) is determined by the integral

$$\tau - \tau_0 = \frac{1}{2(e_3^2 - e_2^2)} \int_{u_0}^{u} \frac{du}{\sqrt{(h_3 + \varkappa u^2)[(1 - h_3) - (1 + \varkappa) u^2]}} \tag{2.17}$$

$$h_3 = \cos^2 \alpha_0 \cos^2 \beta_0 - \varkappa \sin^2 \beta_0, \quad h_3 > 0, \quad 1 + \varkappa < 0, \quad 1 - h_3 \gtrless 0$$

If, for example, $h_3 > 1$, then from (2.17) we find

$$\tau - \tau_0 = \frac{1}{2(e_3^2 - e_2^2)\sqrt{-h_3(1 + \varkappa)}} [F(\varphi_0, k) - F(\varphi, k)]$$

Here $F(\varphi, k)$ is an elliptical integral of the first order

$$\varphi = \arcsin \left( \frac{(h_3 + \varkappa u^2)(1 + \varkappa)}{h_3 + \varkappa} \right)^{1/2}, \quad k = \left( \frac{h_3 + \varkappa}{h_3(1 + \varkappa)} \right)^{1/2}$$

$$\varphi_0 = \arcsin \cos \alpha_0 \left( \frac{1 + \varkappa}{\varkappa + \cos^2 \alpha_0} \right)^{1/2}$$

3. The examined motion of the electrostatic gyroscope rotor is analogous to the motion of the dynamically symmetrical solid body about a fixed point of three Newtonian attracting centers in the case when the center of masses of the body coincides with the fixed point $O_1$, and the attracting centers are located fairly far from the body on three mutually orthogonal axes $\eta_1$, $\eta_2$, $\eta_3$.

Actually, the force function $W$ of the described field has the form

$$W = \frac{3}{2}(C - A)\left[\frac{\mu_1}{R_1^3}\gamma_1^2 + \frac{\mu_2}{R_2^3}\gamma_2^2 + \frac{\mu_3}{R_3^3}\gamma_3^2\right] \tag{3.1}$$

and the expression for the potential energy $V(\alpha, \beta)$ from (2.11) can be presented in the form

$$V = e_1^2\gamma_1^2 + e_2^2\gamma_2^2 + e_3^2\gamma_3^2 \tag{3.2}$$

Here $C$, $A$ are polar and equatorial moments of inertia of the solid body, $\mu_i$ is the graviational constant, $R_i$ is the distance to the attracting center located on axis $\eta_i$, $\gamma_i$ are the controlling cosines of the axis of symmetry of the body in the coordinate system $\eta_1$, $\eta_2$, $\eta_3$.

In the integrated cases (1), (3), (5) the field (3.2) degenerates into a field of one attracting center located respectively at the axes $\eta_1$, $\eta_2$, $\eta_3$. Here the equations of (2.15) with an accuracy to the designations coincide with the equations investigated by V. V. Baletsky in the problem of the motion of a solid body about a fixed point in a Newtonian force field [ref.10].

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