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CONFIGURATION SPACE THREE-BODY SCATTERING THEORY

BY

E. GERJUOY

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APPENDIX A. CONVERGENCE OF INTEGRALS

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A.1. Eq. (63d) at Real Energies

I wish to show that the integral

$$G^{(+)} V G_F^{(+)} = G_F^{(+)} V G^{(+)} \tag{A1}$$

converges. In particular, consider a typical term in (A1), e.g.,

$$\int d\underset{\sim}{r}'' G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'') V_{12}(\underset{\sim}{r}_{12}'') G_F^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}') \tag{A2}$$

which is the $\underset{\sim}{r}$, $\underset{\sim}{r}'$ element of the term involving V_{12} on the left side of (A1).

Because V_{12} is a short range force, the integral (A2) can diverge (at infinity in $\underset{\sim}{r}''$ space) only in directions $\underset{\sim}{\nu}_{12}''$ along which $\underset{\sim}{r}_{12}''$ remains finite. Recalling Eqs. (95) and (112), one sees that the laboratory system analogues of (114) and (115) must imply

$$\lim_{r''\to\infty \parallel \underset{\sim}{\nu}''_{12f}} G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'';E) = \lim_{r''\to\infty \parallel \underset{\sim}{\nu}''_{12f}} G^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r};E)$$

$$= \sum_j C_2(E-\varepsilon_j)\frac{e^{i\rho''\sqrt{E-\varepsilon_j}}}{\rho''^{5/2}} u_j(\underset{\sim}{r}''_{12})\, e^{-i\underset{\sim}{K}''_f\cdot \underset{\sim}{R}}\, \bar{\Psi}^{(-)*}_{12jf}(\bar{\underset{\sim}{r}};\bar{E}) \qquad \text{(A3)}$$

plus [on the right side of (A3)] an integral over the continuum functions $u(\underset{\sim}{r}_{12}'', \hat{\underset{\sim}{k}}_{12})$, as in Eq. (112a). In (A3), since $\underset{\sim}{r}_{12}''$ remains finite, the magnitude of the nine-dimensional vector $\underset{\sim}{\rho}''$ defined by the double-primed analogue of (25d) is the same as the magnitude of the six-dimensional vector $\underset{\sim}{\rho}_{12}''$ defined by

$$\underset{\sim}{\rho}_{12}'' = \left(\frac{2M}{\hbar^2}\right)^{1/2} \underset{\sim}{R}'' \oplus \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{1/2} \underset{\sim}{q}_{12}'' \tag{A4}$$

in the notation of Eq. (102d); also, the two masses appearing in $C_2(E - \varepsilon_j)$ [which is defined by (90b)] are the two effective masses in (A4), namely M and μ_{3R}. The neglected continuum contribution in (A3) is of order ρ''^{-4} [compare Eqs. (112a) and (116a)], i.e., indeed negligible compared to the retained discrete sum in (A3), for the purposes of this section.

The asymptotic behavior of $G_F^{(+)}(\underset{\sim}{r}';\underset{\sim}{r}'')$ in (A2) as $r'' \to \infty$ is given by Eqs. (90). Thus, employing Eqs. (40) along with (A3) [and recognizing there can be at most a finite number of bound states u_j because V_{12} is short range], it is clear that the integral (A2) converges if and only if individual integrals of the form

$$\int d\underset{\sim}{R}''\, d\underset{\sim}{q}_{12}''\, d\underset{\sim}{r}_{12}''\, \frac{e^{i\rho\sqrt{E-\varepsilon_j}}}{\rho''^{5/2}}\, u_j(\underset{\sim}{r}_{12}'')\, e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}}\, \bar{\Psi}_{12jf}^{(-)*}(\bar{\underset{\sim}{r}};\bar{E})\, V_{12}(\underset{\sim}{r}_{12}'') \times \frac{e^{i\rho''\sqrt{E}}}{\rho''^4}\, e^{-i[\underset{\sim}{K}_f''\cdot\underset{\sim}{R}' + \underset{\sim}{K}_{12f}''\cdot\underset{\sim}{q}_{12}']} \tag{A5}$$

converge at large ρ'' for each j. In (A5) the integral over $\underset{\sim}{r}_{12}''$

obviously converges. Introduce

$$\tilde{\underset{\sim}{R}}'' = \left(\frac{2M}{\hbar^2}\right)^{1/2} \underset{\sim}{R}''$$

$$\tilde{\underset{\sim}{q}}''_{12} = \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{1/2} \underset{\sim}{q}''_{12} \tag{A6}$$

(where, of course, the tilde merely distinguishes the vectors on the two sides of (A6) and has nothing to do with the transpose[29]). Then, recalling Eqs. (89) and (A4), one sees that the volume element

$$d\underset{\sim}{R}'' d\underset{\sim}{q}''_{12} \cong \rho''^5 d\rho'' d\underset{\sim}{\nu}''_{12} \tag{A7}$$

where the distinction between ρ'' and ρ_{12}'' can be ignored here. In (A5), moreover, the plane waves in $\underset{\sim}{R}'$ and $\underset{\sim}{q}_{12}'$, as well as the factor $\bar{\Psi}^{(-)*}$, depend--via the double-primed analogues of Eqs. (92)--on the five independent angles specifying $\underset{\sim}{\nu}_{12}''$, but not on the magnitude of $\underset{\sim}{\rho}_{12}''$. It follows that the integrand of (A5) is of order $\rho''^{-3/2}$ at large ρ''. Therefore (A5), and concomitantly (A2), are convergent, Q.E.D.

A.2 Eq. (65b)

Next consider Eq. (65b), whose validity depends on the convergence of

$$G^{(+)}\left[V_{23}+V_{31}\right]G_{12}^{(+)} = \int d\underset{\sim}{r}'' G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')\left[V_{23}(\underset{\sim}{r}_{23}'')+V_{31}(\underset{\sim}{r}_{31}'')\right]G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}') \tag{A8}$$

Consider, e.g., the term involving V_{23}. This term can diverge (at infinity in $\underset{\sim}{r}''$ space) only in directions $\underset{\sim}{\nu}_{23}''$ along which $\underset{\sim}{r}_{23}''$ remains finite. Along such directions, $G_{12}^{(+)}$ cannot propagate in bound states, however, i.e., along directions $\underset{\sim}{\nu}_{23}''$ $G_{12}^{(+)}$ behaves asymptotically essentially like $G_F^{(+)}$ [recall the discussion concerning Eqs. (116)]. Consequently (A8)---like (A2)--converges, as asserted following Eq. (65a).

A.3 Center of Mass Analogues of Eqs. (63)

The convergence of the center of mass analogues of Eqs. (63) now is trivially demonstrable. Consider, e.g., the center of mass analogue of (A2), namely

$$\int d\bar{\underset{\sim}{\tau}}'' \bar{G}^{(+)}(\bar{\underset{\sim}{\tau}}''; \bar{\underset{\sim}{\tau}}) V_{12}(\underset{\sim}{\tau}_{12}'') \bar{G}_F^{(+)}(\bar{\underset{\sim}{\tau}}'; \bar{\underset{\sim}{\tau}}'') \tag{A9a}$$

Using Eqs. (40), (90) and (114), it is clear that (A9a) converges if and only if integrals of the form (omitting inessential factors in the integrand)

$$\int d\underset{\sim}{q}_{12}'' \, d\underset{\sim}{\tau}_{12}'' \frac{1}{q_{12}''} u_j(\underset{\sim}{\tau}_{12}'') V_{12}(\underset{\sim}{\tau}_{12}'') \frac{1}{q_{12}''^{5/2}} \tag{A9b}$$

converge at large q_{12}''. But (A9a) does converge, because its integrand is of order $q_{12}''^{-3/2}$.

A.4 Eq. (52a) with Two-Body Bound States

I now turn to Eq. (52a), in which, as in (46), it now is convenient to suppose the incident plane wave $\psi_i(E')$ may correspond to a different energy than the Green's function $G^{(+)}(E)$. In particular, consider the term in (52a) involving V_{12}, which can be rewritten as

$$\int d\underset{\sim}{R}' d\underset{\sim}{q}_{12}' d\underset{\sim}{r}_{12}' G^{(+)}(\underset{\sim}{r}';\underset{\sim}{r}) V_{12}(\underset{\sim}{r}_{12}') e^{i[\underset{\sim}{K}'\cdot\underset{\sim}{R}' + \underset{\sim}{K}_{12}'\cdot\underset{\sim}{q}_{12}' + \underset{\sim}{k}_{12}'\cdot\underset{\sim}{r}_{12}']} \tag{A10}$$

As above, the integral over $d\underset{\sim}{r}_{12}'$ converges, and it is necessary only to examine the behavior of the integrand along directions $\underset{\sim}{\nu}_{12}'$ on which $\underset{\sim}{r}_{12}'$ remains finite. The behavior of $G^{(+)}$ still is given by (A3), but the plane wave factors in (A10) do not behave like $G_F^{(+)}$. Introduce(29) [in analogy with (A6)]

$$\begin{aligned} \underset{\sim}{\tilde{K}}' &= \left(\frac{\hbar^2}{2M}\right)^{1/2} \underset{\sim}{K}' \\ \underset{\sim}{\tilde{K}}_{12}' &= \left(\frac{\hbar^2}{2\mu_{3R}}\right)^{1/2} \underset{\sim}{K}_{12}' \end{aligned} \tag{A11}$$

Then one can write

$$e^{i(\underset{\sim}{K}'\cdot\underset{\sim}{R}' + \underset{\sim}{K}_{12}'\cdot\underset{\sim}{q}_{12}')} = e^{i\varepsilon_{12}'\rho_{12}'\cos\chi} \tag{A12a}$$

where, recalling Eqs. (28d) and (35),

$$(\varepsilon_{12}')^2 = \tilde{K}'^2 + \tilde{K}_{12}'^2 = E' - \frac{\hbar^2 k_{12}'^2}{2\mu_{12}} \tag{A12b}$$

while χ is the angle between the six-dimensional vectors $\underset{\sim}{\rho}_{12}'$ [defined by (A4)] and $\underset{\sim}{\varepsilon}_{12}'$, with

$$\underset{\sim}{\varepsilon}_{12}' = \underset{\sim}{\tilde{K}}' \oplus \underset{\sim}{\tilde{K}}_{12}' \tag{A12c}$$

For the purposes of this section the awkward subscripts in $\underset{\sim}{\mathcal{E}}'_{12}$ can be dropped, i.e., henceforth I shall write $\underset{\sim}{\mathcal{E}}_{12}' \equiv \underset{\sim}{\mathcal{E}}'$.

When two-body bound states $u_j(\underset{\sim}{r}_{12})$ exist, therefore, the integral (A10) will be a sum of integrals proportional to [again omitting inessential factors in the integrand and ignoring the distinction between ρ' and ρ_{12}']

$$\int d\rho' d\underset{\sim}{\nu}'_{12} d\underset{\sim}{r}'_{12}\, \rho'^5 \frac{e^{i\rho'\sqrt{E-\epsilon_j}}}{\rho'^{5/2}}\, u_j(\underset{\sim}{r}'_{12})\, e^{-i\underset{\sim}{K}'_f\cdot\underset{\sim}{R}}\, \bar{\Psi}^{(-)*}_{12jf}(\underset{\sim}{\tau};E)\, V_{12}(\underset{\sim}{r}'_{12})\, e^{i\mathcal{E}'\rho'\cos\chi} \tag{A13a}$$

where [compare (A5)] $\underset{\sim}{K}_f'$ is determined solely by the direction $\underset{\sim}{\nu}_{12}'$ along which $\underset{\sim}{r}'$ becomes infinite in (A10), and is wholly unrelated to the original $\underset{\sim}{K}'$ in $\psi_i(E')$. In (A13a) it is presumed that propagation in $u_j(\underset{\sim}{r}_{12}')$ is energetically possible[15]; otherwise $\sqrt{E - \epsilon_j}$ is positive imaginary, $e^{i\rho'\sqrt{E - \epsilon_j}}$ is exponentially decreasing at large ρ', and (A13a) assuredly converges. Also, as explained following (A7), the factor $\bar{\Psi}^{(-)*}$ in (A13a) depends on $\underset{\sim}{\nu}_{12}'$, but not on ρ_{12}'. Consequently the oscillations of the integrand in (A13a) at large $\rho_{12}' \cong \rho'$ dominantly are determined by the phase factors $e^{i\rho'\sqrt{E - \epsilon_j}}$ and $e^{i\mathcal{E}'\rho'\cos\chi}$. Correspondingly, letting the polar axis for $\underset{\sim}{\nu}_{12}'$ lie along $\underset{\sim}{\mathcal{E}}'$ of (A12c), at large ρ' the integral (A13a) must behave essentially like

$$\int d\rho' \rho'^5 \frac{e^{i\rho'\sqrt{E-\epsilon_j}}}{\rho'^{5/2}} \int_0^{\pi} d\chi \sin^4\chi\, e^{i\mathcal{E}'\rho'\cos\chi} \tag{A13b}$$

[For further details concerning the form of the volume element

$\rho'^5 d\rho' d\underset{\sim}{\nu}_{12}'$ in six-dimensional spherical coordinates, see section D.1].
The expression (A13b) is proportional to[30]

$$\int d\rho' \rho'^{5/2} e^{i\rho'\sqrt{E-\varepsilon_j}} \frac{J_2(\varepsilon'\rho')}{(\varepsilon'\rho')^2} \cong \int d\rho' \rho'^{5/2} e^{i\rho'\sqrt{E-\varepsilon_j}} \frac{e^{\pm i\varepsilon'\rho'}}{\rho'^{5/2}} \qquad \text{(A13c)}$$

Hence the integral (A10) fails to converge when two-body bound states can occur. Eq. (A13c) indicates that the non-convergence results from contributions to (A10) behaving like

$$\delta\left(\sqrt{E-\varepsilon_j} - \varepsilon'\right) \qquad \text{(A13d)}$$

which, recalling (A12b), is identical with (47).

At large $\varepsilon'\rho'$, the integration over $d\chi$ in (A13b) is determined primarily by the values of the integrand in the neighborhoods of $\chi = 0$ and π, which are points of stationary phase; in fact, writing $\cos\chi = 1 - \chi^2/2$ immediately shows the contribution to the integral (A13b) from the vicinity of $\chi = 0$ is of the order of

$$e^{i\varepsilon'\rho'} / (\rho')^{\frac{n+1}{2}} \qquad \text{(A13e)}$$

where n [in this case equal to 4] is the power of $\sin\chi$ in the integrand of (A13b). At large ρ', therefore, to justify the step from (A13a) to (A13b), it is necessary only to assume that the comparatively slowly varying (with the angles specifying $\underset{\sim}{\nu}_{12}'$) neglected factors in (A13a) are finite, i.e., neither infinite nor zero, at $\chi = 0$ and π. There seems no reason to doubt the general correctness of this assumption.

At this point I caution the reader that in the very frequent subsequent instances (in this and later Appendices) wherein the behavior of integrals like (A13a) must be estimated, the angular dependence of the slowly varying wave functions, etc., in the integrand will be routinely ignored without further explanation, on the grounds of the arguments in this and the preceding paragraph.

A.5 Eq. (52a) with Three-Body Bound States

If bound states do not exist, the dominant asymptotic behavior of $G^{(+)}$ in (A10) is given by the continuum contribution omitted from (A3). Correspondingly, (A13a) is replaced by

$$\int d\rho' d\underset{\sim}{\nu}'_{12} d\underset{\sim}{r}'_{12} \, \rho'^5 \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} \Psi_f^{(-)*}(\underset{\sim}{r};E) V_{12}(\underset{\sim}{r}'_{12}) e^{i\varepsilon'\rho'\cos\chi} \tag{A14a}$$

As above, Eq. (A14a) behaves essentially like

$$\int d\rho' \rho'^5 \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} \frac{J_2(\varepsilon'\rho')}{(\varepsilon'\rho')^2} \cong \int d\rho' \frac{e^{i\rho'\sqrt{E}} e^{\pm i\varepsilon'\rho'}}{\rho'^{3/2}} \tag{A14b}$$

which converges. Thus the aforementioned divergences in (52a) do not appear when bound states do not occur. On the other hand, suppose there exist three-body bound states

$$u_j(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3) \equiv u_j(\underset{\sim}{r}_{12}, \underset{\approx}{r}_{23}) \equiv u_j(\underset{\sim}{r}_{12}, \underset{\sim}{q}_{12})$$

Then, because u_j necessarily is independent of $\underset{\approx}{R}$, the integral (A10) now behaves like

$$\int d\underset{\sim}{R}' dq'_{\underset{\sim}{12}} d\underset{\sim}{r}'_{12} \frac{e^{iK_jR'}}{R'} u_j(\underset{\sim}{r}'_{12}, \underset{\sim}{q}'_{12}) V_{12}(\underset{\sim}{r}'_{12}) e^{i\underset{\sim}{K}'\cdot\underset{\sim}{R}'} . \tag{A15a}$$

at large R', where

$$\frac{\hbar^2 K_j^2}{2M} = E - \varepsilon_j \tag{A15b}$$

Whereas only $\underset{\sim}{r}_{12}'$ remained effectively finite in (A13a), the quadratically integrable $u_j(\underset{\sim}{r}_{12}', \underset{\sim}{q}_{12}')$ keeps both $\underset{\sim}{r}_{12}'$ and $\underset{\sim}{q}_{12}'$ effectively finite in (A15a). Thus only a three-dimensional vector [namely $\underset{\sim}{R}'$] becomes infinite in (A15a), whereas a six-dimensional vector [namely $\underset{\sim}{\rho}_{12}' \equiv \rho_{12}'\underset{\sim}{\nu}_{12}' \cong \rho'\underset{\sim}{\nu}_{12}'$] became infinite in (A13a). Therefore, performing the angular integration over the direction of $\underset{\sim}{R}'$, (A15a) reduces to

$$\int dR' R' e^{iK_jR'} j_0(K'R') \cong \int dR' e^{iK_jR'} e^{\pm iK'R'} \cong \delta(K_j - K') \tag{A15c}$$

The δ-function in (A15c), like the δ-function (A13d), always vanishes on the energy shell because ε_j is negative. I add that one readily sees these 3-body states do not affect the convergence of (A2), (A8) or (A9a). Moreover, it is clear from this section and preceding sections that the absence of the V_{12} term keeps the laboratory system version of (115b) convergent except when three-body bound states exist, in which event an analysis essentially identical to Eqs. (A15a) - (A15c) applies.

A.6 Eq. (52b)

In Eq. (52b), terms in $\bar{G}^{(+)}$ corresponding to three-body bound states are proportional to $u_j(\underset{\sim}{r}_{12},\underset{\sim}{r}_{23})$, and therefore obviously cannot cause divergence. When two-body bound states $u_j(\underset{\sim}{r}_{12})$ exist, the integral involving V_{12} in (52b) [here letting $\bar{E}'$ be different from $\bar{E}$] is a sum of integrals proportional to

$$\int d\underset{\sim}{q}'_{12}\, d\underset{\sim}{r}'_{12} \frac{e^{iK_{12j}q'_{12}}}{q'_{12}} u_j(\underset{\sim}{r}'_{12}) \bar{\Psi}^{(-)*}_{12jf}(\bar{\underset{\sim}{r}};\bar{E}) V_{12}(\underset{\sim}{r}'_{12}) e^{i(\underset{\sim}{K}'_{12}\cdot\underset{\sim}{q}'_{12}+\underset{\sim}{k}'_{12}\cdot\underset{\sim}{r}'_{12})} \tag{A16a}$$

at large q'_{12}. Eq. (A16a), which has employed (114b) and (115a), is the center of mass analogue of Eq. (A13a). The integral (A16a) is basically similar to (A15a), and reduces to the non-convergent integral

$$\int dq'_{12}\, q'^2_{12} \frac{e^{iK_{12j}q'_{12}}}{q'_{12}} j_0(K'_{12}q'_{12}) \cong \delta(K_{12j}-K'_{12}) \cong \delta\left(\sqrt{\bar{E}-\varepsilon_j}-\sqrt{\bar{E}'-\frac{\hbar^2 k'^2_{12}}{2\mu_{12}}}\right) \tag{A16b}$$

which obviously is the center of mass analogue of (47), and necessarily vanishes on the energy shell $\bar{E} = \bar{E}'$; the interpretation of (A16b)--namely that the argument of the δ-function vanishes when the unprimed and primed speeds, of particle 3 relative to the center of mass of particles 1 and 2, are equal--is immediate, recalling the remarks following Eq. (29), and recognizing that $\hbar^2 K^2_{12j}/\mu_{3R}$ is the center of mass frame kinetic energy when 1, 2 are bound in $u_j(\underset{\sim}{r}_{12})$. In the absence of two-body bound states, recalling Eqs. (116), the integral (A16b) would be replaced by [compare Eqs. (A13c) and (A14b)]

$$\int dq_{12}' q_{12}'^2 \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} j_0(K_{12}' q_{12}') \cong \int d\bar{\rho}' \frac{e^{i\bar{\rho}'\sqrt{E}} e^{\pm iK_{12}'\bar{\rho}'}}{\bar{\rho}'^{3/2}} \tag{A16c}$$

which converges.

A.7 Eq. (42)

For ψ_i of Eq. (21a), the integral in Eq. (42) is $G_F^{(+)} V \Psi_i^{(+)}$, whose V_{12} term is

$$\int d\underset{\sim}{r}' G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}';E) V_{12}(\underset{\sim}{r}_{12}') \Psi_i^{(+)}(\underset{\sim}{r}';E) \tag{A17}$$

where now there is no need to differentiate between the energies of the Green's function and the wave function. The incident plane wave part of $\Psi_i^{(+)}$ in (A17) yields an integral behaving essentially like (A14a). In other words, the incident part of $\Psi_i^{(+)}$ makes (A17) a convergent integral. Now the integral (A14a) converges because $e^{i\mathcal{E}'\rho'\cos\chi}$, though of absolute value unity, produces a factor of order $\rho'^{-5/2}$ in (A14b) after integrating over $d\underset{\sim}{\nu}_{12}'$. All the scattered (non-incident) parts of $\Psi_i^{(+)}(\underset{\sim}{r}')$ decrease at least as rapidly as $\psi_i(\underset{\sim}{r}')$ when $\underset{\sim}{r}' \to \infty$, and apparently oscillate sufficiently that after integrating over $d\underset{\sim}{\nu}_{12}'$ they also yield convergent analogues of (A14b). For example, replacing $\Psi_i^{(+)}$ in (A17) by $\Phi_{12}^{(+)}$ of Eqs. (61) and (72) again yields an integral behaving like (A14a); replacing $\Psi_i^{(+)}$ in (A17) by $\Phi_{23}^{(+)}$ yields an integral which is even more rapidly convergent than (A17).

There remains for consideration the result of replacing $\Psi_i^{(+)}$

in (A17) by $\Phi_i^{s(+)}$, as defined in (62). If bound states $u_j(\underset{\sim}{r}_{12})$ do not exist, that part of $\Phi_i^{(+)}$ which is truly three-body and has been termed $\Phi_i^{t(+)}$ obviously causes no difficulty in (A17), because by definition [recall the **introduction** to Chapter 4]

$$\lim_{r'\to\infty \| \underset{\sim}{\nu}'} \Phi_i^{t(+)}(\underset{\sim}{r}') = \lim_{r'\to\infty \| \underset{\sim}{\nu}'} e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'} \bar{\Phi}_i^{t(+)}(\bar{\underset{\sim}{r}}') \cong e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'} \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} \tag{A18a}$$

Eq. (A18a) as it stands, even before angular integration, contains the factor $\bar{\rho}'^{-5/2}$ ($\cong \rho'^{-5/2}$ at infinity along $\underset{\sim}{\nu}_{12}'$) needed to make (A17) convergent. However, $\Phi_i^{t(+)}$ also can represent recombination reactions (17a), i.e., can propagate in bound states. Thus, when bound states $u_j(\underset{\sim}{r}_{12})$ exist, Eq. (A18a) is very seriously incorrect along directions $\underset{\sim}{\nu}_{12}'$, and must be replaced by

$$\lim_{r'\to\infty \| \underset{\sim}{\nu}_{12}'} \Phi_i^{t(+)}(\underset{\sim}{r}') \cong e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'} u_j(\underset{\sim}{r}_{12}') \frac{e^{iK_{12j}q_{12}'}}{q_{12}'} \tag{A18b}$$

This complication was ignored in the preceding paragraph because energy-momentum conservation prevents $\Phi_{12}^{(+)}(\underset{\sim}{r}_{12})$ from propagating in bound states $u_j(\underset{\sim}{r}_{12})$.

Inserting (A18b) into (A17) gives an integral behaving like

$$\int d\underset{\sim}{R}'\, d\underset{\sim}{q}_{12}'\, d\underset{\sim}{r}_{12}' \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} \Psi_f^{(-)*}(\underset{\sim}{r}, E)\, V_{12}(\underset{\sim}{r}_{12}') e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'} u_j(\underset{\sim}{r}_{12}') \frac{e^{iK_{12j}q_{12}'}}{q_{12}'} \tag{A19a}$$

The three-dimensional (rather than six-dimensional) plane wave factor $e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'}$ makes it impractical to immediately introduce spherical coordinates in the six-dimensional space spanned by $\underset{\sim}{\rho}_{12}' \equiv (\underset{\sim}{\tilde{R}}', \underset{\sim}{\tilde{q}}_{12}')$ [recall Eqs. (A4) and (A6)], as has been done in previous integrals in this Appendix. However, the integrations over the directions of $\underset{\sim}{q}_{12}'$ and $\underset{\sim}{R}'$ in (A19a) can be performed immediately, yielding [still omitting inessential factors]

$$\int dR' \int dq' R'^2 q'^2 \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} j_0(KR') \frac{e^{iK_{12j}q_{12}'}}{q_{12}'} \tag{A19b}$$

Next, replace R', q_{12}' by $\tilde{R}'$, $\tilde{q}_{12}'$ respectively, and then introduce polar coordinates in the $\tilde{R}'$, $\tilde{q}_{12}'$ plane, i.e., consistent with Eqs. (A4) and (A6),

$$\begin{aligned} \tilde{R}' &= \rho_{12}' \cos\varphi \\ \tilde{q}_{12}' &= \rho_{12}' \sin\varphi \end{aligned} \tag{A20}$$

As in (A7) and (A13a), the distinction between ρ_{12}' and ρ' now can be ignored, because in effect $\underset{\sim}{r}_{12}'$ is being kept constant as R' and q' become large. Hence Eqs. (A20), with the aid of (A11), reduce (A19b) to a sum of two integrals, of form

$$\int^{\infty} d\rho' \int_0^{\pi/2} d\varphi\, \rho'^5 \sin^2\varphi \cos^2\varphi \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} \frac{e^{\pm i\tilde{K}\rho'\cos\varphi}}{\rho'\cos\varphi} \frac{e^{i\tilde{K}_{12j}\rho'\sin\varphi}}{\rho'\sin\varphi} \tag{A21a}$$

$$= \int^{\infty} d\rho' \int_0^{\pi/2} d\varphi \sin\varphi\cos\varphi \frac{e^{i\rho'\sqrt{E}}}{\rho'} e^{i\varepsilon_j \rho' \cos(\varphi - \varphi_\pm)} \tag{A21b}$$

In Eq. (A21a) I have made it explicit that we are interested only in the behavior of the integral at large ρ'; anyway, at small values of ρ' the substitutions [e.g., (A18b) for $\Phi_i^{t(+)}$] leading to (A21a) are unjustified. Because R', ρ' are each intrinsically positive in (A19b), $0 \leq \phi \leq \pi/2$ in (A21a). In (A21b), the quantity $\mathcal{E}_j \equiv \mathcal{E}_{12j}$ is defined in terms of $\tilde{K}$, $\tilde{K}_{12j}$ by the analogue of (A12b), therewith making the definitions of $\phi_\pm$ obvious.

At large $\mathcal{E}_j\rho'$ the behavior of the integral (A21b) is determined primarily by the values of the integrand in the neighborhood of $\sin(\phi - \phi_\pm) = 0$, which are the points of stationary phase. Referring to the discussion following Eq. (A13d), one sees that if (A21b) has a point of stationary phase at $0 < \phi < \pi/2$, the integral over ϕ in (A21b) is of order $\rho'^{-1/2}$ [recalling (A13e) and recognizing that the factors $\sin\phi, \cos\phi$ now are nonvanishing at the point of stationary phase]. When (A21b) has points of stationary phase at $0 < \phi < \pi/2$, therefore, the integrand--after performing the integration over ϕ--is of order $\rho'^{-3/2}$, which converges. If the points of stationary phase lie at 0 or $\pi/2$, or entirely outside the range $0 \lessgtr \phi \lessgtr \pi/2$, the integration over ϕ in (A21b) will yield a result decreasing even more rapidly than $\rho'^{-1/2}$. Consequently the $\Phi_i^{t(+)}$ contribution to (A17) converges.

I remark that the influence of three-body bound states need not be considered in (A17) because energy-momentum conservation prevents three incident particles 1, 2, 3 from combining into a three-body $u_j(\underset{\sim}{r}_{12}, \underset{\sim}{r}_{23})$, although such three-body terms necessarily appear in the Green's function $G^{(+)}(E)$, and therefore had to be considered in examination of the convergence of, e.g., (A10). Moreover, all parts of $\Phi_i^{(+)}$ which conceivably can propagate in two-body bound states already have been examined. Therefore the preceding arguments in this section are sufficient to show Eq. (42) has no divergences associated with bound states [of the type found in Eq. (52a)]. To complete the demonstration that (A17) converges, it still is necessary to examine the contribution to (A17) made by the double-scattering terms in $\Phi_i^{s(+)}$, which [recall subsection 4.1.3] are of order $\bar{\rho}^{-2}$ in the limit $\bar{r} \to \infty$. In other words

$$\lim_{r' \to \infty \parallel \underset{\sim}{r}'} \Phi_i^{s(+)}(\underset{\sim}{r}') \cong e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'} \frac{e^{i\Lambda(\bar{\underset{\sim}{\rho}}')}}{\bar{\rho}'^2} \tag{A22}$$

where the phase factor $\Lambda(\bar{\underset{\sim}{\rho}}')$ cannot depend on $\underset{\sim}{R}'$. Presumably the exact form of Λ ultimately though arduously could be found via the analysis in section E.3, but for our present purposes (A22) suffices. Replacing $\Psi_i^{(+)}$ in (A17) by (A22) at large $\underset{\sim}{R}'$, q_{12}' and finite $\underset{\sim}{r}_{12}'$ yields the integral [compare (A19a)]

$$\int d\underset{\sim}{R}' d\underset{\sim}{q}_{12}' d\underset{\sim}{r}_{12}' \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} \Psi_f^{(-)*}(\underset{\sim}{r};E) V_{12}(\underset{\sim}{r}_{12}') e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}'} \frac{e^{i\Lambda}}{q_{12}'^2} \tag{A23}$$

Recalling (A7), one sees that the integrand in (A23) already is of order ρ'^{-1}. From Eqs. (A19) - (A21) it is evident that—even if Λ is constant—the integration over five angles (in the six-dimensional space of $\underset{\sim}{R}'$, $\underset{\sim}{q}_{12}'$), which reduces (A23) to an integral over $d\rho'$ alone, cannot fail to produce at least one factor of $\rho'^{-1/2}$. Hence (A17) is convergent, Q.E.D. Similar—rather simpler—arguments show the center of mass version of (42) also is convergent.

A.8 Eq. (43b) and Uniform Convergence

Consider Eq. (43b). Let me introduce, to simplify the notation,

$$Y^{(+)}(\underset{\sim}{r};r';E) = \int d\underset{\sim}{\nu}' r'^8 G_i^{(+)}(\underset{\sim}{r},\underset{\sim}{r}';E) V_i(\underset{\sim}{r}') \Psi_i^{(+)}(\underset{\sim}{r}';E) \tag{A24a}$$

$$Y(\underset{\sim}{r};r';E+i\varepsilon) = \int d\underset{\sim}{\nu}' r'^8 G_i(\underset{\sim}{r};\underset{\sim}{r}';E+i\varepsilon) V_i(\underset{\sim}{r}') \Psi_i(\underset{\sim}{r}';E+i\varepsilon) \tag{A24b}$$

Then, using (89a), one sees Eq. (43b) takes the form

$$\lim_{\varepsilon\to 0}\int_0^\infty dr'\, Y(\underset{\sim}{r};r';E+i\varepsilon) = \int_0^\infty dr'\, Y^{(+)}(\underset{\sim}{r};r';E) \tag{A25a}$$

In Eqs. (A24), the integration over $d\underset{\sim}{\nu}'$ involves a finite angular range only [section D.1]. Thus the integrals (A24) may be presumed to converge; correspondingly, there would be no difficulty in justifying the interchange of order of integration and limit $\varepsilon \to 0$ for the pair of integrals on the right sides of (A24), i.e., in justifying

$$\lim_{\varepsilon\to 0} Y(\underset{\sim}{r};r';E+i\varepsilon) = Y^{(+)}(\underset{\sim}{r};r';E) \tag{A25b}$$

The question whether or not Eq. (43b) is valid [i.e., the question whether or not (A25a) is valid] arises only because (A25a) involves an integration over an infinite range of r'.

The assertion that the integral on the right side of (A25a) converges at large r' for specified $\underset{\sim}{r}'$,E means precisely the following. There exists a number N, depending on $\underset{\sim}{r}'$, E, such that--given any $\eta > 0$ however small--one can find an L_o depending on η for which

$$\left| N(\underset{\sim}{r};E) - \int_0^L dr' \, Y^{(+)}(\underset{\sim}{r};r';E) \right| < \eta \tag{A26a}$$

whenever $L > L_o(\eta)$. The value assigned to the integral on the right side of (A25a) is of course the number $N(\underset{\sim}{r},E)$ in (A26a). But granted this assignment [which now provides a definition of the previously undefined expression on the right side of (A25a)] will be made, introduction of the symbol $N(\underset{\sim}{r},E)$ is superfluous; one may as well symbolize this number by the original expression on the right side of (A25a). With this understanding, (A26a) can be rewritten as

$$\left| \int_0^\infty dr' \, Y^{(+)}(\underset{\sim}{r};r';E) - \int_0^L dr' \, Y^{(+)}(\underset{\sim}{r};r';E) \right| < \eta \quad \text{if} \quad L > L_o(\eta) \tag{A26b}$$

Similarly, the assertion that the integral on the left side of (A25a) converges at large r' for specified $\underset{\sim}{r}'$, E and $\varepsilon > 0$ means

$$\left| \int_0^\infty dr' \, Y(\underset{\sim}{r};r';E+i\varepsilon) - \int_0^L dr' \, Y(\underset{\sim}{r};r';E+i\varepsilon) \right| < \eta \quad \text{if} \quad L > L_\varepsilon(\eta) \tag{A26c}$$

where the subscript ε in $L_\varepsilon(\eta)$ makes explicit the dependence on ε (as well as on η) of the smallest allowed upper limit in the second integral under the absolute value sign. Of course, in general both L_o and L_ε depend also on $\underset{\sim}{r}$,E, but in the subsequent discussion $\underset{\sim}{r}$,E will be held fixed.

The set of integrals on the left side of (A25a) is said to be uniformly convergent[21] in a domain about $\varepsilon = 0$ if there exists an $\varepsilon_m > 0$ such that--given any $\eta > 0$--for any ε in the open interval $0 < \varepsilon \leq \varepsilon_m$ one can find an L_m depending on η but independent of ε for which

$$\left| \int_0^{\infty} dr' \, Y(\underset{\sim}{r}; r'; E+i\varepsilon) - \int_0^{L} dr' \, Y(\underset{\sim}{r}; r'; E+i\varepsilon) \right| < \eta \quad \text{if } L > L_m(\eta), \quad 0 < \varepsilon \leq \varepsilon_m \tag{A27}$$

The point $\varepsilon = 0$ is excluded in (A27) because $Y(\underset{\sim}{r}; r'; E + i\varepsilon)$ may not be well-defined when $\varepsilon = 0$, at which value of ε, therefore, a limiting relation such as (A25b) is required to prescribe $Y(\varepsilon = 0)$ and to assign it a sensible value. Just this situation obtains, of course, for the functions $\Psi_i(E + i\varepsilon)$ and $G_i(E + i\varepsilon)$ in (A24), where Eqs. (8a) and (26a) respectively must be introduced because the relations (8b) and (10b) are **not** prescriptive at $\varepsilon = 0$.

Granting that the integral on the left side of (A25a) converges for every $\varepsilon > 0$, then for otherwise arbitrary functions G_i, V_i, Ψ_i in (A24b) the mere fact that (A25a) converges when $Y^{(+)}$ is defined by (A25b) is not sufficient to ensure that the integrals on the left side of (A25a) are uniformly convergent in a domain about $\varepsilon = 0$. A simple illustration of this assertion is provided by the set of integrals

$$\int_0^{\infty} dx \, f(x, \varepsilon) \equiv \int_0^{\infty} dx \, \frac{2}{\varepsilon x^3} \, e^{-\varepsilon^{-1} x^{-2}} \tag{A28a}$$

The integrals (A28a) converge for every $\varepsilon > 0$; specifically for any such ε

$$\int_0^\infty dx\, f(x,\varepsilon) = 1 \tag{A28b}$$

Moreover, since for all $x \geq 0$

$$f^{(+)}(x) = \lim_{\varepsilon\to 0} f(x,\varepsilon) = \lim_{\varepsilon\to 0} \frac{2}{\varepsilon x^3}\, e^{-\varepsilon^{-1}x^{-2}} = 0 \tag{A28c}$$

we have also

$$\int_0^\infty dx\, f^{(+)}(x) = 0 \tag{A28d}$$

where the integral on the left side of (A28d) surely converges. The analogue of (A27) for the set of integrals (A28a) is

$$\left| 1 - \int_0^L dx\, f(x,\varepsilon) \right| < \eta \quad \text{if } L > L_m(\eta), \; 0 < \varepsilon \leq \varepsilon_m \tag{A29a}$$

But the left side of (A29a) is

$$1 - \left\{ e^{-\varepsilon^{-1}x^{-2}} \Big|_0^L \right\} = 1 - e^{-\frac{1}{\varepsilon L^2}} \tag{A29b}$$

Thus, for (A29a) to be satisfied it is necessary that

$$L > \left[\varepsilon \log\left(\frac{1}{1-\eta}\right) \right]^{-1/2} \tag{A29c}$$

Eq. (A29c) shows that it is not possible to find an $L_m(\eta)$, independent of ε, for which (A29a) will hold as ε becomes arbitrarily close to zero. Correspondingly, because (A29a) fails, the interchange of order of integration and limit $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\int_0^\infty dx\, f(x,\varepsilon) = \int_0^\infty dx \lim_{\varepsilon\to 0} f(x,\varepsilon) \equiv \int_0^\infty dx\, f^{(+)}(x) \tag{A29d}$$

[which is the analogue of (A25a)] need not hold; indeed, (A29d) is not true, as comparison of (A28b) and (A28d) shows.

However, I argue (I am not able to prove) that when G_i, V_i, Ψ_i in (A24b) are respectively the Green's function, potential, scattering wave function defined in this paper, then the mere knowledge that the right side of (A25a) is convergent for $Y^{(+)}$ given by (A25b) is sufficient to ensure the criterion (A27) for uniform convergence holds. In essence, the argument is that the convergence of the integral on the right side of (A25a)--which for ψ_i of Eq. (21a) is identical with the integral in Eq. (42) examined in section A.7--tends to be slower than the convergence (for $\varepsilon > 0$) of the integrals on the left side of (A25a). For $\varepsilon > 0$, $G_i(\underset{\sim}{r}')$ in (A24b) is exponentially decreasing as $r' \to \infty$ along any $\underset{\sim}{\nu}'$; similarly the ε-dependent parts of $\Psi_i(E + i\varepsilon)$ [e.g., the truly three-body part $\Phi_i^{t(+)}(E + i\varepsilon)$ whose limit as $\varepsilon \to 0$ is $\Phi_i^{t(+)}$] will tend to be exponentially decreasing at large r'. Correspondingly, the integral on the left side of (A25a) should converge for $\varepsilon > 0$ even when G_i, Ψ_i in (A24b) are replaced by their absolute values. In (A24a), on the other hand, the functions $G_i^{(+)}$ and $\Psi_i^{(+)}$ are not exponentially decreasing at infinite r', but merely rapidly oscillating. In fact, along directions $\underset{\sim}{\nu}' = \underset{\sim}{\nu}_{12}'$ where $V_i(\underset{\sim}{r}')$ is not small, the factor r'^8 actually causes the absolute magnitude of the integrand in (A24a) to increase as $r' \to \infty$, as section A.7 has made very clear; certainly the right side of (A25a) would not converge if $G_i^{(+)}$, $\Psi_i^{(+)}$ in (A24a) were replaced by their absolute values. When $G_i^{(+)}$, $\Psi_i^{(+)}$ have their actual values, the right side of (A25a) converges only because the aforementioned oscillations of $G_i^{(+)}\Psi_i^{(+)}$, when integrated over a range of $d\underset{\sim}{\nu}'$ in the vicinity of $\underset{\sim}{\nu}' = \underset{\sim}{\nu}_{12}'$, bring down enough powers of r' to make $Y^{(+)}(\underset{\sim}{r}')$ integrable at infinite r' despite the diverging factor r'^8.

In other words, I am claiming: (i) the convergence of (A24b) in some small range $0 < \varepsilon \leq \varepsilon_m$ can be taken for granted; and (ii) although exceptionally cancelling oscillating mathematical functions doubtless can be constructed, in scattering theory one expects that for $0 < \varepsilon \leq \varepsilon_m$ and sufficiently large L the (exponentially decreasing with increasing L) contribution to the left side of (A25a) from $r' > L$ increases as $\varepsilon \to 0$, but does not exceed the corresponding contribution to the right side of (A25a). Or since large L corresponds to small η in Eqs. (A26), the above claim (ii) means that for sufficiently small η it should be possible to find an $L_o(\eta)$ in (A26b) which for $0 < \varepsilon \leq \varepsilon_m$ exceeds the (presumably increasing with decreasing ε) maximum required $L_\varepsilon(\eta)$ in (A26c). But this last assertion is just another way of saying that (A27) holds. Note that in the set of integrals in (A28a), which is not uniformly convergent, the contribution from $x > L$ still increases as $\varepsilon \to 0$ for fixed L, but that this contribution now always exceeds the corresponding contribution (namely, zero) to the integral (A28d). As a result it is not possible to assert that $L_o(\eta)$ can provide an upper bound to $L_\varepsilon(\eta)$; in fact, as (A29c) shows, $L_\varepsilon(\eta)$ is not bounded in $0 < \varepsilon \leq \varepsilon_m$.

Although the discussion in this section thus far has referred specifically to the integrals (A24), which correspond to the known-to-converge integral in (42), it is evident that the same discussion pertains to any integral [e.g., Eq. (65b), examined in section A.2] involving Green's functions, potentials and wave functions at real energies. If this integral converges, then the set of integrals obtained by replacing $G^{(+)}(E)$ by $G(E + i\varepsilon)$, etc., should be uniformly convergent in a domain $0 < \varepsilon \leq \varepsilon_m$, because for sufficiently small η the quantity $L_o(\eta)$ [known to exist because it has been postulated that the integral is convergent at real energies] will provide an upper bound to the maximum required $L_\varepsilon(\eta)$. Actually, the foregoing discussion suggests that the domain of uniform convergence should extend over all $0 < \varepsilon$. As $E + i\varepsilon$ moves sufficiently far into the complex plane, however, $\Psi_i(E + i\varepsilon)$, $\underset{\sim}{T}(E + i\varepsilon)$, etc., can develop singularities which will negate some of the assertions which have been made. Thus it is more accurate to assume merely that, for each E at which the right side of (A25a) converges, there will be an $\varepsilon_m > 0$ such that the integrals on the left side of (A25a) converge uniformly in $0 < \varepsilon \leq \varepsilon_m$, where now the integrands in (A24) correspond to any convergent real-energy integral [e.g., (65b)]. In any event, the existence of such a domain $0 < \varepsilon \leq \varepsilon_m$ of uniform convergence is all that is required for the purposes of this section.

I now go on to show that uniform convergence, Eq. (A27), will guarantee (A25a), i.e., will guarantee the validity of interchange of order of integration and limit $\varepsilon \to 0$ in (A25a) [and, therefore, in the original scattering theory relation to which (A25a) corresponds], provided of course that the limit (A25b) exists and the right side of (A25a) converges. The proof is given here for completeness, because it is fundamental to the approach of this paper, and because I have found it difficult to locate a reference which is readable, readily accessible and wholly pertinent.

To prove (A25a) I must show that—given any $\eta_1 > 0$—I can find an $\varepsilon_1(\eta_1)$ such that [now dropping the awkward and unnecessary variables $\underset{\sim}{r}$,E]

$$\left|\int_0^\infty dr'\, Y^{(+)}(r') - \int_0^\infty dr'\, Y(r';\varepsilon)\right| < \eta_1 \quad \text{if } 0 < \varepsilon < \varepsilon_1(\eta_1) \leq \varepsilon_m \tag{A30}$$

In (A30) one must keep in mind the definitions of the infinite integrals therein, as explained following Eq. (A26a). Thus one cannot immediately write

$$\int_0^\infty dr'\, Y^{(+)}(r') - \int_0^\infty dr'\, Y(r';\varepsilon) = \int_0^\infty dr' \left[Y^{(+)}(r') - Y(r';\varepsilon) \right] \tag{A31a}$$

However, one can write

$$\begin{aligned} \int_0^\infty dr'\, Y^{(+)}(r') - \int_0^\infty dr'\, Y(r';\varepsilon) = \Big\{ &\left[\int_0^\infty dr'\, Y^{(+)}(r') - \int_0^L dr'\, Y^{(+)}(r') \right] \\ &- \left[\int_0^\infty dr'\, Y(r';\varepsilon) - \int_0^L dr'\, Y(r';\varepsilon) \right] \\ &+ \left[\int_0^L dr'\, Y^{(+)}(r') - \int_0^L dr'\, Y(r';\varepsilon) \right] \Big\} \end{aligned} \tag{A31b}$$

In the last bracket in (A31b) it is legitimate to write

$$\int_0^L dr'\, Y^{(+)}(r') - \int_0^L dr'\, Y(r';\varepsilon) = \int_0^L dr' \left[Y^{(+)}(r') - Y(r';\varepsilon) \right] \tag{A32a}$$

Moreover, because the limit (A25b) is postulated to exist, it follows that--given any $\eta_2 > 0$--there exists an $\varepsilon_2(\eta_2)$ such that

$$\left| Y^{(+)}(r') - Y(r';\varepsilon) \right| < \eta_2 \quad \text{if } 0 < \varepsilon < \varepsilon_2(\eta_2) \leq \varepsilon_m \tag{A32b}$$

Now choose η in (A26b) equal to $\eta_1/6$, where η_1 is the assigned value on the right side of (A30). Then, for the first bracket on the right side of (A31b),

$$\left| \int_0^\infty dr'\, Y^{(+)}(r') - \int_0^L dr'\, Y^{(+)}(r') \right| < \frac{\eta_1}{6} \quad \text{if } L > L_0(\eta_1/6) \tag{A33a}$$

Similarly, choose η in (A27) equal to $\eta_1/6$. Then, for the second bracket on the right side of (A31b),

$$\left| \int_0^\infty dr'\, Y(r';\varepsilon) - \int_0^L dr'\, Y(r';\varepsilon) \right| < \eta_1/6 \quad \text{if } L > L_m(\eta_1/6),\ 0 < \varepsilon \leq \varepsilon_m \tag{A33b}$$

Next let L, which was not specified in (A31b), be fixed at some value consistent with both (A33a) and (A33b), i.e., L is fixed at some value exceeding the larger of $L_o(\eta_1/6)$ and $L_m(\eta_1/6)$. The important point is that, because of the postulated uniform convergence, Eqs. (A33a) and (A33b) can be simultaneously satisfied by appropriate choice of an $L(\eta_1/6)$ independent of ε. Finally, in (A32b), choose

$\eta_2 = \eta_1/6L$, where $L \equiv L(\eta_1/6)$. Since L is a finite (though possibly very large) number independent of ε, this choice of η_2 is legitimate. Then, for the third bracket in (A31b), using (A32a),

$$\begin{aligned}
&\left|\int_0^L dr'\,Y^{(+)}(r') - \int_0^L dr'\,Y(r';\varepsilon)\right| \\
&= \left|\int_0^L dr'\left[Y^{(+)}(r') - Y(r';\varepsilon)\right]\right| \\
&< \int_0^L dr'\left|Y^{(+)}(r') - Y(r';\varepsilon)\right| \\
&< \int_0^L dr'\,\frac{\eta_1}{6L} = \frac{\eta_1}{6} \qquad \text{if } 0 < \varepsilon < \varepsilon_2(\eta_1/6L) \leq \varepsilon_m
\end{aligned} \tag{A33c}$$

Hence, if $\varepsilon < \varepsilon_2(\eta_1/6L)$, Eqs. (A31b) and the three inequalities (A33) imply

$$\left|\int_0^\infty dr'\,Y^{(+)}(r') - \int_0^\infty dr'\,Y(r';\varepsilon)\right| < \frac{\eta_1}{6} + \frac{\eta_1}{6} + \frac{\eta_1}{6} = \frac{\eta_1}{2} < \eta_1 \quad \text{if } 0<\varepsilon<\varepsilon_2(\eta_1/6) \leq \varepsilon_m \tag{A34}$$

Eq. (A34) demonstrates that the desired inequality (A30) will hold, provided $\varepsilon_1(\eta_1)$ in (A30) is chosen $\leq \varepsilon_2(\eta_1/6L)$.

A.9 Alternative Criterion for Interchange

In previous publications[2,16] it was found that the vanishing of a surface integral at infinity--of the type (44b) or (66)--typically is the condition for the validity (at real energies E) of identities obtained via the operator manipulations (at complex energies E + iε) commonly employed in scattering theory[33]. In practice one sees that use of these operator manipulations at real energies $\varepsilon = 0$ almost invariably involves the implicit assumption that interchange of order of integration and limit $\varepsilon \to 0$ is permissible. Recalling section A.8 and the discussion following Eqs. (52), it follows that in configuration space scattering theory the aforementioned surface integral at infinity should vanish whenever the corresponding (volume) integral--wherein interchange of order of integration and limit $\varepsilon \to 0$ is being questioned--converges at $\varepsilon = 0$; otherwise the results of the present publication and previous work[2,16] might be inconsistent.

It is awkward to attempt a general proof that convergence of the volume integral at $\varepsilon = 0$ indeed is associated with vanishing of the surface integral examined previously. I have examined a number of cases, however, and--when the surface integral can be evaluated at all--invariably have found that this postulated association indeed occurs. In other words, it gratifyingly appears to be true that the present and previous work are not inconsistent--at least insofar as the legitimacy of interchange of order of integration and limit $\varepsilon \to 0$ is concerned. The following two subsections (of this present section A.9) discuss a few simple illustrative examples of this (perhaps surprising) association, with the intent of making its existence more believable. I stress that more complicated examples are not difficult to find, starting from various previously demonstrated[2,16] identities involving surface integrals at infinity.

A.9.1 Validity of Eqs. (44)

The most immediate illustration of the postulated association is provided by comparison of Eqs. (44) and (50), where the incident wave ψ_i of Eq. (9) now need not be a plane wave, i.e., where the associated Green's function G_i of Eqs. (10) now is not necessarily identical with the free space G_F. According to sections 2.2 and A.8, recalling especially the discussion of Eqs. (51), $\Phi_i^{(+)}(\underset{\sim}{r})$ from (44a) is identical with the "true" $\Phi_i^{(+)}(\underset{\sim}{r})$ defined by Eqs. (8) and (11a) whenever the integral (44a) converges. On the other hand, if as is always presumed[17] $\Psi_i^{(+)}$ defined by Eq. (8a) satisfies the original Schrodinger equation (7), then[16] directly from Eqs. (7), (9) and (27d), the "true" scattered part $\Phi_i^{(+)}$ of Eq. (11a) satisfies

$$\Phi_i^{(+)}(\underset{\sim}{r}) = -\int d\underset{\sim}{r}'\, G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')V_i(\underset{\sim}{r}')\psi_i(\underset{\sim}{r}') + \mathcal{J}\left[G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'), \Phi_i^{(+)}(\underset{\sim}{r}')\right] \qquad \text{(A35)}$$

where the last term in (A35) is the surface integral defined by Eq. (44b), but integrated here over the sphere at infinity in $\underset{\sim}{r}'$-space. Eq. (A35) implies, as stated in the text, that Eq. (44a) holds only when Eq. (44b) holds, and vice versa. However, the preceding remarks in this paragraph now further imply that for consistency Eq. (44b) must hold, i.e., $\mathcal{J}$ in (A35) must vanish, whenever the integral (44a)--which is also the volume integral in (A35)--converges. Of course--as already remarked in section 2.2 [preceding Eqs. (49)]--when the integral in (A35) does not converge, Eq. (A35) is not really meaningful, and the manipulations leading to (A35) cannot have been mathematically acceptable.

The preceding paragraph has explained the necessity of the association--in this case between the convergence of $G^{(+)}V_i\psi_i$ and the vanishing of $\mathcal{J}(G^{(+)},\Phi_i^{(+)})$--postulated in the opening paragraphs of the present section A.9. To specifically demonstrate the association, however, i.e., to demonstrate the desired consistency, one would like to actually evaluate $\mathcal{J}$ in (A35), so as to verify that it really does vanish when the integral in (A35) converges. For the three-particle systems on which this publication is concentrating, this desired verification is readily accomplished when the collision is two-body, as, e.g., in Eq. (17b). In other words, let me consider the case that particle 1 is incident on an initially bound state of 2 and 3. Then, instead of Eqs. (33b) and (21b) respectively,

$$\bar{\psi}_i = e^{i\underset{\sim}{K}_{23}\cdot\underset{\sim}{q}_{23}} u_j(\underset{\sim}{r}_{23}) \tag{A36a}$$

$$V_i = V - V_{23} = V_{12} + V_{31} \tag{A36b}$$

where [although the precaution really is not necessary] to avoid any possibility of evidently irrelevant divergences associated with the usual total momentum conserving $e^{i\underset{\sim}{K}\cdot\underset{\sim}{R}}$ factors in ψ_i and $\Phi_i^{(+)}$, I shall consider the center of mass frame version of (A35). With Eqs. (A36) holding, the volume integral in [the center of mass frame version of] (A35) surely converges, as explained in connection with Eq. (115b). Correspondingly, with the incident wave (A36a) the surface integral $\mathcal{J}(\bar{G}^{(+)},\bar{\Phi}_i^{(+)})$ surely vanishes because in the limit $\bar{\underset{\sim}{r}} \to \infty \parallel$ any $\bar{\underset{\sim}{\nu}}_f$--even directions $\bar{\underset{\sim}{\nu}}_f = \bar{\underset{\sim}{\nu}}_{\alpha\beta}$ keeping $r_{\alpha\beta}$ finite--$\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ behaves like $\bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$, i.e., because, as is physically obvious anyway, the center of mass frame two-body scattered part $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ really is everywhere outgoing. A formal proof of this asserted outgoing property of $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ can be given along the lines of section C.4 below; one readily sees that with Eqs. (A36) the contribution to $\bar{G}^{(+)}\bar{V}_i\bar{\psi}_i$ of (A35) from the region $\bar{r}' \gg \bar{r}$ is utterly negligible compared to the contribution from the region $\bar{r}' < \bar{r}$, implying that the order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \parallel \bar{\underset{\sim}{\nu}}_f$ legitimately can be interchanged in the present $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}}) = -\bar{G}^{(+)}V_i\bar{\psi}_i$ of Eq. (A35).

For three-body collisions of particles 1, 2, 3, where ψ_i and V_i in (A35) are given by Eqs. (21a) and (21b) respectively, the desired verification is more difficult to achieve. Recalling sections A.4 - A.5, we need examine only the situation that two-body and three-body bound states do not occur; otherwise the integral in (A35) does not converge. Unfortunately, even when no bound states occur, the surface integral $\mathcal{S}(G^{(+)},\Phi_i^{(+)})$ now is not easily evaluated, because the asymptotic behavior of $\Phi_i^{(+)}$ now is so complicated, as has been discussed throughout the body of the text. Indeed, even in the center of mass frame with bound states absent, it is not immediately obvious that the surface integral $\mathcal{S}(G^{(+)},\Phi_i^{(+)})$ has been meaningfully defined, although it is evident from the derivation[16] of (A35) that if the integral $G^{(+)}V_i\psi_i$ truly converges, then the surface integral (44b) must approach a limit as the radius of the spherical surface in $\underset{\sim}{r}$-space approaches infinity.

For ψ_i of (21a), therefore, I am not able to specifically demonstrate the necessary association between the convergence of $G^{(+)}V_i\psi_i$ and the vanishing of $\mathcal{S}(G^{(+)},\Phi_i^{(+)})$ in (A35), when bound states do not occur. On the other hand, I have no reason to doubt the fact that Eq. (44b) holds in the case of present interest, namely, no bound states and ψ_i given by (21a). In fact, various manipulations [some of which resemble those in subsection A.9.2 below] indicate that a whole host of improbable inconsistencies could be proved if--still with ψ_i of (21a)--$\mathcal{S}(G^{(+)},\Phi_i^{(+)})$ failed to vanish when $G^{(+)}V_i\psi_i$ converges.

A.9.2 Uniqueness of Solution to Eq. (42)

Let ψ_i be specified as at the beginning of the previous subsection, namely as a solution to Eq. (9), but not necessarily a plane wave. Similarly, $\psi_f(E)$ solves

$$(H_f - E)\psi_f \equiv (H - V_f - E)\psi_f = 0 \tag{A37}$$

where H_f need not be identical with H_i. Let $\Psi_f(E + i\varepsilon)$ be the solution to the Lippmann-Schwinger integral equation corresponding to the incident $\psi_f(E)$. Then, comparing with Eqs. (8b) and (49b),

$$\Psi_f(E+i\epsilon) = \psi_f(E) - G_f(E+i\epsilon)V_f\Psi_f(E+i\epsilon) \tag{A38a}$$

implies

$$(H-E-i\epsilon)\Psi_f(E+i\epsilon) = -i\epsilon\,\psi_f(E) \tag{A38b}$$

Eq. (A38b) can be rewritten in the form

$$(H_i-E-i\epsilon)\Psi_f(E+i\epsilon) = -i\epsilon\,\psi_f(E) - V_i\Psi_f(E+i\epsilon) \tag{A39a}$$

which permits the inference that Ψ_f also satisfies the alternative [to (A38a)] integral equation

$$\Psi_f(E+i\epsilon) = -i\epsilon\,G_i(E+i\epsilon)\psi_f(E) - G_i(E+i\epsilon)V_i\Psi_f(E+i\epsilon) \tag{A39b}$$

Taking the limit of (A39b) as $\varepsilon \to 0$, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\Psi_f(E+i\epsilon) &\equiv \Psi_f^{(+)}(E)\\ &= \lim_{\epsilon\to 0}\left[-i\epsilon\,G_i(E+i\epsilon)\psi_f(E)\right] - \lim_{\epsilon\to 0}G_i(E+i\epsilon)V_i\Psi_f(E+i\epsilon)\end{aligned} \tag{A40}$$

Consider first the last term in (A40). Referring to sections A.4 - A.7, one sees that--for three-particle collisions of present interest, at any rate--the integral $G_i^{(+)}(E)V_i\Psi_f^{(+)}(E)$ always converges, whether or not the i and f channels are identical, i.e., whether or not $H_f = H_i$. In essence, the reason for this convergence is that--by virtue of the way the quantities H_i and V_i are defined--$G_i^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ cannot propagate in bound states along directions $\underset{\sim}{\nu}'$ wherein $V_i(\underset{\sim}{r}')$ can remain finite as $\underset{\sim}{r}' \to \infty \;||\; \underset{\sim}{\nu}'$. For instance, if $\psi_i(E)$ is identical with ψ_i of (21a), $G_i^{(+)} \equiv G_F^{(+)}$ which never propagates in bound states, so that the arguments in section A.7 can be taken over directly to show $G_F^{(+)}V_F\Psi_f^{(+)}$ converges, whatever channel f may represent. Similarly, if i represents the channel with 2, 3 initially bound, as in Eqs. (A36), then

$$G_i^{(+)}V_i\Psi_f^{(+)} \equiv \int d\underset{\sim}{r}'\, G_{23}^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')\left[V_{12}(\underset{\sim}{r}_{12}') + V_{31}(\underset{\sim}{r}_{31}')\right]\Psi_f(\underset{\sim}{r}') \tag{A41}$$

converges by the same arguments of section A.7 because, e.g., in the V_{12} term of (A41), the only possibility of divergence is along directions $\underset{\sim}{\nu}_{12}'$ wherein $\underset{\sim}{r}_{12}'$ remains finite; along $\underset{\sim}{\nu}_{12}'$ $G_{23}^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ behaves like $G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$, i.e., the V_{12} term in (A41) behaves essentially like the integral (A17) previously shown to be convergent. It follows, according to section A.8, that the last term in (A40) always can be replaced by $G_i^{(+)}V_i\Psi_f^{(+)}$.

Next consider the first term on the right side of Eq. (A40). If $\lim G_i(E + i\epsilon)\psi_f(E)$ as $\epsilon \to 0$ exists, then

$$\lim_{\epsilon\to 0}\left[-i\epsilon G_i(E+i\epsilon)\psi_f(E)\right] = \left[\lim_{\epsilon\to 0}(-i\epsilon)\right]\left[\lim_{\epsilon\to 0} G_i(E+i\epsilon)\psi_f(E)\right] = 0 \tag{A42a}$$

But according to sections 2.2 and A.8, if the integral $G_i^{(+)}(E)\psi_f(E)$ converges, then

$$\lim_{\epsilon \to 0} G_i(E+i\epsilon)\psi_f(E) = G_i^{(+)}(E)\psi_f(E) \tag{A42b}$$

In the three-particle case of present interest, the right side of (A42b) converges whenever f represents a channel which differs from i, provided f is not merely a dissociation of i. For instance, if $\psi_i(E)$ is identical with ψ_i of (21a), then $f \neq i$ means $\psi_f(E)$ must be proportional to a bound state, e.g., to $u_j(\underset{\sim}{r}_{12})$; in this event $G_i^{(+)}\psi_f(E)$ behaves like $G_F^{(+)}(E)V_{12}(\underset{\sim}{r}_{12})\psi_i(E)$, which behaves like the integral (A17) previously shown to be convergent. On the other hand, if $\psi_f(E) = \psi_i(E) = \psi_i$ of (21a) the integrand of $G_i^{(+)}\psi_f \equiv G_F^{(+)}\psi_i$ contains no rapidly decreasing factor, and diverges. As a second example, if ψ_i represents the channel wherein 2, 3 are bound, while ψ_f represents the channel wherein 1, 2 are bound, $G_i^{(+)}(E)\psi_f(E)$ obviously converges, because $G_i^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ can't propagate in bound states as $\underset{\sim}{r} \to \infty$ along directions $\underset{\sim}{\nu}_{12}$ keeping $\underset{\sim}{r}_{12}$ finite. On the other hand, if $\psi_f = \psi_i$ = the (laboratory frame version of the) wave function (A36a), then $G_i^{(+)}\psi_f = G_i^{(+)}\psi_i$ behaves like the integral (A13a) previously shown to be divergent; nor is the divergence removed by going to the center of mass frame [compare sections A.4 and A.6]. Similarly, if ψ_i continues to be given by (A36a), but $E > 0$ so that one can choose $\psi_f(E)$ to be a free three-particle wave (21a)--in other words, if the f channel can represent a dissociation (without any recombination) of the particles 2, 3 bound in the i channel--then $G_i^{(+)}\psi_f$ again behaves like (A13a) and diverges, in the laboratory or center of mass frames. That the limit (A42b) cannot exist when f = i also can be seen as follows. From Eq. (9),

$$(H_i - E - i\epsilon)\psi_i(E) = -i\epsilon\psi_i(E) \tag{A43a}$$

implying, for every $\varepsilon > 0$

$$\psi_i(E) = -i\epsilon G_i(E+i\epsilon)\psi_i(E) \tag{A43b}$$

which in turn implies

$$\lim_{\epsilon \to 0} \left[-i\epsilon G_i(E+i\epsilon)\psi_i(E) \right] = \psi_i(E) \neq 0 \tag{A43c}$$

The preceding two paragraphs imply that--in three-particle systems at least--if the general argument of section A.8 is correct then: (i) when $f \neq i$ and is not merely a dissociation of i, $G_i^{(+)}(E)\psi_f(E)$ converges and

$$\Psi_f^{(+)}(E) = -G_i^{(+)}(E)V_i\Psi_f^{(+)}(E); \tag{A44a}$$

(ii) when $f = i$, $G_i^{(+)}(E)\psi_f(E)$ diverges and

$$\Psi_i^{(+)}(E) = \psi_i(E) - G_i^{(+)}(E)V_i\Psi_i^{(+)}(E) \tag{A44b}$$

Now directly from Eqs. (9) and the real energy version of (10b) [much as in the derivation of (A35)] one obtains[16]

$$\Psi_f^{(+)}(\underset{\sim}{r};E) = -G_i^{(+)}(E)V_i\Psi_f^{(+)}(E) + \mathcal{L}\left[G_i^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'),\ \Psi_f^{(+)}(\underset{\sim}{r}')\right] \tag{A44c}$$

Thus to be sure Eqs. (A44) are consistent, I must be able to show: (i) when $f \neq i$ and is not merely a dissociation of i,

$$\mathcal{L}\left[G_i^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'),\ \Psi_f^{(+)}(\underset{\sim}{r}')\right] = 0; \tag{A45a}$$

(ii) when f = i

$$\mathcal{S}\left[G_i^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'),\ \Psi_i^{(+)}(\underset{\sim}{r}')\right] = \psi_i(\underset{\sim}{r}) \tag{A45b}$$

Let me verify Eqs. (A45) in the three-particle circumstances considered in the penultimate paragraph. If the f channel is one in which particles 1, 2 are bound in $u_j(\underset{\sim}{r}_{12})$, then [as explained in subsection A.9.1]

$$\Phi_f^{(+)} = \Psi_f^{(+)} - \psi_f$$

surely is everywhere outgoing in the center of mass frame. Now suppose ψ_i is given by (21a). In this event the left side of (A45a) becomes

$$\begin{aligned}\mathcal{S}\left[G_i^{(+)},\ \Psi_f^{(+)}\right] &\equiv \mathcal{S}\left[G_F^{(+)},\ \psi_f + \Phi_f^{(+)}\right] \\ &= \mathcal{S}\left[G_F^{(+)},\ \psi_f\right] + \mathcal{S}\left[G_F^{(+)},\ \Phi_f^{(+)}\right]\end{aligned} \tag{A46a}$$

Again as explained in subsection A.9.1, the last term in (A46a) assuredly vanishes, in the center of mass frame at any rate. Moreover, because $G_F^{(+)}$ cannot propagate in bound states, it can be seen that the surface integral

$$\mathcal{S}\left[G_F^{(+)},\ \psi_f\right] = 0 \tag{A46b}$$

Therefore (A45a) indeed holds when i denotes the channel in which all three particles are unbound and freely propagating. Similarly, because $G_{23}^{(+)}$ cannot propagate in bound states other than $u_j(\underset{\sim}{r}_{23})$, one sees that (A45a) also holds when, e.g., ψ_i is given by Eqs. (A36) and ψ_f is as chosen in the second sentence of this paragraph. As for Eq. (A45b), it is difficult to verify directly when ψ_i is given by (21a) [once more for reasons discussed in subsection A.9.1] although there is no reason to doubt its validity. When ψ_i is given by (A36), however, Eq. (A45b) reduces to

$$\mathcal{J}\left[G_i^{(+)}, \psi_i\right] = \psi_i \tag{A46c}$$

because [see subsection A.9.1] $\mathcal{J}[G_i^{(+)}, \Phi_i^{(+)}]$ now surely vanishes since $\Phi_i^{(+)}$ now surely is outgoing--in the center of mass frame at any rate. With ψ_i of (A36), Eq. (A46c) can be confirmed by direct calculation, most easily (though not necessarily) in the center of mass frame. Of course, when the collision is two-body, as with the incident wave (A36a), the validity of the real energy Lippmann-Schwinger equation (42) hardly can be challenged, in which event (A45b) can be inferred from Eqs. (42) and (A44c) without any reference to Eqs. (A40) and (A43c).

The above paragraph has achieved our original purpose in this subsection, namely to illustrate the postulated association between the convergence of a (volume) integral at real E and the vanishing of a related surface integral at infinity. However, it also has been shown that when circumstances are such that both Eqs. (A44a) and (A44b) can hold for specified ψ_i, Eq. (A44b) [i.e., Eq. (42)] does not have a unique solution, since if $\Psi_f^{(+)}$ and $\Psi_i^{(+)}$ satisfy Eqs. (A44a) and (A44b) respectively,

$$\Psi^{(+)} = \Psi_i^{(+)} + \Psi_f^{(+)} \tag{A47}$$

also satisfies Eq. (A44b). In particular, therefore [as asserted in section 2.2], Eq. (42) does not have a unique solution when $\psi_i(E)$ is given by (21a) and when at least one rearrangement channel of i exists, i.e., when bound states $u_j(\underset{\sim}{r}_{12})$, $u_j(\underset{\sim}{r}_{23})$ or $u_j(\underset{\sim}{r}_{31})$ exist and can be reached via recombination reactions like (17a).

I conclude this subsection with the remark that the customary operator manipulations yield

$$
\begin{aligned}
G_i(E+i\epsilon)\psi_f(E) &\equiv \frac{1}{H_i - E - i\epsilon}\,\psi_f(E) \\
&= \frac{1}{H_f + V_f - V_i - E - i\epsilon}\,\psi_f(E) \\
&= \frac{1}{V_f - V_i - i\epsilon}\,\psi_f(E)
\end{aligned}
\tag{A48}
$$

which seems to imply: (i) whenever $f \neq i$, so that $V_f \neq V_i$, the first term on the right side of (A40) vanishes, i.e., Eq. (A42a) holds; (ii) whenever $f = i$, Eq. (A43c) holds. I do not question the conclusion (ii) immediately above, namely that when $f = i$, Eq. (A43c) quite generally holds. As explained earlier in this subsection, however, the conclusion (i) immediately above is questionable when f is merely a dissociation of i, and as a matter of fact one can show that in this circumstance the first term on the right side of (A40) need not vanish. More specifically, choose ψ_i as in Eq. (A36a), and let the f channel represent free propagation under no forces. Suppose also that $E > 0$ [otherwise $\psi_f(E)$ isn't even defined, and Eq. (A48) as well as (A42a) are essentially meaningless], and suppose $V_{12} = V_{31} = 0$, i.e., suppose the total Hamiltonian H is identical with H_{23} of Eq. (56b). Then in (A40): $V_i = 0$; $G_i = G_{23}$; and $\Psi_f^{(+)}$ is identical with $\Psi_{23}^{(+)}$ defined by the 2, 3 analogues of Eqs. (58a) and (72) [recall the present defining equation (A38a) for Ψ_f]. Therefore, in the circumstances just described, Eq. (A40) implies

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\left[-i\epsilon\, G_i(E+i\epsilon)\psi_f(E)\right] \\
&\equiv \lim_{\epsilon\to 0}\left[-i\epsilon\, G_{23}(E+i\epsilon)\psi_f(E)\right] = \Psi_{23}^{(+)}
\end{aligned}
\tag{A49}
$$

In other words, in the present circumstances the first term on the right side of (A40) does not vanish. This result (A40) demonstrates the possible dangers of uncritical reliance on operator manipulations such as employed in (A48).

A.10 Alternative Derivation of Eq. (205b)

The derivation of Eq. (175b), and the discussion in section 5.1, make it clear that the function F_{12} defined by Eq. (201b) is given by

$$F = \lim_{R\to\infty}\int_0^R d\underset{\sim}{r}\left\{e^{-i\underset{\sim}{r}\cdot\underset{\sim}{A}}\,\varphi^{(+)}(\underset{\sim}{r}) + \frac{i\mu}{4\pi\hbar^2}\frac{e^{ikr}}{Ar^2}\left[\langle k\underset{\sim}{\nu}_A|\underset{\sim}{t}|\underset{\sim}{k}\rangle e^{-iAr} - \langle -k\underset{\sim}{\nu}_A|\underset{\sim}{t}|\underset{\sim}{k}\rangle e^{iAr}\right]\right\} \tag{A50}$$

where for simplicity I have dropped the here unnecessary subscripts 1, 2 and i. Let

$$u_c^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) \equiv u_c^{(+)}(\underset{\sim}{r}) = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + \varphi^{(+)}(\underset{\sim}{r}) \tag{A51}$$

be the continuum solution to the Schrodinger equation in the center of mass frame of particles 1 and 2, containing the scattered wave $\phi^{(+)}$ of (A50) when the interaction is $V_{12}(\underset{\sim}{r}_{12}) \equiv V(\underset{\sim}{r})$ and the incident wave is given by (74a). Suppose the short range potential $V(\underset{\sim}{r})$ can be considered negligible for $r > a$, i.e., suppose in effect $V(\underset{\sim}{r}) = 0$ for $r > a$. Let me further assume that $V(\underset{\sim}{r})$ is spherically symmetric; this assumption does not significantly detract from the objective of this section, which is to confirm the validity of (204b) via an argument not employing interchange of order of integration and limit $\varepsilon \to 0$, as in Eqs. (203).

With the above assumptions we know[40] that

$$u_c^{(+)}(\underset{\sim}{r}) = \sum_{\ell} C_\ell \left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right] P_\ell(\underset{\sim}{k},\underset{\sim}{r}) \qquad r > a \tag{A52}$$

where $P_\ell(\underset{\sim}{k},\underset{\sim}{r})$ is the Legendre polynomial in the angle between $\underset{\sim}{r}$ and $\underset{\sim}{k} \equiv \underset{\sim}{k}_{12i}$ of (201b); the j_ℓ and n_ℓ are the usual spherical Bessel functions[34]; and the numerical coefficients C_ℓ are given in terms of the phase shifts δ_ℓ by

$$C_\ell = i^\ell (2\ell+1) e^{i\delta_\ell} \tag{A53}$$

In (A50),

$$\int_0^R d\underset{\sim}{r}\, e^{-i\underset{\sim}{r}\cdot\underset{\sim}{A}} \varphi^{(+)}(\underset{\sim}{r}) = \int_0^a d\underset{\sim}{r}\, e^{-i\underset{\sim}{r}\cdot\underset{\sim}{A}} \varphi^{(+)}(\underset{\sim}{r}) + \int_a^R d\underset{\sim}{r}\, e^{-i\underset{\sim}{r}\cdot\underset{\sim}{A}} \varphi^{(+)}(\underset{\sim}{r}) \tag{A54}$$

The second integral in (A54) is evaluated as follows. Using the expansion of $e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$ in spherical harmonics[40] [which is quoted in Eq. (E31) below], Eqs. (A51) - (A52) can be rewritten in the form

$$\varphi^{(+)}(\underset{\sim}{r}) = u_c^{(+)}(\underset{\sim}{r}) - e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = \sum_\ell F_\ell(r) P_\ell(\underset{\sim}{k},\underset{\sim}{r}) \tag{A55a}$$

where

$$F_\ell(r) = C_\ell \sin\delta_\ell \left[i\, j_\ell(kr) - n_\ell(kr)\right] \qquad r > a \tag{A55b}$$

Substituting (A55a) into (A54), and again employing the expansion (E31) [which is written out explicitly for $e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}}$ in Eq. (E45) below] yields

$$\begin{aligned}&\int_a^R d\underset{\sim}{r}\, e^{-i\underset{\sim}{r}\cdot\underset{\sim}{A}}\varphi^{(+)}(\underset{\sim}{r})\\&= 4\pi\sum_{\ell}(-i)^{\ell}P_{\ell}(\underset{\sim}{k},\underset{\sim}{A})\int_a^R dr\,r^2 F_{\ell}(r)j_{\ell}(Ar)\end{aligned} \tag{A56}$$

Next use the formula[41]

$$\begin{aligned}&\int dr\,r^2 j_{\ell}(Ar)f_{\ell}(kr)\\&= \frac{r^2}{A^2-k^2}\left[k\,j_{\ell}(Ar)f_{\ell-1}(kr) - A\,j_{\ell-1}(Ar)f_{\ell}(kr)\right]\end{aligned} \tag{A57a}$$

where $f_{\ell}(kr)$ is any linear combination of $j_{\ell}(kr)$ and $n_{\ell}(kr)$ with coefficients independent of ℓ or kr, and where it is understood that

$$\begin{aligned}j_{-1}(\rho) &= -n_0(\rho)\\ n_{-1}(\rho) &= +j_0(\rho)\end{aligned} \tag{A57b}$$

Then in (A56)

$$\int_a^R dr\, r^2 F_\ell(r) j_\ell(Ar)$$

$$= \Big\{ i C_\ell \sin\delta_\ell \frac{R^2}{A^2-k^2}\left[k j_\ell(AR) j_{\ell-1}(kR) - A j_{\ell-1}(AR) j_\ell(kR)\right]$$

$$- C_\ell \sin\delta_\ell \frac{R^2}{A^2-k^2}\left[k j_\ell(AR) n_{\ell-1}(kR) - A j_{\ell-1}(AR) n_\ell(kR)\right]\Big\}$$

$$- i C_\ell \sin\delta_\ell \frac{a^2}{A^2-k^2}\left[k j_\ell(Aa) j_{\ell-1}(ka) - A j_{\ell-1}(Aa) j_\ell(ka)\right] \tag{A58a}$$

$$+ C_\ell \sin\delta_\ell \frac{a^2}{A^2-k^2}\left[k j_\ell(Aa) n_{\ell-1}(ka) - A j_{\ell-1}(Aa) n_\ell(ka)\right]$$

$$\equiv Q_\ell(R) + W_\ell \tag{A58b}$$

where $Q_\ell(R)$ denotes the R-dependent terms enclosed within the braces in (A58a), and W_ℓ (independent of R) denotes the remaining terms in (A58a).

The terms in (A50) involving the matrix elements of $\underset{\sim}{t}$ are trivially integrable. Thus, employing (A54), (A56) and (A58), the formula (A50) becomes

$$F = \lim_{R\to\infty} \Big\{ \int_0^a d\underset{\sim}{r}\, e^{-i\underset{\sim}{r}\cdot\underset{\sim}{A}} \varphi^{(+)}(\underset{\sim}{r})$$

$$+ 4\pi \sum_\ell (-i)^\ell P_\ell(\underset{\sim}{k},\underset{\sim}{A}) \left[\varphi_\ell(R) + W_\ell\right]$$

$$+ \frac{i\mu}{\hbar^2 A} \langle \underset{\sim}{k}\underset{\sim}{\nu}_A | \underset{\sim}{t} | \underset{\sim}{k} \rangle \frac{\left(e^{i(k-A)R} - 1\right)}{i(k-A)}$$

$$- \frac{i\mu}{\hbar^2 A} \langle -\underset{\sim}{k}\,\underset{\sim}{\nu}_A | \underset{\sim}{t} | \underset{\sim}{k} \rangle \frac{\left(e^{i(k+A)R} - 1\right)}{i(k+A)} \Big\} \tag{A59}$$

In (A59) it can be seen that

$$\lim_{R\to\infty} 4\pi \sum_\ell (-i)^\ell P_\ell(\underset{\sim}{k},\underset{\sim}{A}) \varphi_\ell(R)$$

$$= \frac{4\pi}{2Ak} \sum_\ell (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \left[(-)^{\ell+1} \frac{e^{i(k+A)R}}{A+k} - \frac{e^{i(k-A)R}}{A-k} \right] P_\ell(\underset{\sim}{k},\underset{\sim}{A}) \tag{A60}$$

plus terms of order 1/R, negligible as $R \to \infty$. Moreover, from Eqs. (A55),

$$\lim_{r\to\infty \parallel \underset{\sim}{\nu}_A} \varphi^{(+)}(\underset{\sim}{r}) = \sum_\ell (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\underset{\sim}{k},\underset{\sim}{A})\, \frac{e^{ikr}}{kr} \tag{A61a}$$

while from Eq. (131g)

$$\lim_{r\to\infty \parallel \underset{\sim}{\nu}_A} \varphi^{(+)}(\underset{\sim}{r}) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2} \langle k\underset{\sim}{\nu}_A | \underset{\sim}{t} | \underset{\sim}{k} \rangle \frac{e^{ikr}}{r} \tag{A61b}$$

Hence

$$\langle k\underset{\sim}{\nu}_A | \underset{\sim}{t} | \underset{\sim}{k} \rangle = -\frac{2\pi\hbar^2}{\mu k} \sum_\ell (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\underset{\sim}{k},\underset{\sim}{A}) \tag{A62a}$$

and

$$\begin{aligned}\langle -k\underset{\sim}{\nu}_A | \underset{\sim}{t} | \underset{\sim}{k} \rangle &= -\frac{2\pi\hbar^2}{\mu k} \sum_\ell (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\underset{\sim}{k},-\underset{\sim}{A}) \\ &= -\frac{2\pi\hbar^2}{\mu k} \sum_\ell (-)^\ell (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\underset{\sim}{k},\underset{\sim}{A}) \end{aligned} \tag{A62b}$$

Eqs. (A60) and (A62) imply that the sum of the R-dependent terms in (A59) approaches zero in the limit R → ∞, i.e., that the limit in (A59) really does exist. In this fashion we have reduced (A59) [i.e., (A50)] to

$$F = \int_0^a d\mathbf{r}\, e^{-i\mathbf{r}\cdot\mathbf{A}}\,\varphi^{(+)}(\mathbf{r}) + 4\pi \sum_\ell (-i)^\ell W_\ell P_\ell(\mathbf{k},\mathbf{A})$$

$$-\frac{\mu}{\hbar^2 A}\left[\frac{\langle \mathbf{k}\nu_A|\mathbf{t}|\mathbf{k}\rangle}{k-A} - \frac{\langle -\mathbf{k}\nu_A|\mathbf{t}|\mathbf{k}\rangle}{k+A}\right] \tag{A63}$$

Now we can return at last to Eq. (205b). Evidently the result (A63) will be identical with (205b) if we can show

$$\int_0^a d\mathbf{r}\, e^{-i\mathbf{r}\cdot\mathbf{A}}\,\varphi^{(+)}(\mathbf{r}) + 4\pi \sum_\ell (-i)^\ell W_\ell P_\ell(\mathbf{k},\mathbf{A})$$

$$= -\frac{2\mu}{\hbar^2}\,\frac{\langle \mathbf{A}|\mathbf{t}|\mathbf{k}\rangle}{A^2-k^2} \tag{A64}$$

From the fundamental definitions of the quantities involved [recall Eqs. (131e) - (131i)]

$$u_c^{(+)}(\underset{\sim}{r}) = (1 - g^{(+)}V)e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$$

$$Vu_c^{(+)} = \underset{\sim}{t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = \int_0^\infty d\underset{\sim}{r}'\, \underset{\approx}{t}(\underset{\sim}{r};\underset{\sim}{r}')e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}$$

$$\int_0^\infty d\underset{\sim}{r}\, e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}} V(\underset{\sim}{r})\, u_c^{(+)}(\underset{\sim}{r}) = \int_0^\infty d\underset{\sim}{r}\, d\underset{\sim}{r}'\, e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}}\, \underset{\sim}{t}(\underset{\sim}{r};\underset{\sim}{r}')e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'} \tag{A65}$$

$$= \langle \underset{\sim}{A} | \underset{\sim}{t} | \underset{\sim}{k} \rangle$$

But we are assuming $V(\underset{\sim}{r})$ is negligible for $r > a$. Also,

$$\left(\nabla^2 + k^2 - \frac{2\mu V}{\hbar^2}\right) u_c^{(+)} = 0$$

$$(\nabla^2 + A^2)\, e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}} = 0 \tag{A66a}$$

so that

$$\frac{2\mu}{\hbar^2} \langle \underset{\sim}{A} | \underset{\sim}{t} | \underset{\sim}{k} \rangle = \frac{2\mu}{\hbar^2} \int_0^a d\underset{\sim}{r}\, e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}} V(\underset{\sim}{r})\, u_c^{(+)}(\underset{\sim}{r})$$

$$= (k^2 - A^2) \int_0^a d\underset{\sim}{r}\, e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}} u_c^{(+)} + \int_0^a d\underset{\sim}{r} \left[e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}} \nabla^2 u_c^{(+)} - u_c^{(+)} \nabla^2 e^{-i\underset{\sim}{A}\cdot\underset{\sim}{r}} \right] \tag{A66b}$$

Therefore, comparing Eqs. (A64) - (A66), we see that demonstrating

$$(A^2-k^2)\int_0^a d\mathbf{r}\, e^{-i\mathbf{r}\cdot\mathbf{A}}\varphi^{(+)}(\mathbf{r}) \quad + 4\pi(A^2-k^2)\sum_\ell (-i)^\ell W_\ell P_\ell(\mathbf{k},\mathbf{A})$$

$$= (A^2-k^2)\int_0^a d\mathbf{r}\, e^{-i\mathbf{r}\cdot\mathbf{A}} u_c^{(+)} + \int_0^a d\mathbf{r}\left[u_c^{(+)}\nabla^2 e^{-i\mathbf{A}\cdot\mathbf{r}} - e^{-i\mathbf{A}\cdot\mathbf{r}}\nabla^2 u_c^{(+)}\right] \tag{A67}$$

is equivalent to demonstrating (A64). Eliminating $u_c^{(+)}$ in favor of $\phi^{(+)}$ via (A51), Eq. (A67) becomes

$$4\pi(A^2-k^2)\sum_\ell (-i)^\ell W_\ell P_\ell(\mathbf{k},\mathbf{A})$$

$$= \int_0^a d\mathbf{r}\left[\varphi^{(+)}\nabla^2 e^{-i\mathbf{A}\cdot\mathbf{r}} - e^{-i\mathbf{A}\cdot\mathbf{r}}\nabla^2\varphi^{(+)}\right]$$

$$= 4\pi\sum_\ell (-i)^\ell P_\ell(\mathbf{k},\mathbf{A})\left\{r^2\left[F_\ell\frac{dj_\ell(Ar)}{dr} - j_\ell(Ar)\frac{dF_\ell}{dr}\right]\right\}_{r=a} \tag{A68}$$

where the value of the right side of (A68) vanishes at the lower limit $r = 0$ because we know $F_\ell(r)$ defined by (A55a) is well-behaved at $r = 0$.

At $r = a$, $F_\ell(r)$ and dF_ℓ/dr are continuous; hence these quantities can be calculated from (A55b). Thus (A68) reduces to demonstrating

$$-i\left[k j_\ell(Aa) j_{\ell-1}(ka) - A j_{\ell-1}(Aa) j_\ell(ka)\right]$$
$$+\left[k j_\ell(Aa) n_{\ell-1}(ka) - A j_{\ell-1}(Aa) n_\ell(ka)\right]$$

$$= i\left[A j_\ell'(Aa) j_\ell(ka) - k j_\ell(Aa) j_\ell'(ka)\right]$$
$$-\left[A j_\ell'(Aa) n_\ell(ka) - k j_\ell(Aa) n_\ell'(ka)\right] \qquad \text{(A69)}$$

However, the relation[34]

$$\frac{d}{d\rho}\left[\rho^{\ell+1} j_\ell(\rho)\right] = \rho^{\ell+1} j_{\ell-1}(\rho)$$

can be put in the form

$$\rho j_\ell'(\rho) = \rho j_{\ell-1}(\rho) - (\ell+1) j_\ell(\rho) \qquad \text{(A70a)}$$

Similarly

$$\rho n_\ell'(\rho) = \rho n_{\ell-1}(\rho) - (\ell+1) n_\ell(\rho) \qquad \text{(A70b)}$$

Moreover, it can be verified that Eqs. (A70) hold for $\ell = 0$ with the interpretations (A57b). From (A70a) we infer

$$\rho_2 j_\ell(\rho_1) j_\ell'(\rho_2) - \rho_1 j_\ell'(\rho_1) j_\ell(\rho_2)$$

$$= \rho_2 j_\ell(\rho_1) j_{\ell-1}(\rho_2) - \rho_1 j_\ell(\rho_2) j_{\ell-1}(\rho_1) \tag{A71}$$

Letting $\rho_1 = ka$, $\rho_2 = Aa$, we see that (A71) ensures the equality of the terms proportional to i on the left and right sides of (A69). One similarly shows the remaining terms on the left and right sides of (A69) are equal. Therefore, we have demonstrated (A67) and (A64) hold, i.e., without interchanging order of integration and limit $\varepsilon \to 0$ we have demonstrated that Eq. (A50) reduces to Eq. (205b), Q.E.D.

APPENDIX B. CONVERGENCE OF ITERATED EXPRESSIONS

B.1 Eq. (69)

A typical term in (69) is

$$\bar{G}^{(+)} V_{23} \bar{\Phi}_{12}^{(+)} = \int d\underset{\sim}{q}'_{23}\, d\underset{\sim}{r}'_{23}\, \bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')\, V_{23}(\underset{\sim}{r}'_{23})\, \bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}') \tag{B1}$$

As discussed in section A.1, the integral (B1) can diverge (at infinity in $\bar{\underset{\sim}{r}}'$ space) only in directions $\bar{\underset{\sim}{\nu}}_{23}'$ along which $\underset{\sim}{r}_{23}'$ remains finite. Moreover, divergence can occur at infinite $\underset{\sim}{q}_{23}'$ only, so that terms in $\bar{G}^{(+)}$ corresponding to three-body bound states (wherein both $\underset{\sim}{q}_{23}'$ and $\underset{\sim}{r}_{23}'$ remain finite) obviously cannot cause (B1) to diverge.

The asymptotic behavior of $\bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}')$ at large $\underset{\sim}{q}_{23}'$ is found from (72) and (73), using

$$\begin{aligned} \underset{\sim}{r}_{12} &= \underset{\sim}{q}_{23} - \frac{m_3}{m_2+m_3}\underset{\sim}{r}_{23} \\ \underset{\sim}{q}_{12} &= \frac{-m_1}{m_1+m_2}\underset{\sim}{q}_{23} - \frac{m_2 M}{(m_1+m_2)(m_2+m_3)}\underset{\sim}{r}_{23} \end{aligned} \tag{B3}$$

Thus remembering $\underset{\sim}{r}_{23}'$ remains finite,

$$\begin{aligned} \lim_{q_{23}'\to\infty \| \bar{\underset{\sim}{\nu}}_{23}'} \bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}';E) &\cong e^{\frac{-i m_1}{m_1+m_2}\underset{\sim}{K}_{12}\cdot\underset{\sim}{q}_{23}'} \frac{e^{ik_{12}\left|\underset{\sim}{q}_{23}' - \frac{m_3}{m_2+m_3}\underset{\sim}{r}_{23}'\right|}}{\left|\underset{\sim}{q}_{23}' - \frac{m_3}{m_2+m_3}\underset{\sim}{r}_{23}'\right|}\, a\!\left(\frac{\underset{\sim}{r}_{12}'}{r_{12}'}\right) \\ &\cong e^{\frac{-i m_1}{m_1+m_2}\underset{\sim}{K}_{12}\cdot\underset{\sim}{q}_{23}'} \frac{e^{ik_{12}q_{23}'}}{q_{23}'} e^{-\frac{i m_3}{m_2+m_3}k_{12}\bar{\underset{\sim}{\nu}}_{23}'\cdot\underset{\sim}{r}_{23}'}\, a(\bar{\underset{\sim}{\nu}}_{23}') \end{aligned} \tag{B3}$$

In (B3), factors depending on $\underset{\sim}{r}_{23}'$ only have been dropped, because for the purposes of this section they play no role in (B1). The factor $a(\underset{\sim}{r}_{12}/r_{12}')$ in (B3) of dourse is proportional to the amplitude for elastic scattering in the two-body collision of particles 1 and 2.

Now, omitting factors irrelevant to the question of its convergence, if bound states $u_j(\underset{\sim}{r}_{23})$ exist the integral (B1) behaves like [compare section A.6, and recall the discussion at the end of section A.4]

$$\int d\underset{\sim}{q}_{23}' \, d\underset{\sim}{r}_{23}' \frac{e^{iK_{23j}q_{23}'}}{q_{23}'} u_j(\underset{\sim}{r}_{23}') V_{23}(\underset{\sim}{r}_{23}') e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12}\cdot\underset{\sim}{q}_{23}'} \frac{e^{ik_{12}q_{23}'}}{q_{23}'} e^{-i\frac{m_3}{m_2+m_3}k_{12}\bar{\underset{\sim}{\nu}}_{23}'\cdot\underset{\sim}{r}_{23}'} \tag{B4a}$$

Similarly, if bound states do not exist, the integral (B1) behaves like

$$\int d\underset{\sim}{q}_{23}' \, d\underset{\sim}{r}_{23}' \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} V_{23}(\underset{\sim}{r}_{23}') e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12}\cdot\underset{\sim}{q}_{23}'} \frac{e^{ik_{12}q_{23}'}}{q_{23}'} e^{-i\frac{m_3}{m_2+m_3}k_{12}\bar{\underset{\sim}{\nu}}_{23}'\cdot\underset{\sim}{r}_{23}'} \tag{B4b}$$

After integrating over $d\underset{\sim}{r}_{23}'$, the integrand in (B4b) will be of order $q_{23}'^{-3/2}$ even before integrating over $d\bar{\underset{\sim}{\nu}}_{23}'$; thus (B4b) is obviously convergent. The integral (B4a) reduces to

$$\int dq_{23}' \, e^{i(K_{23j}+k_{12})q_{23}'} j_0\left(\frac{m_1}{m_1+m_2}K_{12}q_{23}'\right) \tag{B5a}$$

which converges unless

$$K_{23j} + k_{12} - \frac{m_1}{m_1 + m_2} K_{12} = 0 \tag{B5b}$$

Now, from Eq. (29d)

$$\underset{\sim}{k}_{12} = \underset{\sim}{K}_{23} + \frac{m_1}{m_1 + m_2} \underset{\sim}{K}_{12} \tag{B6a}$$

implying the triangular inequalities

$$\left| k_{12} - \frac{m_1}{m_1 + m_2} K_{12} \right| \leq K_{23} \leq k_{12} + \frac{m_1}{m_1 + m_2} K_{12} \tag{B6b}$$

The quantity K_{23j} is defined by the analogue of (114b), so that, as in the analogous case of Eq. (A16b)

$$K_{23} < K_{23j} \tag{B7a}$$

It follows that

$$\left| k_{12} - \frac{m_1}{m_1 + m_2} K_{12} \right| < K_{23j} \tag{B7b}$$

permitting the inference that (B5b) cannot hold on the energy shell.

In the foregoing, various slowly varying (independent of q_{23}') angular dependent (on $\underset{\sim}{\nu}_{23}'$) factors have been omitted from Eqs. (B4), in accordance with the discussion at the end of section A.4. Comparison with sections A.4 and A.7, plus a little thought, shows that inclusion of those factors could not have worsened the convergence of the integrals (B4a) and (B4b). For instance, the $q_{23}'^{-1}$ dependence of the j_o factor in (B5a) is understandable on the basis of the principle of stationary phase, recalling (A13e) and recognizing that [in spherical coordinates with polar axis along $\underset{\sim}{K}_{12}$] $d\bar{\underset{\sim}{v}}'_{23}$ in (B4) is proportional to sin $\bar{\theta}'_{23}$. If, for example, the omitted $a(\bar{\underset{\sim}{v}}'_{23})$ factor from (B3) also is proportional to sin $\bar{\theta}'_{23}$ with this choice of polar axis, the integrand in (B5a) actually will be of order $q'^{-3/2}$, and will converge independently of the criterion (B5b). I conclude that the integral (B1) surely converges on the energy shell, i.e., the right side of (69) surely converges on the energy shell, Q.E.D.

Recalling section A.4, it is not difficult to see that the laboratory system term corresponding to (B1)--namely the term $G^{(+)}V_{23}\Phi_{12}^{(+)}$ in (67c)--has essentially the same behavior as Eqs. (B4), i.e., is convergent on the energy shell whether or not $G^{(+)}$ can propagate in bound states $u_j(\underset{\sim}{r}_{23})$. On the other hand, when bound states $u_j(\underset{\sim}{r}_{12},\underset{\sim}{r}_{23})$ exist, the integral $G^{(+)}V_{23}\Phi_{12}^{(+)}$ obviously [recall Eqs. (A10) and (A15a)] behaves like the integral (A15c), i.e., contains a δ-function vanishing on the energy shell.

If, as in section A.4, the energy associated with ψ_i in (60) is permitted to be $E' \neq E$, the limit on the right side of (60) yields [instead of (72)]

$$\Phi_{12}^{(+)}(E;E') = -\lim_{\epsilon\to 0} G_{12}(E+i\epsilon)V_{12}\psi_i(E') = e^{i\underset{\sim}{K}'\cdot\underset{\sim}{R}}\, e^{i\underset{\sim}{K}'_{12}\cdot\underset{\sim}{q}_{12}}\,\varphi_{12}^{(+)}(\underset{\sim}{r}_{12};E''_{12};\underset{\sim}{k}'_{12}) \tag{B8a}$$

where

$$\varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{E}_{12}'';\underset{\sim}{k}_{12}') = -\int d\underset{\sim}{r}_{12}'\, g_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{r}_{12}';E_{12}'')\, V_{12}(\underset{\sim}{r}_{12}')\, e^{i\underset{\sim}{k}_{12}'\cdot\underset{\sim}{r}_{12}'} \tag{B8b}$$

and

$$E_{12}'' = E - \frac{\hbar^2 K'^2}{2M} - \frac{\hbar^2 K_{12}'^2}{2\mu_{3R}} \tag{B8c}$$

Because V_{12} is short range, the integral (B8b) obviously remains convergent for all E_{12}'', $\underset{\sim}{k}_{12}'$. Hence the δ-function singularities of

$$\bar{\Phi}_i^{(+)}(\bar{E};\bar{E}') = -\bar{G}^{(+)}(\bar{E})V\bar{\Psi}_i(\bar{E}') \tag{B9a}$$

found in section A.6 are absent from a correctly performed continuation--to energies $\bar{E}' \neq \bar{E}$--of the two-body scattering terms $\bar{\Phi}_{\alpha\beta}^{(+)}$ in $\bar{\Phi}_i^{(+)}$. Of course, if $\bar{\Phi}_{12}^{(+)}(\bar{E};\bar{E}')$ were to be computed from the incorrect [because it is non-convergent] analogue of (B9a), namely from

$$\bar{\Phi}_{12}^{(+)}(\bar{E};\bar{E}') = -\bar{G}_{12}^{(+)}(\bar{E})V_{12}\bar{\Psi}_i(\bar{E}') \tag{B9b}$$

rather than from (B8a), the δ-function singularities of section A.6 would reappear, since $\bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$ behaves like $\bar{G}_{12}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$ as $\underset{\sim}{r}' \to \infty$ along directions $\bar{\underset{\sim}{\nu}}_{12}'$ which keep $\underset{\sim}{r}_{12}'$ finite.

A corresponding continuation of (69) might be to define

$$\bar{\Phi}_i^{s(+)}(\bar{E};\bar{E}') = -\bar{G}^{(+)}(\bar{E})\Big[(V_{23}+V_{31})\bar{\Phi}_{12}^{(+)}(\bar{E}') + (V_{31}+V_{12})\bar{\Phi}_{23}^{(+)}(\bar{E}') + (V_{12}+V_{23})\bar{\Phi}_{31}^{(+)}(\bar{E}')\Big] \qquad \text{(B10)}$$

In this event, the preceding analysis in this section is essentially unaltered, except that the criterion for divergence of, e.g., the $\bar{G}^{(+)}(\bar{E})V_{23}\bar{\Phi}_{12}^{(+)}(\bar{E}')$ term in (B10) corresponding to (B1) becomes

$$K_{23j} + k'_{12} - \frac{m_1}{m_1+m_2}K'_{12} = 0 \qquad \text{(B11)}$$

Eq. (B11) can hold, as the following argument shows. Note first that K_{23j}, k_{12}' and K_{12}' all are intrinsically positive. Thus (B11) is a segment of a straight line in the K_{12}', k_{12}' plane, starting at $k_{12}' = 0$. Eq. (B11) can be satisfied when this straight line intersects the ellipse [recall Eq. (35)]

$$\frac{\hbar^2 K_{12}'^2}{2\mu_{3R}} + \frac{\hbar^2 k_{12}'^2}{2\mu_{12}} = \bar{E}' \tag{B12}$$

Now, starting from small values of $\bar{E}'$, one can see that the ellipses (B12) first intersect (B11) at $k_{12}' = 0$, i.e., at

$$\bar{E}' = \frac{\hbar^2 K_{12}'^2}{2\mu_{3R}} = \frac{\hbar^2}{2\mu_{3R}} \frac{(m_1+m_2)^2}{m_1^2} K_{23j}^2 = \frac{\hbar^2}{2} \frac{M(m_1+m_2)}{m_3 m_1^2} K_{23j}^2 = \frac{(m_1+m_2)(m_2+m_3)}{m_3 m_1} \bar{E}_{kin} \tag{B13a}$$

where [recall Eq. (114b)]

$$\bar{E}_{kin} = \frac{\hbar^2 K_{23j}^2}{2\mu_{1R}} = \bar{E} - \varepsilon_j \tag{B13b}$$

denotes the kinetic energy in the center of mass system when the total center of mass system energy is $\bar{E}$, and when particles 2, 3 are propagating (relative to 1) in a bound state $u_j(\mathbf{r}_{23})$ of energy ε_j. At larger values of $\bar{E}'$ than given by (B13a), the ellipse (B12) always intersects (B11). Hence (B1) [with $\bar{E}' \neq \bar{E}$] can be divergent at real energies

$$\bar{E}' \geqslant \frac{(m_1+m_2)(m_2+m_3)}{m_3 m_1} \bar{E}_{kin} > \bar{E}_{kin} > \bar{E} \tag{B14a}$$

when bound states $u_j(\underset{\sim}{r}_{23})$ exist. The divergence is at worst logarithmic, however; certainly there are no signs of the δ-functions associated with $u_j(\underset{\sim}{r}_{23})$ obtained in section A.6. Moreover, the 2, 3 analogue of Eq. (A16b) implies that—as $\bar{E}'$ increases from small values—δ-functions associated with $u_j(\underset{\sim}{r}_{23})$ occur at energies

$$\bar{E}' \gg \bar{E}_{kin} \tag{B14b}$$

a result not the same as (B14a).

Replacing $\bar{\Phi}_{\alpha\beta}^{(+)}(\bar{E})$ in (B10) by $\bar{\Phi}_{\alpha\beta}^{(+)}(\bar{E};\bar{E})$ from (B8a) [where now, in the center of mass system, $\underset{\sim}{K}'$ still equals the original $\underset{\sim}{K}$ in $\psi_i(E)$] perhaps yields a continuation of (69) more analogous to the continuation (B9a) of section A.6 than the continuation (B10) just examined. With this alternative form for $\bar{\Phi}_i^{s(+)}(\bar{E};\bar{E}')$, one sees that the criterion for divergence of, e.g., the integral $\bar{G}^{(+)}(\bar{E})V_{23}\bar{\Phi}_{12}^{(+)}(\bar{E};\bar{E}')$ is

$$K_{23j} + k_{12}'' - \frac{m_1}{m_1+m_2} K_{12}' = 0 \tag{B15a}$$

instead of (B11), where k_{12}'' is defined in terms of $\bar{E}$ by

$$\frac{\hbar^2 k_{12}''^2}{2\mu_{12}} = \bar{E} - \frac{\hbar^2 K_{12}'^2}{2\mu_{3R}} \tag{B15b}$$

A little algebra shows that (B15a) cannot be satisfied for real k_1', k_2', k_3', though it can hold for complex $\underset{\sim}{k}$. For real $\underset{\sim}{k}'$, therefore, replacing $\bar{\Phi}_{\alpha\beta}^{(+)}(\bar{E})$ in (B10) by $\bar{\Phi}_{\alpha\beta}^{(+)}(\bar{E};\bar{E}')$ yields a continuation of $\bar{\Phi}_i^{s(+)}$ which—like the original formula (69)—is always convergent.

The foregoing considerations justify the remarks made following Eq. (75).

B.2 Eq. (133b)

A typical term in the integral on the right side of (133b) is

$$\bar{\Psi}_f^{(-)*} V_{23} \bar{\Phi}_{12}^{(+)} = \int d\underset{\sim}{q}_{23} d\underset{\sim}{r}_{23} \bar{\Psi}_f^{(-)*}(\bar{\underset{\sim}{r}}) V_{23}(\underset{\sim}{r}_{23}) e^{i\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}_{12}} \varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) \qquad \text{(B16)}$$

using Eq. (72) and dropping the irrelevant prime on $\bar{\underset{\sim}{r}}'$. The convergence of (B16) depends only on the asymptotic behavior of the integrand at large q_{23}. Consider now the contribution to (B16) from the part of $\bar{\Psi}_f^{(-)*}$ denoted by $\bar{\Psi}_{23f}^{(-)*}$, Eq. (136). Recalling the argument leading from (B1) [which (B16) very much resembles] to (B4b), one sees [using the 2, 3 analogues of (33b) and (105b)] that at large q_{23} the $\bar{\Psi}_{23f}^{(-)*}$ contribution to (B16) is proportional to

$$\int d\underset{\sim}{q}_{23}\, e^{-i\underset{\sim}{K}_{23f}\cdot\underset{\sim}{q}_{23}} e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}_{23}} \frac{e^{ik_{12i}q_{23}}}{q_{23}}. \qquad \text{(B17a)}$$

$$\cong \int dq_{23}\, q_{23}\, e^{ik_{12i}q_{23}}\, j_0\!\left(q_{23}\left|\underset{\sim}{K}_{23f}+\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\right|\right) \qquad \text{(B17b)}$$

which contains a term proportional to the δ-function (135a). I add that the validity of this result--and therefore, in effect, of the procedures which I have been employing, especially the procedure (explained in section A.4) of neglecting the angular dependence of slowly varying factors in integrands--is borne out by explicit calculation in the later section E.4.

The contribution to (B16) from propagation in bound states $u_j(\underset{\sim}{r}_{23})$, the only bound states that possibly can cause (B16) to diverge, behaves like

$$\int d\underset{\sim}{q}_{23} d\underset{\sim}{r}_{23} \frac{e^{iK_{23j}q_{23}}}{q_{23}} u_j(\underset{\sim}{r}_{23}) V_{23}(r_{23}) e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}_{23}} \frac{e^{ik_{12i}q_{23}}}{q_{23}}. \tag{B18a}$$

$$= \int dq_{23} e^{iK_{23j}q_{23}} e^{ik_{12i}q_{23}} j_0\left(\frac{m_1}{m_1+m_2} K_{12i} q_{23}\right) \tag{B18b}$$

The integral (B18b) is identical with (B5a). Therefore the formula (133b) is convergent on the energy shell, and contains at worst logarithmic singularities, as discussed in section B.1.

Evidently the result (B17b) can be associated with the fact that $\bar{\Phi}_{12}^{(+)}$ in (B16) decreases asymptotically as q_{23}^{-1} at large q_{23}. If $\bar{\Phi}_{12}^{(+)}$ were to be replaced by a quantity decreasing as $\bar{\rho}^{-2} \cong q_{23}^{-2}$ at large q_{23}, then (B17b) would be replaced by

$$\int dq_{23} \frac{e^{iq_{23}F}}{q_{23}} \tag{B19}$$

where $F(\underset{\sim}{k}_i, \underset{\sim}{k}_f)$ will depend on the particular asymptotic behavior [i.e., on the phase at large q_{23}] of the quantity which replaced $\bar{\Phi}_{12}^{(+)}$. The integral (B19) is logarithmically divergent when $F = 0$. Comparing (133b) and (165b), it can be seen that (B19) is the form to which the large q_{23} contribution from the $\bar{\Psi}_f^{(-)*}V_{23}\bar{\Phi}_{12}^{s(+)}$ term in (165b) will reduce, remembering that $\bar{\Phi}_{12}^{s(+)}(\bar{\underset{\sim}{r}})$ behaves like $\bar{\rho}^{-2}$ at large $\bar{\rho}$ [as discussed in subsection 4.3.1]. However, as section E.3 below shows, the phase of $\bar{\Phi}_{12}^{s(+)}(\underset{\sim}{r})$ at large q_{23} will be a very complicated function of $\underset{\sim}{k}_i$ and the direction $\bar{\underset{\sim}{\nu}}_{23}$ along which $\bar{\underset{\sim}{r}} \to \infty$ keeping $\underset{\sim}{r}_{23}$ finite. Thus it is very difficult (and for the purposes of section 4.3 not worth while) to determine the form of F in the case of present interest, i.e., it is very difficult to write down the relations between $\underset{\sim}{k}_i, \underset{\sim}{k}_f$ determining the values of $\underset{\sim}{k}_f$ (for given $\underset{\sim}{k}_i$) along which the various integrals in (165b) can be logarithmically divergent. Fortunately, logarithmic divergences of this sort in scattering amplitudes, e.g., in the $\bar{\Psi}_f^{(-)*}V_{23}\bar{\Phi}_{12}^{(+)}$ term of (165b), apparently do not make measurable contributions to the scattered current. In fact, referring to section E.2, it seems that a contribution to $\bar{\Psi}_f^{(-)*}V_{23}\bar{\Phi}_{12}^{(+)}$ of form (B19) is associated with the result that, in the integral $\bar{G}^{(+)}V_{23}\bar{\Phi}_{12}^{(+)}$ leading to the amplitude $\bar{\Psi}_f^{(-)*}V_{23}\bar{\Phi}_{12}^{(+)}$, the $\bar{r}' > \bar{r}$ contribution is of order $\bar{r}^{-5/2}$ along special directions. But an $\bar{r}^{-5/2}$ contribution for $\bar{r}' > \bar{r}$--though sufficient to generate the transition amplitude divergence indicating interchange of order of integration and limit $\underset{\sim}{r} \to \infty$ is not wholly justified for the integral $\bar{G}^{(+)}V_{23}\bar{\Phi}_{12}^{(+)}$--is not sufficient to produce divergences in the integrand for the center of mass frame probability current flow [recall Eqs. (118)]. Therefore, because these special directions $\bar{\underset{\sim}{\nu}}_f$ along which (165a) fails for $\bar{G}^{(+)}V_{23}\bar{\Phi}_{12}^{(+)}$ form at most a four-dimensional manifold in the five-dimensional manifold of physically allowed $\underset{\sim}{k}_f$ [recall subsection 5.3.3], the contributions along $\bar{\underset{\sim}{\nu}}_f$ to the probability current flow are inconsequential.

APPENDIX C. ASYMPTOTIC BEHAVIOR OF UNITERATED INTEGRALS

Page

C.1 Eq. (99)

I wish to determine the contribution to the left side of (99) from the region $r'' > r$, in the limit that $r \to \infty$. In particular, I wish to show that when bound states exist this contribution decreases no more rapidly than r^{-4} at large r. I note that for the present purpose, namely to examine the validity of using (99) to deduce Eqs. (100), it is sufficient to suppose $r \to \infty$ along directions $\underset{\sim}{\nu}$ such that no $r_{\alpha\beta}$ remains finite[27]; it is clear from section 3.2 that Eq. (97a) [which led to (99)] generally is not a useful starting point for evaluation of $\lim G(\underset{\sim}{r};\underset{\sim}{r}')$ as $r \to \infty$ along directions $\underset{\sim}{\nu}_{\alpha\beta}$ in which $\underset{\sim}{r}_{\alpha\beta}$ remains finite.

Consider, e.g., the V_{12} term in (99), assuming particles 1, 2 can propagate in bound states. Then, recalling Eq. (A5), at large $r'' > r$, along directions $\underset{\sim}{\nu}_{12}''$ wherein r_{12}'' remains finite, the term in question behaves like

$$\int d\underset{\sim}{R}'' d\underset{\sim}{q}_{12}'' d\underset{\sim}{r}_{12}'' \frac{e^{i\rho''\sqrt{E}}}{\rho''^{4}} e^{-i(\underset{\sim}{K}_f''\cdot\underset{\sim}{R}+\underset{\sim}{K}_{12f}''\cdot\underset{\sim}{q}_{12})} V_{12}(\underset{\sim}{r}_{12}'') \frac{e^{i\rho''\sqrt{E-\epsilon_j}}}{\rho''^{5/2}}$$

$$\times u_j(\underset{\sim}{r}_{12}'') e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}'} \bar{\Psi}^{(-)*}_{12jf}(\bar{\underset{\sim}{r}}';E) \qquad \text{(C1a)}$$

In Eq. (C1a), as in Eq. (A5), the defining relation (90c) makes $k_{12}'' = 0$ when r_{12}'' remains finite as $r'' \to \infty$. Correspondingly,

$$\underset{\sim}{K}_f'' \cdot \underset{\sim}{R} + \underset{\sim}{K}_{12f}'' \cdot \underset{\sim}{q}_{12} = \frac{2M}{\hbar^2} \frac{\sqrt{E}\,\underset{\sim}{R}'' \cdot \underset{\sim}{R}}{\rho''} + \frac{2\mu_{3R}\sqrt{E}}{\hbar^2} \frac{\underset{\sim}{q}_{12}'' \cdot \underset{\sim}{q}_{12}}{\rho''}$$
$$= \frac{\sqrt{E}\,\underset{\sim}{\ell}_{12}'' \cdot \underset{\sim}{\ell}_{12}}{\rho''} = \frac{\sqrt{E}\,\rho_{12}\,\underset{\sim}{\gamma}_{12}'' \cdot \underset{\sim}{\rho}_{12}}{\rho''} \tag{C1b}$$

using Eq. (A4). As in section A.1, the distinction between the magnitudes of ρ_{12}'' and ρ'' can be neglected when r_{12}'' remains finite while $\rho'' \to \infty$. Thus, now recalling (A7), Eq. (C1a) can be replaced by

$$\int d\rho'' d\underset{\sim}{\nu}_{12}'' \rho''^5 \frac{e^{i\rho''\sqrt{E}}}{\rho''^4} e^{-i\sqrt{E}\underset{\sim}{\nu}_{12}''\cdot\underset{\sim}{\rho}_{12}} \frac{e^{i\rho''\sqrt{E-\epsilon_j}}}{\rho''^{5/2}} e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}'} \bar{\Psi}_{12jf}^{(-)*}(\underset{\sim}{\bar{r}}';\bar{E}) \tag{C2}$$

where $\bar{\Psi}_{12jf}^{(-)*}$ depends on $\underset{\sim}{\nu}_{12}''$, but not on the magnitude of $\underset{\sim}{\rho}_{12}'' \cong \rho''\underset{\sim}{\nu}_{12}''$. The magnitude of $\underset{\sim}{r}'$ also remains finite in (C2), so that (for the purpose of estimating the behavior at large r) the integration over $d\underset{\sim}{\nu}_{12}''$ can be replaced by $\rho_{12}^{-2}J_2(\rho_{12}\sqrt{E})$ as in Eqs. (A13) and the discussion subsequent thereto. Note that in (C2) the magnitude of the six-dimensional vector $\underset{\sim}{\rho}_{12}$ defined by Eq. (A4) cannot be equated to the magnitude of the nine-dimensional vector $\underset{\sim}{\rho}$ at large r, because r_{12}/r does not approach zero as $r \to \infty \,||\, \underset{\sim}{\nu}$. For given $\underset{\sim}{\nu}$, however, the ratio r/ρ_{12} remains constant and finite as $r \to \infty \,||\, \underset{\sim}{\nu}$, excepting those very special $\underset{\sim}{\nu}$ (surely ignorable for present purposes) wherein q_{12} and R remain finite--i.e., wherein r_{12} alone becomes infinite--as $r \to \infty$.

At large r, therefore, the contribution--to the V_{12} term on the left side of (99)--from values of r" large compared to r is estimated by

$$\int_{\gamma\rho}^{\infty} d\rho'' \, e^{i\rho''\sqrt{E}} \frac{e^{i\rho''\sqrt{E-\epsilon_j}}}{\rho''^{3/2}} \frac{J_2(\rho_{12}\sqrt{E})}{\rho_{12}^2} \tag{C3}$$

where γ is some number > 1, sufficiently large to ensure that the factors preceding $V_{12}(\underset{\sim}{r}_{12}'')$ in the integrand of (C1a) represent $G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ in the range $\rho'' > \rho$ to within, e.g., one percent. The reason for this choice of lower limit is to be understood as follows. Strictly speaking, we want to compare the contribution from the region $r'' > r$ with the contribution from $r'' < r$, in which event the integration in (C3) should run from ρ to ∞. However the aforementioned asymptotic representation of $G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ is only valid for r'' sufficiently large compared to r. On the other hand, if (99) really is valid, then the contribution to the left side of (99) from every region $r'' > \gamma r$, $\gamma > 1$ (i.e., from every region $\rho'' > \gamma\rho$, $\gamma > 1$) must be negligible compared to the contribution from $r'' < r$. Thus, in (C3), I am estimating the contribution to the left side of (99) from that region $\rho'' > \gamma\rho > \rho$ wherein one can be confident that (C3) reasonably accurately estimates that contribution.

Letting $\rho'' = \rho x$, (C3) becomes

$$\frac{J_2(\rho_{12}\sqrt{E})}{\rho_{12}^2\,\rho^{1/2}}\int_{\gamma}^{\infty} dx\,\frac{e^{i\rho x(\sqrt{E}+\sqrt{E-\varepsilon_j})}}{x^{3/2}} \qquad \text{(C4)}$$

But the integral in (C4) is of order ρ^{-1} by the Riemann-Lebesgue lemma[31]; also, $J_2(\rho_{12}\sqrt{E}) \sim \rho_{12}^{-1/2}$. Hence [recalling the remarks at the end of the next to the last paragraph] the entire expression (C4)—and, consequently, the expression (C1a)—is of order $\rho_{12}^{-5/2}\,\rho^{-3/2} \cong r^{-4}$, when bound states $u_j(\underset{\sim}{r}_{12})$ exist.

In the absence of bound states $u_j(\underset{\sim}{r}_{12})$, or equivalently when $r'' \to \infty$ along directions $\underset{\sim}{\nu}''$ wherein every $r_{\alpha\beta}'' \to \infty$, (C1a) would be replaced by [neglecting inessential factors]

$$\int d\underset{\sim}{R}'' d\underset{\sim}{q}_{12}'' d\underset{\sim}{r}_{12}'' \frac{e^{i\rho''\sqrt{E}}}{\rho''^4} e^{-i(\underset{\sim}{K}_f''\cdot\underset{\sim}{R}+\underset{\sim}{K}_{12f}''\cdot\underset{\sim}{q}_{12})} V_{12}(\underset{\sim}{r}_{12}'') \frac{e^{i\rho''\sqrt{E}}}{\rho''^4} \cong \rho_{12}^{-5/2}\rho^{-3} \cong r^{-11/2} \tag{C5a}$$

because the only essential difference between (C5a) and (C1a) is the extra factor $\rho''^{3/2}$ in the denominator of (C5a). Thus use of (99) should be legitimate in the absence of bound states. On the other hand, when $r'' \to \infty$ along those very special directions $\underset{\sim}{\nu}_{12}''$ wherein propagation in three-body bound states is possible, (C1a) would be replaced by [recall Eqs. (A15)]

$$\int d\underset{\sim}{R}'' d\underset{\sim}{q}_{12}'' d\underset{\sim}{r}_{12}'' \frac{e^{i\rho''\sqrt{E}}}{\rho''^4} e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}} V_{12}(\underset{\sim}{r}_{12}'') \frac{e^{iK_jR''}}{R''} u_j(\underset{\sim}{r}_{12}'', q_{12}'') \tag{C5b}$$

In (C5b), both $\underset{\sim}{q}_{12}''$ and $\underset{\sim}{r}_{12}''$ remain finite, so that Eqs. (25d) and (90c) imply $K_f'' = (2ME/\hbar^2)^{1/2}$ at large $r'' \cong R''$. Thus, making use of (A6), the $r'' > r$ contribution to (C5b) is estimated by

$$\int_{\gamma_\rho}^{\infty} d\rho''\, \rho''^2 \frac{e^{i\rho''\sqrt{E}}}{\rho''^4} \frac{e^{i\rho''\sqrt{E-\varepsilon_j}}}{\rho''} j_0(\tilde{R}\sqrt{E}) \cong \rho^{-4} \cong r^{-4} \tag{C5c}$$

comparing with Eqs. (C3) – (C4) and recognizing $j_o(\tilde{R}\sqrt{E})$ is $\sim \rho^{-1}$ as $r \to \infty$ || fixed $\underset{\sim}{\nu}$.

C.2 Eq. (101)

Next consider Eq. (101). The right side of (101) is of order $\bar{r}^{-5/2}$ at large $\bar{r}$. However, much as in Eq. (C1a), when $u_j(\underset{\sim}{r}_{12})$ occur the contribution—from $\bar{r}'' > \bar{r}$—to the V_{12} term on the left side of (101) behaves like

$$\int d\underset{\sim}{q}''_{12}\, d\underset{\sim}{r}''_{12} \frac{e^{i\bar{\rho}''\sqrt{\bar{E}}}}{\bar{\rho}''^{5/2}} e^{-i\underset{\sim}{K}''_{12f}\cdot\underset{\sim}{q}_{12}} V_{12}(\underset{\sim}{r}''_{12}) \frac{e^{iK''_{12j}q''_{12}}}{q''_{12}} u_j(\underset{\sim}{r}''_{12}) \bar{\Psi}^{(-)*}_{12jf}(\underset{\sim}{\bar{r}}'; \bar{E}) \qquad \text{(C6a)}$$

where $\underset{\sim}{\bar{\rho}}''$ is defined by the analogue of (102d), and approaches infinity along $\underset{\sim}{\bar{\nu}}_{12}''$ keeping r_{12}'' finite. The fact that r_{12}'' remains finite as $\bar{\rho}'' \to \infty$ now implies that the magnitudes of the six-dimensional $\underset{\sim}{\bar{\rho}}''$ and the three-dimensional $\underset{\sim}{q}_{12}''$ are related by the analogue of (116b). Also, we now have, in analogy with (C1b),

$$\underset{\sim}{K}''_{12f}\cdot\underset{\sim}{q}_{12} = \frac{2\mu_{3R}}{\hbar^2}\sqrt{\bar{E}}\, q''_{12} \frac{\underset{\sim}{\bar{\nu}}''_{12}\cdot\underset{\sim}{q}_{12}}{\bar{\rho}''} \cong \left(\frac{2\mu_{3R}\bar{E}}{\hbar^2}\right)^{1/2} \underset{\sim}{\bar{\nu}}''_{12}\cdot\underset{\sim}{q}_{12} \qquad \text{(C6b)}$$
$$\cong \sqrt{\bar{E}}\,\underset{\sim}{\bar{\nu}}''_{12}\cdot\underset{\sim}{\tilde{q}}_{12}\,.$$

using (A6). Hence, recalling (114b), the integral (C6a) can be replaced by

$$\int d\bar{\rho}''\,\bar{\rho}''^2\, d\underset{\sim}{\bar{\nu}}''_{12} \frac{e^{i\bar{\rho}''\sqrt{\bar{E}}}}{\rho''^{5/2}} e^{-i\sqrt{\bar{E}}\,\underset{\sim}{\bar{\nu}}''_{12}\cdot\underset{\sim}{\tilde{q}}_{12}} \frac{e^{i\bar{\rho}''\sqrt{\bar{E}-\varepsilon_j}}}{\bar{\rho}''} \qquad \text{(C7)}$$

In (C7), again because r_{12}/r does not approach zero as $\bar{r} \to \infty \parallel \bar{\underset{\sim}{\nu}}$, Eq. (116b) does not hold for the unprimed variables; however, as in (C2), for given $\bar{\underset{\sim}{\nu}}$ the ratio $\bar{r}/\tilde{q}_{12}$ remains constant and finite. Hence at large $\bar{r}$ the contribution (C6a) to the left side of (101)--at large $\bar{r}'' > \bar{r}$--is estimated by [compare Eq. (C5c)]

$$\int_{\gamma\bar{\rho}}^{\infty} d\bar{\rho}'' \frac{e^{i\bar{\rho}''\sqrt{E}}\, e^{i\bar{\rho}''\sqrt{E-\varepsilon_j}}}{\bar{\rho}''^{3/2}} j_0(\tilde{q}_{12}\sqrt{E}) \cong \frac{1}{\bar{\rho}^{3/2}} \cdot \frac{1}{\tilde{q}_{12}} \cong \bar{r}^{-5/2} \tag{C8}$$

In other words, when two-body bound states exist, the contribution to the left side of (101) from $\bar{r}'' > \bar{r}$ is of the same order of magnitude as the right side of (101), implying that Eq. (101) is invalid. On the other hand, when there are no two-body bound states, the continuum contribution to the left side of (101), estimated by the center of mass analogue of (C5a), readily is found to be of order $\bar{r}^{-4}$; correspondingly, Eq. (101) should be valid when two-body bound states do not occur. Of course, the presence of three-body bound states also does not adversely affect the validity of Eq. (101).

I add that since $G^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}')$ behaves essentially like $G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}')$ when $r'' \to \infty$ along $\underset{\sim}{\nu}_{12}''$, the analysis of this section--relative to interchange of the order of integration and limit $r \to \infty$ in the $G_F^{(+)}V_{12}G^{(+)}$ term of Eq. (99)--applies also to the corresponding interchange in the integral on the left side of (103).

C.3 Comparison of Eqs. (97a) and (97b)

It is of interest to compare the $r'' > r$ contributions to the integrals in (97a) and (97b). In particular, I shall compare

$$\lim_{r \to \infty \| \underset{\sim}{\nu}} \int d\underset{\sim}{r}'' G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'') V_{12}(\underset{\sim}{r}''_{12}) G_F^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}') \tag{C9}$$

from (97b) with the corresponding contribution [evaluated in section C.1] from (97a). As in section C.1, no $r_{\alpha\beta}$ remains finite as $r \to \infty \,||\, \underset{\sim}{\nu}$.

In the integrand of (C9), $G_F^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}')$ is of order ρ''^{-4} as $r'' \to \infty$. Thus, recalling Eq. (C5a), it is clear that the $r'' > r$ contribution to (C9) will be of order $r^{-11/2}$ unless there are important contributions from $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ as $r'' \to \infty$ along directions $\underset{\sim}{\nu}_{12}''$ corresponding to propagation in two-body or three-body bound states. Suppose, first, that two-body bound states $u_j(\underset{\sim}{r}_{12}'')$ exist. Then, recalling Eq. (A3), along such directions $\underset{\sim}{\nu}_{12}''$, $\lim G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ is proportional to $\rho''^{-5/2}\, e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}}$ and to $\bar{\Psi}_{12jf}^{(-)*}(\bar{\underset{\sim}{r}})$. But, now referring to Eqs. (115), $\bar{\Psi}_{12jf}^{(-)*}(\bar{\underset{\sim}{r}})$ is composed of two terms: (i) its incident part, which is proportional to $u_j^*(\underset{\sim}{r}_{12})$, and thus vanishes exponentially because it has been hypothesized that $r_{12} \to \infty$ with r; (ii) its scattered part, which—because of the $u_j^*V_{23}$ and $u_j^*V_{31}$ products in the integral on the right side of (115b)—clearly behaves like $\bar{\rho}^{\,-5/2}$ at large $\bar{r}$. At large r and $r'' > r$, therefore, the two-body bound state contribution to (C9) from $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ is at worst $\cong e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}}\, \rho''^{-5/2}\, \bar{\rho}^{-5/2}$. This is equivalent to replacing $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ in (C9) by a factor of order $e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}}/\rho''^5$, which is small compared even to the ρ''^{-4} contribution made by propagation of $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$ in unbound states and which [after performing the angular integration over $\underset{\sim}{\nu}_{12}''$] obviously makes a contribution to (C9) decreasing more rapidly than r^{-4}. Similarly, there are negligibly small (compared to r^{-4}) contributions to (C9) from propagation in three-body states $u_j(\underset{\sim}{r}_{12}'',\underset{\sim}{r}_{23}'')$.

It follows that the processes of integration and lim as $r \to \infty \,||\, \underset{\sim}{\nu}$ can be legitimately reversed in (C9), and therefore in (97b). Evidently the fundamental difference between (97a) and (97b) is that we have $G^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}')$ in (97a), while in (97b) we have $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')$. As a result, because all the complicating contributions at large $\underset{\sim}{r}''$ arise from propagation in bound states, and because $G_F^{(+)}$ never propagates in bound states, these contributions are more important in (97a)[where $\underset{\sim}{r}'$ remains finite] than in (97b) [where all $r_{\alpha\beta}$ are becoming infinite with r].

C.4 Eq. (123) and Related Expressions

I wish to show that the interchange of order of integration and limit $r \to \infty$ is unjustified in (123), even when the integral on the left side of (123) is convergent, i.e., even when bound states do not exist. In other words, I wish to show that the contribution to the left side of (123) from the region $r' > r$ is not negligible compared to r^{-4} as $r \to \infty$, even when $G^{(+)}$ cannot propagate in bound states. Consider, e.g., the V_{12} term on the left side of (123). Then, under the circumstances described the $r' > r$ contribution to this term is estimated by an integral of form

$$\int d\underset{\sim}{R}' dq'_{\sim 12} d\underset{\sim}{r}'_{12} \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} \Psi_f^{(-)*}(\underset{\sim}{r}) V_{12}(\underset{\sim}{r}'_{12}) e^{i(\underset{\sim}{K}_i \cdot \underset{\sim}{R}' + \underset{\sim}{K}_{12i} \cdot \underset{\sim}{q}'_{12} + \underset{\sim}{k}_{12i} \cdot \underset{\sim}{r}'_{12})} \tag{C10a}$$

where the subscript f corresponds to directions $\underset{\sim}{\nu}_f' \equiv \underset{\sim}{\nu}_{12f}'$. Replacing $\Psi_f^{(-)*}$ by its ψ_f^* part, Eq. (C10a) reduces to

$$\int d\underset{\sim}{R}' d\underset{\sim}{q}'_{12} \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} e^{-i(\underset{\sim}{K}_f' \cdot \underset{\sim}{R} + \underset{\sim}{K}_{12f}' \cdot \underset{\sim}{q}_{12})} e^{i(\underset{\sim}{K}_i \cdot \underset{\sim}{R}' + \underset{\sim}{K}_{12i} \cdot \underset{\sim}{q}'_{12})} \tag{C10b}$$

Next employ Eqs. (90c), (A4), (A6), and (A11), and recall Eqs. (A12). Therewith, much as in (C1b), it is possible to rewrite (C10b) as

$$\int d\rho' d\underset{\sim}{\nu}'_{12} \frac{\rho'^5 e^{i\rho'\sqrt{E}}}{\rho'^4} e^{i\tilde{\underset{\sim}{R}}' \cdot [\tilde{\underset{\sim}{K}}_i - \frac{\sqrt{E}\tilde{\underset{\sim}{R}}}{\rho'}]} e^{i\tilde{\underset{\sim}{q}}'_{12} \cdot [\tilde{\underset{\sim}{K}}_{12i} - \frac{\sqrt{E}\,\tilde{\underset{\sim}{q}}_{12}}{\rho'}]} \tag{C11a}$$

$$= \int d\rho' d\underset{\sim}{\nu}'_{12} \frac{\rho'^5 e^{i\rho'\sqrt{E}}}{\rho'^4} e^{i\underset{\sim}{\rho}' \cdot \underset{\sim}{B}} \tag{C11b}$$

where the definition of $\underset{\sim}{B}$ [not to be confused with $\underset{\sim}{B}$ of Eqs. (171a)] is

$$\underset{\sim}{B} = \left(\underset{\sim}{\tilde{K}}_i - \frac{\sqrt{E}}{\rho'}\underset{\sim}{\tilde{R}}\right) \oplus \left(\underset{\sim}{\tilde{K}}_{12i} - \frac{\sqrt{E}}{\rho'}\underset{\sim}{\tilde{q}}_{12}\right) \qquad \text{(C11c)}$$

Thus, as in the transition from (C2) to (C3), one can replace (C11a) by

$$\int_{\gamma_\rho}^{\infty} d\rho'\rho' e^{i\rho'\sqrt{E}} \frac{J_2(B\rho')}{(B\rho')^2} \cong \int_{\gamma_\rho}^{\infty} d\rho'\rho' \frac{e^{i(\rho'\sqrt{E} \pm B\rho')}}{(B\rho')^{5/2}} \qquad \text{(C12)}$$

From the results of Appendix F [or directly from the defining relation (25d) for the components of the nine-dimensional $\underset{\sim}{\rho}$],

$$\rho^2 = \tilde{R}^2 + \tilde{q}_{12}^{\,2} + \tilde{r}_{12}^{\,2} \qquad \text{(C13a)}$$

where

$$\underset{\sim}{\tilde{r}}_{12} = \left(\frac{2\mu_{12}}{\hbar^2}\right)^{1/2} \underset{\sim}{r}_{12} \qquad \text{(C13b)}$$

For given $\underset{\sim}{\nu}_f$ in (123), therefore, I can write

$$\begin{aligned} \tilde{R} &= \rho\cos\theta \\ \tilde{q}_{12} &= \rho\sin\theta\cos\varphi \\ \tilde{r}_{12} &= \rho\sin\theta\sin\varphi \end{aligned} \qquad \text{(C13c)}$$

where the angles θ, ϕ are determined by the given direction $\underset{\sim}{\nu}_f$.

Hence, from (C11c)

$$B\rho' = \left[\left(\underset{\sim}{\tilde{K}}_i\rho' - \sqrt{E}\rho\cos\theta\, \underset{\sim}{n}_R\right)^2 + \left(\underset{\sim}{\tilde{K}}_{12i}\rho' - \sqrt{E}\rho\sin\theta\cos\varphi\, \underset{\sim}{n}_q\right)^2\right]^{1/2} \qquad \text{(C13d)}$$

where $\underset{\sim}{n}_R$, $\underset{\sim}{n}_q$ are respectively the directions along which $\underset{\sim}{R}$, $\underset{\sim}{q}$ approach infinity in three-dimensional physical space.

Now, finally, let $\rho' = \rho x$ in (C12). Thus, as in (C4), and using the Riemann-Lebesgue lemma, (C12) is estimated by

$$\frac{\rho^2}{\rho^{5/2}} \int_\gamma^\infty dx \frac{e^{i\rho[x\sqrt{E} \pm B_1(x)]}}{[B_1(x)]^{5/2}} \cong \frac{1}{\rho^{3/2}} \tag{C14a}$$

where

$$B_1(x) = \left[\left(x\tilde{\underset{\sim}{K}}_i - \sqrt{E}\cos\theta\, \underset{\sim R}{n}\right)^2 + \left(x\tilde{\underset{\sim}{K}}_{12i} - \sqrt{E}\ \sin\theta\cos\varphi\, \underset{\sim q}{n}\right)^2 \right]^{1/2} \tag{C14b}$$

The reader is reminded that $\underset{\sim}{K}_i$, $\underset{\sim}{K}_{12i}$ are fixed by the incident wave ψ_i in (123), and that $\underset{\sim}{n}_R$, $\underset{\sim}{n}_q$ are determined (along with θ, ϕ) by the given direction $\underset{\sim}{\nu}_f$. If (C14a) has a point of stationary phase in the range of integration over x, which is conceivable, the left side of (C14a) could only decrease even less rapidly than $\rho^{-3/2}$. Consequently it has been shown that the continuum (non-bound state) contribution to the left side of (123) from the region $r' > r$ assuredly is non-negligible compared to r^{-4}, which suffices to demonstrate Eq. (123) cannot be valid even in the absence of bound states.

The preceding analysis also immediately demonstrates that Eq. (124a) has been deduced via unjustified manipulations. For (124a) to be justified, it is necessary that the $r' > r$ contribution to Eq. (42) be negligible compared to r^{-4}. Thus, for ψ_i of (21a), and for the V_{12} interaction in V_i, I must estimate the magnitude of

$$\int d\underset{\sim}{R}' d\underset{\sim}{q}_{12}' d\underset{\sim}{r}_{12}' \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} e^{-i[\underset{\sim}{K}_f' \cdot \underset{\sim}{R} + \underset{\sim}{K}_{12f}' \cdot \underset{\sim}{q}_{12} + \underset{\sim}{k}_{12f}' \cdot \underset{\sim}{r}_{12}]} V_{12}(\underset{\sim}{r}_{12}') \Psi_i^{(+)}(\underset{\sim}{r}') \quad \text{(C15)}$$

in the range $r' > r$. Replacing $\Psi_i^{(+)}$ by its incident part ψ_i and performing the integration over $d\underset{\sim}{r}_{12}'$, one arrives at precisely the expression (C10b), which has been shown to be of order $\rho^{-3/2} \cong r^{-3/2}$, Q.E.D.

Furthermore, the aforementioned preceding analysis is almost as immediately applicable to the validity of (125), and to the corresponding interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty$ in the center of mass version of (42). Evidently the center of mass integral which now must be evaluated will reduce to the analogue of (C10b), namely

$$\int d\underset{\sim}{q}_{12}' \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} e^{-i\underset{\sim}{K}_{12f}' \cdot \underset{\sim}{q}_{12}} e^{i\underset{\sim}{K}_{12i} \cdot \underset{\sim}{q}_{12}'} \quad \text{(C16a)}$$

For the center of mass interchanges under present consideration to be valid, the contribution to (C16a) from the range $\bar{r}' > \bar{r}$ will have to be negligible compared to $\bar{r}^{-5/2}$ at large $\bar{r}$. In (C16a), moreover, (116b) relates $\bar{\rho}'$ and q_{12}', so that now $\tilde{q}_{12}' = \bar{\rho}'$; also, using (90), now

$$K'_{12f} = \frac{2\mu_{3R}}{\hbar^2}\sqrt{\bar{E}}\,\frac{q'_{12}}{\bar{\rho}'} = \left(\frac{2\mu_{3R}\bar{E}}{\hbar^2}\right)^{1/2} \tag{C16b}$$

consistent with (116c). Thus (C16a) reduces to

$$\int d\bar{\rho}'\, d\underset{\sim}{\bar{\nu}}'_{12}\, \bar{\rho}'^2 \frac{e^{i\bar{\rho}'\sqrt{\bar{E}}}}{\bar{\rho}'^{5/2}}\, e^{i\underset{\sim}{\bar{\nu}}'_{12}\cdot[\bar{\rho}'\underset{\sim}{\tilde{K}}_{12i} - \sqrt{\bar{E}}\,\underset{\sim}{\tilde{q}}_{12}]} \tag{C17a}$$

$$= \int_{\gamma\bar{\rho}}^{\infty} d\bar{\rho}' \frac{e^{i\bar{\rho}'\sqrt{\bar{E}}}}{\bar{\rho}'^{1/2}}\, j_0\left(|\bar{\rho}'\underset{\sim}{\tilde{K}}_{12i} - \sqrt{\bar{E}}\,\underset{\sim}{\tilde{q}}_{12}|\right) \tag{C17b}$$

Letting $\bar{\rho}' = x\bar{\rho}$, and comparing with Eq. (C8), as well as with the arguments reducing Eq. (C12) to (C14a), one sees that (C17b) is of order $\bar{\rho}^{-3/2}$, i.e., non-negligible compared to $\bar{r}^{-5/2}$.

It is completely obvious that the $\bar{r}' > \bar{r}$ contribution to the integral (131a) involves precisely the integral (C16a), so that the result just obtained also implies the $\bar{r}' > r$ contribution to (131a) is of order $\bar{\rho}^{-3/2} \cong \bar{r}^{-3/2}$.

C.5 Bound State Effects in Transition Amplitudes

To begin this section, I examine the $r' > r$ contribution to the left side of (123) associated with propagation in two-body bound states; such contributions were deliberately ignored in the preceding section C.4. In particular, consider, e.g., the V_{12} term on the left side of (123), and suppose a bound state $u_j(\underset{\sim}{r}_{12})$ exists. Then the contribution in question is given by [compare Eqs. (A13a) and (C10a)]

$$\int d\underset{\sim}{R}' d\underset{\sim}{q}'_{12} d\underset{\sim}{r}'_{12} \frac{e^{i\rho'\sqrt{E-\epsilon_j}}}{\rho'^{5/2}} u_j(\underset{\sim}{r}'_{12}) e^{-i\underset{\sim}{K}'\cdot\underset{\sim}{R}} \bar{\Psi}^{(-)*}_{12j}(\bar{\underset{\sim}{r}};\bar{E}) V_{12}(r'_{12}) e^{i(\underset{\sim}{K}_i\cdot\underset{\sim}{R}'+\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}'_{12}+\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}'_{12})} \tag{C18}$$

The notation in (C18) is complicated by the fact that $\underset{\sim}{r} \to \infty \mid\mid \underset{\sim}{\nu}_f$ [recall (123)], while $\underset{\sim}{r}'$ is becoming infinite along $\underset{\sim}{\nu}_{12}'$. I will retain the subscript f for the $\underset{\sim}{r} \to \infty \mid\mid \underset{\sim}{\nu}_f$ process, and therefore drop this subscript from $\bar{\Psi}^{(-)*}$, with the understanding that the incident wave $\bar{\psi}^*$ associated with $\bar{\Psi}_{12j}^{(-)*}$ is [recall Eqs. (115)]

$$\bar{\psi}^*(\bar{\underset{\sim}{r}}) = e^{-i\underset{\sim}{K}'_{12j}\cdot\underset{\sim}{q}_{12}} u_j^*(\underset{\sim}{r}_{12}) \tag{C19}$$

where the primed wave vectors $\underset{\sim}{K}'$, $\underset{\sim}{K}_{12j}'$ are associated with the $\underset{\sim}{r}' \to \infty \mid\mid \underset{\sim}{\nu}_{12}'$ process; moreover, $\underset{\sim}{K}_{12j}' = K_{12j}\underset{\sim}{\nu}_{12}'$, where K_{12j} satisfies (114b).

The incident part (C19) of $\bar{\Psi}_{12j}^{(-)*}$ in (C18) vanishes exponentially as $\underset{\sim}{r} \to \infty \mid\mid \underset{\sim}{\nu}_f$, so that $\bar{\Psi}_{12j}^{(-)*}$ in (C18) can be replaced by its scattered part. Thus (C18) reduces to

$$\int d\rho' d\underset{\sim}{\nu}'_{12}\, \rho'^5 \frac{e^{i\rho'\sqrt{E-\epsilon_j}}}{\rho'^{5/2}} e^{-i\underset{\sim}{K}'\cdot\underset{\sim}{R}} \frac{e^{i\bar{\rho}\sqrt{\bar{E}}}}{\bar{\rho}^{5/2}} e^{i(K_i\cdot\underset{\sim}{R}'+\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}'_{12})} \tag{C20a}$$

where $\bar{\rho}$ is given by (102d), while $\rho' \equiv \rho_{12}'$ is defined by (A4). Eq. (C20a) can be rewritten as [recall Eqs. (C10) - (C11)]

$$\frac{e^{i\bar{\rho}\sqrt{E}}}{\bar{\rho}^{5/2}} \int d\rho' d\underset{\sim}{\nu}'_{12}\, \rho'^{5} \frac{e^{i\rho'\sqrt{E-\varepsilon_j}}}{\rho'^{5/2}} e^{i\underset{\sim}{\rho}'\cdot \underset{\sim}{C}} \tag{C20b}$$

where $\underset{\sim}{C}$ [not to be confused with $\underset{\approx}{C}$ of Eqs. (171b)] is

$$\underset{\sim}{C} = \left(\underset{\sim}{\tilde{K}}_i - \frac{\sqrt{E}}{\rho'} \underset{\approx}{\tilde{R}} \right) \oplus \underset{\approx}{\tilde{K}}_{12i} \tag{C20c}$$

Consequently (C20b) is estimated by

$$\frac{1}{\bar{\rho}^{5/2}} \int_{\gamma_\rho}^{\infty} d\rho' \rho'^{5/2} e^{i\rho'\sqrt{E-\varepsilon_j}} \frac{J_2(C\rho')}{(C\rho')^2} \cong \frac{1}{\bar{\rho}^{5/2}} \int_{\gamma_\rho}^{\infty} d\rho' \rho'^{5/2} e^{i\rho'\sqrt{E-\varepsilon_j}} \frac{e^{\pm iC\rho'}}{(C\rho')^{5/2}} \cong \frac{1}{\rho^{5/2}} \tag{C21}$$

where the argument proceeds along the lines of Eqs. (C12) - (C14). The result (C21) is small compared to the continuum contribution examined in section C.4, but definitely is not negligible compared to r^{-4}.

Note that if one pursues the above argument for the case of a three-body bound state $u_j(\underset{\sim}{r}_{12};\underset{\sim}{r}_{23})$, the $r' > r$ contribution is negligible, for then [among other changes] $\bar{\Psi}_{12j}^{(-)*}$ in (C18) is replaced by precisely $e^{-i\underset{\sim}{K}'\cdot\underset{\sim}{R}} u_j(\underset{\sim}{r}_{12},\underset{\sim}{r}_{23})$; in other words $\bar{\Psi}$ in (C18) now will contain no scattered part, a result associated with the fact that the Hamiltonian is independent of $\underset{\sim}{R}$. To this result corresponds the additional fact that there are no δ-functions in (120b) associated with three-body bound states. I also note that in the center of mass system the analogues of (C18) - (C21) yield an $\bar{r}' > \bar{r}$ contribution of order $\bar{\rho}^{-5/2}$, as is readily seen; consequently the possibility of propagation in two-body bound states also invalidates Eq. (125).

The bound state $r' > r$ contribution to the integral (42), for ψ_i of (21a), is estimated by

$$\int d\underset{\sim}{R}' d\underset{\sim}{q}'_{12} d\underset{\sim}{r}'_{12} \frac{e^{i\rho'\sqrt{E}}}{\rho'^4} e^{-i(\underset{\sim}{K}'\cdot\underset{\sim}{R}+\underset{\sim}{k}'_{12}\cdot\underset{\sim}{r}_{12}+\underset{\sim}{K}'_{12}\cdot\underset{\sim}{q}_{12})} V_{12}(\underset{\sim}{r}'_{12}) \frac{e^{i\rho'\sqrt{E-\varepsilon_j}}}{\rho'^{5/2}} u_j(\underset{\sim}{r}'_{12}) \tag{C22}$$

where the first factors arise from $G_i^{(+)} = G_F^{(+)}$ in (42), and the last factors from $\Psi_i^{(+)}$. The integral (C22) is essentially of the type (C10b), except for an extra factor $\rho'^{-5/2}$ in the integrand. Therefore, recalling (C12), the integral (C22) behaves like $\rho^{-3/2}\,\rho^{-5/2} = \rho^{-4}$ at large ρ. The corresponding $\bar{r}' > \bar{r}$ contribution to the center of mass version of (42) can be seen to be of order $\bar{\rho}^{-5/2}$, still not sufficiently rapidly decreasing to justify interchange of order of integration and limit $\underset{\sim}{\bar{r}} \to \infty \mid\mid \bar{\underset{\sim}{\nu}}_f$.

APPENDIX D. APPLICATIONS OF STATIONARY PHASE METHOD

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D.1 Eqs. (105)

The first step is to derive Eq. (103). By the same reasoning as is used to derive (53a), one can see that the free-space Green's function has the expansion

$$G_F(\underset{\sim}{r};\underset{\sim}{r}'';E+i\varepsilon)=\frac{1}{(2\pi)^6}\int d\underset{\sim}{K}\,d\underset{\sim}{K}_{12}\,e^{i\underset{\sim}{K}\cdot(\underset{\sim}{R}-\underset{\sim}{R}'')}\,e^{i\underset{\sim}{K}_{12}\cdot(\underset{\sim}{q}_{12}-\underset{\sim}{q}''_{12})}\,g_F\left(\underset{\sim}{r}_{12};\underset{\sim}{r}''_{12};\frac{\hbar^2k_{12}^2}{2\mu_{12}}+i\varepsilon\right) \tag{D1}$$

where Eqs. (28d) and (35) define k_{12} in terms of the integration variables $\underset{\sim}{K}$, $\underset{\sim}{K}_{12}$. Here g_F is the one-dimensional Green's function introduced in Eq. (103) which, directly from the defining equation (24) is

$$g_F(\underset{\sim}{r}_{12};\underset{\sim}{r}''_{12};\lambda)=\left(\frac{2\mu_{12}}{\hbar^2}\right)^{3/2}\frac{i}{4}\left(\sqrt{\frac{\lambda}{2\pi}}\right)^{1/2}\frac{H^{(1)}_{1/2}(\sqrt{\lambda}\,|\rho-\rho''|)}{|\underset{\sim}{\rho}-\underset{\sim}{\rho}''|^{1/2}} \tag{D2}$$

$$=\frac{2\mu_{12}}{\hbar^2}\frac{1}{4\pi}\frac{e^{i\sqrt{\frac{2\mu_{12}\lambda}{\hbar^2}}|\underset{\sim}{r}_{12}-\underset{\sim}{r}''_{12}|}}{|\underset{\sim}{r}_{12}-\underset{\sim}{r}''_{12}|}$$

Similarly

$$G_{12}(\underset{\sim}{r}'';\underset{\sim}{r}';E+i\varepsilon)=\frac{1}{(2\pi)^6}\int d\underset{\sim}{K}\,d\underset{\sim}{K}_{12}\,e^{i\underset{\sim}{K}\cdot(\underset{\sim}{R}''-\underset{\sim}{R}')}\,e^{i\underset{\sim}{K}_{12}\cdot(\underset{\sim}{q}''_{12}-\underset{\sim}{q}'_{12})}\,g_{12}\left(\underset{\sim}{r}''_{12};\underset{\sim}{r}'_{12};\frac{\hbar^2 k_{12}^2}{2\mu_{12}}+i\varepsilon\right) \quad \text{(D3)}$$

with g_{12} defined as in (75). Thus

$$\int d\underset{\sim}{r}'' G_F(\underset{\sim}{r};\underset{\sim}{r}'';E+i\varepsilon)\, V_{12}(\underset{\sim}{r}_{12}'')\, G_{12}(\underset{\sim}{r}'';\underset{\sim}{r}';E+i\varepsilon)$$

$$= \frac{1}{(2\pi)^6}\int d\underset{\sim}{r}_{12}'' d\underset{\sim}{K}\, d\underset{\sim}{K}_{12}\, e^{i\underset{\sim}{K}\cdot(\underset{\sim}{R}-\underset{\sim}{R}')} e^{i\underset{\sim}{K}_{12}\cdot(\underset{\sim}{q}_{12}-\underset{\sim}{q}_{12}')} g_F(\underset{\sim}{r}_{12};\underset{\sim}{r}_{12}'';\frac{\hbar^2 k_{12}^2}{2\mu_{12}}+i\varepsilon) V_{12}(\underset{\sim}{r}_{12}'') g_{12}(\underset{\sim}{r}_{12}'';\underset{\sim}{r}_{12}';\frac{\hbar^2 k_{12}^2}{2\mu_{12}}+i\varepsilon) \quad \text{(D4)}$$

Employing the notation of (A6) and (A11), the integral on the right side of (D4) takes the form[29]

$$\frac{1}{(2\pi)^6}\left(\frac{2M}{\hbar^2}\right)^{3/2}\left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2}\int d\underset{\sim}{r}_{12}'' d\underset{\approx}{\tilde{K}}\, d\underset{\approx}{\tilde{K}}_{12}\, e^{i\underset{\approx}{\tilde{K}}\cdot(\underset{\approx}{\tilde{R}}-\underset{\approx}{\tilde{R}}')} e^{i\underset{\approx}{\tilde{K}}_{12}\cdot(\underset{\sim}{\tilde{q}}_{12}-\underset{\sim}{\tilde{q}}_{12}')}$$

$$\times g_F(\underset{\sim}{r}_{12};\underset{\sim}{r}_{12}'';\hat{E}_{12}+i\varepsilon) V_{12}(\underset{\sim}{r}_{12}'') g_{12}(\underset{\sim}{r}_{12}'';\underset{\sim}{r}_{12}';\hat{E}_{12}+i\varepsilon) \quad \text{(D5)}$$

where

$$\hat{E}_{12} = \frac{\hbar^2 k_{12}^2}{2\mu_{12}} \doteq E - \tilde{K}^2 - \tilde{K}_{12}^2 \quad \text{(D6)}$$

Now introduce the six-dimensional vector $\underset{\approx}{\mathcal{E}}_{12} \equiv \underset{\sim}{\mathcal{E}}$ of Eq. (A12c), and define the corresponding six-dimensional vector $\underset{\sim}{s}$ by

$$\underset{\sim}{s} = (\underset{\sim}{\tilde{R}}-\underset{\sim}{\tilde{R}}') \oplus (\underset{\sim}{\tilde{q}}_{12}-\underset{\sim}{\tilde{q}}_{12}') \equiv \underset{\sim}{\rho}_{12}-\underset{\sim}{\rho}_{12}' \quad \text{(D7)}$$

where $\underset{\sim}{\rho}_{12}$, $\underset{\sim}{\rho}_{12}'$ are defined as in (A4). Then Eq. (104b) holds; also, as in (A12a),

$$e^{i\left[\underset{\sim}{K}\cdot(\underset{\sim}{R}-\underset{\sim}{R}')+\underset{\sim}{K}_{12}\cdot(\underset{\sim}{q}_{12}-\underset{\sim}{q}'_{12})\right]} = e^{is\mathcal{E}\cos\theta_1} \tag{D8}$$

where θ_1 is the angle between the six-dimensional vectors $\underset{\sim}{s}$ and $\underset{\sim}{\mathcal{E}}$.

I now introduce spherical coordinates in the six-dimensional $\tilde{\underset{\sim}{K}}$, $\tilde{\underset{\sim}{K}}_{12}$ space, with polar axis along $\underset{\sim}{s}$. If $\hat{\underset{\sim}{v}}_1$ is the unit vector along $\underset{\sim}{s}$,

$$\underset{\sim}{\mathcal{E}} = \mathcal{E}\cos\theta_1\,\hat{\underset{\sim}{v}}_1 \oplus \underset{\sim}{a}_1 \tag{D9}$$

where $\underset{\sim}{a}_1$ is a vector of magnitude $\mathcal{E}\sin\theta_1$, in the 5-dimensional subspace [of the original space subtended by $\tilde{\underset{\sim}{K}}_1$, $\tilde{\underset{\sim}{K}}_{12}$] orthogonal to $\hat{\underset{\sim}{v}}_1$. In this subspace pick some other unit vector $\hat{\underset{\sim}{v}}_2$ as polar axis for $\underset{\sim}{a}_1$. Then, as in (D9)

$$\underset{\sim}{a}_1 = a_1\cos\theta_2\,\hat{\underset{\sim}{v}}_2 \oplus \underset{\sim}{a}_2 = \mathcal{E}\sin\theta_1\cos\theta_2\,\hat{\underset{\sim}{v}}_2 + \underset{\sim}{a}_2 \tag{D10}$$

where θ_2 is the angle between $\underset{\sim}{a}_2$ and $\hat{\underset{\sim}{v}}_2$. **Proceeding similarly, it** is obvious that

$$\begin{aligned}\underset{\sim}{\mathcal{E}} = \mathcal{E}\Big[&\cos\theta_1\,\hat{\underset{\sim}{v}}_1 + \sin\theta_1\cos\theta_2\,\hat{\underset{\sim}{v}}_2 + \sin\theta_1\sin\theta_2\cos\theta_3\,\hat{\underset{\sim}{v}}_3 \\ &+ \sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4\,\hat{\underset{\sim}{v}}_4 + \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos\theta_5\,\hat{\underset{\sim}{v}}_5 \\ &+ \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\,\hat{\underset{\sim}{v}}_6\Big]\end{aligned} \tag{D11a}$$

where $\hat{\underset{\sim}{v}}_1, \hat{\underset{\sim}{v}}_2, \ldots, \hat{\underset{\sim}{v}}_6$ are orthogonal unit vectors spanning the space of $\underset{\sim}{\tilde{K}}$, $\underset{\sim}{\tilde{K}}_{12}$, and where the angles $\theta_1, \ldots, \theta_4$ range from 0 to π, but

$$0 \leq \theta_5 \leq 2\pi \tag{D11b}$$

One then sees that the six-dimensional volume element

$$\begin{aligned} d\underset{\sim}{\tilde{K}}\, d\underset{\sim}{\tilde{K}}_{12} = d\mathcal{E}\, dS = \; & d\mathcal{E}\,(\mathcal{E}\, d\theta_1)(\mathcal{E} \sin\theta_1 d\theta_2) \\ & \times (\mathcal{E} \sin\theta_1 \sin\theta_2 d\theta_3)(\mathcal{E} \sin\theta_1 \sin\theta_2 \sin\theta_3 d\theta_4)(\mathcal{E} \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 d\theta_5) \\ = \; & \mathcal{E}^5 \sin^4\theta_1 \sin^3\theta_2 \sin^2\theta_3 \sin\theta_4\, d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\theta_5 d\mathcal{E} \end{aligned} \tag{D12}$$

Using Eqs. (D8) and (D12), and performing the elementary integrals over $\theta_5, \theta_4, \theta_3, \theta_2$ the integral (D5) reduces to

$$\frac{1}{(2\pi)^6} \frac{8\pi^2}{3} \left(\frac{2M}{\hbar^2}\right)^{3/2} \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2} \int d\underset{\sim}{r}''_{12}\, d\mathcal{E}\, \mathcal{E}^5 d\theta_1 \sin^4\theta_1\, e^{is\mathcal{E}\cos\theta_1}\, g_F(\underset{\sim}{r}_{12}, \underset{\sim}{r}''_{12}; \hat{E}_{12} + i\varepsilon) V_{12}(\underset{\sim}{r}''_{12})\, g_{12}(\underset{\sim}{r}''_{12}, \underset{\sim}{r}'_{12}; \hat{E}_{12} + i\varepsilon) \tag{D13}$$

The integral over $d\theta_1$ in (D13) is known[30]. Consequently, Eqs. (D4) – (D7) and (D13) are seen to yield Eq. (103) in the limit $\varepsilon \to 0$, remembering the discussion following Eq. (65a), and recognizing that the integral over $d\underset{\sim}{r}_{12}''$ on the right side of (103) surely converges because V_{12} is short range.

Replacing J_2 and $g_F^{(+)}$ on the right side of (103) by their asymptotic forms(32) at large $\underset{\sim}{R}$, $\underset{\sim}{r}_{12}$, $\underset{\sim}{q}_{12}$, gives [as justified preceding Eq. (105a)]

$$\lim_{r\to\infty\|\underset{\sim}{\nu}_f}\int d\underset{\sim}{r}''\, G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'';E)\, V_{12}(\underset{\sim}{r}_{12}'')\, G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}';E)$$

$$= \frac{1}{(2\pi)^{9/2}}\frac{2\mu_{12}}{\hbar^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2}\int d\underset{\sim}{r}_{12}''\int d\mathcal{E}\,\frac{\mathcal{E}^{5/2}}{\rho^{5/2}}\cos\left(\rho\mathcal{E}-\frac{5\pi}{4}\right)\frac{e^{ik_{12}r_{12}}}{r_{12}}\, e^{-ik_{12}\underset{\sim}{n}_{12f}\cdot\underset{\sim}{r}_{12}''}$$
$$\times V_{12}(\underset{\sim}{r}_{12}'')\, g_{12}^{(+)}(\underset{\sim}{r}_{12}'';\underset{\sim}{r}_{12}';\hat{E}_{12}) \qquad \text{(D14)}$$

where $\underset{\sim}{n}_{12f}$ represents the (three-dimensional) unit vector along which $\underset{\sim}{r}_{12}$ becomes infinite in magnitude. In (D14), moreover, at large $\underset{\sim}{R}$ and $\underset{\sim}{q}_{12}$, we have from (104a)

$$\cos\left(\rho\mathcal{E}-\frac{5\pi}{4}\right) = \frac{e^{-i\frac{5\pi}{4}}e^{i\rho\mathcal{E}} + e^{i\frac{5\pi}{4}}e^{-i\rho\mathcal{E}}}{2}$$

$$\cong \frac{e^{-i\frac{5\pi}{4}}}{2}e^{i\rho_{12}\mathcal{E}}e^{-i\frac{\mathcal{E}}{\rho_{12}}(\tilde{\underset{\sim}{R}}\cdot\tilde{\underset{\sim}{R}}'+\tilde{\underset{\sim}{q}}_{12}\cdot\tilde{\underset{\sim}{q}}_{12}')} + \frac{e^{i\frac{5\pi}{4}}}{2}e^{-i\rho_{12}\mathcal{E}}e^{i\frac{\mathcal{E}}{\rho_{12}}(\tilde{\underset{\sim}{R}}\cdot\tilde{\underset{\sim}{R}}'+\tilde{\underset{\sim}{q}}_{12}\cdot\tilde{\underset{\sim}{q}}_{12}')} \qquad \text{(D15)}$$

wherein ρ_{12} now is the magnitude of the vector defined by the unprimed analogue of (A4).

The integral over $\mathcal{E}$ on the right side of (D14) now can be evaluated by the method of stationary phase, remembering Eqs. (D6) and (104b) - (104c) still hold. Of course, in this section [unlike the stationary phase estimate (A13e) of the χ-integral in (A13b)] all constant factors must be carefully retained. Because of (D15), and because $V_{12}(\underset{\sim}{r}_{12}'')$ keeps r_{12}'' effectively finite, one sees that—at large $\underset{\sim}{R}_{12}$, $\underset{\sim}{r}_{12}$, $\underset{\sim}{q}_{12}$—the $\mathcal{E}$-integrand in (D14) is a sum of two terms, each of which is slowly varying except for a rapidly oscillating factor $e^{i\phi_\pm}$, where

$$\phi_\pm(\mathcal{E}) = r_{12}k_{12} \pm \rho_{12}\mathcal{E} \tag{D16}$$

Therefore, the points of stationary phase $\mathcal{E}_\pm$ satisfy

$$\frac{d\phi_\pm}{d\mathcal{E}} = 0 = r_{12}\frac{dk_{12}}{d\mathcal{E}} \pm \rho_{12} = -\left(\frac{2\mu_{12}}{\hbar^2}\right)^{1/2}\frac{r_{12}\mathcal{E}}{\sqrt{E-\mathcal{E}^2}} \pm \rho_{12} \tag{D17}$$

Eq. (D17) shows that there is a point of stationary phase only for the factor involving $e^{i\phi_+}$. This stationary phase point is the single positive root of (D17), namely

$$\mathcal{E}_+ = \left[\frac{\rho_{12}^2}{\rho_{12}^2 + \frac{2\mu_{12}}{\hbar^2}r_{12}^2}E\right]^{1/2} = \frac{\sqrt{E}\,\rho_{12}}{\rho} \tag{D18}$$

where I have used (A4) and, e.g., Eq. (25d). Thus, recalling the fundamental definition (90c) of the final wave vectors, one sees that

$$\mathcal{E}_+ = \left[\frac{\hbar^2 K_f^2}{2M} + \frac{\hbar^2 K_{12f}^2}{2\mu_{3R}} \right]^{1/2} \tag{D19}$$

where, in (D19), K_f and K_{12f} have their usual definitions (28c) and (29c) in terms of the final wave vectors $\underset{\sim}{k}_{1f}$, $\underset{\sim}{k}_{2f}$, $\underset{\sim}{k}_{3f}$.

Hence the estimate of the right side of (D14) via the method of stationary phase is

$$\begin{aligned} &\frac{1}{(2\pi)^{9/2}} \frac{2\mu_{12}}{\hbar^2} \left(\frac{2M}{\hbar^2}\right)^{3/2} \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2} \left(\frac{\sqrt{E}}{\rho}\right)^{5/2} \frac{e^{-i5\pi/4}}{2r_{12}} e^{-i(\underset{\sim}{K}_f \cdot \underset{\sim}{R}' + \underset{\sim}{k}_{12f} \cdot \underset{\sim}{r}'_{12})} \\ &\times \int d\underset{\sim}{r}''_{12} \int_0^\infty d\mathcal{E}\, e^{i\varphi_+} e^{-i\underset{\sim}{k}_{12f} \cdot \underset{\sim}{r}''_{12}} V_{12}(\underset{\sim}{r}''_{12}) g_{12}^{(+)}(\underset{\sim}{r}''_{12}, \underset{\sim}{r}'_{12}; E_{12f}) \end{aligned} \tag{D20}$$

where, in accordance with the principal of stationary phase

$$\varphi_+ \equiv \varphi_+(\mathcal{E}) \cong \varphi_+(\mathcal{E}_+) + \frac{1}{2} \left.\frac{d^2\varphi_+}{d\mathcal{E}^2}\right|_{\mathcal{E}=\mathcal{E}_+} (\mathcal{E} - \mathcal{E}_+)^2 \tag{D21}$$

and where, in (D20), E_{12f} is specified as in Eqs. (105), with $\underset{\sim}{k}_{12f}$ the usual final wave vector defined by (29d). To obtain (D20) I have used

$$k_{12}(\mathcal{E}_+) = \left(\frac{2\mu_{12}}{\hbar^2}\right)^{1/2} \sqrt{E - \mathcal{E}_+^2} = k_{12f} \tag{D22a}$$

$$\left(\frac{\varepsilon}{s}\right)^{5/2} \cong \left(\frac{\varepsilon_+}{\rho_{12}}\right)^{5/2} = \left(\frac{\sqrt{E}}{\rho}\right)^{5/2} \tag{D22b}$$

$$\frac{\varepsilon}{\rho_{12}} \tilde{\underset{\sim}{R}} \cdot \tilde{\underset{\sim}{R}}' \cong \frac{2M}{\hbar^2} \frac{\varepsilon_+}{\rho_{12}} \underset{\sim}{R} \cdot \underset{\sim}{R}' = \frac{2M\sqrt{E}}{\hbar^2} \frac{\underset{\sim}{R} \cdot \underset{\sim}{R}'}{\rho} = \underset{\sim}{K}_f \cdot \underset{\sim}{R}' \tag{D22c}$$

$$\frac{\varepsilon}{\rho_{12}} \tilde{\underset{\sim}{q}}_{12} \cdot \tilde{\underset{\sim}{q}}'_{12} \cong \frac{2\mu_{3R}}{\hbar^2} \frac{\varepsilon_+}{\rho_{12}} \underset{\sim}{q}_{12} \cdot \underset{\sim}{q}'_{12} = \underset{\sim}{K}_{12f} \cdot \underset{\sim}{q}'_{12} \tag{D22d}$$

Furthermore, one has

$$\phi_+(\varepsilon_+) = r_{12} k_{12f} + \rho_{12} \varepsilon_+ = \frac{2\mu_{12}}{\hbar^2} \frac{r_{12}^2}{\rho} \sqrt{E} + \frac{\rho_{12}^2}{\rho} \sqrt{E} = \rho\sqrt{E} \tag{D23a}$$

$$\left.\frac{d^2\varphi_+}{d\varepsilon_+^2}\right|_{\varepsilon=\varepsilon_+} = -\left(\frac{2\mu_{12}}{\hbar^2}\right)^{1/2} \frac{r_{12} E}{(E-\varepsilon_+^2)^{3/2}} = -\left(\frac{2\mu_{12}}{\hbar^2}\right)^2 \frac{r_{12} E}{k_{12f}^3} = -\frac{\rho^2}{r_{12} k_{12f}} \tag{D23b}$$

In (D20), therefore

$$\int_0^\infty d\mathcal{E}\, e^{i\phi_+(\mathcal{E})} \cong e^{i\rho\sqrt{E}} \int_0^\infty d\mathcal{E}\, e^{-\frac{i}{2}\frac{\rho^2}{r_{12}k_{12f}}(\mathcal{E}-\mathcal{E}_+)^2} \cong$$

$$e^{i\rho\sqrt{E}}\int_{-\infty}^{\infty} d\chi\, e^{-\frac{i}{2}\frac{\rho^2}{r_{12}k_{12f}}\chi^2} = e^{i\rho\sqrt{E}}\sqrt{\frac{\pi}{\frac{i\rho^2}{2r_{12}k_{12f}}}} = e^{i\rho\sqrt{E}}e^{-\frac{i\pi}{4}}\frac{\sqrt{2\pi r_{12}k_{12f}}}{\rho} \tag{D24}$$

Substituting (D24) into (D20), and again using (90c) to eliminate k_{12f}, the stationary phase estimate of the right side of (D14) becomes

$$\frac{1}{(2\pi)^4}\left(\frac{2\mu_{12}}{\hbar^2}\right)^{3/2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2} e^{-i3\pi/2}\frac{E^{3/2}}{2}\frac{e^{i\rho\sqrt{E}}}{\rho^4}e^{-i(\underset{\sim}{K}_f\cdot\underset{\sim}{R}'+\underset{\sim}{K}_{12f}\cdot\underset{\sim}{q}'_{12})}$$

$$\times\int d\underset{\sim}{r}''_{12}\, e^{-i\underset{\sim}{k}_{12f}\cdot\underset{\sim}{r}''_{12}} V_{12}(\underset{\sim}{r}''_{12})\, g_{12}^{(+)}(\underset{\sim}{r}''_{12};\underset{\sim}{r}'_{12};E_{12f}) \tag{D25}$$

The result (D25) is equivalent to Eqs. (105), noting that, from Eqs. (90a) - (90b)

$$C_3(E) = \left(\frac{2m_1}{\hbar^2}\right)^{3/2}\left(\frac{2m_2}{\hbar^2}\right)^{3/2}\left(\frac{2m_3}{\hbar^2}\right)^{3/2}\frac{e^{-i\frac{3\pi}{2}}}{2\sqrt{E}}\left(\frac{\sqrt{E}}{2\pi}\right)^4$$

$$= \frac{1}{(2\pi)^4}\left(\frac{2\mu_{12}}{\hbar^2}\right)^{3/2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2}e^{-i\frac{3\pi}{2}}\frac{E^{3/2}}{2} \tag{D26}$$

D.2 Eq. (116a)

If r_{12} is kept finite, the term $\mathbf{r}_{12}\mathbf{k}_{12}$ must be dropped from the equation (D16) for the rapidly oscillating phase factor $e^{i\phi_\pm}$ in (D14); in fact, I should not have replaced $g_F(\underset{\sim}{r}_{12};\underset{\sim}{r}_{12}'')$ in (D13) by its asymptotic form. However, $\phi_\pm(\mathcal{E}) = \pm\, \rho_{12}\mathcal{E}$ has no point of stationary phase. For directions $\underset{\sim}{\nu}_f = \underset{\sim}{\nu}_{12f}$, therefore, where r_{12} remains finite, the preceding derivation of Eqs. (105) is incorrect, even when no bound states $u_j(\underset{\sim}{r}_{12})$ exist. On the other hand, in the absence of such bound states, the desired asymptotic behavior of $\bar{G}_{12}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{E})$ can be deduced from (112a). I note first of all that the integral in (112a) appears to converge when $g_F(\bar{\mathcal{E}}^2 + i\varepsilon)$ is replaced by $g_F^{(+)}(\bar{\mathcal{E}}^2)$. Thus use in (112a) of the analogue of (D2) for $g_F(\underset{\sim}{q}_{12};\underset{\sim}{q}_{12}';\lambda)$ yields

$$\lim_{q_{12}\to\infty\parallel\bar{\underset{\sim}{\nu}}_{12f}} \bar{G}_{12}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{E}) = \frac{2\mu_{3R}}{4\pi\hbar^2}\int d\hat{\underset{\sim}{k}}_{12}\, u(\underset{\sim}{r}_{12};\hat{\underset{\sim}{k}}_{12})\,u^*(\underset{\sim}{r}_{12}';\hat{\underset{\sim}{k}}_{12}) \times \frac{e^{i\hat{K}_{12}q_{12}}}{q_{12}}\, e^{-i\hat{K}_{12}\bar{\underset{\sim}{\nu}}_{12f}\cdot\underset{\sim}{q}_{12}'} \tag{D27}$$

where, from (112c), $\hat{K}_{12}$ is defined by

$$\bar{\mathcal{E}}^2 = \frac{\hbar^2\hat{K}_{12}^2}{2\mu_{3R}} = \bar{E} - \frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}} \tag{D28}$$

Because $\underset{\sim}{r}_{12}$, $\underset{\sim}{r}_{12}'$ and $\underset{\sim}{q}_{12}'$ all now are being held finite, application of the method of stationary phase in (D27) now involves merely finding the point of stationary phase of

$$e^{i\hat{K}_{12}q_{12}} = \exp\left[iq_{12}\left(\frac{2\mu_{3R}}{\hbar^2}\right)^{1/2}\left(\bar{E} - \frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}\right)^{1/2}\right] \tag{D29}$$

This point of stationary phase $\hat{k}_{12} = k_{12f}$ satisfies

$$\frac{d}{d\hat{k}_{12}}\left(\bar{E} - \frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}\right)^{1/2} = 0 = -\frac{\hbar^2\hat{k}_{12}}{2\mu_{12}}\left[\bar{E} - \frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}\right]^{-1/2} \tag{D30}$$

Eq. (D30) does have a solution, namely $k_{12f} = 0$, to which corresponds, according to (D28), a value of $\hat{K}_{12} \equiv \hat{K}_{12f}$ precisely identical with (116c). Moreover, at $\hat{k}_{12} = k_{12f} = 0$,

$$\frac{d^2}{d\hat{k}_{12}^2}\left(\bar{E} - \frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}\right)^{1/2} = -\frac{\hbar^2}{2\mu_{12}}\left.\frac{\bar{E}}{\left(\bar{E} - \frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}\right)^{3/2}}\right|_{\hat{k}_{12}=0} = -\frac{\hbar^2}{2\mu_{12}\bar{E}^{1/2}} \tag{D31}$$

Consequently the right side of (D27) reduces to [recall Eqs. (D21) and (D24)]

$$\frac{2\mu_{3R}}{4\pi\hbar^2}\,4\pi\, u(\underset{\sim}{r}_{12};0)\,u^*(\underset{\sim}{r}'_{12};0)\,\frac{e^{iK_{12f}q_{12}}}{q_{12}}\,e^{-iK_{12f}\underset{\sim}{\nu}_{12f}\cdot\underset{\sim}{q}'_{12}}\int_0^\infty d\hat{k}_{12}\,\hat{k}_{12}^2\,e^{-ibq_{12}\hat{k}_{12}^2} \tag{D32}$$

wherein b is a constant, namely

$$b = \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{1/2}\frac{\hbar^2}{4\mu_{12}\bar{E}^{1/2}} \tag{D33}$$

But

$$\int_0^\infty dx^2\, x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{4a^{3/2}} \tag{D34}$$

Employing (D34) in (D33) with $a = ibq_{12}$, and recalling (116b), one sees (D32) is precisely identical with the right side of (116a), with [directly from Eqs. (90)]

$$C_2(\bar{E}) = \left(\frac{2\mu_{12}}{\hbar^2}\right)^{3/2} \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{3/2} \frac{e^{-i3\pi/4}}{2\sqrt{\bar{E}}} \left(\frac{\sqrt{\bar{E}}}{2\pi}\right)^{5/2} \tag{D35}$$

I note that an attempt to derive Eqs. (105) starting from (112a) as in this section, but now letting $\underset{\sim}{r}_{12} \to \infty$ along with $\underset{\sim}{q}_{12}$, runs into difficulties which possibly may cast some doubt on the exactness of the result (116a), but which I do not believe can alter the fundamental (for the purposes of this work) $e^{i\bar{\rho}\sqrt{\bar{E}}}/\bar{\rho}^{5/2}$ dependence.

APPENDIX E. ASYMPTOTIC BEHAVIOR OF SINGLY ITERATED INTEGRALS

Page

E.1 Eqs. (106)

I wish to examine the validity of employing Eqs. (105) in (80c), so as to yield the asymptotic form of $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ at large $\underset{\sim}{r} \equiv$ large $\underset{\sim}{R}$, $\underset{\sim}{q}_{12}$ and $\underset{\sim}{r}_{12}$. The asymptotic forms of the first four terms on the right side of (80c) already are known, by virtue of Eqs. (90) and (105). The problem is to justify the use of (105) in the iterated expressions on the right side of (80c). Consider, e.g., the $\underset{\sim}{r},\underset{\sim}{r}'$ element of the term

$$\left\{G_F^{(+)}V_{12}G_{12}^{(+)}\right\}V_{23}G^{(+)} = \int d\underset{\sim}{r}''' \int d\underset{\sim}{r}'' G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')V_{12}(\underset{\sim}{r}'')G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}''')V_{23}(\underset{\sim}{r}''')G^{(+)}(\underset{\sim}{r}''';\underset{\sim}{r}') \tag{E1}$$

where it is understood that the integral over $\underset{\sim}{r}''$ is to be performed first. Eqs. (105) yield the asymptotic form of the $\underset{\sim}{r},\underset{\sim}{r}'''$ element of $G_F^{(+)}V_{12}G_{12}^{(+)}$ in the limit that $\underset{\sim}{R}$, $\underset{\sim}{q}_{12}$ and $\underset{\sim}{r}_{12}$ approach infinity holding $\underset{\sim}{r}'''$ finite. Recalling Appendix C, it is clear that use of (105) in (E1) will be justified provided the contribution to the right side of (E1) from values of $r''' > r$ is negligible compared to ρ^{-4} at large r.

The argument is very much as in section C.1. I note first that because of the factor $V_{23}(\underset{\sim}{r}''')$ in (E1), significant contributions to (E1) at large r''' can come only from those $\underset{\sim}{r}'''$ approaching infinity along directions $\underset{\sim}{\nu}_{23}'''$ wherein $\underset{\sim}{r}_{23}'''$ remains finite. Next, using (95) and (A1), I note that

$$\int d\underset{\sim}{r}''\, G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')V_{12}(\underset{\sim}{r}'')G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}''') = \int d\underset{\sim}{r}''\, G_F^{(+)}(\underset{\sim}{r}''';\underset{\sim}{r}'')V_{12}(\underset{\sim}{r}'')G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}) \tag{E2}$$

Thus the limit of the left side of (E2) as r''' becomes infinite can be found from Eqs. (105). Examination of the derivation of Eqs. (105) [in section D.1] will make it clear that Eqs. (105) remain valid as they stand even when the direction $\underset{\sim}{\nu}_f \equiv \underset{\sim}{\nu}_{23}$ along which $r \to \infty$ corresponds to holding $\underset{\sim}{r}_{23}$ finite; in particular, because $\underset{\sim}{r}_{12}$ becomes infinite along $\underset{\sim}{\nu}_{23}$, the results of section D.2 are not germane.

In other words, with $\underset{\sim}{r}$ held fixed, Eqs. (105) imply

$$\lim_{r'''\to\infty \| \underset{\sim}{\nu}_{23}'''} \int d\underset{\sim}{r}''\, G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}'')V_{12}(\underset{\sim}{r}'')G_{12}^{(+)}(\underset{\sim}{r}'';\underset{\sim}{r}''')$$
$$= -C_3(E)\frac{e^{i\rho'''\sqrt{E}}}{\rho'''^4}\,\Phi_{12f}^{(-)*}(\underset{\sim}{r};E) \tag{E3a}$$

where

$$\Phi_{12f}^{(-)*}(\underset{\sim}{r};E) = e^{-i\underset{\sim}{K}_f'''\cdot\underset{\sim}{R}}\, e^{-i\underset{\sim}{K}_{12f}'''\cdot\underset{\sim}{q}_{12}}\, \varphi_{12f}^{(-)*}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12f}''') \tag{E3b}$$

and (105c) continues to hold; however, because $\underset{\sim}{r}_{23}'''$ has been kept finite, the vectors $\underset{\sim}{K}_f'''$, $\underset{\sim}{k}_{12f}'''$, $\underset{\sim}{K}_{12f}'''$ are interrelated by the condition $\underset{\sim}{k}_{23f}''' = 0$. I now employ Eqs. (E3) in (E1), supposing first that bound states $u_j(\underset{\sim}{r}_{23})$ exist. Then the behavior of (E1) at large $r''' > r$ is estimated by

$$\int d\underset{\sim}{R}'''\, d\underset{\sim}{q}_{23}'''\, d\underset{\sim}{r}_{23}'''\, \frac{e^{i\rho'''\sqrt{E}}}{\rho'''^{4}}\, e^{-i\underset{\sim}{K}_f'''\cdot\underset{\sim}{R}}\, e^{-i\underset{\sim}{k}_{12f}'''\cdot\underset{\sim}{q}_{12}}$$

$$\times\ \varphi_{12f}^{(-)*}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12f}''')\,V_{23}(\underset{\sim}{r}_{23}''')\,\frac{e^{i\rho'''\sqrt{E-\epsilon_j}}}{\rho'''^{5/2}}\,e^{-i\underset{\sim}{K}_f''\cdot\underset{\sim}{R}'}\,\Psi_{123f}^{(-)*}(\underset{\sim}{r};E) \tag{E4}$$

In obtaining (E4) from (E1) the laboratory system analogues of Eqs. (115) have been used. In (E4), however, the details of Eqs. (115)--which give the precise dependence of $\lim G^{(+)}(\underset{\sim}{r}''';\underset{\sim}{r}')$ on $\underset{\sim}{r}'$ as r''' becomes infinite along $\underset{\sim}{\nu}'''$--are irrelevant; the analysis which follows uses only the fact that this limit is proportional to $e^{i\rho'''\sqrt{E-\epsilon_j}}/\rho'''^{5/2}$, a result which hardly can be questioned. In effect, therefore, the estimate (E4) of (E1) could have been written down even without foreknowledge of Eqs. (115).

In (E4), $\rho''' \cong \rho_{23}'''$, where $\underset{\sim}{\rho}_{23}'''$ is the 2, 3 analogue of the six-dimensional vector defined in (A4), i.e.,

$$\underset{\sim}{\rho}_{23}''' = \left(\frac{2M}{\hbar^2}\right)^{1/2}\underset{\sim}{R}''' \oplus \left(\frac{2\mu_{1R}}{\hbar^2}\right)^{1/2}\underset{\sim}{q}_{23}''' \tag{E5}$$

Moreover, making use of Eqs. (90c) and (B2), and remembering $\underset{\sim}{r}_{23}'''$ is remaining finite as $\underset{\sim}{R}'''$, $\underset{\sim}{q}_{23}'''$ become infinite,

$$e^{-i\underset{\sim}{K}_{12f}'''\cdot\underset{\sim}{q}_{12}} = \exp\left[-\frac{i2\mu_{3R}\sqrt{E}}{\hbar^2}\ \frac{\underset{\sim}{q}_{12}'''\cdot\underset{\sim}{q}_{12}}{\rho'''}\right] \cong \exp\left[\frac{i2m_3m_1\sqrt{E}}{\hbar^2 M}\ \frac{\underset{\sim}{q}_{23}'''\cdot\underset{\sim}{q}_{12}}{\rho_{23}'''}\right] \tag{E6}$$

to the order of accuracy needed to estimate the dominant asymptotic behavior of (E4). Similarly, at large $\underset{\sim}{r}_{12}$—where $\phi_{12f}^{(-)*}$ in (E4) ultimately is to be evaluated—

$$\varphi_{12f}^{(-)*}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12f}''') \cong \frac{e^{ik_{12f}'''r_{12}}}{r_{12}} \cong \frac{1}{r_{12}}\exp\left[\frac{i2\mu_{12}\sqrt{E}}{\hbar^2}\ \frac{q_{23}'''}{\rho_{23}'''}\ r_{12}\right] \tag{E7}$$

The right sides of Eqs. (E6) - (E7) show that the expressions on the left sides of those equations—which occur explicitly in (E4)—depend only on the direction of $\underset{\sim}{\rho}_{23}''' = \rho_{23}'''\,\underset{\sim}{\nu}_{23}'''$, but not on its magnitude. Therefore, now dropping the distinction between ρ''' and ρ_{23}''', the behavior of (E4) at large $r''' > r$ is estimated by

$$\int d\rho''' d\underset{\sim}{\nu}_{23}'''\,\rho'''^5\,\frac{e^{i\rho'''\sqrt{E}}}{\rho'''^4}\,e^{-i\underset{\sim}{K}_f'''\cdot\underset{\sim}{R}}\,e^{-i\underset{\sim}{K}_{12f}'''\cdot\underset{\sim}{q}_{12}}\,\frac{e^{ik_{12f}'''r_{12}}}{r_{12}}\,\frac{e^{i\rho'''\sqrt{E-\varepsilon_j}}}{\rho'''^{5/2}} \tag{E8}$$

where $\underset{\sim}{K}_f'''$, $\underset{\sim}{K}_{12f}'''$, k_{12f}''' are independent of ρ'''.

I now claim [see below] that integration over $d\underset{\sim}{\nu}_{23}'''$ brings down a factor decreasing no less rapidly than $r^{-5/2}$, so that [compare

Eqs. (C2) - (C3)] the integral (E8) behaves like

$$\int_{\gamma_\rho}^{\infty} d\rho''' \frac{e^{i\rho'''\sqrt{E}}\, e^{i\rho'''\sqrt{E-\varepsilon_j}}}{\rho'''^{3/2}\, r_{12}} \frac{1}{r^{5/2}} \tag{E9}$$

In (C3), the factor $J_2(\rho_{12}\sqrt{E})/\rho_{12}^{\,2}$, resulting from explicitly performing the integration over $d\underset{\sim}{\nu}_{12}''$ in (C2), also behaves like $\rho_{12}^{-5/2} \cong r^{-5/2}$. Thus (E9) behaves essentially like (C3), except that (E9) has an extra factor $r_{12}^{-1} \cong r^{-1}$ (remembering R, r_{12}, q_{12} each have fixed $\underset{\sim}{\nu}$-dependent ratios to r as $\underset{\sim}{r} \to \infty \,||\, \underset{\sim}{\nu}$). But (C3) has been shown to be of order r^{-4}. Hence (E9)--and consequently also the contribution to (E1) represented by (E4)--is of order r^{-5} at large r, which indeed is negligible compared to $\rho^{-4} \cong r^{-4}$.

An even simpler argument [which I shall not bother to give in detail here] shows the contribution to (E1) from $r''' > r$ associated with three-body bound state terms $u_j(\underset{\sim}{r}_{23},\underset{\sim}{q}_{23})$ also is of order r^{-5} at large r. As in section C.1 [compare Eqs. (C1a) and (C5a)] in the absence of bound states the $r''' > r$ contribution to the right side of (E1) from $r'''> r$ will decrease even more rapidly than r^{-5}. Therefore interchange of order of integration and limit $r \to \infty$ is valid in (E1), i.e., direct insertion of (105) into (80c) is justified.

E.1.1 Reduction of (E8) to (E9)

I wish to justify the claim that integration over $d\underset{\sim}{\nu}_{23}'''$ in (E8) brings down a factor decreasing no less rapidly than $r^{-5/2}$. This integration over the six-dimensional element of solid angle actually involves integration over five angles [recall Eq. (D12)]. In general, each of these angles will be represented in the $\underset{\sim}{\nu}_{23}'''$-dependent quantities $\underset{\sim}{K}_f'''$, $\underset{\sim}{K}_{12f}'''$, k_{12f}''' appearing in the exponentials in (E8). Moreover, R, q_{12}, and r_{12}--which enter linearly in the exponents of (E8)--each are large and proportional to r. In general, therefore--appealing once again to the principle of stationary phase--each of the five angular integrations over $d\underset{\sim}{\nu}_{23}'''$ should bring down a factor decreasing no more slowly than $r^{-1/2}$. Failure to find a point of stationary phase in the allowed angular ranges [quoted in section D.1], or extra heretofore ignored angular-dependent factors in the integrand [recall the remarks following Eq. (B7b)], can only cause each angular integration to decrease more rapidly than $r^{-1/2}$. The sole way to avoid bringing down an $r^{-1/2}$ factor with each integration over $\theta_a (a = 1,\ldots,5)$ is to have the integrand in (E8) independent of that θ_a. Such independence is perfectly possible, especially if the polar axis in the triple-primed space is favorably chosen relative to the direction $\underset{\sim}{\nu}$ along which $\underset{\sim}{r} \rightarrow \infty$. On the other hand, the basic asymptotic behavior of the integral cannot be altered merely by choice of polar axis; if, as seems clear from the foregoing argument, an $r^{-5/2}$ dependence is expected with unfavorable or arbitrary choice of polar axis, then--with favorable choice of polar axis eliminating an angle (say θ_5) from the exponentials--the apparently lost $r^{-1/2}$ factor from the integration over θ_5 must be regained during the remaining integrations over $\theta_1,\ldots,\theta_4$. For example, in section A.4, with the aid of an appropriate choice of polar axis for $\underset{\sim}{\nu}_{12}'$, it was found possible to write

$$\underset{\sim}{K}' \cdot \underset{\sim}{R}' + \underset{\sim 12}{K}' \cdot \underset{\sim}{q}'_{12} = \mathcal{E}'\rho'\cos\chi$$

i.e., the relevant (rapidly oscillating) exponentials in (A10) were made to depend on a single angle only. But in so doing the points of stationary phase for the integration over χ became $\chi = 0$ and π, where the $\sin^4\chi$ factor in the integrand of (A13b) vanished. Thus [recall (A13e)] in (A13b) the expected $\rho'^{-5/2}$ dependence resulted from a single angular integration.

I believe the foregoing argument that (E8) must reduce to (E9) is quite generally correct. However, to eliminate all doubt, I shall detail the derivation of (E9) from the specific form (E8). In (E8), the volume element is the same as in (E4), i.e.,

$$d\rho''' d\underset{\sim}{\nu}'''_{23} \rho'''^5 \cong d\rho'''_{23} d\underset{\sim}{\nu}'''_{23} \rho'''^5_{23} = d\underset{\sim}{R}''' d\underset{\sim}{q}'''_{23}$$

Recollecting Eqs. (E6) and (E7), one now sees that the integrations over the directions of $\underset{\sim}{R}'''$ and $\underset{\sim}{q}_{23}'''$ can be performed explicitly. Thus (E8) becomes, using (90c) for K_f'''

$$\int dR''' dq'''_{23} R'''^2 q'''^2_{23} \frac{e^{i\rho'''\sqrt{E}}}{r_{12}\rho'''^4} \frac{e^{i\rho'''\sqrt{E-\epsilon_j}}}{\rho'''^{5/2}} j_0\left(\frac{2M\sqrt{E}}{\hbar^2}\frac{R'''R}{\rho'''}\right)$$
$$\times j_0\left(\frac{2m_3 m_1\sqrt{E}}{\hbar^2 M}\frac{q'''_{23}q_{12}}{\rho'''}\right)\exp\left(\frac{i2\mu_{12}\sqrt{E}}{\hbar^2}\frac{q'''_{23}r_{12}}{\rho'''}\right) \quad \text{(E10)}$$

where I still am using $\rho_{23}''' \cong \rho'''$. Next introduce $\tilde{R}'''$ and $\tilde{q}_{23}'''$ defined in analogy with (A6), so that (E5) yields

$$\rho_{23}^{\prime\prime\prime 2} = \tilde{R}^{\prime\prime\prime 2} + \tilde{q}_{23}^{\prime\prime\prime 2} \tag{E11a}$$

Hence, one can write, as in (A20)

$$\begin{aligned} \tilde{R}''' &= \rho''' \cos\phi \\ \tilde{q}_{23}''' &= \rho''' \sin\phi \end{aligned} \tag{E11b}$$

where ρ''', ϕ now are polar coordinates in the (two-dimensional) $\tilde{R}'''$, $\tilde{q}_{23}'''$ plane. In this fashion (remembering $\tilde{R}'''$ and $\tilde{q}_{23}'''$ are intrinsically positive), the integral (E10) in the domain $r''' > r$ of present interest can be rewritten as

$$\int_{\gamma_\rho}^{\infty} d\rho''' \rho'''^5 \frac{e^{i\rho'''\sqrt{E}}}{\rho'''^4} \frac{e^{i\rho'''\sqrt{E-\varepsilon_j}}}{\rho'''^{5/2} r_{12}} \int_0^{\pi/2} d\phi \cos^2\phi \sin^2\phi \, e^{ia_1 r_{12}\sin\varphi} \times j_0(a_2 R\cos\varphi) j_0(a_3 q_{12}\sin\varphi) \tag{E12a}$$

where a_1, a_2, a_3 are positive constants, depending on the masses and on E, e.g., $a_2 = (2ME/\hbar^2)^{1/2}$. The integral (E12a) reduces to a sum of four integrals of form

$$\int_{\gamma_\rho}^{\infty} d\rho''' \frac{e^{i\rho'''\sqrt{E}} e^{i\rho'''\sqrt{E-\varepsilon_j}}}{\rho'''^{3/2} r_{12} R q_{12}} \int_0^{\pi/2} d\phi \cos\varphi \sin\varphi \, e^{ia_1 r_{12}\sin\varphi} \times e^{\pm i a_2 R\cos\varphi} e^{\pm i a_3 q_{12}\sin\varphi} \tag{E12b}$$

Now write

$$r_{12} = c_1 r$$
$$R = c_2 r \qquad \text{(E13)}$$
$$q_{12} = c_3 r$$

where c_1, c_2, c_3 are positive numbers depending on the direction (in nine-dimensional space) with which $\underset{\sim}{r} \to \infty$. Then the integral over ϕ in (E12b) is a sum of four integrals of form

$$\int_0^{\pi/2} d\phi \cos\phi \sin\phi \, e^{iAr\cos(\varphi - \varphi_1)} \qquad \text{(E14a)}$$

with A and ϕ_1 dependent on the a's and c's but not on r. In particular,

$$A = \left[(a_1 c_1 \pm a_3 c_3)^2 + a_2^2 c_2^2 \right]^{1/2} \qquad \text{(E14b)}$$

But the integral (E14a) has a point of stationary phase at $\sin(\phi - \phi_1) = 0$, which generally will not coincide with 0 or $\pi/2$, and which should lie in the allowed range $0 < \phi < \pi/2$ for at least one of the four integrals represented by (E14a). Therefore the angular integration in (E12b) generally is of order $r^{-1/2}$, which reduces (E12b) to

$$\int_{\gamma_\rho}^{\infty} d\rho''' \frac{e^{i\rho'''\sqrt{E}} \, e^{i\rho'''\sqrt{E - \varepsilon_j}}}{\rho'''^{3/2} \, r_{12} R q_{12} r^{1/2}} \qquad \text{(E15)}$$

which in turn, using (E13), is precisely of the form (E9), Q.E.D.

E.2 Eq. (132)

I wish to show that the interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty$ in (132) is unjustified, even in the absence of bound states. In other words [as in section C.4] for my present purposes I can assume that bound states do not exist. Under these circumstances the $\bar{r}' > \bar{r}$ contribution to the left side of (132) is estimated by

$$\int d\underset{\sim}{q}'_{23}\, d\underset{\sim}{r}'_{23} \frac{e^{i\bar{\rho}'\sqrt{\bar{E}}}}{\bar{\rho}'^{5/2}} \bar{\Psi}_f^{(-)*}(\bar{\underset{\sim}{r}})\, V_{23}(\underset{\sim}{r}'_{23})\, \bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}') \tag{E16}$$

In (E16), because of the $V_{23}(\underset{\sim}{r}_{23}')$ factor, I need to be concerned only with $\bar{\underset{\sim}{r}}' \to \infty$ along directions $\bar{\underset{\sim}{\nu}}' = \bar{\underset{\sim}{\nu}}_{23}'$. Thus, as in (C16b),

$$\tilde{q}'_{23} = \left(\frac{2\mu_{1R}}{\hbar^2}\right)^{1/2} q'_{23} = \bar{\rho}' \tag{E17a}$$

and

$$K'_{23f} = \left(\frac{2\mu_{1R}\bar{E}}{\hbar^2}\right)^{1/2} \tag{E17b}$$

where the subscript f on K_{23f}' [and on $\bar{\Psi}_f^{(-)*}$ in (E16)] here corresponds to directions $\bar{\underset{\sim}{\nu}}' = \bar{\underset{\sim}{\nu}}_{23}'$, and bears no relation to $\bar{\underset{\sim}{\nu}}_f$ of (132). Now, in (E16), replace $\bar{\Psi}_f^{(-)*}$ by $\bar{\psi}_f^{\,*}$, and use Eqs. (B2) - (B3). Eq. (E16) then reduces to [after integrating over $\underset{\sim}{r}_{23}'$ and proceeding as in Eqs. (C16) - (C17)]

$$\int d\underset{\sim}{q}'_{23} \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} e^{-i\underset{\sim}{K}'_{23f}\cdot\underset{\sim}{q}_{23}} e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}'_{23}} \frac{e^{ik_{12i}q'_{23}}}{q'_{23}} . \qquad \text{(E18a)}$$

$$\cong \int d\bar{\rho}'\, d\underset{\sim}{\bar{\nu}}'_{23}\, \bar{\rho}'^2 \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} e^{-i\underset{\sim}{\bar{\nu}}'_{23}\cdot\left[\sqrt{E}\,\underset{\sim}{\tilde{q}}_{23}+\frac{m_1}{m_1+m_2}\bar{\rho}'\underset{\sim}{\tilde{K}}_{12i}\right]} \times \frac{\exp\left[i\left(\frac{\hbar^2}{2\mu_{1R}}\right)^{1/2} k_{12i}\bar{\rho}'\right]}{\bar{\rho}'} \qquad \text{(E18b)}$$

Hence the $\bar{r}' > \bar{r}$ contribution to (E16) is estimated by

$$\int_{\gamma\bar{\rho}}^{\infty} d\bar{\rho}' \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{3/2}} \exp\left[i\left(\frac{\hbar^2}{2\mu_{1R}}\right)^{1/2} k_{12i}\bar{\rho}'\right] j_0\left(\left|\sqrt{E}\,\underset{\sim}{\tilde{q}}_{23}+\frac{m_1}{m_1+m_2}\bar{\rho}'\underset{\sim}{\tilde{K}}_{12i}\right|\right) \qquad \text{(E19)}$$

Comparing with Eq. (C17b), one sees that (E19) is of order $\bar{\rho}^{-5/2}$, i.e., not sufficiently rapidly decreasing for (132) to be valid. This result is all that is needed for the purposes of the present section. For the purposes of the next section, however, and also as a matter of interest in connection with the physical significances of the δ-functions (135) and (141), it is worth noting that the $\bar{r} > \bar{r}$ contribution to the left side of (132) associated with bound state propagation is negligible compared to $\bar{\rho}^{-5/2}$. In the particular term (132) from (69), only states $u_j(\underset{\sim}{r}_{23})$ are of possible interest.

For propagation in such states, (E16) is replaced by [recall Eqs. (115), or compare Eq. (A16a)]

$$\int d\underset{\sim}{q}'_{23}\, d\underset{\sim}{r}'_{23} \frac{e^{iK_{23j}q'_{23}}}{q'_{23}} u_j(\underset{\sim}{r}'_{23}) \bar{\Psi}^{(-)*}_{23jf}(\bar{\underset{\sim}{r}}) V_{23}(\underset{\sim}{r}'_{23}) \bar{\Phi}^{(+)}_{12}(\bar{\underset{\sim}{r}}') \tag{E20}$$

One now follows the lines of an argument given in sections C.3 and C.5. From Eqs. (115) one sees $\bar{\Psi}_{23jf}^{(-)*}(\bar{\underset{\sim}{r}})$ is a sum of two terms: (i) its incident part, proportional to $u_j^*(\underset{\sim}{r}_{23})$, and thus vanishing exponentially as $\bar{r} \to \infty$ along $\bar{\underset{\sim}{\nu}}_f$ of (132), for which $r_{23} \to \infty$ as $\bar{r} \to \infty$; (ii) its scattered part, proportional to $e^{i\bar{\rho}\sqrt{E}}/\bar{\rho}^{5/2}$. At large $\bar{r}$, therefore, (E20) behaves like

$$\int d\underset{\sim}{q}'_{23} \frac{e^{iK_{23j}q'_{23}}}{q'_{23}} \frac{e^{i\bar{\rho}\sqrt{E}}}{\bar{\rho}^{5/2}} e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\cdot \underset{\sim}{q}'_{23}} \frac{e^{ik_{12i}q'_{23}}}{q'_{23}}$$

$$\cong \frac{e^{i\bar{\rho}\sqrt{E}}}{\bar{\rho}^{5/2}} \int_{\gamma'\bar{\rho}}^{\infty} dq'_{23}\, e^{i(K_{23j}+k_{12i})q'_{23}}\, j_0\!\left(\frac{m_1}{m_1+m_2}K_{12i}q'_{23}\right) \cong \bar{\rho}^{-7/2} \tag{E21}$$

Because of (E17a), which still holds here, writing the lower limit of the q_{23}' integration in the form $\gamma'\bar{\rho}$—where γ' is a constant of dimensionality $(\hbar^2/2\mu_{1R})^{1/2}\gamma$, with γ as in (E9)--is admissible. The integral (E21) is identical with (B5a), which [as shown in section B.1] can be logarithmically divergent off-shell, but is convergent on-shell, the circumstance of present interest. Thus the result (E21) is valid, and demonstrates that the $\bar{r}' > \bar{r}$ contribution to the left side of (132) indeed is negligible compared to $\bar{\rho}^{-5/2}$. In this connection I note that the considerations of the following section could not make (E21) decrease less rapidly than $\bar{\rho}^{-3}$; actually the $\bar{\rho}^{-7/2}$ result probably is correct, however, because (E21) shows no sign of a point of stationary phase.

I close this section with some parenthetical remarks bearing on the interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$ in the integrals (162). According to section E.3 below, the quantities $\bar{\Phi}_{\alpha\beta}^{s(+)}(\bar{\underset{\sim}{r}})$ in (162) are of order $\bar{\rho}^{-2}$ at large $\bar{r}$. Comparing Eqs. (162) and (132), and referring to the argument leading from (E16) to (E19), it is clear that the $\bar{r}' \gg \bar{r}$ contribution to (162) normally will be of order $\bar{\rho}^{-7/2}$, i.e., sufficiently rapidly decreasing to justify interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$ in (162); the existence of a point of stationary phase, as in section E.3 below, will make the $\bar{r}' \to \bar{r}$ contribution of order $\bar{\rho}^{-3}$, still sufficiently rapid for the aforementioned interchange in (162) to be valid. However, there is a difficulty with this attempt to justify Eqs. (165) via an appeal to (E19). As section E.3 below discusses, $\bar{\Phi}_{\alpha\beta}^{s(+)}(\bar{\underset{\sim}{r}})$ actually is of order $\bar{\rho}^{-2}e^{i\Lambda(\bar{\underset{\sim}{\rho}})}$ at large $\bar{\rho}$, where the precise expression for Λ is very complicated and not known. Therefore, the $\bar{r}' > \bar{r}$ contribution to (162) actually reduces to integrals like

$$\int_{\gamma\bar{\rho}}^{\infty} d\bar{\rho}' \, \frac{e^{iF(\bar{\rho}')}}{\bar{\rho}'^{7/2}} \qquad \text{(E22)}$$

where F [which depends on $\underset{\sim}{k}_i$ and $\bar{\underset{\sim}{\nu}}_f$] presumably is $\sim c\rho'$ at large ρ', but where the possibility that c = 0 along certain special $\bar{\underset{\sim}{\nu}}_f$ [for given $\underset{\sim}{k}_i$] cannot be ruled out. When c = 0, the Riemann-Lebesgue lemma(31) is irrelevant [compare Eqs. (C4) and (C12) - (C14)] and (E22) is of order $\bar{\rho}^{-5/2}$ at large $\bar{r}$. In other words, the $\bar{r}' > \bar{r}$ contribution to (162) may be of order $\bar{\rho}^{-5/2}$ as $\bar{\underset{\sim}{r}} \to \infty$ along certain special $\bar{\underset{\sim}{\nu}}_f$, in which event interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$ would not be legitimate for such $\bar{\underset{\sim}{\nu}}_f$. Note that we merely have not eliminated the possibility of such special $\bar{\underset{\sim}{\nu}}_f$; there is no particular reason to think these special $\bar{\underset{\sim}{\nu}}_f$ actually occur.

E.3 Asymptotic Form of $\bar{\Phi}_i^{s(+)}$

In this section I shall show that $\bar{\Phi}_i^{s(+)}$ given by (69) has parts behaving like $\bar{r}^{-2} \cong \bar{\rho}^{-2}$ at large $\bar{r}$. Consider a typical term in (69), e.g.,

$$\bar{G}^{(+)} V_{23} \bar{\bar{\Phi}}_{12}^{(+)} = \int d\underset{\sim}{q}'_{23} d\underset{\sim}{r}'_{23} \, \bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}') V_{23}(r'_{23}) \bar{\bar{\Phi}}_{12}^{(+)}(\bar{\underset{\sim}{r}}') \equiv \bar{D}(\bar{\underset{\sim}{r}}) \tag{E23}$$

Because [using (102a)] the contribution to (E23) from the integration domain $\bar{r}' < \bar{r}$ behaves like $\bar{\rho}^{-5/2}$ as $\bar{r} \to \infty$, any contribution to (E23) behaving like $\bar{\rho}^{-2}$ must come from integration over $\bar{r}' > \bar{r}$. In other words, in the present section I am examining the very same contribution as in the previous section E.2. Now in section E.2 it was shown that the contribution in question is of order $\bar{\rho}^{-5/2}$, which was sufficient to show (132) is invalid. However, as has been pointed out earlier [section C.4], the condlusion that (E19) behaves like $\bar{\rho}^{-5/2}$ assumes that there is no point of statšonary phase in the integration range $\gamma \leq x < \infty$ (after introducing the new integration variable $x = \bar{\rho}'/\bar{\rho}$); if there is such a point of stationary phase, the integral (E19)--and therefore the $\bar{r}' > \bar{r}$ contribution to (E23)--will decrease more slowly than $\bar{\rho}^{-5/2}$. However, to demonstrate that such a point of stationary phase really exists, the analysis in the preceding section is too crude. In particular, the parameter γ [first introduced and defined in section C.1] is insufficiently well-determined to permit a decision that the point of stationary phase lies within--rather than without--the range $\gamma \leq x < \infty$. For the purposes of the present section, therefore, it is not useful to proceed as in section E.2, where $\bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$ in (E23) was immediately replaced by its asymptotic form at $\bar{r}' \gg \bar{r}$ (in the absence of bound states), thus yielding the previous section's starting point (E16).

Instead, I argue as follows. I am interested in the behavior of (E23) as $\bar{\underset{\sim}{r}}' \to \infty$ along directions $\bar{\underset{\sim}{\nu}}_{23}'$, along which $\bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$ behaves essentially like $\bar{G}_{23}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$. In other words, for my present purposes, $\bar{G}^{(+)}$ in (E23) can be replaced by $\bar{G}_{23}^{(+)}$ [as a matter of fact by $\bar{G}_F^{(+)}$ according to section E.2, where it was shown that the dominant asymptotic behavior of (E23) stems from the continuum propagation of $\bar{G}^{(+)}$]. Equivalently, and more rigorously, I can iterate $\bar{G}^{(+)}$ in (E23) via the center of mass analogue of (63b), yielding

$$\bar{G}^{(+)} V_{23} \bar{\bar{\Phi}}^{(+)}_{12} = \lim_{\varepsilon \to 0} \bar{G}(\bar{E}+i\varepsilon) V_{23} \bar{\bar{\Phi}}^{(+)}_{12}(\bar{E})$$

$$= \lim_{\varepsilon \to 0} \left[\bar{G}_{23}(\bar{E}+i\varepsilon) - \bar{G}_{23}(\bar{E}+i\varepsilon)(V_{31}+V_{12})\bar{G}(\bar{E}+i\varepsilon) \right] V_{23} \bar{\bar{\Phi}}^{(+)}_{12}(\bar{E})$$

$$= \lim_{\varepsilon \to 0} \left[\bar{G}_{23}(\bar{E}+i\varepsilon) V_{23} \bar{\bar{\Phi}}^{(+)}_{12}(\bar{E}) - \bar{G}_{23}(\bar{E}+i\varepsilon)(V_{31}+V_{12}) \left\{ \bar{G}(\bar{E}+i\varepsilon) V_{23} \bar{\bar{\Phi}}^{(+)}_{12}(\bar{E}) \right\} \right]$$

$$= \bar{G}^{(+)}_{23} V_{23} \bar{\bar{\Phi}}^{(+)}_{12} - \bar{G}^{(+)}_{23}(V_{31}+V_{12}) \left\{ \bar{G}^{(+)} V_{23} \bar{\bar{\Phi}}^{(+)}_{12} \right\} \tag{E24}$$

where the integrals in the braces are to be performed first; as usual in this work, the manipulations in (E24), like the manipulations in the derivation of (67c) from (64b), are considered legitimate because all the integrals involved are convergent at $\varepsilon = 0$. I next claim [see subsection E.3.2 below] that any $\bar{\rho}^{-2}$ dependence at large $\bar{r}$ in the first term on the right side of (E24) cannot be cancelled by the last two terms in (E24). My starting point for this section, therefore, is

$$\bar{G}^{(+)}_{23} V_{23} \bar{\bar{\Phi}}^{(+)}_{12} = \lim_{\varepsilon \to \infty} \bar{G}_{23}(\bar{E}+i\varepsilon) V_{23} \bar{\bar{\Phi}}^{(+)}_{12}(\bar{E})$$

$$= \lim_{\varepsilon \to 0} \int d\bar{\underset{\sim}{r}}' \, \bar{G}_{23}(\underset{\sim}{r}; \underset{\sim}{r}'; \bar{E}+i\varepsilon) V_{23}(\underset{\sim}{r}'_{23}) \bar{\bar{\Phi}}^{(+)}_{12}(\underset{\sim}{r}') \tag{E25a}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^3} \int d\bar{\underset{\sim}{r}}' \, d\underset{\sim}{K}_{23} \, e^{i \underset{\sim}{K}_{23} \cdot (\underset{\sim}{q}_{23} - \underset{\sim}{q}'_{23})} g_{23}\left(\underset{\sim}{r}_{23}; \underset{\sim}{r}'_{23}; \frac{\hbar^2 k^2_{23}}{2\mu_{23}} + i\varepsilon\right)$$

$$\times V_{23}(\underset{\sim}{r}'_{23}) e^{i \underset{\sim}{K}_{12i} \cdot \underset{\sim}{q}'_{12}} \phi^{(+)}_{12}(\underset{\sim}{r}'_{12}; \underset{\sim}{k}_{12i}) \tag{E25b}$$

where I have used (72) and the center of mass analogue of (D3), and where the 2, 3 analogue of (35) defines k_{23} in terms of the integration variable $\underset{\sim}{K}_{23}$. In (E25b), and in the remainder of this section, the dummy wave vectors $\underset{\sim}{\bar{k}} \equiv \underset{\sim}{K}_{23}, \underset{\sim}{k}_{23}$ are to be distinguished from the incident wave vectors $\underset{\sim}{\bar{k}}_i$, from the final wave vectors $\underset{\sim}{\bar{k}}_f$ [corresponding to $\underset{\sim}{\bar{\nu}}_f$ of (132)] and from the wave vectors $\underset{\sim}{\bar{k}}_f'$ [corresponding to $\underset{\sim}{\bar{r}}' \to \infty$ along some $\underset{\sim}{\bar{\nu}}'$].

Recalling Eq. (40b),

$$d\underset{\sim}{\bar{r}}' = d\underset{\sim}{r}'_{23}\, d\underset{\sim}{q}'_{23} = d\underset{\sim}{r}'_{23}\, d\underset{\sim}{r}'_{12}$$

Thus, using also (E2), the integral on the right side of (E25b) takes the form

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^3}\int d\underset{\sim}{r}'_{23}\, d\underset{\sim}{r}'_{12}\, d\underset{\sim}{K}_{23}\Big\{e^{i\underset{\sim}{K}_{23}\cdot \underset{\sim}{q}_{23}}\; e^{-i\underset{\sim}{r}'_{12}\cdot\left[\underset{\sim}{K}_{23}+\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\right]}$$

$$\times\, e^{-i\underset{\sim}{r}'_{23}\cdot\left[\frac{m_3}{m_2+m_3}\underset{\sim}{K}_{23}+\underset{\sim}{K}_{12i}\right]}\, \mathcal{G}_{23}\Big(\underset{\sim}{r}_{23};\underset{\sim}{r}'_{23};\frac{\hbar^2 k_{23}^2}{2\mu_{23}}+i\varepsilon\Big)V_{23}(\underset{\sim}{r}'_{23})\,\varphi^{(+)}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12i})\Big\} \tag{E26a}$$

At this point I shall interchange the order of integration over $d\underset{\sim}{K}_{23}$ and $d\underset{\sim}{r}_{23}'d\underset{\sim}{r}_{12}'$, on the basis (sections 2.2 and A.8) that the result of this interchange leads to a convergent expression; the integral (E26a), which is identical with (E25a), is of course known to be convergent even at $\varepsilon = 0$ [section B.1]. Once the above interchange is performed, it is obvious that the limit $\varepsilon \to 0$ again can be taken under the integral sign, yielding, in place of (E26a)

$$\frac{1}{(2\pi)^3}\int d\underset{\sim}{K}_{23}\, e^{i\underset{\sim}{K}_{23}\cdot\underset{\sim}{q}_{23}} \int d\underset{\sim}{r}_{12}'\, e^{-i\underset{\sim}{r}_{12}'\cdot\underset{\sim}{Q}_1}\, \phi_{12}^{(+)}(\underset{\sim}{r}_{12}';\underset{\sim}{k}_{12i}) \int d\underset{\sim}{r}_{23}'\, g_{23}^{(+)}\left(\underset{\sim}{r}_{23};\underset{\sim}{r}_{23}';\frac{\hbar^2 k_{23}^2}{2\mu_{23}}\right) \times V_{23}(\underset{\sim}{r}_{23}')\, e^{-i\underset{\sim}{r}_{23}'\cdot\underset{\sim}{Q}_2} \tag{E26b}$$

where

$$\begin{aligned} \underset{\sim}{Q}_1 &= \underset{\sim}{K}_{23} + \frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i} \\ \underset{\sim}{Q}_2 &= \frac{m_3}{m_2+m_3}\underset{\sim}{K}_{23} + \underset{\sim}{K}_{12i} \end{aligned} \tag{E26c}$$

I desire the asymptotic value of (E26b) at large $\underset{\sim}{\bar{r}} \equiv \underset{\sim}{r}_{23}, \underset{\sim}{q}_{23}$. In pursuit of this objective, I need not--and should not--make any immediate approximation to the value of the integrand in (E26b) at large $\underset{\sim}{q}_{23}$ [as was done in section E.2]. However, because of the factor $V_{23}(\underset{\sim}{r}_{23}')$, the Green's function $g_{23}^{(+)}$ in (E26b) can be replaced by its asymptotic form at large $\underset{\sim}{r}_{23}$. In this fashion one sees that in (E26b)

$$\int d\underset{\sim}{r}_{23}'\, g_{23}^{(+)}\left(\underset{\sim}{r}_{23};\underset{\sim}{r}_{23}';\frac{\hbar^2 k_{23}^2}{2\mu_{23}}\right) V_{23}(\underset{\sim}{r}_{23}')\, e^{-i\underset{\sim}{r}_{23}'\cdot\underset{\sim}{Q}_2} \cong \frac{e^{ik_{23}r_{23}}}{r_{23}} F_1(k_{23}\underset{\sim}{n}_{23f};\underset{\sim}{Q}_2) \tag{E27a}$$

where F_1 is a well-behaved function, and $\underset{\sim}{n}_{23f}$ denotes the direction, in three-dimensional physical space, along which $\underset{\sim}{r}_{23}$ becomes infinite. Similarly, it must be true that

$$\int d\underset{\sim}{r}'_{12}\, e^{-i\underset{\sim}{r}'_{12}\cdot\underset{\sim}{Q}_1}\varphi^{(+)}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12i}) = F_2(\underset{\sim}{k}_{12i};\underset{\sim}{Q}_1) \tag{E27b}$$

Hence (E26b) has the form

$$\int d\underset{\sim}{K}_{23}\, e^{i\underset{\sim}{K}_{23}\cdot\underset{\sim}{q}_{23}}\,\frac{e^{ik_{23}r_{23}}}{r_{23}}F_1(k_{23}\underset{\sim}{n}_{23f};\underset{\sim}{Q}_2)\,F_2(\underset{\sim}{k}_{12i};\underset{\sim}{Q}_1) \tag{E28}$$

which is kept convergent as $K_{23} \to \infty$ by the fact that

$$k_{23} = \left(\frac{2\mu_{23}}{\hbar^2}\right)^{1/2}\left(\bar{E} - \frac{\hbar^2 K_{23}^2}{2\mu_{1R}}\right)^{1/2} \tag{E29}$$

is defined to be positive imaginary when $\bar{E} < \hbar^2 K_{23}{}^2/2\mu_{1R}$.

E.3.1 Singularity in (E28)

Actually F_2 is a singular function of the angles in $d\underset{\sim}{K}_{23}$, as is shown below. Suppose for the moment, however, that F_2 is a well-behaved function. Then at large $\bar{\underset{\sim}{r}}$ the (comparatively) slow angular dependence of F_1F_2 in (E28) can be ignored, in accordance with the remarks made at the end of section A.4. Thus the integral (E28) behaves like

$$\int dK_{23}K_{23}^2\frac{e^{ik_{23}r_{23}}}{r_{23}}j_0(K_{23}q_{23}) \cong \frac{1}{r_{23}q_{23}}\int dK_{23}K_{23}\,e^{ik_{23}r_{23}}e^{\pm iK_{23}q_{23}} \tag{E30}$$

Alternatively, I can insert the expansion[40]

$$e^{i\underset{\sim}{K}_{23}\cdot\underset{\sim}{q}_{23}} = \sum_{\ell} i^{\ell}(2\ell+1)P_{\ell}(\underset{\sim}{K}_{23},\underset{\sim}{q}_{23})j_{\ell}(K_{23}q_{23}) \tag{E31}$$

into (E28), where the P_ℓ are Legendre polynomials in the angle between $\underset{\sim}{K}_{23}$ and $\underset{\sim}{q}_{23}$. Then, performing the angular integration in (E28), and replacing the spherical Bessel functions by their asymptotic forms at large $K_{23}q_{23}$, again yields (E30). By either route, having arrived at (E30), the integral over K_{23} therein can be evaluated by the method of stationary phase. The rapidly oscillating factor $e^{i\phi_{\pm}}$ in (E30) is essentially of the same form as the corresponding factor treated in section D.1 [compare Eq. (D16)], and the analysis in that section is applicable. There is no need to go through all the details here, however, because obviously the integral on the right side of (E30) can decrease no less slowly than $\bar{r}^{-5/2}$.

It follows that any $\bar{\rho}^{-2}$ dependence of (E28) can arise only from the singular part of $F_2(\underset{\sim}{k}_{12i};\underset{\sim}{Q}_1)$. Since

$$\lim_{r'_{12}\to\infty} \varphi^{(+)}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12i}) = a_1(\underset{\sim}{n}'_{12})\frac{e^{ik_{12i}r'_{12}}}{r'_{12}} \tag{E32}$$

where a_1 can be presumed to be well-behaved, I conclude, just as in the above transition from (E28) to (E30), that

$$\int d\underset{\sim}{r}'_{12}\, e^{-i\underset{\sim}{r}'_{12}\cdot\underset{\sim}{Q}_1}\varphi^{(+)}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12i}) \cong \int d\underset{\sim}{r}'_{12}\, e^{-i\underset{\sim}{r}'_{12}\cdot\underset{\sim}{Q}_1}\frac{e^{ik_{12i}r'_{12}}}{r'_{12}}$$
$$= \int dr'_{12}\, r'_{12}\, j_0(Q_1 r'_{12})e^{ik_{12i}r'_{12}} \tag{E33a}$$

Therefore the function F_2 contains a contribution which can be put in the form(42)

$$F_3(\underset{\sim}{k}_{12i};\underset{\sim}{Q}_1)\,\delta(Q_1 - k_{12i}) \qquad \text{(E33b)}$$

where F_3 presumably is well-behaved. Correspondingly, the integral (E26b) has a contribution which cannot be put in the form (E28), but instead must be written as

$$\int d\underset{\sim}{K}_{23}\, e^{i\underset{\sim}{K}_{23}\cdot\underset{\sim}{q}_{23}} \frac{e^{ik_{23}r_{23}}}{r_{23}} F_1(k_{23}\underset{\sim}{n}_{23f};Q_2) \times F_3(\underset{\sim}{k}_{12i};\underset{\sim}{Q}_1)\,\delta(Q_1 - k_{12i}) \qquad \text{(E34a)}$$

I remark that [as has been discussed in section 5.1] the non-convergent integral (E33a) usually is interpreted(42) so as to add to (E34a) the principal part of a second integral, whose integrand contains the singular factor $(Q_1 - k_{12i})^{-1}$. Ignoring this possible principal part contribution to (E34a) does not affect the subsequent analysis, however, as the comments at the very end of this section E.3.1 make clear.

In (E34a)—unlike (E28) where F_2 was assumed well-behaved—I cannot yet claim that the angular-dependent factors are comparatively slowly varying at large $\bar{\underset{\sim}{r}}$; the integral over the δ-function must be performed first. To do so, introduce $\underset{\sim}{Q}_1$ as a new integration variable, replacing $\underset{\sim}{K}_{23}$. Then (E34a) becomes [with the aid of (E26c)]

$$e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}_{23}}\int d\underset{\sim}{Q}_1\, e^{i\underset{\sim}{Q}_1\cdot\underset{\sim}{q}_{23}}\frac{e^{ik_{23}r_{23}}}{r_{23}}F_1(k_{23}\underset{\sim}{n}_{23f};\underset{\sim}{Q}_2)$$
$$\times F_3(\underset{\sim}{k}_{12i};\underset{\sim}{Q}_1)\,\delta(Q_1-k_{12i}) \qquad \text{(E34b)}$$

$$=e^{-i\frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}_{23}}k_{12i}^2\int d\hat{\underset{\sim}{n}}_1\, e^{ik_{12i}\hat{\underset{\sim}{n}}_1\cdot\underset{\sim}{q}_{23}}\frac{e^{ik_{23}r_{23}}}{r_{23}} \qquad \text{(E34c)}$$
$$\times F_1(k_{23}\underset{\sim}{n}_{23f};\underset{\sim}{Q}_2)\,F_3(\underset{\sim}{k}_{12i};k_{12i}\hat{\underset{\sim}{n}}_1)$$

In going from (E34b) to (E34c) I have written

$$\underset{\sim}{Q}_1 \equiv Q_1 \hat{\underset{\sim}{n}}_1 = k_{12i} \hat{\underset{\sim}{n}}_1 \tag{E35a}$$

Correspondingly, in (E34c)

$$k_{23} = \left(\frac{2\mu_{23}}{\hbar^2}\right)^{1/2} \left[\bar{E} - \frac{\hbar^2}{2\mu_{1R}} \left| k_{12i} \hat{\underset{\sim}{n}}_1 - \frac{m_1}{m_1+m_2} \underset{\sim}{K}_{12i} \right|^2 \right]^{1/2} \tag{E35b}$$

$$\underset{\sim}{Q}_2 = \frac{m_3}{m_2+m_3} k_{12i} \hat{\underset{\sim}{n}}_1 + \frac{m_2 M}{(m_2+m_3)(m_1+m_2)} \underset{\sim}{K}_{12i} \tag{E35c}$$

The integral over $d\hat{\underset{\sim}{n}}_1$ in (E34c) now can be evaluated as follows. Choose $\underset{\sim}{K}_{12i}$ as the polar axis for $\hat{\underset{\sim}{n}}_1$ (and other vectors), so that k_{23} is independent of $\hat{\phi}_1$, the azimuth angle of $\hat{\underset{\sim}{n}}_1$. Then--now legitimately ignoring the slowly varying $F_1 F_3$ factor--the behavior of (E34c) at large $\underset{\sim}{\bar{r}}$ is seen to be given by

$$\frac{e^{-i \frac{m_1}{m_1+m_2} \underset{\sim}{K}_{12i} \cdot \underset{\sim}{q}_{23}}}{r_{23}} \int d\hat{\theta}_1 \sin\hat{\theta}_1 d\hat{\varphi}_1 e^{ik_{23} r_{23}} e^{ik_{12i} q_{23} \left[\cos\hat{\theta}_1 \cos\theta_{qf} + \sin\hat{\theta}_1 \sin\theta_{qf} \cos(\hat{\varphi}_1 - \varphi_{qf})\right]} \tag{E36a}$$

where, of course, θ_{qf}, ϕ_{qf} are the spherical coordinate angles of $\underset{\sim}{q}_{23}$. The integral (E36a) reduces to [neglecting inessential factors]

$$\frac{1}{r_{23}}\int d\hat{\theta}_1 \sin\hat{\theta}_1 e^{ik_{23}r_{23}} e^{ik_{12i}q_{23}\cos\hat{\theta}_1\cos\theta_{qf}} J_0(k_{12i}q_{23}\sin\hat{\theta}_1\sin\theta_{qf}) \tag{E36b}$$

which at large q_{23} is a sum of two integrals proportional to

$$\cong \frac{1}{r_{23}q_{23}^{1/2}}\int d\hat{\theta}_1 \sin\hat{\theta}_1 e^{ik_{23}r_{23}} e^{ik_{12i}q_{23}\cos\hat{\theta}_1\cos\theta_{qf}} \frac{e^{\pm ik_{12i}q_{23}\sin\hat{\theta}_1\sin\theta_{qf}}}{(\sin\hat{\theta}_1)^{1/2}} \tag{E36c}$$

If the factors F_1F_3 had been retained in (E36a), the integral over $\hat{\phi}_1$ in (E36a) would be performed using the method of stationary phase. The points of stationary phase are at

$$\sin(\hat{\varphi}_1 - \varphi_{qf}) = 0 \tag{E37a}$$

i.e., at

$$\cos(\hat{\varphi}_1 - \varphi_{qf}) = \pm 1 \tag{E37b}$$

Thus one would regain (E36c), except that the integrands would be multiplied by F_1F_3 evaluated at the values of $\hat{\phi}_1$ satisfying (E37b).

The integrals (E36c) have the form

$$\frac{1}{r_{23}q_{23}^{1/2}}\int d\hat{\theta}_1 (\sin\hat{\theta}_1)^{1/2} e^{ik_{23}r_{23}} e^{ik_{12i}q_{23}\cos(\hat{\theta}_1 \pm \theta_{qf})} \tag{E38}$$

which again can be evaluated by the method of stationary phase. I first demonstrate—as is important and not obvious—that k_{23} in (E38) is real in the integration range $0 \leq \hat{\theta}_1 \leq \pi$. From (E35b),

$$k_{23}^2 = \left(\frac{2\mu_{23}}{\hbar^2}\right)\left[\bar{E} - \frac{\hbar^2}{2\mu_{1R}}\left(k_{12i}^2 + \left(\frac{m_1}{m_1+m_2}\right)^2 K_{12i}^2 - \frac{2m_1 k_{12i} K_{12i}\cos\hat{\theta}_1}{m_1+m_2}\right)\right] \qquad \text{(E39a)}$$

which, using Eqs. (29) and (35), can be rewritten as

$$k_{23}^2 = \frac{m_2 m_3}{(m_2+m_3)^2}\left[\frac{m_3}{m_2}k_{12i}^2 + \frac{m_2 M^2}{m_3(m_1+m_2)^2}K_{12i}^2 + \frac{2M}{m_1+m_2}k_{12i}K_{12i}\cos\hat{\theta}_1\right] \qquad \text{(E39b)}$$

The minimum value of (E39b), at $\cos\hat{\theta}_1 = -1$, is

$$(k_{23}^2)_{min} = \frac{m_2 m_3}{(m_2+m_3)^2}\left[\left(\frac{m_3}{m_2}\right)^{1/2} k_{12i} - \left(\frac{m_2}{m_3}\right)^{1/2}\frac{M}{m_1+m_2}K_{12i}\right]^2 \geq 0 \qquad \text{(E39c)}$$

Consequently k_{23} in (E38) always is real, as asserted.

Hence the points of stationary phase in (E38) are the roots of

$$r_{23}\frac{dk_{23}}{d\hat{\theta}_1} - k_{12i}\,q_{23}\sin(\hat{\theta}_1 \pm \theta_{qf}) = 0 \qquad \text{(E40a)}$$

or, using (E39b), at

$$-k_{12i}q_{23}\sin(\hat{\theta}_1 \pm \theta_{qf}) - \frac{(m_2m_3)^{1/2}}{(m_2+m_3)} \frac{\frac{M}{m_1+m_2} k_{12i} K_{12i} \sin\hat{\theta}_1}{\left[\frac{m_3}{m_2}k_{12i}^2 + \frac{m_2}{m_3}\frac{M^2}{(m_1+m_2)^2}K_{12i}^2 + \frac{2M}{m_1+m_2}k_{12i}K_{12i}\cos\hat{\theta}_1\right]^{1/2}} = 0 \quad \text{(E40b)}$$

Now consider the plus sign in (E40b). At $\hat{\theta}_1 = 0$ the left side of (E40b) is negative; at $\hat{\theta}_1 = \pi$, however, the left side of (E40b) is positive (since by definition $0 \le \theta_{qf} \le \pi$). Consequently, (E40b) with the plus sign surely has a root in the range $0 \le \hat{\theta}_1 \le \pi$. Similarly (E40b) with the lower (minus) sign has a root in the integration range. I conclude that the integral in (E38) has contributions proportional to $\bar{r}^{-1/2}$, and therefore that the expression (E38) is proportional to $\bar{r}^{-2} \cong \bar{\rho}^{-2}$ at large $\bar{r}$. Actually, the integral (E38) really should have included the angular dependent product F_1F_3 from (E34c), which would be evaluated at the various pairs of stationary phase points satisfying (E37b) and (E40b). But the roots of (E37b) and (E40b) depend only on the direction (specified by the angles θ_{qf}, ϕ_{qf}) along which $\underset{\sim}{q}_{23} \to \infty$, whereas F_1 is a function of the direction $\underset{\sim}{n}_{23f}$ along which $\underset{\sim}{r}_{23} \to \infty$. Thus it does not seem possible that the aforementioned various pairs of stationary phase points can yield a set of $\bar{\rho}^{-2}$ contributions to (E34a)—i.e., to (E26a) and (E23)—which cancel for $\bar{\underset{\sim}{r}} \equiv (\underset{\sim}{r}_{23}, \underset{\sim}{q}_{23})$ approaching infinity along arbitrary $\bar{\underset{\sim}{\nu}}_f$.

I conclude this subsection with the remark [recall subsection E.1.1] that in effect the role of the δ-function in (E34a) is to reduce the number of stationary phase integrations over $d\underset{\sim}{K}_{23}$ from three [the conventionally expected number, corresponding to the three dimensions represented in $d\underset{\sim}{\Pi}_{23}$] to two. Thus, whereas the integrations in (E28) bring down an extra factor of order $\bar{\rho}^{-3/2}$, the integrations in (E25a) bring down a factor only of order $\bar{\rho}^{-1}$.

E.3.2 Neglected Terms in Eq. (E24)

The preceding subsection completes the demonstration that the $\delta(Q_1 - k_{12i})$ part of F_2, Eq. (E27b), makes a contribution of order $\bar{\rho}^{-2}$ to (E23). To achieve the primary objective of this section, there remains only to show that these apparently not self-cancelling $\bar{\rho}^{-2}$ contributions also cannot be cancelled by the heretofore neglected last two terms on the right side of (E24). Consider, e.g., the term

$$\bar{G}_{23}^{(+)} V_{31} \left\{ \bar{G}^{(+)} V_{23} \bar{\Phi}_{12}^{(+)} \right\} = \int d\underset{\sim}{\bar{r}}' \, \bar{G}_{23}^{(+)}(\underset{\sim}{\bar{r}}; \underset{\sim}{\bar{r}}') V_{31}(\underset{\sim}{r}'_{31}) \bar{D}(\underset{\sim}{\bar{r}}') \tag{E41a}$$

from (E24), where $\bar{D}(\underset{\sim}{\bar{r}})$ is defined by (E23). As in the case of (E23) itself, any $\bar{\rho}^{-2}$ contribution to (E41a) must come from integration over the domain $\bar{r}' > \bar{r}$. I now can proceed as in section E.2. In (E41a), because of the $V_{31}(\underset{\sim}{r}_{31}')$ factor, I need be concerned only with $\bar{r}' \to \infty$ along directions $\underset{\sim}{\bar{\nu}}' = \bar{\nu}_{31}'$. Along such directions, $\bar{G}_{23}^{(+)}$ cannot propagate in bound states. Thus the $\bar{r}' > \bar{r}$ contribution to (E41a) is estimated by

$$\int d\underset{\sim}{q}'_{31} \, d\underset{\sim}{r}'_{31} \frac{e^{i\bar{\rho}'\sqrt{E}}}{\bar{\rho}'^{5/2}} \bar{\Psi}_{23f}^{(-)*}(\underset{\sim}{\bar{r}}) V_{31}(\underset{\sim}{r}'_{31}) \frac{1}{\bar{\rho}'^{2}} \tag{E41b}$$

where, in accordance with the results of the preceding subsection, I have replaced $\bar{D}(\bar{\underset{\sim}{r}}')$ by $\bar{r}'^{-2}$, ignoring the additional oscillatory factors in $\bar{D}(\bar{\underset{\sim}{r}})$. This replacement certainly gives the leading term in (E41a); if the possible cancellation under present discussion were to occur, it only would make $\bar{D}(\bar{\underset{\sim}{r}})$ decrease more rapidly than $\bar{r}^{-2}$. Now, comparing Eqs. (E16) – (E19) with (E41b), it is clear that (E41b) will decrease no more slowly than $\bar{\rho}^{-3}$, even assuming that—as was the case for $\bar{D}(\bar{\underset{\sim}{r}})$ itself—the crude analysis along the lines of section E.2, which leads to the conclusion (E41b) is of order $\bar{\rho}^{-7/2}$, has overlooked the presence of a point of stationary phase.

It has been demonstrated, therefore, that there are contributions to $\bar{\Phi}_f^{s(+)}$ given by (69) which behave like $\bar{r}^{-2} \cong \bar{\rho}^{-2}$ at large $\bar{r}$, Q.E.D.

E.4 Derivation of Eqs. (175)

I start from Eq. (170c), to which Eq. (170a) has been reduced. Recalling Eqs. (130) and (131e) – (131i), we see that in (170c)

$$\int d\underset{\sim}{r}_{23}\, u^{(-)*}_{23cf}(\underset{\sim}{r}_{23};\underset{\sim}{k}_{23f})\, V_{23}\, e^{-i\underset{\sim}{Q}_{23}\cdot\underset{\sim}{r}_{23}}$$

$$= \langle \underset{\sim}{k}_{23f} \,|\, \underset{\sim}{t}_{23f} \,|\, -\underset{\sim}{B} \rangle \tag{E42}$$

where we have employed the notation of Eqs. (171a) and (172c). Moreover, from Eq. (131g)

$$\lim_{r_{12}\to\infty \,\|\, \underset{\sim}{\nu}_{12}} \varphi^{(+)}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i})$$

$$= -\frac{1}{4\pi}\frac{2\mu_{12}}{\hbar^2}\frac{e^{ik_{12i}r_{12}}}{r_{12}}\langle k_{12i}\underset{\sim}{\nu}_{12} \,|\, \underset{\sim}{t}_{12i} \,|\, \underset{\sim}{k}_{12i}\rangle \tag{E43}$$

Thus Eq. (170a) can be rewritten in the form

$$\bar{\bar{\Psi}}^{(-)*}_{23f} V_{23} \bar{\bar{\Phi}}^{(+)}_{12}$$

$$= -\frac{1}{4\pi}\frac{2\mu_{12}}{\hbar^2}\langle \boldsymbol{k}_{23f}|\boldsymbol{t}_{23f}|-\boldsymbol{B}\rangle \int d\boldsymbol{r}_{12}\, e^{-i\boldsymbol{r}_{12}\cdot\boldsymbol{A}}\, \frac{e^{ik_{12i}r_{12}}}{r_{12}} \langle k_{12i}\boldsymbol{\nu}_{12}|\boldsymbol{t}_{12i}|\boldsymbol{k}_{12i}\rangle$$

$$+\langle \boldsymbol{k}_{23f}|\boldsymbol{t}_{23f}|-\boldsymbol{B}\rangle \int d\boldsymbol{r}_{12}\, e^{-i\boldsymbol{r}_{12}\cdot\boldsymbol{A}} \Big\{ \varphi^{(+)}_{12}(\boldsymbol{r}_{12};\boldsymbol{k}_{12i})$$

$$+\frac{1}{4\pi}\frac{2\mu_{12}}{\hbar^2}\frac{e^{ik_{12i}r_{12}}}{r_{12}}\langle k_{12i}\boldsymbol{\nu}_{12}|\boldsymbol{t}_{12i}|\boldsymbol{k}_{12i}\rangle\Big\} \tag{E44}$$

The second integral in (E44) is convergent [except at $A^2 = k_{12i}^2$, as explained in the text in connection with the integral of the quantity inside the braces of Eq. (172a)], provided the quantity within the braces of (E44) is treated as a single $\boldsymbol{r}_{12}$-dependent function. The non-convergent part of the first integral in (E44) is subtracted away via the following manipulations. We know

$$e^{-i\boldsymbol{r}_{12}\cdot\boldsymbol{A}} = \sum_{\ell}(-i)^{\ell}(2\ell+1)\, j_{\ell}(Ar_{12})\, P_{\ell}(\boldsymbol{\nu}_{12},\boldsymbol{\nu}_A) \tag{E45}$$

where P_{ℓ}, the Legendre polynomial of order ℓ, depends only on the cosine of the angle between $\boldsymbol{\nu}_{12}$ and $\boldsymbol{\nu}_A$, with $\boldsymbol{A} = A\boldsymbol{\nu}_A$. Using well known properties of P_{ℓ}, as well as the asymptotic (large r_{12}) dependence[34] of the spherical Bessel functions

$$j_\ell(A r_{12}) \sim \frac{\cos\left[A r_{12} - \frac{1}{2}(\ell+1)\pi\right]}{A r_{12}} \tag{E46a}$$

it can be seen[35] that

$$\lim_{r_{12}\to\infty \,\|\, \underset{\sim}{\nu}_{12}} e^{-i \underset{\sim}{r}_{12}\cdot \underset{\sim}{A}} = \frac{2\pi i}{A}\,\delta(\underset{\sim}{\nu}_{12} - \underset{\sim}{\nu}_A)\frac{e^{-iAr_{12}}}{r_{12}} - \frac{2\pi i}{A}\,\delta(\underset{\sim}{\nu}_{12} + \underset{\sim}{\nu}_A)\frac{e^{iAr_{12}}}{r_{12}} \tag{E46b}$$

Thus

$$\int d\underset{\sim}{r}_{12}\, e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\, \frac{e^{ik_{12i}r_{12}}}{r_{12}} \langle k_{12i}\underset{\sim}{\nu}_{12}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle$$

$$= \int d\underset{\sim}{r}_{12}\, \frac{e^{ik_{12i}r_{12}}}{r_{12}} \langle k_{12i}\underset{\sim}{\nu}_{12}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle \Big\{ e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}} - 2\pi i\,\delta(\underset{\sim}{\nu}_{12} - \underset{\sim}{\nu}_A)\frac{e^{-iAr_{12}}}{Ar_{12}} + 2\pi i\,\delta(\underset{\sim}{\nu}_{12} + \underset{\sim}{\nu}_A)\frac{e^{iAr_{12}}}{Ar_{12}} \Big\}$$

$$+ \int d\underset{\sim}{r}_{12}\, \frac{e^{ik_{12i}r_{12}}}{r_{12}} \langle k_{12i}\underset{\sim}{\nu}_{12}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle \left[2\pi i\,\delta(\underset{\sim}{\nu}_{12} - \underset{\sim}{\nu}_A)\frac{e^{-iAr_{12}}}{Ar_{12}} - 2\pi i\,\delta(\underset{\sim}{\nu}_{12} + \underset{\sim}{\nu}_A)\frac{e^{iAr_{12}}}{Ar_{12}} \right] \tag{E47}$$

Provided the quantity within the braces is treated like a single $\underset{\sim}{r}_{12}$-dependent function, the first integral in (E47) [like the second integral in (E44)] converges except at $A^2 = k_{12i}{}^2$. Combining Eqs. (E42), (E44) and (E47)--after performing the trivial angular integrations over $\underset{\sim}{\nu}_{12}$ in the $\delta(\underset{\sim}{\nu}_{12} \pm \underset{\sim}{\nu}_A)$ terms of (E47)--then immediately yields Eq. (172a).

To see how the divergent one-dimensional integrals in (172a) should be interpreted, one argues as follows.

$$\int_0^\infty dx\, e^{ikx} = \lim_{R\to\infty} \int_0^R dx\, e^{ikx} = \lim_{R\to\infty} \frac{e^{ikR}-1}{ik}$$

$$= \lim_{R\to\infty} \left\{ \frac{\sin kR}{k} + \frac{\cos kR - 1}{ik} \right\} \tag{E48}$$

Now consider the behavior--as a function of k--of the quantity inside the braces of (E48) when R becomes very large. Evidently the term $k^{-1}\sin kR$ becomes very rapidly oscillating, with average value zero, except in the vicinity of k = 0. Therefore, since

$$\int_{-\infty}^{\infty} dk\, \frac{\sin kR}{k} = \pi$$

it is reasonable to write

$$\lim_{R\to\infty} \frac{\sin kR}{k} = \pi\,\delta(k) \tag{E49a}$$

as is well known[43]. Similarly, the quantity (cos kR − 1) in (E48) is very rapidly oscillating, with average value − 1, except for values of k in the range $|k| < \sim R^{-1}$, where for every R the average value of (cos kR − 1) tends to zero as $k \to 0$. Consequently it also is reasonable to write

$$\lim_{R\to\infty} \frac{\cos kR - 1}{ik} = \frac{i}{k} \qquad k > 0$$

$$\lim_{R\to\infty} \frac{\cos kR - 1}{ik} = 0 \qquad k = 0 \tag{E49b}$$

Eqs. (49) are equivalent to Eqs. (173), recognizing that $\delta(k) = 0$ for $k \neq 0$. Of course, because of the infinitely rapid oscillations as $R \to \infty$, the limits on the left sides of (E49a) and (E49b) do not really exist in a mathematically strict sense. However, under circumstances when these oscillations average out [as, e.g., when integrating over k] or for other reasons can be disregarded, the right sides of (E49a) - (E49b) are the only plausible values one can assign to the corresponding quantities on the left sides.

When Eqs. (173) are combined with Eqs. (E42), (E44) and (172a), one obtains [for $A^2 \neq k_{12i}{}^2$]

$$\bar{\Psi}_{23f}^{(-)*} V_{23} \bar{\Phi}_{12}^{(+)} = \bar{T}^{t}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{T}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{E50}$$

where the quantities on the right side of (E50) are given by the right sides of (175b) and (176b). But since the right side of (176b) is a sum of δ-functions, i.e., has no finite part, the discussion connected with Eq. (169a) indicates the right side of (176b) indeed must be wholly a contribution to that part of $\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ which has been termed $\bar{T}^a(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$. Having come to this conclusion, the long-sought formulas (175) for $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ follow immediately.

APPENDIX F. THE NINE-DIMENSIONAL CONFIGURATION SPACE

A number of the remarks in the text concerning the nine-dimensional space spanned by $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, as well as its six-dimensional subspace "orthogonal to $\mathbf{R}$" deserve some clarification. In particular, the relationship between $\boldsymbol{\rho}$ defined by (25d) and $\bar{\boldsymbol{\rho}}$ given by Eq. (102d) is not wholly apparent from anything said in the text thus far.

Points in the nine-dimensional configuration space of particles 1, 2, 3 are specified by the three-dimensional vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$. That is to say, the nine-dimensional space can be thought to contain nine mutually orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$, $\mathbf{i}_2$, $\mathbf{j}_2$, $\mathbf{k}_2$, $\mathbf{i}_3$, $\mathbf{j}_3$, $\mathbf{k}_3$, in terms of which the nine-dimensional vector $\boldsymbol{r}$ can be written as

$$\boldsymbol{r} = x_1\mathbf{i}_1 \oplus y_1\mathbf{j}_1 \oplus z_1\mathbf{k}_1 \oplus x_2\mathbf{i}_2 \oplus y_2\mathbf{j}_2 \oplus z_2\mathbf{k}_2 \oplus x_3\mathbf{i}_3 \oplus y_3\mathbf{j}_3 \oplus z_3\mathbf{k}_3 \qquad \text{(F1)}$$

where the $\oplus$ specifies vector addition in the nine-dimensional space, to be distinguished from the ordinary plus sign signifying addition of three-dimensional vectors, as, e.g., in Eqs. (28) -(29). It is convenient to denote the coordinates in this nine-dimensional space by s_a, $a = 1,\ldots,9$, in the order respectively x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3. Similarly, the nine-dimensional unit vectors will be denoted by $\mathbf{v}_a$, $a = 1,\ldots,9$, in the order $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, etc. Then (F1) can be rewritten as

$$\boldsymbol{r} = \sum_a \mathbf{v}_a s_a = \tilde{\mathbf{v}}\, s \qquad \text{(F2)}$$

where I now use matrix notation, with s a column matrix composed of $s_1, \ldots, s_9$, and $\tilde{\underset{\sim}{v}}$—the transpose[22] of the column $\underset{\sim}{v}$—now composed of the vectors $\underset{\sim}{v}_1, \ldots \underset{\sim}{v}_9$.

In the above configuration space, the six-dimensional subspace independent of $\underset{\sim}{R}$ is found by writing (F2) in terms of, e.g., $\underset{\sim}{R}$, $\underset{\sim}{r}_{12}$, $\underset{\sim}{q}_{12}$. More precisely, define the primed quantities $s_1', \ldots, s_9'$ by

$$\begin{aligned} \underset{\sim}{R} &\equiv (X, Y, Z) \equiv (s_1', s_4', s_7') \\ \underset{\sim}{r}_{12} &\equiv (x_{12}, y_{12}, z_{12}) \equiv (s_2', s_5', s_8') \\ \underset{\sim}{q}_{12} &\equiv (q_x, q_y, q_z) \equiv (s_3', s_6', s_9') \end{aligned} \qquad \text{(F3)}$$

Then s' and s are related by a nine-dimensional linear transformation

$$s' = Ws = \begin{pmatrix} w & & \\ & w & \\ & & w \end{pmatrix} s \qquad \text{(F4a)}$$

where, from Eqs. (12), (28a) and (29b), the 3 x 3 matrix

$$w = \begin{pmatrix} \frac{m_1}{M} & \frac{m_2}{M} & \frac{m_3}{M} \\ 1 & -1 & 0 \\ -\frac{m_1}{m_1+m_2} & -\frac{m_2}{m_1+m_2} & 1 \end{pmatrix} \qquad \text{(F4b)}$$

Substituting in (F2)

$$\underset{\sim}{r} = \tilde{\underset{\sim}{v}} W^{-1} s' = \tilde{\underset{\sim}{v}}' s' \qquad \text{(F5a)}$$

when [remembering the transpose of a row is a column]

$$\begin{aligned} \tilde{\underset{\sim}{v}}\, W^{-1} &= \tilde{\underset{\sim}{v}}' \\ \underset{\sim}{v}' &= \tilde{W}^{-1} \underset{\sim}{v} \end{aligned} \tag{F5b}$$

Putting $\underset{\sim}{R} \equiv (s_1', s_4', s_7') = 0$ in (F5a) gives the six-dimensional subspace of vectors depending only on $\underset{\sim}{r}_{12}$, $\underset{\sim}{q}_{12}$. Putting $\underset{\sim}{r}_{12}, \underset{\sim}{q}_{12} = 0$ in (F5a) yields the three-dimensional subspace of vectors depending only on $\underset{\sim}{R}$. However, because W from Eqs. (F4) is not an orthogonal transformation, the subspace independent of $\underset{\sim}{R}$ is not orthogonal in the usual sense to the subspace depending only on $\underset{\sim}{R}$. To be exact, the scalar product between a vector depending only on $\underset{\sim}{R}$ and a vector independent of $\underset{\sim}{R}$ need not be zero.

For this reason, it is useful to introduce a new nine-dimensional space spanned by $\underset{\sim}{R}$, $\underset{\sim}{r}_{12}$, $\underset{\sim}{q}_{12}$. Points $\underset{\sim}{r}'$ in this second nine-dimensional space can be written as

$$\begin{aligned} \underset{\sim}{r}' &= X\underset{\sim}{i}_1' \oplus Y\underset{\sim}{j}_1' \oplus Z\underset{\sim}{k}_1' \oplus x_{12}\underset{\sim}{i}_2' \oplus y_{12}\underset{\sim}{j}_2' \oplus z_{12}\underset{\sim}{k}_2' \oplus q_x\underset{\sim}{i}_3' \oplus q_y\underset{\sim}{j}_3' \oplus q_z\underset{\sim}{k}_3' \\ &= \sum_a \underset{\sim}{v}_a' s_a' = \tilde{\underset{\sim}{v}}' s' \end{aligned} \tag{F6}$$

where $\underset{\sim}{i}_1'$, $\underset{\sim}{j}_1'$, etc., are a set of nine mutually orthogonal unit vectors in the second space, whose ordering in terms of $\underset{\sim}{v}_a'$ is obvious. The difference between (F6) and (F5a) lies solely in the fact that $\underset{\sim}{v}'$ from (F5b) are not an **orthonormal** set, whereas $\underset{\sim}{v}'$ in (F6) **have** been chosen **orthonormal**. As a result, the primed space is essentially different from the unprimed space, i.e., in general points $\underset{\sim}{r}'$ and $\underset{\sim}{r}$ corresponding to the same original three-dimensional locations of particles 1, 2, 3 cannot be identical (having the same magnitude and direction) nine-dimensional vectors. For any choice of $\underset{\sim}{v}'$ in (F6), however, there is a one-to-one correspondence between points in the unprimed and primed spaces. In the primed space, moreover, the subspaces independent of $\underset{\sim}{r}$ and depending only on $\underset{\sim}{r}$ obviously are orthogonal.

On the other hand, the primed and unprimed spaces can be made identical by simple transformations. Specifically, to every point in the nine-dimensional $\underset{\sim}{r}$-space there corresponds a nine-dimensional point

$$\underset{\sim}{\rho} = \underset{\sim}{\rho}_1 \oplus \underset{\sim}{\rho}_2 \oplus \underset{\sim}{\rho}_3 = \tilde{\underset{\sim}{v}} F s \tag{F7a}$$

where

$$F = \left(\frac{2}{\hbar^2}\right)^{1/2} \begin{pmatrix} f & & \\ & f & \\ & & f \end{pmatrix} \tag{F7b}$$

$$f = \begin{pmatrix} \sqrt{m_1} & & \\ & \sqrt{m_2} & \\ & & \sqrt{m_3} \end{pmatrix} \tag{F7c}$$

Evidently the components of $\underset{\sim}{\rho}$ in (F7a) are identical with those defined by (25d). Similarly, to every point in the nine dimensional $\underset{\sim}{r}'$-space there corresponds

$$\underset{\sim}{\rho}' = \left(\frac{2M}{\hbar^2}\right)^{1/2} \underset{\sim}{R} \oplus \left(\frac{2\mu_{12}}{\hbar^2}\right)^{1/2} \underset{\sim}{r}_{12} \oplus \left(\frac{2\mu_{3R}}{\hbar^2}\right)^{1/2} \underset{\sim}{q}_{12} = \underset{\sim}{\tilde{v}}' G s' \tag{F8a}$$

where

$$G = \left(\frac{2}{\hbar^2}\right)^{1/2} \begin{pmatrix} g & & \\ & g & \\ & & g \end{pmatrix} \tag{F8b}$$

$$g = \begin{pmatrix} \sqrt{M} & & \\ & \sqrt{\mu_{12}} & \\ & & \sqrt{\mu_{3R}} \end{pmatrix} \tag{F8c}$$

Then $\underset{\sim}{\rho}$ and $\underset{\sim}{\rho}'$ can be made identical nine-dimensional vectors by appropriate choice of the **orthonormal** set $\underset{\sim}{v}'$.

To prove this last assertion I note that, directly from their defining equations, one can verify that

$$m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = MR^2 + \mu_{12} r_{12}^2 + \mu_{3R} q_{12}^2 \tag{F9a}$$

which in our nine-dimensional matrix notation is

$$\tilde{s} F^2 s = \tilde{s}' G^2 s' \tag{F9b}$$

Hence

$$F s = U' G s' \tag{F10a}$$

where U' is an **orthogonal** transformation. Using (F4a)

$$W = G^{-1} U'^{-1} F \tag{F10b}$$

$$U'^{-1} = G W F^{-1} \tag{F10c}$$

which yields

$$U'^{-1} = \begin{pmatrix} u'^{-1} & & \\ & u'^{-1} & \\ & & u'^{-1} \end{pmatrix} \tag{F11a}$$

$$u'^{-1} = \begin{pmatrix} \sqrt{\frac{m_1}{M}} & \sqrt{\frac{m_2}{M}} & \sqrt{\frac{m_3}{M}} \\ \sqrt{\frac{m_2}{m_1+m_2}} & -\sqrt{\frac{m_1}{m_1+m_2}} & 0 \\ -\sqrt{\frac{m_3 m_1}{M(m_1+m_2)}} & -\sqrt{\frac{m_3 m_2}{M(m_1+m_2)}} & \sqrt{\frac{m_1+m_2}{M}} \end{pmatrix} \tag{F11b}$$

One easily verifies that (F11b) indeed is orthogonal. Thus (F7a) and (F8a) will be identical if

$$\tilde{\underset{\sim}{\upsilon}} Fs = \tilde{\underset{\sim}{\upsilon}}' G s' = \tilde{\underset{\sim}{\upsilon}}' G W s = \tilde{\underset{\sim}{\upsilon}}' G G^{-1} U'^{-1} F s = \tilde{\underset{\sim}{\upsilon}}' U'^{-1} F s \tag{F12a}$$

i.e., if

$$\tilde{\underset{\sim}{\upsilon}}' = \tilde{\underset{\sim}{\upsilon}} U' \tag{F12b}$$

$$\underset{\sim}{\upsilon}' = U'^{-1} \underset{\sim}{\upsilon} \tag{F12c}$$

The above proves that $\underset{\sim}{\rho}'$ indeed can be made identical with $\underset{\sim}{\rho}$. Once this result, which does require proof, is in hand, it follows that $\underset{\sim}{\rho}$ defined by (25d) can be written in the form

$$\underset{\sim}{\rho} = \left(\frac{2M}{\hbar^2}\right)^{1/2} \underset{\sim}{R} \oplus \underset{\sim}{\bar{\rho}} \tag{F13}$$

where $\underset{\sim}{\bar{\rho}}$ is given by (102c), and therefore really has the desired properties of being independent of—as well as orthogonal to—$\underset{\sim}{R}$. The second equality in (102d) then follows by symmetry, recognizing that the orientations of the basis vectors in the 1, 2 and 2, 3 representations of course must be consistent with the foregoing.

References and Footnotes

1. In this paragraph, and in subsequent paragraphs, the Greek letters α, β are used as running subscripts over the electron indices 1, 2, 3, while the letters i, f are employed to denote respectively initial and final states. Furthermore, barred and unbarred symbols regularly will denote corresponding quantities in the center of mass and laboratory systems respectively.

2. E. Gerjuoy, Ann. Phys. 5, 58 (1958).

3. One can infer the two-particle version of Eq. (3) from, e.g., Eq. (XIX.19) of A. Messiah, "Quantum Mechanics" (Wiley 1962), volume II, p. 806.

4. Cf., e.g., K. M. Watson and J. Nuttall, "Topics in Several Particle Dynamics" (Holden-Day, San Francisco, 1967), especially section 1.2 and Chapter 4. The δ-functions in the matrix elements of $\underset{\sim}{\bar{T}}$ stem primarily from the fact that V [defined in Eq. (21b) below] contains purely two-body interactions $V_{\alpha\beta}$.

5. M. Rubin, R. Sugar and G. Tiktopoulos, Phys. Rev. 146, 1130 (1966); 159, 1348 (1967); 162, 1555 (1967).

6. L. D. Faddeev, Soviet Phys. JETP 12, 1014 (1961).

7. B. A. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950); M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91, 398 (1953).

8. S. Weinberg, Phys. Rev. 133, B232 (1964).

9. Cf., e.g., C. Lovelace, Phys. Rev. 135, B1225 (1964), as well as reference 6 itself.

10. W. Zickendraht, Phys. Rev. 159, 1448 (1967).

11. J. Nuttall, Phys. Rev. Letters 19, 473 (1967); also J. Nuttall and J. G. Webb, "The Asymptotic Form of the Wave Function for Three Particle Scattering", to be published.

12. At energies E corresponding to bound states of the total Hamiltonian, or to thresholds of inelastic processes, one should not expect the limit (8a) to exist. Similarly, $G^{(+)}(E)$ defined by Eq. (27a) need not exist at such exceptional energies.

13. Here $\bar{G}^{(+)}(\bar{E})$ is the center of mass analogue of the everywhere outgoing Green's function $G^{(+)}(E)$ appearing in Eq. (5); for a more precise definition of this Green's function, see Eqs. (36) and (39a).

14. Gary Doolen, Phys. Rev. 166, 1651 (1968).

15. To be rigorous by mathematicians' standards requires very much more elaborate analysis than I am able to or desire to give. Cf., e.g., L. D. Faddeev, "Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory" (Daniel Davey, Inc., New York 1965).

16. E. Gerjuoy, Phys. Rev. 109, 1806 (1958).

17. It is not obvious that $G^{(+)}(E)$ defined by (27a) satisfies (27d), because demonstrating (27d) from (27a) and (27c) involves the demonstration that interchange of the order of differentiation and limit $\varepsilon \to 0$ is justified. Clearly such interchange need not be valid, e.g.,

$$\frac{d}{d\tau}\left\{\lim_{\varepsilon\to 0} \varepsilon \sin(\tau/\varepsilon)\right\} \neq \lim_{\varepsilon\to 0}\left\{\frac{d}{d\tau}\varepsilon \sin(\tau/\varepsilon)\right\}$$

Similarly, it is not obvious that $\Psi_i^{(+)}(E)$ defined by Eq. (8a) satisfies Eq. (7). Nevertheless, on physical grounds, it appears unlikely that either $G(E + i\varepsilon)$ or $\Psi_i(E + i\varepsilon)$ becomes so wildly a fluctuating function of r as $\varepsilon \to 0$ that interchange of order of differentiation and $\lim \varepsilon \to 0$ becomes unjustified. In any event, excluding exceptional energies E (footnote 12), the theorem that $\Psi_i^{(+)}(E)$ exists and satisfies Eq. (7) apparently is proved by Faddeev in reference 15, sections 8 and 9, subject to some probably inessential restrictions concerning the number of discrete eigenvalues of the associated two-particle Hamiltonians $T_1 + T_2 + V_{12}$, etc. (in their individual two-particle center of mass systems). Under the same restrictions, moreover, $G^{(+)}(E)$ apparently exists and satisfies (27d). However, I shall not pretend that I have completely mastered all intricacies and implications of Faddeev's mathematics. Cf. also L. D. Faddeev, Soviet Phys. Doklady, 6, 384 (1961) and 7, 600 (1963), as well as references 4, 8 and 9.

18. L. L. Foldy and W. Tobocman, Phys. Rev. 105, 1099 (1957).
19. With E regarded as a parameter not necessarily > 0, the condition for $u_j(\underset{\sim}{r}_{12})$ to propagate, i.e., to reach infinity relative to particle 3, is of course $E - \varepsilon_j > 0$.
20. It is supposed that the energy ε_j of the three-body state is < 0. If there ever should be any need to consider (perhaps very long-lived) three-body or two-body states u_j having energy $\varepsilon_j > 0$, the assertion referenced--and similar assertions throughout the text--will remain valid provided "exist" is replaced by "can propagate" [see footnote 19].
21. E. T. Whittaker and G. N. Watson, "Modern Analysis" (Cambridge, 1946), p. 70.

22. Cf., e.g., reference 4, Chapter 1. For a configuration space derivation without appeal to operator algebra, see reference 16.

23. The key theorems seem to be contained in sections 7 and 9 of reference 15, especially theorem 7.1.

24. Presumably Faddeev's proofs (footnote 23) mean the order of integration and limit $\varepsilon \to 0$ can be interchanged in Eq. (79), but performance of this interchange is unnecessary for the purposes of the present discussion of Faddeev's equations.

25. Note my $G = (H - E)^{-1}$ is the negative of Weinberg's $G = (E - H)^{-1}$.

26. The kernel in (84c) does have some undesirable properties, however, cf., Roger G. Newton, Phys. Rev. **153**, 1502 (1967).

27. In a correctly formulated theory, those very special directions $\underset{\sim}{\nu}_f = \underset{\sim}{\nu}_{\alpha\beta}$ (in nine-dimensional configuration space) corresponding to unbound particles α, β moving to infinity (in physical space) with identical velocities (magnitudes and directions) surely must make a negligible contribution to the total three-body elastic scattering rate. Thus, for our present purpose of computing three-body elastic scattering coefficients $\bar{w}$, Eq. (1), it is not necessary to examine $\lim G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ as $r \to \infty$ along any $\underset{\sim}{\nu}_{\alpha\beta}$, although knowledge of this limit (along $\underset{\sim}{\nu}_{\alpha\beta}$) is essential for predicting the rate of two-body bound state $u_j(\underset{\sim}{r}_{\alpha\beta})$ production in the three-body collision.

28. If, e.g., particle 3 is incident on a bound state $u_j(\underset{\sim}{r}_{12})$, $\bar{\psi}_i$ is proportional to $u_j(\underset{\sim}{r}_{12})$, while $V_i = V - V_{12}$, so that (126b) surely converges for short range forces. Of course, to be wholly mathematically correct for reactions causing breakup of the initial $u_j(\underset{\sim}{r}_{12})$, $\bar{\Psi}_f^{(-)*}$ in (126b) must be

correctly prescribed, e.g., via the center of mass analogues of Eqs. (106).

29. I apologize for this awkward notation, but easily printable boldface quantities are in short supply. I do not believe use of the tilde to denote the transpose, as I do on numerous occasions, should cause any confusion in applications of (A6). In particular, the tilde denotes the transpose throughout Appendix F, but does not denote the transpose anywhere in any of the other Appendices.

30. G. N. Watson, "Bessel Functions" (Cambridge University Press, 1944), p. 48.

31. Ref. 21, p. 172.

32. Ref. 30, p. 199.

33. Cf., e.g., M. L. Goldberger and K. M. Watson, "Collision Theory" (Wiley, 1964), Chapter 5.

34. L. I. Schiff, "Quantum Mechanics" (McGraw Hill, 1955), pp. 77-79.

35. E. Gerjuoy and D. S. Saxon, Phys. Rev. 94, 1445 (1954).

36. H. Pierre Noyes, Phys. Rev. Letters 23, 1201 (1969).

37. M.Lieber, L. Rosenberg and L. Spruch, Bull. Am. Phys. Soc. 14, 937 (1969).

38. L. Rosenberg, M. Lieber and L. Spruch, Bull. Am. Phys. Soc. 14, 937 (1969).

39. Cf., e.g., reference 33, pp. 51-57 and pp. 413-414.

40. Cf., e.g., reference 34, section 19.

41. P. M. Morse and H. Feshbach, "Methods of Theoretical Physics" (McGraw Hill, 1953), p. 1574.

42. Cf., e.g., W. Brenig and R. Haag, "General Quantum Theory of Collision Processes", in Marc Ross, "Quantum Scattering Theory" (Indiana University Press, Bloomington, Indiana, 1963), pp. 106-108.

43. Cf., e.g., reference 34, p. 51.

44. C. Moller, "General Properties of the Characteristic Matrix in the Theory of Elementary Particles", Det. Kgl. Danske Videnskabernes Selskab, Mat-Fys. Meddelelser 23, no. 1 (1945), reprinted in Marc Ross, reference 42, pp. 109 ff.

45. Daniel Iagolnitzer, J. Math. Phys. 6, 1576 (1965). I am indebted to Dr. Roland Omnes for calling my attention to this reference.

46. A. Messiah, "Quantum Mechanics" (Wiley, 1962), esp. Chapter 19.