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ABSTRACT

The Krylov-Bogoluibov transformation is used to simplify the guiding center equations for a charged particle in electric and magnetic fields. The resulting transformation of variables is then applied to the Vlasov distribution function, yielding a magnetic Vlasov equation which describes the low frequency behavior of a system with no statistical effects. The equation is generalized to include effects of high and low frequency fluctuations by a procedure developed by Klimontovich and Dupree. A consistent treatment of the conservation laws and Maxwell's equations is given to complete the kinetic description.

I. INTRODUCTION

Many experiments dealing with plasmas in a magnetic field take place in the difficult regime where the collision mean free path of a particle is comparable to, or longer than, the characteristic scale lengths (gradient, curvature, etc.) of the system. For these situations it is desirable to have a theory which contains both a partial solution of the equations of motion of a single particle and the corrections due to statistical fluctuations, including collisions due to particle discreteness.

The standard method of solving the particle equations for motion for

$r(t), v(t)$ introduces complications when a strong magnetic field is present, for the rapid oscillation about the field line produce a lengthy expression for the orbit. This in turn causes great complexity in the statistical theory of many particles, even for the most simple geometry.[1]

Recently, Wilson[2] has carried the analysis of guiding center motion through order ϵ , and has written the conservation equation for guiding centers. This work is conceptually similar to his, and to that of Hastie and Taylor[3], but with considerable simplification in detail and with a development which connects with earlier work in statistical kinetic theory.[4] In addition, Wilson[5] has reviewed much of the guiding center work to date.

In Section II by using the Krylov-Boglouibov expansion technique and several modifications of the definition of initial guiding center variables we produce economical expressions for single particle motion. The Krylov-Bogoluibov method has the virtue that the new variables are defined to be equal on the average (over the phase angle about the magnetic field) to the old variables, so that an intuitive identification is possible.

In Section III this change of variables is applied to the Vlasov equation, yielding a magnetic Vlasov equation which is valid for low frequency disturbances if statistical effects may be ignored. Because the definition of the distribution function is precise we are able to identify the charge and current for Maxwell's equations without the need to carry auxiliary moment equations as in Wilson's work.[6]

Section IV develops the effect of statistical fluctuations, including the definition of the statistical transformation of variables, and the equation of motion for fluctuations. Although the method described earlier[4] is now applicable, we defer evaluation of the kinetic equation until a specific problem makes further approximation possible.

II. SINGLE PARTICLE MOTION

The motion of a particle of charge q and mass m is described by the equations

$$\dot{r} = v$$

$$\dot{v} = \frac{1}{\epsilon}\left[c\,E(r,t) + v \times B(r,t)\right]$$

where the fields are in Gausian units and $\epsilon \equiv \frac{mc}{q}$. In the usual way[7] we wish to expand in ϵ, treating the revolution about the magnetic field lines as the lowest order effect. Throughout the paper we work only to first order in ϵ, so that we may omit the explicit rescaling of time which limits the guiding center theory to frequencies far below the cyclotron frequency.[8] We define the guiding center variables $(R, v_{\parallel}, v_{\perp}, \theta)$

$$r = R + \frac{\epsilon v_{\perp}}{B(R,t)}\left[N(R,t)\cos\theta - M(R,t)\sin\theta\right] = R - \frac{\epsilon\, v' \times L(R,t)}{B(R,t)}$$

$$v_{\parallel} = v \cdot L(R,t)$$

$$v_{\perp}^2 = (v')^2 - v_{\parallel}^2$$

$$\theta = \tan^{-1}\left[v' \cdot N(R,t) / v' \cdot M(R,t)\right], \qquad \text{where} \qquad v' = v - U \tag{1}$$

Here L, M, N are orthogonal unit vectors with $L(R,t)$ parallel to the magnetic field at R, and M and N chosen in any convenient way with $N = L \times M$. The vector U satisfying $U \cdot L = 0$ remains to be determined.

These variables satisfy the equations

$$\dot{R} = U + v_{\parallel} L + \frac{(A+\Delta)\times L}{B} + \epsilon v_{\perp} (M\cos\theta + N\sin\theta) \times \frac{d}{dt}\left(\frac{L}{B}\right)$$

$$\dot{v}_{\parallel} = \frac{(A+\Delta)\cdot L}{\epsilon} + v_{\perp} (M\cos\theta + N\sin\theta)\cdot \frac{dL}{dt}$$

$$\dot{v}_{\perp} = \frac{(M\cos\theta + N\sin\theta)}{\epsilon} \cdot (A+\Delta)$$

$$\dot{\theta} = -\left[\frac{B}{\epsilon} + N\cdot\frac{dM}{dt}\right] + \frac{(N\cos\theta - M\sin\theta)}{\epsilon v_{\perp}}\cdot(A+\Delta)$$

(2)

where for a quantity $C(R,t)$, $\frac{dC}{dt} = \frac{\partial C}{\partial t} + \dot{R}\,\nabla C$ and

$$\Delta = c\left[E(r,t) - E(R,t)\right] + v \times \left[B(r,t) - B(R,t)\right]$$

$$A = c\,E(R,t) + U \times B(R,t) - \epsilon\left[\frac{dU}{dt} + v_{\parallel}\frac{dL}{dt}\right]$$

The parallel component of E is assumed to be of order ϵ, i.e., $L(R,t)\cdot E(R,t) = O(\epsilon)$

We now choose U such that the perpendicular component of A equals O.

Thus $U = U_0 + \epsilon U_1 + \cdots$ where

$$U_0 = \frac{c\, E(R,t) \times L(R,t)}{B(R,t)}$$

$$U_1 = \epsilon \frac{L(R,t)}{B(R,t)} \times \left[\frac{dU_0}{dt} + v_\parallel \frac{dL}{dt} \right]$$

Also we Taylor expand Δ about R using the relation $r - R = \frac{\epsilon v_\perp}{B}(N\cos\theta - M\sin\theta)$

in order to eliminate r. For the accuracy we require it is sufficient to keep angle independent terms through order ϵ, and periodic terms through order 1.

$$\dot{R} = U + v_\parallel L + \frac{\epsilon v_\perp^2}{2B^2} L \times \nabla B$$

$$\dot{v}_\parallel = \frac{E\cdot L}{\epsilon} + U\cdot\frac{dL}{dt} - \frac{v_\perp^2}{2B} L\cdot\nabla B + \frac{\epsilon v_\perp^2}{2B}\Big[(MN:\nabla L)(NN:\nabla U) - (NN:\nabla L)(MN:\nabla U) + (MM:\nabla L)(NM:\nabla U) - (NM:\nabla L)(MM:\nabla U)\Big] + v_\perp\Big[ML:\nabla(U + v_\parallel L)\cos\theta + NL:\nabla(U+v_\parallel L)\sin\theta\Big] - \frac{v_\perp^2}{2}\Big[(MM-NN):\nabla L\cos 2\theta + (MN+NM):\nabla L\sin 2\theta\Big]$$

$$\dot{v}_\perp = \frac{v_\perp}{2B}\left[-c\,L\cdot\nabla\times E + (U + v_\parallel L)\cdot\nabla B\right] + \frac{v_\perp}{2}\left[(MM-NN)\cdot\nabla(U+v_\parallel L)\cos 2\theta + (MN+NM):\nabla(U+v_\parallel L)\sin 2\theta\right]$$

$$\dot{\theta} = -\left[\frac{B}{\epsilon} + N\cdot\frac{dM}{dt} + \frac{1}{2}L\cdot\nabla\times(U+v_\parallel L)\right] - \frac{v_\perp}{B}(N\cos\theta - M\sin\theta)\cdot\nabla B + \frac{1}{2}\left[(MN+NM):\nabla(U+v_\parallel L)\cos 2\theta - (MM-NN):\nabla(U+v_\parallel L)\sin 2\theta\right] + \mathcal{O}(\epsilon) \qquad (3)$$

The order ϵ part of $\dot{\theta}$ is lengthy but will not be needed. Also, the angle independent terms of $\dot{v}_\perp$ are equal to $(v_\perp/2B)\frac{dB}{dt}$.

We now use the Krylov-Bogoluibov method[10,11] to define new variables

$$(R, v_\parallel, v_\perp, \theta) \rightarrow (P, \mathcal{H}, \Sigma, \Phi)$$

$$R = P$$

$$v_{\parallel} = \mathcal{H} - \frac{\epsilon\Sigma}{B}\Big[ML : \nabla(U+\mathcal{H}L)\sin\Phi - NL : \nabla(U+\mathcal{H}L)\cos\Phi\Big] +$$

$$\frac{\epsilon\Sigma^2}{4B}\Big[(MM-NN):\nabla L \sin 2\Phi - (MN+NM):\nabla L \cos 2\Phi\Big]$$

$$v_{\perp} = \Sigma - \frac{\epsilon\Sigma}{4B}\Big[(MM-NN):\nabla(U+\mathcal{H}L)\sin 2\Phi - (MN+NM):\nabla(U+\mathcal{H}L)\cos 2\Phi\Big]$$

$$\theta = \Phi + \frac{\epsilon\Sigma}{B^2}(N\sin\Phi + M\cos\Phi)\cdot\nabla B - \frac{\epsilon}{4B}\Big[(MN+NM):\nabla(U+\mathcal{H}L)\sin 2\Phi +$$

$$(MM-NN):\nabla(U+\mathcal{H}L)\cos 2\Phi\Big]$$

(4)

where all quantities on the right are located at P , e.g. $L\ (P,t)$. These variables satisfy the Φ independent equations

$$\dot{P} = U + \mathcal{H}L + \frac{\epsilon\Sigma^2}{2B^2}L\times\nabla B \qquad \text{(a)}$$

$$\dot{\mathcal{H}} = \frac{E\cdot L}{\epsilon} + U\cdot\frac{dL}{dt} - \frac{\Sigma^2}{2B}L\cdot\nabla B + \frac{\epsilon\Sigma^2}{2B}L\cdot\left(L\cdot\nabla L + \frac{\nabla B}{B}\right)\times\left[L\cdot\nabla(U+\mathcal{H}L)\right] +$$

$$\frac{\epsilon\Sigma^2}{4B}\left\{[\nabla\cdot L][L\cdot\nabla\times(U+\mathcal{H}L)] + \left[\frac{1}{B}\frac{dB}{dt}\right][L\cdot\nabla\times L]\right\} \qquad \text{(b)}$$

$$\dot{\Sigma} = \frac{\Sigma}{2B}\frac{dB}{dt} \tag{c}$$

$$\dot{\Phi} = -\left[\frac{B}{e} + N\cdot\frac{dM}{dt} + \frac{1}{2}L\cdot\nabla\times(U+\mathcal{H}L)\right] + \mathcal{O}(\epsilon) \tag{d}$$

(5)

where $U = U_0 + \epsilon U_1$ with $U_0 = \dfrac{c\,E(P,t)\times L(P,t)}{B(P,t)}$,

$U_1 = \dfrac{L}{B}\times\left[\dfrac{dU_0}{dt} - \mathcal{H}\dfrac{dL}{dt}\right]$ and $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \dot{P}\cdot\nabla$.

These transformations of variables lead to considerable simplification in the kinetic description of a plasma.

III. MAGNETIC VLASOV EQUATION

If the distribution function $\mathcal{F}(r, v, t)$ normalized to volume V satisfies the equation

$$\frac{\partial \mathcal{F}}{\partial t} + v \cdot \nabla \mathcal{F} + \frac{1}{\epsilon}\left[c\, E + v \times B\right] \cdot \frac{\partial \mathcal{F}}{\partial v} = 0 \tag{6}$$

where the fields satisfy the restriction of Section II, then the transformations (1), (4) produce a different functional form. Because of equation 5c and the fact that Σ is positive we make the further substitution $\mathcal{M} = \Sigma^2 / 2B(P,t)$, and define

$$\mathcal{F}\left(r(P, \mathcal{H}, \mathcal{M}, \Phi, t), v(P, \mathcal{H}, \mathcal{M}, \Phi, t), t\right) \equiv F(P, \mathcal{H}, \mathcal{M}, \Phi, t) \tag{7}$$

By chain rule differentation F satisfies the equation

$$\frac{\partial F}{\partial t} + \dot{P} \cdot \nabla F + \dot{\mathcal{H}} \frac{\partial F}{\partial \mathcal{H}} + \dot{\Phi} \frac{\partial F}{\partial \Phi} = 0 \tag{8}$$

where the coefficients of the derivatives are given by 5, valid to order ϵ. Typically we assume F varies slowly in time, so that the $\dot{\Phi}$ term may be dropped.

Equation 8 plus Maxwell's equations describe the behavior of a system where the initial conditions are known and for times short enough so that statistical fluctuations are not important. In order to complete Maxwell's equations we require the charge density and current, which may be developed from 7 in the following way.

We carry out the velocity integration[12] in the original guiding center variables $v_\perp, v_\parallel, \theta$.

$$\begin{Bmatrix} \text{Charge density} \\ \\ \text{Current density} \end{Bmatrix} \equiv \begin{Bmatrix} K(r,t) \\ \gamma(r,t) \end{Bmatrix} = 4\pi\bar{n}q \int dv \begin{Bmatrix} 1 \\ v \end{Bmatrix} \mathcal{F}$$

$$= 4\pi\bar{n}q \int v_\perp\, dv_\perp\, dv_\parallel\, d\theta \begin{Bmatrix} 1 \\ [1+(P-r)\cdot\nabla][U(r,t)+v_\parallel L(r,t)+v_\perp(M(r,t)\cos\theta+N(r,t)\sin\theta)] \end{Bmatrix} F$$

Here $\bar{n}$ is the system average density.

Now we expand F about the variables $r, v_\parallel, m=\frac{v_\perp^2}{2B(r,t)}, \theta$. Thus

$$\left\{\begin{matrix} K \\ T \end{matrix}\right\} = 4\pi \bar{n} q B(r,t) \int dm\, dv_{\parallel}\, d\theta \left\{ \begin{matrix} 1 \\ [1 + (P-r)\cdot\nabla][U(r,t) + v_{\parallel} L(r,t)] \end{matrix} + \right.$$

$$\left. \sqrt{2mB(r,t)}\,[1 + (P-r)\cdot\nabla][M(r,t)\cos\theta + N(r,t)\sin\theta] \right\} \Big[F(r, v_{\parallel}, m, \theta, t) +$$

$$\left\{ (P-r)\cdot\nabla + (\mathcal{M}-m)\frac{\partial}{\partial m} + (\mathcal{H}-v_{\parallel})\frac{\partial}{\partial v_{\parallel}} + (\Phi-\theta)\frac{\partial}{\partial \theta} \right\} F \Big] \qquad (10)$$

where the order ϵ quantities $(P-r)$, $(\mathcal{H}-v_{\parallel})$, etc., are given by the transformation equations 1 and 4 and the definitions of $\mathcal{M}$ and m, and $F(r, v_{\parallel}, m, \theta, t)$ is simply a relabeling of $F(P, \mathcal{H}, \mathcal{M}, \Phi, t)$. Since F is periodic in θ it may be expanded in Fourier series; the requirement that fields vary slowly in time means that the contribution to K and T from $F_n e^{in\theta}$ must vanish.

Finally we rename the dummy variables of integration $\mathcal{M}$, $\mathcal{H}$, Φ in order to produce a notation consistent with 8. This is not a further change of variables, but simply a relabeling. We have

$$K(r,t) = 8\pi^2 \bar{n} q B(r,t) \int d\mathcal{M}\, d\mathcal{H}\, F(r,\mathcal{H},\mathcal{M},t)$$

$$T(r,t) = 8\pi^2 \bar{n} q B(r,t) \int d\mathcal{M}\, d\mathcal{H} \Big\{ \Big[U + \mathcal{H} L + \epsilon \mathcal{M} (M\cdot\nabla N - N\cdot\nabla M)$$

$$+ \frac{2\epsilon\mathcal{M}}{B} L\times\nabla B - \epsilon\mathcal{M} L\times(L\cdot\nabla L)\Big] F + \epsilon\mathcal{M} L\times\nabla F \Big\}$$

(11)

where in both cases F is the angle independent (average) part of the total distribution. Maxwell's equations in r, t are given by

$$\nabla\times B = T + \frac{1}{c}\frac{\partial E}{\partial t} \qquad \nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla\cdot E = K$$

(12)

IV. STATISTICAL THEORY

A. Conservation Laws

In reference 4 a procedure was developed for treating the Klimontovich-Dupree hierarchy of equations. In order to correct a defect in the conservation properties of that work we write the equation for the single particle distribution of species μ, neglecting electromagnetic effects in the fluctuating fields, i.e., $\delta B = 0$.

$$\frac{\partial f_\mu}{\partial t} + v \cdot \nabla f_\mu + \frac{q_\mu}{m_\mu}\left[\langle E\rangle + \frac{v \times \langle B\rangle}{c}\right] \cdot \frac{\partial f_\mu}{\partial v} = - \frac{q_\mu}{m_\mu} \frac{\partial}{\partial v} \cdot \langle \delta f_\mu \, \delta E\rangle \tag{13}$$

Here the brackets $\langle \ \rangle$ mean a statistical or ensemble average. Although we shall change variables in order to eliminate rapid phase dependence, equation 13 as written is convenient for developing conservation of energy and momentum in the system. In this paper the state of the system is described by the one particle distribution, and by the distribution of electrostatic energy in <u>local modes</u>, i.e., fluctuations in which the perturbed electric field may be approximated locally by $\sum_k |\delta E_k\rangle \exp(i k \cdot r - i \omega_k t)$

with $\omega_k = \Omega_k + i \gamma_k$. Thus we assume knowledge of $f(r, v, t)$ and $\langle \delta E_k^2(r, \omega_k, t)\rangle$, at the initial time $t = 0$, and for all time on any physical boundaries.

Multiplying the right side of 13 by $m_\mu v$ and $\frac{1}{2} m_\mu v^2$, integrating by parts, and using Maxwell's equations in the longitudinal approximation yields[14]

$$\nabla \cdot \left(\frac{\langle \delta E_k \delta E_{-k} \rangle}{8\pi} \right) = \sum_\mu 4\pi \bar{n}_\mu q_\mu \int dv \langle \delta f_\mu(k)\, \delta E_{-k} \rangle$$

$$\frac{\partial}{\partial t}\left(\frac{\langle \delta E_k \delta E_{-k} \rangle}{8\pi} \right) \equiv 2\gamma_k \left(\frac{\langle \delta E_k \delta E_{-k} \rangle}{8\pi} \right) + \frac{\partial}{\partial t}\left(\frac{\langle \delta E_k \delta E_{-k} \rangle}{8\pi} \right)_{\substack{\text{particle} \\ \text{discreteness}}}$$

$$= \sum_\mu 4\pi \bar{n}_\mu q_\mu \int dv\, v \cdot \langle \delta f_\mu(k)\, \delta E_{-k} \rangle \tag{14}$$

We adopt these equations for the determination of Ω_k and γ_k instead of determining them from the dielectric function. In the homogeneous field free case these equations reduce to the imaginary and real parts of the usual dielectric function.[15] These relations guarantee momentum and energy conservation in the system, where the energy is given by $\int dv \frac{1}{2} m v^2 f + \sum_k \frac{\langle \delta E_k \delta E_{-k} \rangle}{8\pi}$

B. Statistical Variables

In this section we utilize the transformations of Section II to develop the statistical theory of plasma in a magnetic field. Because of fluctuations in the electric field we define average variables before following the procedure of Section III.

We assume that instruments which measure electric fields are able to disregard or average over high frequency ($\omega \gtrsim \omega_c$) or short wavelength fluctuations, but that they respond to fluctuations which meet the requirements of the guiding center theory. Then we may define ensemble average variables by splitting the electric field E (measured) = $\langle E \rangle^{16} + \delta E_G$ where δE_G is the fluctuating guiding center field. We define

$$\bar{R} = R - \delta E_G \frac{\partial R}{\partial E}$$

$$\bar{v}_\parallel = v_\parallel - \delta E_G \frac{\partial v_\parallel}{\partial E}$$

$$\bar{v}_\perp = v_\perp - \delta E_G \frac{\partial v_\perp}{\partial E}$$

$$\bar{\Theta} = \theta - \delta E_G \frac{\partial \theta}{\partial E} \tag{15}$$

We shall disregard terms which lead to results of order $\epsilon \langle \delta E_G \, \delta E_G \rangle$ in the final equations.

As in Section II we calculate $(\dot{\bar{R}})$, $(\dot{\bar{v}}_\parallel)$, etc., and use the Krylov-Bogoluibov transformation to define new variables $(\rho, \eta, \sigma, \phi)$

$$\bar{R} = \rho$$

$$\bar{v}_{\parallel} = \eta - \frac{\epsilon\sigma}{B}\left[ML : \nabla(U+\eta L)\sin\phi - NL : \nabla(U+\eta L)\cos\phi\right]$$

$$+ \frac{\epsilon\sigma^2}{4B}\left[(MM-NN) : \nabla L \sin 2\phi - (MN+NM) : \nabla L \cos 2\phi\right]$$

$$\bar{v}_{\perp} = \sigma - \frac{\epsilon\sigma}{4B}\left[(MM-NN) : \nabla(U+\eta L)\sin 2\phi - (MN+NM) : \nabla(U+\eta L)\cos 2\phi\right]$$

$$\bar{\Theta} = \phi + \frac{\epsilon\sigma}{B^2}(N\sin\phi + M\cos\phi)\cdot\nabla B - \frac{\epsilon}{4B}\left[(MN+NM) : \nabla(U+\eta L)\sin 2\phi + \right.$$

$$\left.(MM-NN) : \nabla(U+\eta L)\cos 2\phi\right]$$

(16)

These satisfy the equations of motion

$$\dot{\rho} = \dot{P}(\rho, \eta, \sigma, t)$$

$$\dot{\eta} = \dot{H}(\rho, \eta, \sigma, t)$$

$$\dot{\sigma} = \dot{\Sigma}(\rho, \eta, \sigma, t)$$

$$\dot{\phi} = \dot{\Phi}(\rho, \eta, \sigma, t) - \frac{\langle \delta E_G^2 \rangle}{2\sigma^2 B^2}$$

(17)

where the notation indicates simply the relabeling of the right side of equation 5. However, the electric field is the sum of the average field and fluctuations which satisfy the guiding center restrictions. We ensemble average 17 to find

$$\langle\dot{\rho}\rangle = U + \eta L + \frac{\epsilon\sigma^2}{2B^2} L\times\nabla B$$

$$\langle\dot{\eta}\rangle = \frac{\langle E\rangle\cdot L}{\epsilon} + U\cdot\frac{dL}{dt} - \frac{\sigma^2}{2B}L\cdot\nabla B + \frac{\epsilon\sigma^2}{2B}L\cdot\left(L\cdot\nabla L + \frac{\nabla B}{B}\right)\times\left[L\cdot\nabla(U+\eta L)\right]$$

$$+ \frac{\epsilon\sigma^2}{4B}\left\{\left[\nabla\cdot L\right]\left[L\cdot\nabla\times(U+\eta L)\right] + \left[\frac{1}{B}\frac{dB}{dt}\right]\left[L\cdot\nabla\times L\right]\right\}$$

$$\langle\dot{\sigma}\rangle = \frac{\sigma}{2B}\frac{dB}{dt}$$

$$\langle\dot{\phi}\rangle = -\left[\frac{B}{\epsilon} + N\cdot\frac{dM}{dt} + \frac{1}{2}L\cdot\nabla\times(U+\eta L)\right] - \frac{\langle\delta E_G^2\rangle}{2\sigma^2B^2}$$

where $\quad U = U_0 + \epsilon U_1 \quad$, with $\quad U_0 = \dfrac{c\langle E(\rho,t)\rangle \times L(\rho,t)}{B(\rho,t)}$

$$U_1 = \frac{L}{B} \times \left(\frac{dU_0}{dt} + n\frac{dL}{dt} \right), \text{ and } \frac{d}{dt} = \frac{\partial}{\partial t} + \langle \dot{\rho} \rangle \cdot \nabla$$

By subtracting (18) from (17) and defining $\delta\dot{\rho} \equiv \dot{\rho} - \langle\dot{\rho}\rangle$, etc.,

$$\delta\dot{\rho} = \frac{c\,\delta E_G \times L}{B}, \qquad \delta\dot{n} = \frac{c\,\delta E_G \cdot L}{\epsilon}$$

$$\delta\dot{\sigma} = \frac{\sigma c\,(\delta E_G \times L)\cdot\nabla B}{2B^2}, \qquad \delta\dot{\phi} = -N\left(\frac{c\,\delta E_G \times L}{B}\right):\nabla M + \frac{1}{2} L \cdot \nabla \times \left(\frac{c\,\delta E_G \times L}{B}\right)$$

(19)

C. Statistical Equations

We now use the method developed by Klimontovich and Dupree[17] and the transformations developed in this paper to write statistical equations describing the behavior of a plasma in a magnetic field. The exact one particle distribution

$$F(r,v,t) = \frac{1}{n} \sum_{i=1}^{N} \delta(r - r_i(t))\,\delta(v - v_i(t))$$

satisfies the equation

$$\frac{\partial F}{\partial t} + v \cdot \nabla F + \frac{q}{m}\left(E + \frac{v}{c} \times B\right) \cdot \frac{\partial F}{\partial v} = 0$$

Here E is the exact electric field, including, for example, particle discreteness effects and high frequency collective fields δE_N . Defining $E \equiv \langle E \rangle + \delta E_G + \delta E_N$ and using the change of variables 15, 16, followed by $\mu = v^2/2B(\rho, t)$ we have

$$\left[\frac{\partial}{\partial t} + \dot{\rho} \cdot \nabla + \dot{\eta}\frac{\partial}{\partial \eta} + \dot{\phi}\frac{\partial}{\partial \phi}\right] F(\rho, \eta, \mu, \phi, t) + c\, \delta E_N \left(\rho + \epsilon\sqrt{\frac{2\mu}{B}}(N\cos\phi - M\sin\phi), t\right) \cdot$$

$$\left[\frac{L \times \nabla F}{B} + L\frac{\partial F}{\partial \eta} + \frac{(N\cos\phi - M\sin\phi)}{\epsilon\sqrt{2\mu B}}\frac{\partial F}{\partial \phi} + \left\{\sqrt{\frac{2\mu}{B}}\,\frac{(M\cos\phi + N\sin\phi)}{\epsilon} - \right.\right.$$

$$\left.\left.\mu\frac{L \times \nabla B}{B^2}\right\}\frac{\partial F}{\partial \mu}\right] = 0 \tag{20}$$

We define the ensemble average to have no ϕ dependence

$$\langle F(r,v,t)\rangle \equiv f(\rho,\eta,\mu,t)$$

and take the average of 20, assuming high frequency and low frequency modes are separable (e.g., by Fourier analysis) so that $F - f = \delta f_G + \delta f_N$

$$\left[\frac{\partial}{\partial t} + \langle\dot\rho\rangle\cdot\nabla + \langle\dot\eta\rangle\frac{\partial}{\partial\eta}\right] f = -\langle\delta\dot\rho\cdot\nabla\delta f_G\rangle - \left\langle\delta\dot\eta\,\frac{\partial\delta f_G}{\partial\eta}\right\rangle$$

$$-\left\langle\frac{\delta E_N\times L}{B}\cdot\nabla\delta f_N\right\rangle - \left\langle\frac{\delta E_N\cdot L}{\epsilon}\,\frac{\partial\delta f_N}{\partial\eta}\right\rangle - \left\langle\frac{\delta E_N\cdot(N\cos\phi - M\sin\phi)}{\epsilon\sqrt{2\mu B}}\,\frac{\partial\delta f_N}{\partial\phi}\right\rangle$$

$$+\left\langle\left[\frac{\mu\,\delta E_N\times L\cdot\nabla B}{B^2} - \sqrt{\frac{2\mu}{B}}\,\frac{\delta E_N\cdot(M\cos\phi + N\sin\phi)}{\epsilon}\right]\frac{\partial\delta f_N}{\partial\mu}\right\rangle \tag{21}$$

where

$$\delta E_N = \delta E_N \left(\rho + \epsilon \sqrt{\frac{2\mu}{B}} (N\cos\phi - M\sin\phi), t \right)$$

In the notation of reference 4 we subtract to find the equations for the fluctuations.

$$\left[\frac{\partial}{\partial t} + \langle \dot{\rho} \rangle \cdot \nabla + \langle \dot{\eta} \rangle \frac{\partial}{\partial \eta} \right] | \delta f_G \rangle = - \delta \dot{\rho} \cdot \nabla f - \delta \dot{\eta} \frac{\partial f}{\partial \eta} \tag{22}$$

$$\left[\frac{\partial}{\partial t} + \langle \dot{\rho} \rangle \cdot \nabla + \langle \dot{\eta} \rangle \frac{\partial}{\partial \eta} + \langle \dot{\phi} \rangle \frac{\partial}{\partial \phi} \right] | \delta f_N \rangle = - \frac{| \delta E_N \rangle \times L}{B} \cdot \nabla f$$

$$- \frac{| \delta E_N \rangle \cdot L}{\epsilon} \frac{\partial f}{\partial \eta} - \left[\sqrt{\frac{2\mu}{B}} \frac{| \delta E_N \rangle \cdot (M\cos\phi + N\sin\phi)}{\epsilon} - \mu \frac{| \delta E_N \rangle \times L \cdot \nabla B}{B^2} \right] \frac{\partial f}{\partial \mu} \tag{23}$$

Maxwell's equation may be developed as in Section III. For the longitudinal fluctuations we require only the charge density,

$$\nabla \cdot |\delta E_G\rangle = 8\pi^2 \bar{n} g\, B(r,t) \int d\mu\, d\eta\, |\delta f_G(r,\mu,\eta,t)\rangle$$

$$\nabla \cdot |\delta E_N\rangle = 4\pi \bar{n} g\, B(r,t) \int d\mu\, d\eta\, d\phi\, |\delta f_N(r,\mu,\eta,\phi,t)\rangle$$

We may now use methods developed earlier[4,18] to solve equations 22 and 23 by integration along the characteristics, and insert the results into 21. The low frequency terms represent a generalized form of Dupree's work,[19] while the high frequency terms generalize the result of Rostoker[1]. Because of the great length of the resulting equation we omit the expression until application to a specific problem makes further approximation possible.

References

[1]N. Rostoker, Phys. Fluids 3, 922 (1960).

[2]G. E. Wilson, Phys. Fluids 12, 1673 (1969).

[3]J. B. Taylor and R. J. Hastie, Plasma Phys. 10, 479 (1968).

[4]J. C. Price, Phys. Fluids 12, 539 (1969).

[5]G. E. Wilson, Phys. Fluids 13, 1372 (1970).

[6]Reference 2, Section III.

[7]T. Northrop, The Adiabatic Motion of Charged Particles (John Wiley and Sons, Inc., New York 1963) Chpt. 1, Part A.

[8]Reference 2, Section II.

[9]At this point we note the Galilean invariance property. If the primed system moves with velocity V, then $B' = B$, $E' = E + V \times B$, and the substitution $U_0 \to U_0 + V_\perp$, $v_\parallel \to v_\parallel + V_\parallel$ provides results for the moving system. Inclusion of relativistic corrections is much more complicated because B is modified. See for example W. H. Panofsky and M. Phillips, Classical Electricity and Magneticism, (Addison-Wesley Publishing Co. Inc., Reading, Mass. 1955), p. 283.

[10]N. N. Bogoliubov and Y. A. Mitrospolsky, Asymptotic Method in the Theory Nonlinear Oscillations, (Gordon and Breach, New York, 1961).

[11]P. Musen, J. Astronautical Sci. 12, 123 (1965). This paper gives a clear and concise statement of the theory including the effect of higher order terms.

[12]Integration implies summation over particle index, and g and ϵ depend on particle species.

[13]Note that the fluctuating fields may include forces which do not satisfy the assumptions of the guiding center theory.

[14]More properly we operate on $\langle \delta f_\mu(r_1, v, t)\, \delta E(r_2, t) \rangle$

which justifies the separation of Fourier components.

[15]L. Landau, J. Phys. (USSR) 10, 25 (1946).

[16]In the spirit of reference 4 $\langle E \rangle$ should represent the average force experienced by a particle, including electron-ion drag, gravity, etc.

[17]T. H. Dupree, Phys. Fluids 6, 1714 (1963).

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