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APPLICATION OF BROUWER'S ARTIFICIAL-SATELLITE THEORY TO COMPUTATION OF THE STATE TRANSITION MATRIX

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SUMMARY

Brouwer's solution of the artificial-satellite problem without drag is used to obtain analytical state-transition-matrix expressions that include secular as well as long- and short-period effects of planetary oblateness. A comparison of the accuracy of several different models of the state transition matrix is made. These models include (1) a Keplerian model, (2) a model based on first- and second-order secular terms in Brouwer's theory, and (3) a model including first- and second-order secular terms in addition to first-order long- and short-period perturbations. It is demonstrated that the accuracy of the Keplerian model degenerates rapidly in comparison to either of the models based on Brouwer's theory. In addition, it is shown with numerical results that including the effects of long- and short-period perturbations improves the accuracy of the transition matrix by one to two orders of magnitude over the secular model and three to four orders of magnitude over the Keplerian model.

INTRODUCTION

Determination of the state transition matrix is a necessity in orbit determination and navigation theory. Numerous methods that can be used to compute the state transition matrix exist. These methods include direct integration of the variational equations, numerical differentiation, and various analytical formulations. For example, a Keplerian model for any type of conic motion has been developed by Goodyear (ref. 1). Recently, Ditto (ref. 2) applied the Peano-Baker method to the integration of the variational equations. In this method, numerical integration is applied only to the departure from a simplified analytical model.

The objective of this study was to obtain an analytical formulation of the state transition matrix that accounted for the effects of planetary oblateness for use in long-term satellite navigation studies. The method used was to differentiate the solutions in Brouwer's artificial-satellite theory (ref. 3) to obtain the transition matrix in terms of Keplerian elements. Although only the oblateness was considered in this study, the inclusion of long-period and secular effects for the remaining harmonics in Brouwer's theory would be a simple extension.

A numerical comparison of the accuracy of several different models of the transition matrix was made. These models included (1) a Keplerian model; (2) a model based on first- and second-order secular perturbations in Ω, ω, and M and mean values of a, e, and I as given by Brouwer's theory; and (3) a model including first- and second-order secular perturbations in Ω, ω, and M in addition to long- and short-period perturbations in all the orbital elements. The terminology first- and second-order perturbations means that the perturbations are proportional to J_{20} and J_{20}^2, respectively.

In this study, model 2 will be referred to as the secular model, and model 3 will be called the complete model. Both models require the use of mean elements to be accurate through the first order. The reason epochal elements are not used in the secular model, which frequently is done erroneously, is discussed in the section on analysis.

The authors gratefully acknowledge the assistance of Mr. Claude E. Hildebrand, Jr., of The University of Texas at Austin, who helped with the analysis.

SYMBOLS

A	matrix of time-dependent coefficients
a	semimajor axis
$\hat{a}$	semimajor axis based on total energy
B	generic term for transition matrix
e	eccentricity
F_2^{**}	second-order portion of doubly averaged Hamiltonian
f	true anomaly
G	$(\mu a)^{1/2}\left(1 - e^2\right)^{1/2}$
H	G cos I
I	inclination
J_{20}	oblateness coefficient
K_2	$\frac{J_{20} r_e^2}{2}$

L	$(\mu a)^{1/2}$
M	mean anomaly
N	dimension of the state vector
n	mean motion
$\bar{n}$	mean of mean motion
$\hat{n}$	mean motion based on total energy
R	relative error
r	radius
r_e	mean radius of planet
S	column vector of Cartesian coordinates
t	time
V	velocity
X	perturbation vector
x, y, z	Cartesian coordinates
ϵ	column vector of orbital elements
η	$\sqrt{1 - e^2}$
θ	$\cos I$
μ	gravitational constant
$\Phi(t, t_o)$	state transition matrix
Ω	longitude of ascending node
ω	argument of pericenter

Subscripts:

o	initial value
p	evaluated at pericenter
s	secular rate

Operators:

$(\cdot)$ derivative with respect to time

$(')$ first-order averaged variables

$('')$ mean elements

$(*)$ evaluated on the reference orbit

ANALYSIS

Theoretical Development of State Transition Matrix

The matrix differential equation for the state transition matrix is given by

$$\dot{\Phi}\left(t, t_o\right) = A\Phi\left(t, t_o\right) \tag{1}$$

where

$$\Phi\left(t, t_o\right) = \frac{\partial \epsilon(t)}{\partial \epsilon\left(t_o\right)} \tag{2}$$

An element of $\Phi(t, t_o)$ is given by

$$\Phi_{ij}\left(t, t_o\right) = \frac{\partial \epsilon_i(t)}{\partial \epsilon_j\left(t_o\right)} \tag{3}$$

where $(i, j = 1, 2, \ldots, N)$, ϵ_i is a generic symbol for any of the orbital elements, and N is the dimension of the state vector. Also

$$A = \frac{\partial \dot{\epsilon}(t)}{\partial \epsilon(t)} \tag{4}$$

Equation (1) represents a system of N by N linear differential equations with known time-dependent coefficients. This so-called variational equations system may be solved by numerical integration.

One alternative to integration of the variational equations is the differentiation of an analytical solution, such as Brouwer's, with respect to the initial conditions to obtain the state transition matrix directly. Brouwer's theory separates the secular and periodic terms; therefore, contributions to the state transition matrix of the secular and periodic portions also are computed separately. This separation permits the computation of a transition matrix based on any combination of secular and periodic terms desired.

Because Brouwer's theory is written in terms of mean elements and because differentiation with respect to epochal elements is desired, the chain rule must be used; that is,

$$\Phi(t, t_o) = \left[\frac{\partial\epsilon(t)}{\partial\epsilon''(t_o)}\right]\left[\frac{\partial\epsilon''(t_o)}{\partial\epsilon(t_o)}\right] \tag{5}$$

where

$$\left[\frac{\partial\epsilon''(t_o)}{\partial\epsilon(t_o)}\right] = \left[\frac{\partial\epsilon(t_o)}{\partial\epsilon''(t_o)}\right]^{-1} \tag{6}$$

and

$$\left(\frac{\partial\epsilon}{\partial\epsilon''}\right) = \begin{bmatrix} \frac{\partial a}{\partial a''} & \frac{\partial a}{\partial e''} & \cdots & \frac{\partial a}{\partial M''} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial M}{\partial a''} & \frac{\partial M}{\partial e''} & \cdots & \frac{\partial M}{\partial M''} \end{bmatrix} \tag{7}$$

The matrix $\left[\frac{\partial\epsilon''(t_o)}{\partial\epsilon(t_o)}\right]$ is a constant matrix that needs only to be calculated once. If the transition matrix in terms of Cartesian coordinates is desired, equation (5) must be multiplied by the appropriate transformation matrices as follows

$$\Phi(t, t_o) = \left[\frac{\partial S(t)}{\partial\epsilon(t)}\right]\left[\frac{\partial\epsilon(t)}{\partial\epsilon''(t_o)}\right]\left[\frac{\partial\epsilon''(t_o)}{\partial\epsilon(t_o)}\right]\left[\frac{\partial\epsilon(t_o)}{\partial S(t_o)}\right] \tag{8}$$

where S represents the vector of Cartesian coordinates. The matrix $\partial S/\partial \epsilon$ is obtained by differentiation of the expressions that relate the Cartesian coordinates and the orbital elements. These expressions may be found in most fundamental celestial mechanics texts (e.g., ref. 4). Because the transformation that relates the Cartesian coordinates to the orbital elements is unique,

$$\left[\frac{\partial \epsilon(t_o)}{\partial S(t_o)}\right] = \left[\frac{\partial S(t_o)}{\partial \epsilon(t_o)}\right]^{-1} \tag{9}$$

and this matrix is formed only once.

The differentiation of Brouwer's solutions with respect to the mean elements so that the results are accurate through the first order in the periodic terms and through the second order in the secular terms is straightforward but lengthy. Consequently, the results are not given in this study. However, a copy of the computer program containing the results will be supplied by the authors on request.

The analytical models discussed are based on Keplerian elements and, therefore, are unacceptable for circular orbits or for orbits with zero inclination. In addition, Brouwer's theory is invalid within approximately $\pm 2^\circ$ of the critical inclination. The problems associated with zero eccentricity and inclination may be eliminated with a redefinition of the orbital elements as suggested by Lyddane (ref. 5).

Computation of Mean Elements

The mean elements for Brouwer's theory are obtained accurate to the first order in J_{20} by evaluation of Brouwer's solutions at epoch and by replacement of mean elements with epochal elements in the first-order terms. For example, the mean value of a is given by

$$a'' = a_o - \frac{K_2}{a_o}\left[\left(-1 + 3\theta_o^{\,2}\right)\left(\frac{a_o^{\,3}}{r_o^{\,3}} - \eta_o^{\,-3}\right) + 3\left(1 - \theta_o^{\,2}\right)\frac{a_o^{\,3}}{r_o^{\,3}}\cos\left(2\omega_o + 2f_o\right)\right] \tag{10}$$

Computing mean elements with epochal elements in the first-order terms results in second-order errors in the mean elements. As noted in Breakwell and Vagners (ref. 6), iteration of Brouwer's solution for the mean elements cannot reduce this error because the solutions do not contain second-order periodic terms. However, the secular portion of Brouwer's theory is accurate to the second order; therefore, the mean value of a must be accurate to the second order to avoid second-order secular errors being introduced in the mean value of the mean motion through the term $\dfrac{\mu^{1/2}}{a''^{3/2}}$.

The mean value of the mean motion correct to the second order may be computed according to Breakwell and Vagners (ref. 6) or Hildebrand (ref. 7). In terms of the total energy, $\hat{n}$ is defined as

$$\hat{n} = \frac{\mu^{1/2}}{\hat{a}^{3/2}} \tag{11}$$

where

$$\frac{\mu}{2\hat{a}} = \frac{\mu}{2a_o} + \frac{\mu K_2}{r_o^3}\left[\left(-\frac{1}{2} + \frac{3}{2}\theta_o^{\,2}\right) + \frac{3}{2}\left(1 - \theta_o^{\,2}\right)\cos\left(2\omega_o + 2f_o\right)\right] \tag{12}$$

Replacing equation (12) with its average through second order and using equation (11) results in equation (13).

$$\hat{n} = \bar{n}\left[1 + 3\frac{\mu^2 K_2}{L'G''^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2}\right) + \frac{3L'^2}{\mu^2}F_2{}^{**} + \frac{3}{2}\frac{\mu^4 K_2{}^2}{L'^2 G''^6}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G''^2}\right)^2\right] \tag{13}$$

The quantity $F_2{}^{**}$ is the second-order portion of the Hamiltonian averaged with respect to mean anomaly and argument of pericenter. It is given by equation (29) of Brouwer (ref. 3) as

$$F_2{}^{**} = \frac{\mu^6 K_2{}^2}{L'^{10}}\left[\frac{15}{32}\frac{L'^5}{G'^5}\left(1 - \frac{18}{5}\frac{H^2}{G'^2} + \frac{H^4}{G'^4}\right) + \frac{3}{8}\frac{L'^6}{G'^6}\left(1 - \frac{6H^2}{G'^2} + \frac{9H^4}{G'^4}\right) - \frac{15}{32}\frac{L'^7}{G'^7}\left(1 - \frac{2H'^2}{G'^2} - \frac{7H^4}{G'^4}\right)\right] \tag{14}$$

Equation (13) now may be solved for $\bar{n}$, which is the desired result.

The use of mean elements is necessary to maintain the state transition matrix accurate to the first order. If epochal elements are used in a first-order secular model for the transition matrix, as commonly is done, a first-order secular error in the mean anomaly will exist. In fact, inclusion of the first-order secular term in the computation of the mean motion may result in a transition matrix that is less accurate than a Keplerian model. To demonstrate this situation, the solution for the mean value of the mean motion through the first order is considered.

$$\bar{n} = \frac{\mu^{1/2}}{a''^{3/2}} + \frac{3}{4}\frac{\mu^{1/2} r_e^{2} J_{20}}{a''^{7/2}\eta^{3}}\left(-1 + 3\theta^2\right) \tag{15}$$

To the first order

$$a'' = a_o - O\left(J_{20}\right) \tag{16}$$

Denoting the term of order J_{20} by J_{20}' and substituting equation (16) into equation (15) yields

$$\bar{n} = \frac{\mu^{1/2}}{\left(a_o - J_{20}'\right)^{3/2}} + \frac{\mu^{1/2}}{\left(a_o - J_{20}'\right)^{7/2}}\left[\frac{3}{4}\frac{J_{20} r_e^{2}}{\eta^{3}}\left(-1 + 3\theta^2\right)\right] \tag{17}$$

Expansion of equation (17) and retention of first-order terms result in

$$\bar{n} = \mu^{1/2}\left\{a_o^{-3/2} + \frac{3}{2}a_o^{-5/2}J_{20}' + a_o^{-7/2}\left[\frac{3}{4}\frac{J_{20} r_e^{2}}{\eta^{3}}\left(-1 + 3\theta^2\right)\right] + O\left(J_{20}^{2}\right)\right\} \tag{18}$$

The first-order error introduced by the use of $\bar{n} = \frac{\mu^{1/2}}{a_o^{3/2}}$ is

$$\mu^{1/2}\left\{\frac{3}{2}a_o^{-5/2}J_{20}' + a_o^{-7/2}\left[\frac{3}{4}\frac{J_{20} r_e^{2}}{\eta^{3}}\left(-1 + 3\theta^2\right)\right]\right\} \tag{19}$$

The first-order error introduced by use of equation (15) with the epochal value for a is

$$\frac{3}{2}\mu^{1/2} a_o^{-5/2} J_{20}' \tag{20}$$

Therefore, if the initial conditions are such that the two error terms of equation (19) are of opposite sign, the Keplerian model will be more accurate than a secular model using epochal elements.

RESULTS

Discussion of Comparison Criteria

The three analytical models considered, the Keplerian, the secular, and the complete model, were evaluated by comparison with numerical integration of the variational equations. The numerical integration was performed with a double-precision fourth-order Runge Kutta algorithm. The test cases used were a highly elliptical orbit around Mars and a near-circular orbit around the Earth. The Mars-orbital data were run because the results could be applied to long-term navigation studies for a Mars orbiter. All results presented are for transition matrices transformed to Cartesian coordinates. The disturbing force in all cases was planetary oblateness.

It was assumed that Brouwer's theory for an Earth satellite is valid for a Mars orbiter. The fundamental assumption for Brouwer's theory was that all perturbing effects are second order compared to oblateness.

To compare the accuracy of the various models, the relative errors as defined by Ditto (ref. 2) were used. The relative error is defined as follows. If $\|A\|$ is the norm of the transition matrix obtained by numerical integration of the variational equations; that is,

$$\|A\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}| \tag{21}$$

and $\|B\|$ is the norm of the transition matrix computed from any of the analytical solutions, then the relative error R is defined as

$$R = \frac{\|A - B\|}{\|A\|} \tag{22}$$

Discussion of Figures

The relative error was computed at 0.02-day increments for 25 days for the Mars satellite and at 10-minute increments for 14 400 minutes for the Earth satellite. The results for the Mars orbiter are shown in figure 1. The initial conditions for the Mars orbiter are as follows.

$$\left.\begin{array}{ll} a_o = 0.414245 \times 10^8 \text{ feet} & \Omega_o = 28.66^\circ \\ e_o = 0.7 & \omega_o = 318.116^\circ \\ I_o = 19.583^\circ & M_o = 0^\circ \\ \multicolumn{2}{c}{J_{20} = 0.002011} \end{array}\right\} \tag{23}$$

The exponent of R is approximately equal to the number of digits of agreement between the analytical model and the integration of the variational equations. As seen from figure 1, the complete model agrees to four places; the secular model agrees to two places; and, after the first 2 days, the Keplerian model fails to agree even to one digit with integration of the variational equations.

The relative errors for an Earth orbiter are presented in figure 2. The initial conditions for this orbit are as follows.

$$\left.\begin{array}{ll} a_o = 0.2378932 \times 10^8 \text{ feet} & \Omega_o = 45^\circ \\ e_o = 0.1 & \omega_o = 0^\circ \\ I_o = 75^\circ & M_o = 0^\circ \\ \multicolumn{2}{c}{J_{20} = 0.001083} \end{array}\right\} \tag{24}$$

The agreement of the various models was approximately the same as for the Mars orbiter except that agreement to one more place was obtained with the secular model.

The nominal trajectory for each model was generated with the state-propagation equations corresponding to the particular model. For example, the Keplerian results were based on a Keplerian nominal, while the secular-model nominal was generated from the following relationships.

$$\left.\begin{aligned}
a(t) &= a'' \\
e(t) &= e'' \\
I(t) &= I'' \\
\Omega(t) &= \Omega'' + \dot{\Omega}_s \Delta t \\
\omega(t) &= \omega'' + \dot{\omega}_s \Delta t \\
M(t) &= M'' + \dot{M}_s \Delta t
\end{aligned}\right\} \tag{25}$$

To demonstrate the magnitude of errors encountered by propagating a perturbation vector with the various models, a small perturbation vector was mapped with the analytical transition matrices and with the transition matrix obtained by numerical integration of the variational equations. These results were compared to actual integration of the equations of motion with the perturbed initial conditions.

For all results shown, the initial conditions were perturbed by the addition of the following perturbation vector to the initial state vector.

$$\Delta X(t_o) = \begin{bmatrix} 500 \text{ ft} \\ 500 \text{ ft} \\ 500 \text{ ft} \\ 1.0 \text{ ft/sec} \\ 1.0 \text{ ft/sec} \\ 1.0 \text{ ft/sec} \end{bmatrix} \tag{26}$$

This perturbation vector was propagated for both the Mars and Earth satellite orbits. The results for the Mars orbit are shown in figures 3 to 16. The magnitudes of the position- and velocity-perturbation vectors for the Mars orbiter computed from

$$\Delta X(t) = \Phi(t, t_o) \Delta X(t_o) \tag{27}$$

are shown in figures 3 and 10, respectively. The results shown in these two figures are computed from the Keplerian transition matrix; however, differences between all models are indiscernible when plotted on this scale. It should be noted that the magnitudes of these perturbations grow very rapidly. Hence, the linear state transition matrix will propagate this perturbation vector accurately for only a short period of time.

The result shown in figure 4 is the difference in the magnitude of the perturbation vector as computed by the Keplerian transition matrices and the numerical integration of the equations of motion with perturbed initial conditions. The same result is shown in figures 5 to 7 for the secular model, the complete model, and integration of the variational equations, respectively.

A comparison of figure 4 with figure 5 shows that the maximum error in the magnitude of the perturbation vector after one revolution (0.5 day) is 10 nautical miles for the Keplerian model and 3 nautical miles for the secular model. The complete model and integration of the variational equations both have errors of about 1.2 nautical miles as shown by figures 6 and 7. The primary source of error for the Keplerian model is the error in mean motion, which results in a secular-error growth in the mean anomaly. Consequently, at a given time, the Keplerian transition matrix is computed at an incorrect orbital position. This problem is virtually eliminated by using a mean or average value for the mean motion as evidenced by the secular-model results in figure 5.

The errors resulting from numerical integration of the variational equations are presented in figure 7. Neglecting the effects of numerical integration errors (round off and truncation), the errors shown in figure 7 are caused solely by nonlinearities. It should be noted that the error histories for the complete model and for the variational equations are identical because the transition matrices agree to four places as shown by figure 1.

A plot of the magnitude of the angle between the perturbation vectors in position computed from the Keplerian model and from integration of the equations of motion is presented in figure 8. The corresponding results for the variational equations are presented in figure 9. The results for the complete model were identical to those of the variational equations. Because the angular deviations for the secular model were only slightly larger than those for the complete model, the deviations are not shown. Once again, the primary source of error for the Keplerian model is the use of an improper mean motion.

Figures 10 to 16 correspond to figures 3 to 9, respectively, except that these results are presented for the velocity-perturbation vector. The same comments made for the position-perturbation-vector errors apply to these figures.

The magnitude of the position-perturbation vector as propagated by the Keplerian transition matrix for the Earth orbiter is shown in figure 17. The errors in the magnitude of this vector for the Keplerian and complete models are given in figures 18 and 19, respectively. The results from the complete model were virtually identical to the results from integration of the variational equations. Errors in the secular model were larger than for the complete model but much smaller than those of the Keplerian model.

Theoretical Analysis of Results

It should be noted from figures 3 and 10 that the magnitude of the position-perturbation vector has grown to approximately 190 nautical miles in position and 800 ft/sec in velocity after 0.5 day, which is the pericenter passage of the first revolution. Although seemingly rather large, these numbers are results of the large eccentricity and can be verified easily analytically. Assume that it is desired to compute the perturbations in position and velocity at successive pericenter points because of a small perturbation in the state vector at the initial time. The perturbation in the radius vector Δr_p will have components Δr and $r\,\Delta f$ along and normal to the radius vector, respectively. At the pericenter, $r = a(1 - e)$; therefore, $\Delta r = \Delta a\,(1 - e) - a\,\Delta e$. The component $r\,\Delta f$ is now computed from $df = \frac{\partial f}{\partial e}\,de + \frac{\partial f}{\partial M}\,dM$ where

$$\frac{\partial f}{\partial e} = \left(\frac{1}{1 - e^2} + \frac{a}{r}\right) \sin f \tag{28}$$

and

$$\frac{\partial f}{\partial M} = \frac{a^2}{r^2} \sqrt{1 - e^2} \tag{29}$$

Therefore, for small perturbations in the orbital elements,

$$\begin{aligned} \Delta f &= \left(\frac{1}{1 - e^2} + \frac{a}{r}\right) \sin f\,\Delta e + \frac{a^2}{r^2} \sqrt{1 - e^2}\,\Delta M \\ &= \frac{2 + e \cos f}{1 - e^2} \sin f\,\Delta e \\ &\quad + \left(1 - e^2\right)^{-3/2} (1 + e \cos f)^2 \left(\Delta M_o - \frac{3n}{2a}\,\Delta a\,t\right) \end{aligned} \tag{30}$$

When $r\,\Delta f$ is evaluated at pericenter,

$$(r\,\Delta f)_p = \left(\frac{1 + e}{1 - e}\right)^{1/2} \left(a\,\Delta M_o - 3\pi m\,\Delta a\right) \qquad (m = 0,\ 1,\ 2,\ \ldots) \tag{31}$$

where m denotes the number of pericenter passages. Because of the dominance of the secular term in $(r\,\Delta f)_p$, Δr_p may be written as follows.

$$\Delta r_p = -\left(\frac{1+e}{1-e}\right)^{1/2} 3\pi m\ \Delta a \tag{32}$$

An expression for Δa in terms of the perturbations in r and V can be obtained from the energy expression $-\frac{\mu}{2a} = \frac{V^2}{2} - \frac{\mu}{r}$. Equation (33) results when the first variation is determined.

$$\Delta a = \frac{2a^2 V\ \Delta V}{\mu} + \frac{2a^2}{r^2}\ \Delta r \tag{33}$$

where

$$V\ \Delta V = \dot{x}\ \Delta\dot{x} + \dot{y}\ \Delta\dot{y} + \dot{z}\ \Delta\dot{z} \tag{34}$$

and

$$\Delta r = \frac{1}{r}(x\ \Delta x + y\ \Delta y + z\ \Delta z) \tag{35}$$

An expression for ΔV_p is obtained by solving equation (33) for ΔV and evaluating for Δr_p.

$$\Delta V_p = \frac{\mu}{2V}\left(\frac{\Delta a}{a^2} - \frac{2\ \Delta r_p}{r^2}\right) \tag{36}$$

When a substitution is made for Δr_p, the following equation results.

$$\Delta V_p = \frac{\mu}{2V}\left[\frac{\Delta a}{a^2} + \left(\frac{1+e}{1-e}\right)^{1/2}\frac{6\pi m\ \Delta a}{r^2}\right] \tag{37}$$

When the term $\Delta a/a^2$ is neglected, the following equation results.

$$\Delta V_p = \left(\frac{1 + e}{1 - e}\right)^{1/2} \frac{3\pi m \ \Delta a}{r^2 V} \tag{38}$$

The magnitudes of the position- and velocity-perturbation vectors at pericenter computed with equations (32) and (38) are 185 nautical miles and 775 ft/sec, respectively, which are in very good agreement with the results shown in figures 3 and 10.

CONCLUDING REMARKS

It has been shown by a numerical example that Brouwer's artificial-satellite theory can be used to generate transition matrices that agree closely with numerical integration of the variational equations for long periods of time. It has been demonstrated that inaccuracies in the mean motion cause the accuracy of a Keplerian transition matrix to degenerate rapidly. However, the accuracy of a secular model based on mean elements or of a complete model containing secular and periodic terms did not decrease for the time period considered. Although the complete model, of course, is more accurate than the secular model, using the complete model involves many more terms and requires more computer time. Hence, the secular model should be used whenever possible.

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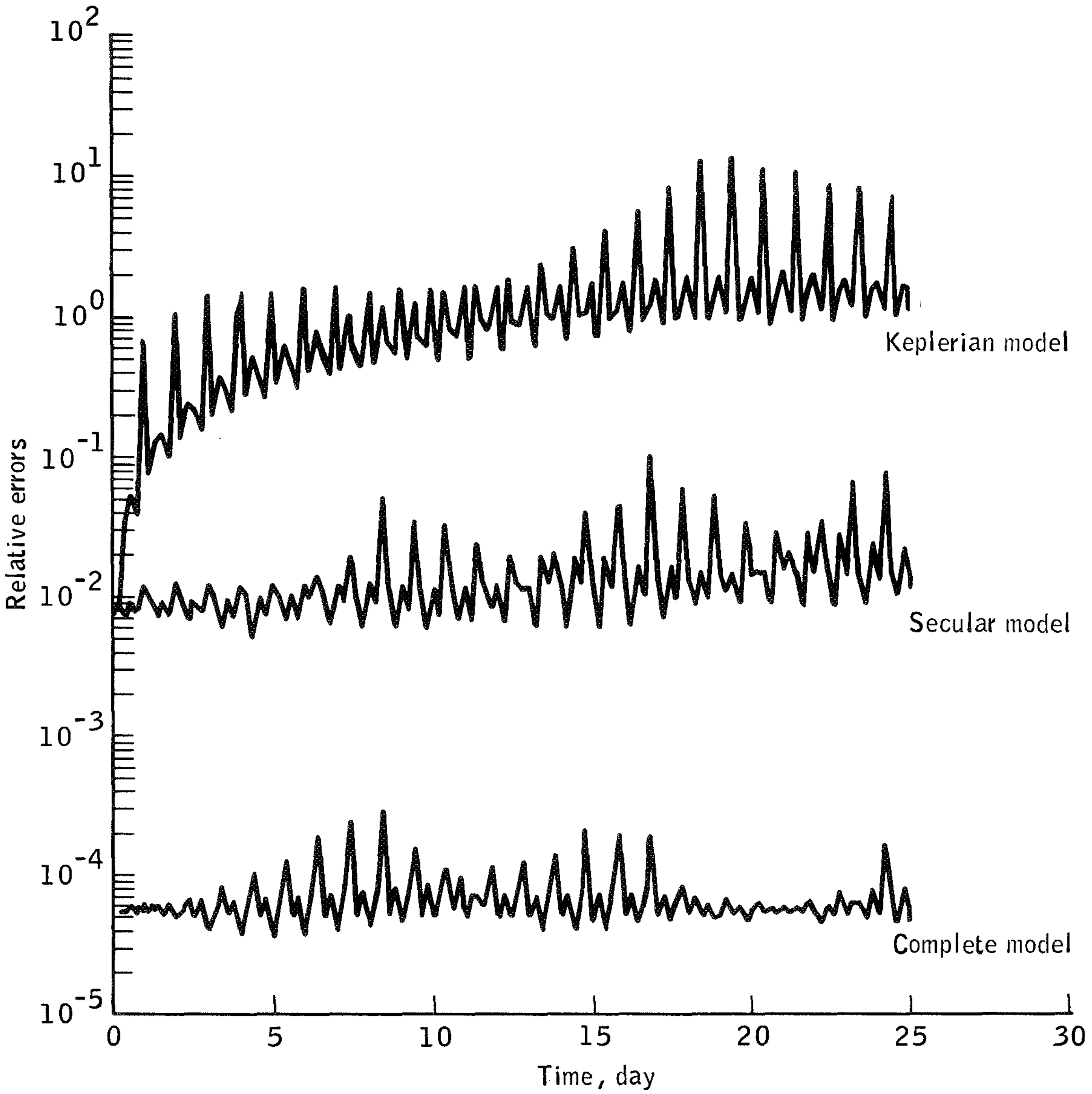

Figure 1. - Relative errors for Mars orbiter.

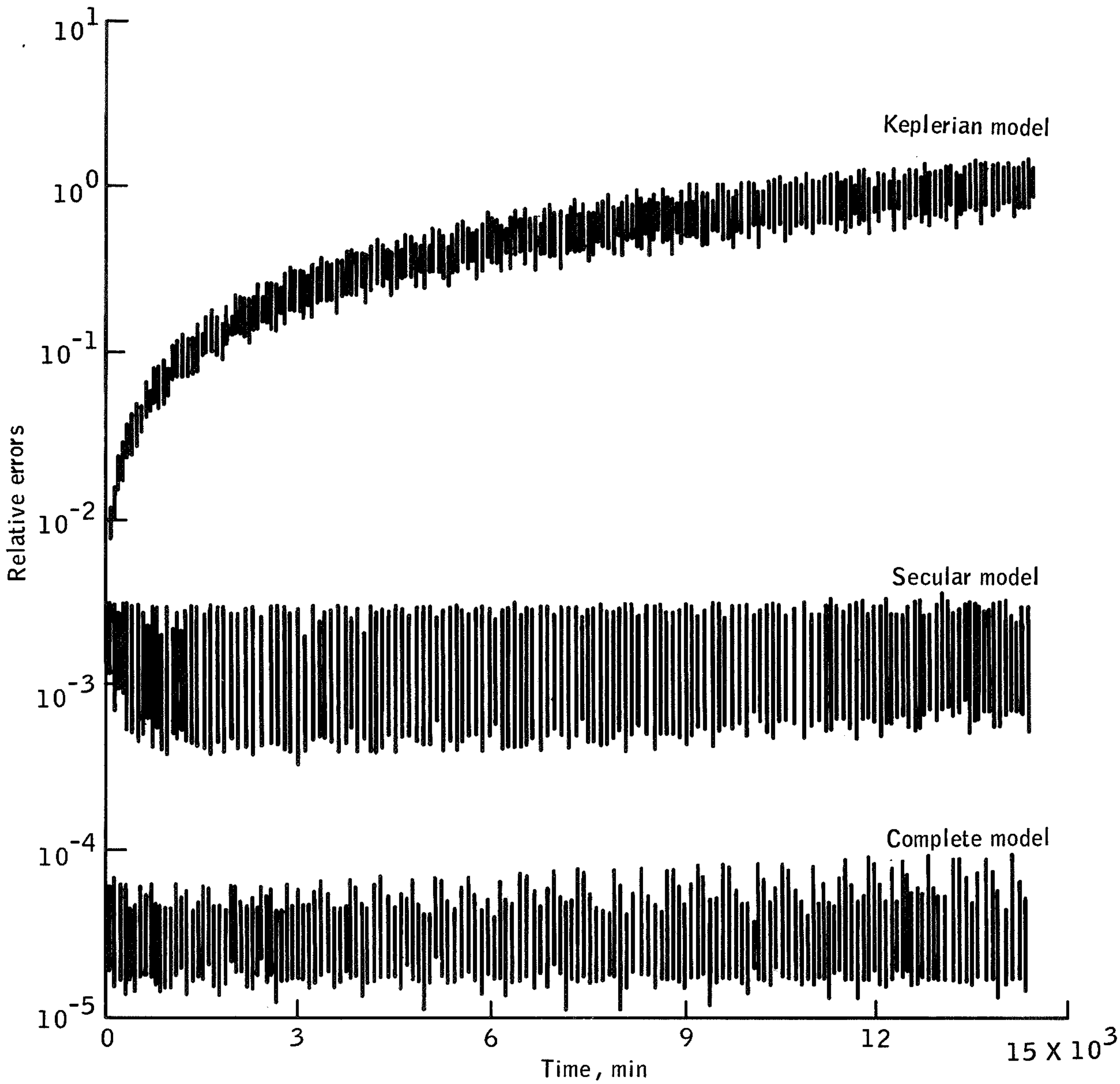

Figure 2. - Relative errors for Earth orbiter.

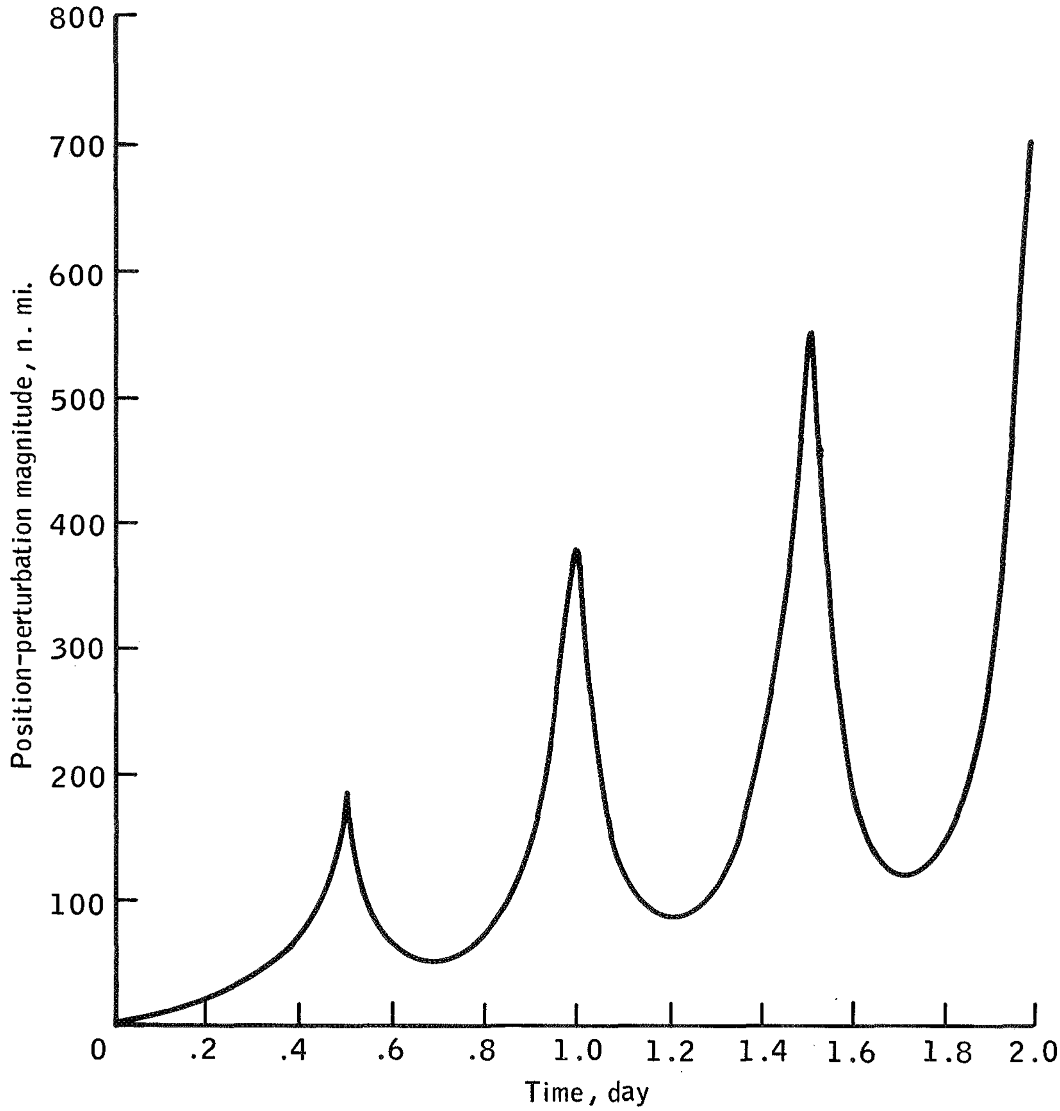

Figure 3. - Position-perturbation magnitude for Keplerian model, Mars orbiter.

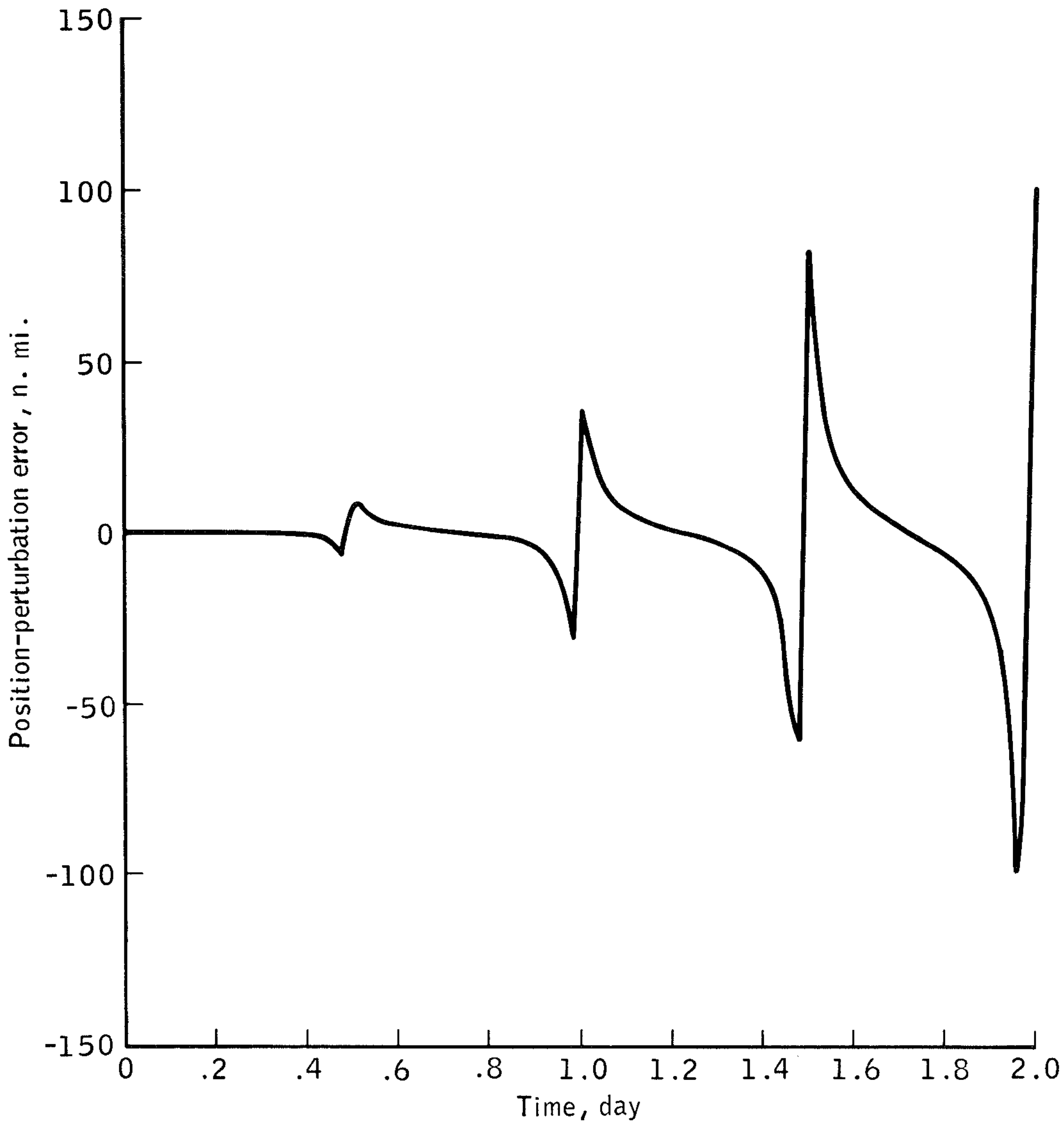

Figure 4. - Position-perturbation error for Keplerian model, Mars orbiter.

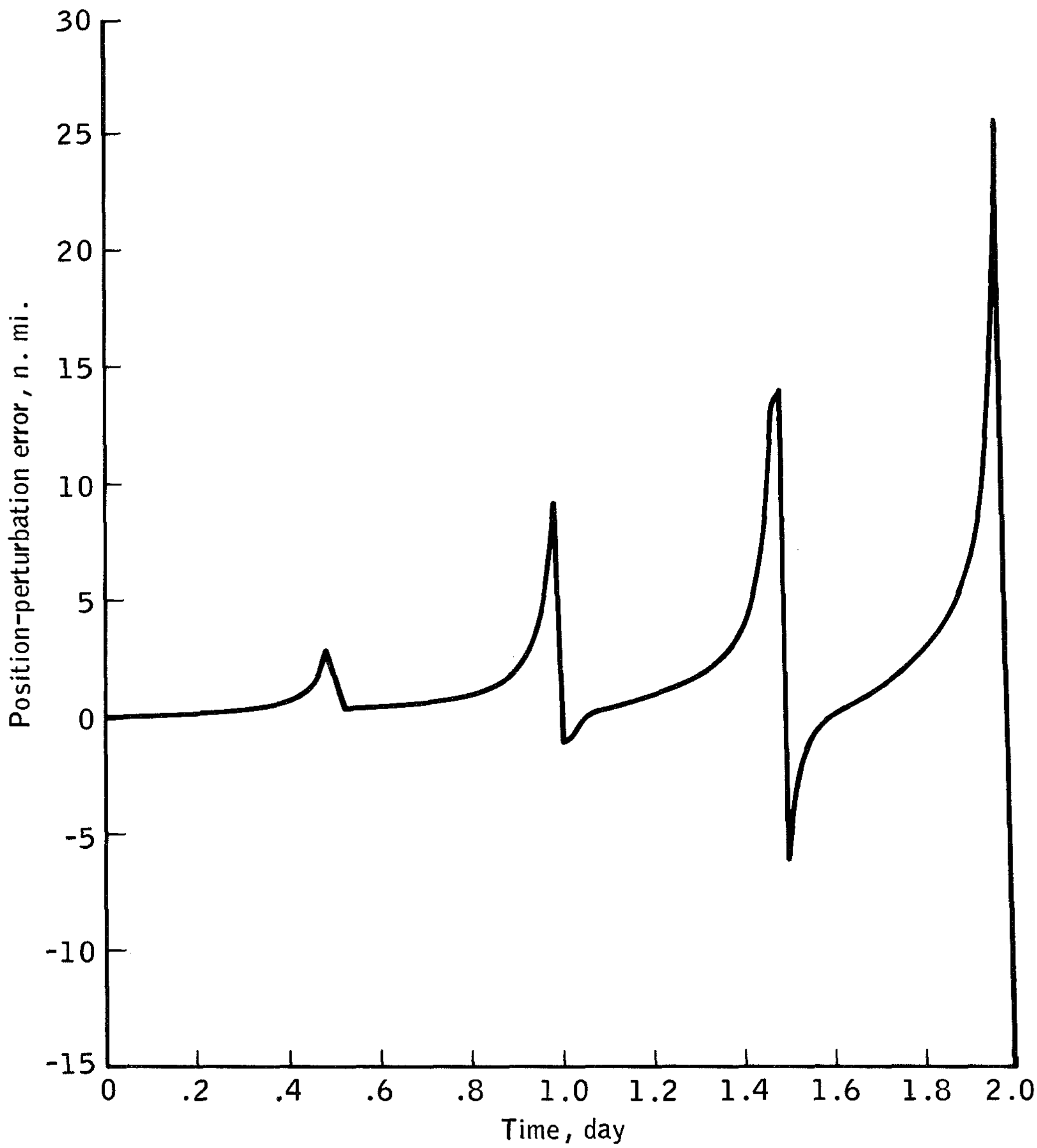

Figure 5. - Position-perturbation error for secular model, Mars orbiter.

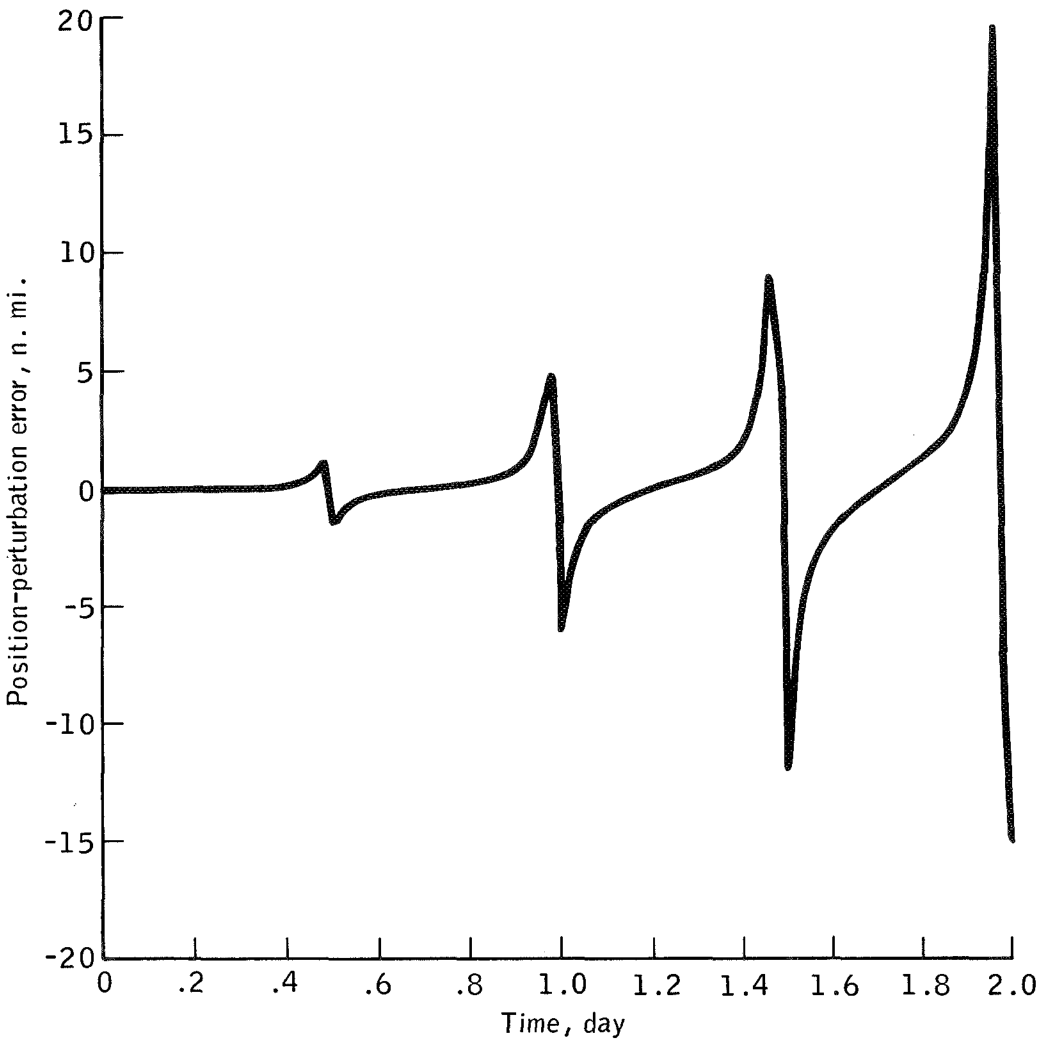

Figure 6. - Position-perturbation error for complete model, Mars orbiter.

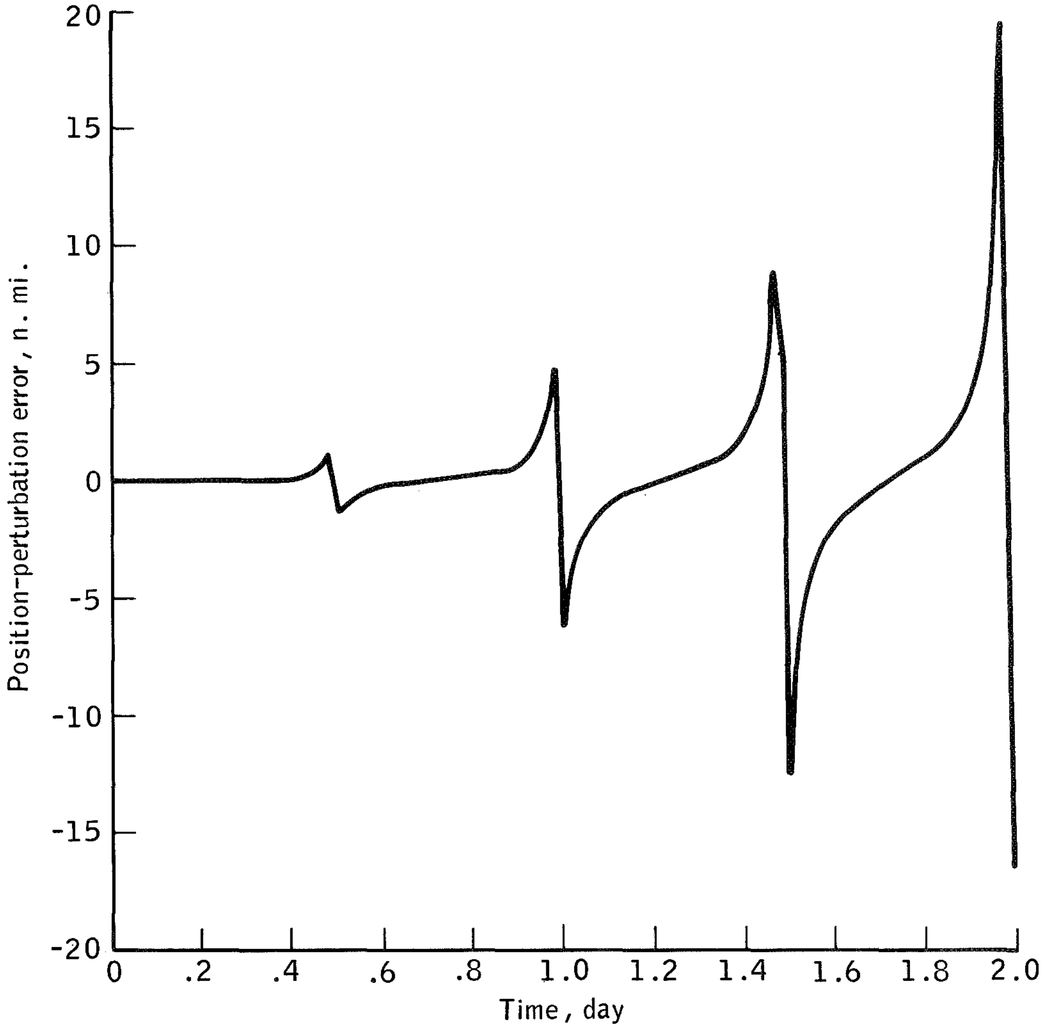

Figure 7. - Position-perturbation error for integrated model, Mars orbiter.

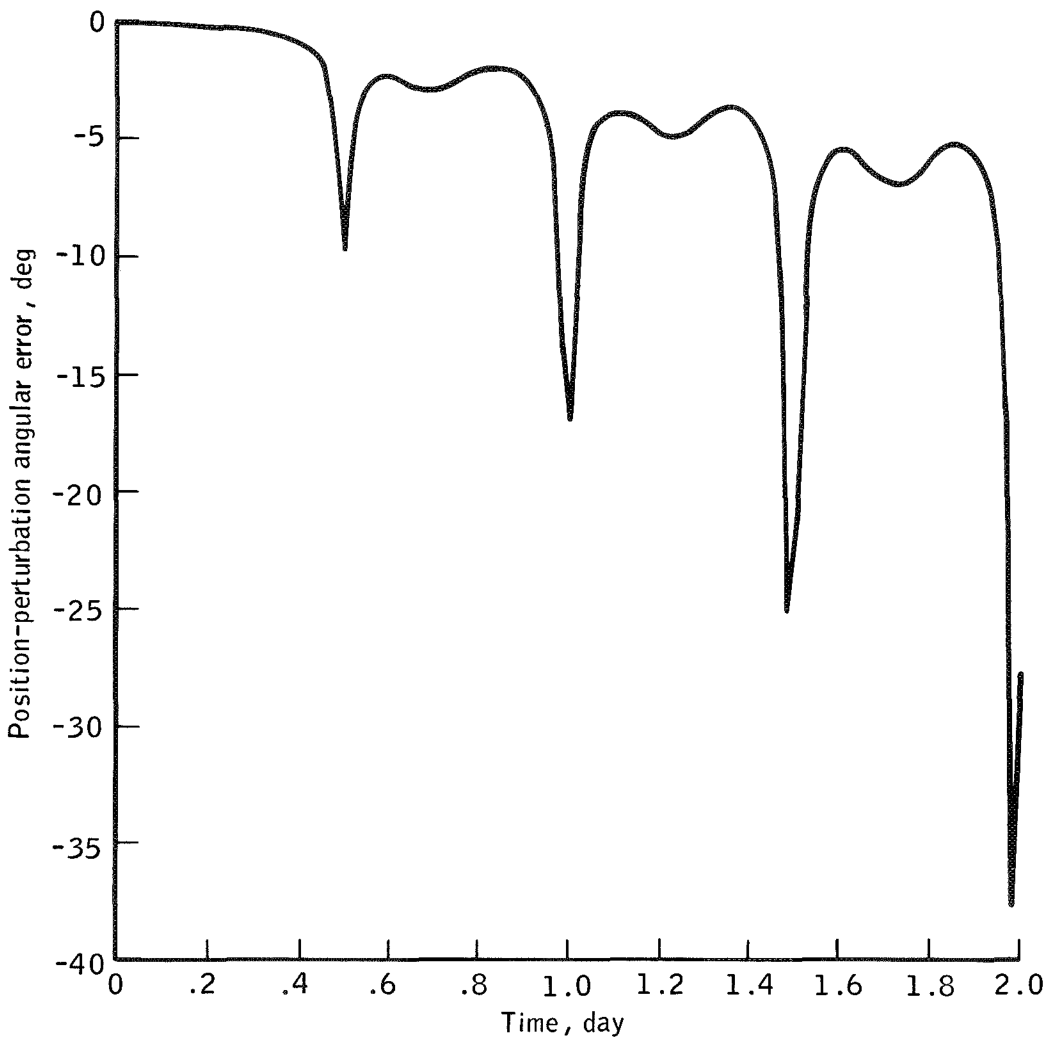

Figure 8. - Position-perturbation angular error for Keplerian model, Mars orbiter.

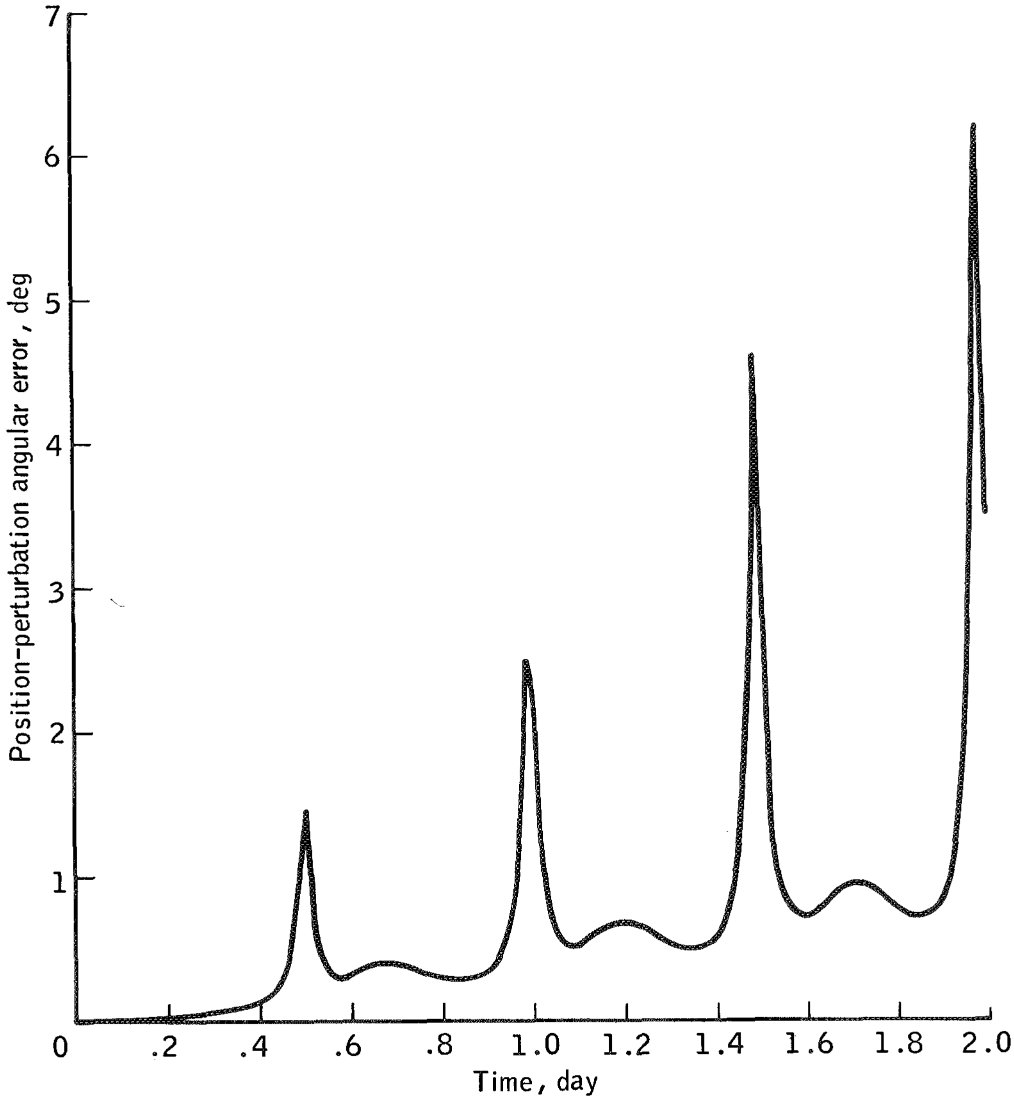

Figure 9. - Position-perturbation angular error for integrated model, Mars orbiter.

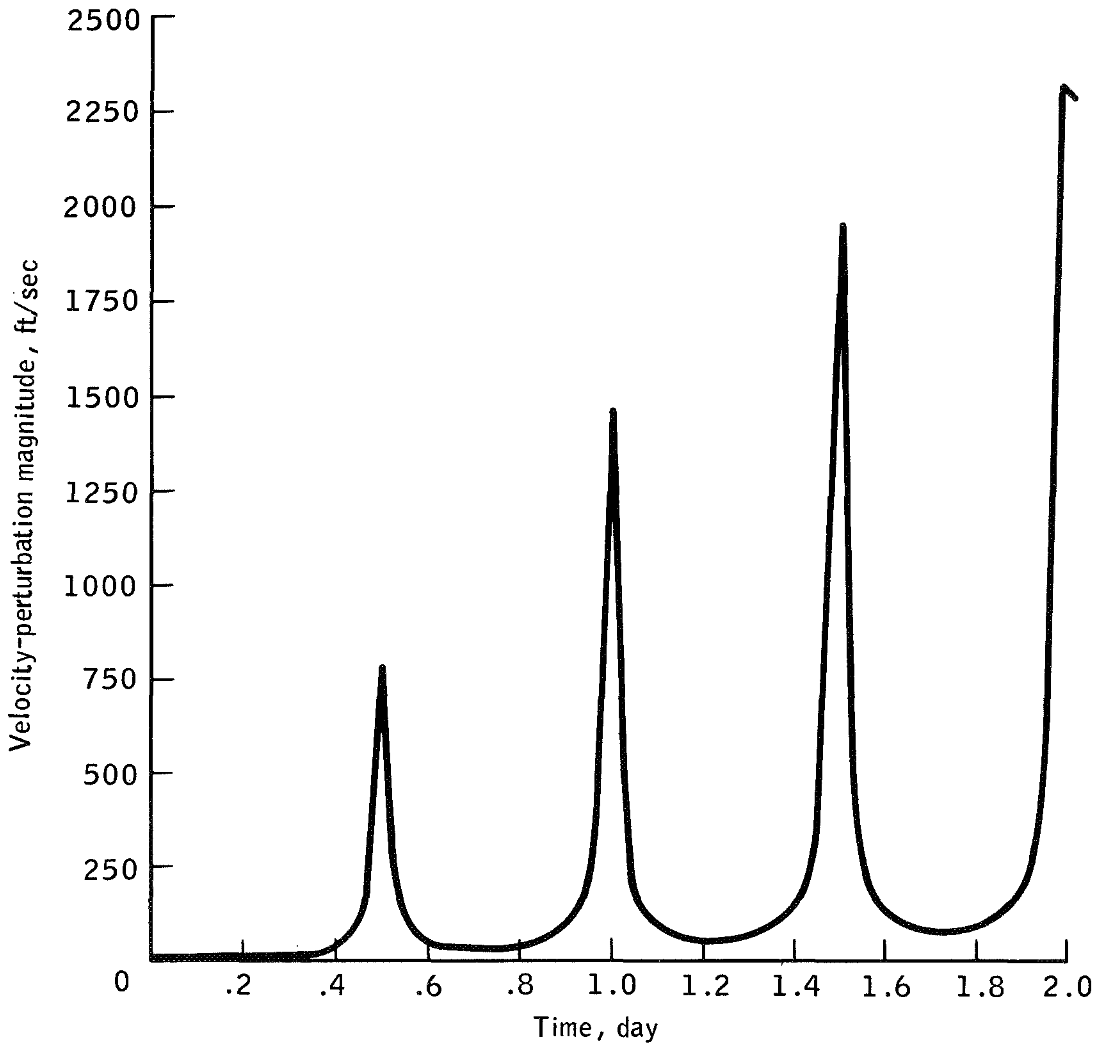

Figure 10. - Velocity-perturbation magnitude for Keplerian model, Mars orbiter.

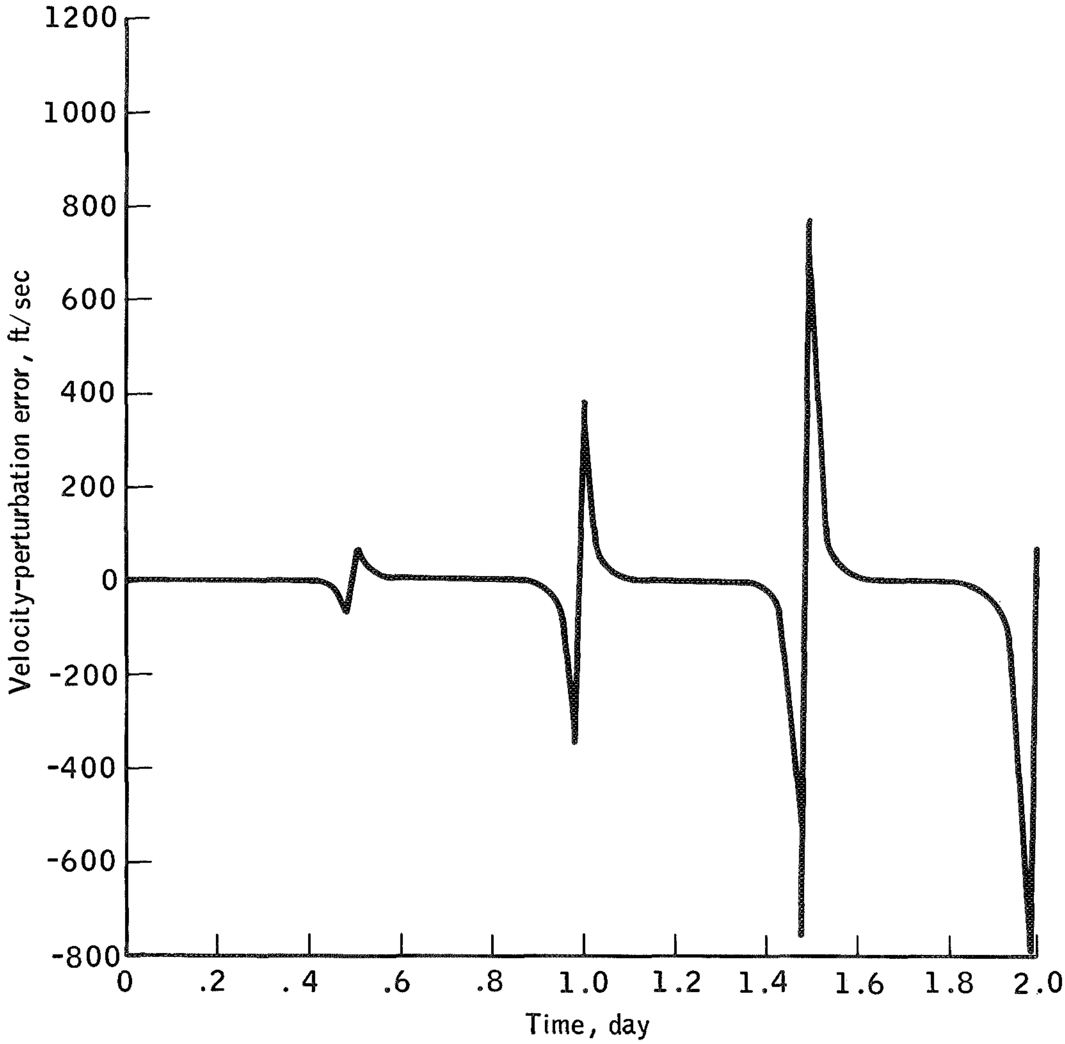

Figure 11. - Velocity-perturbation error for Keplerian model, Mars orbiter.

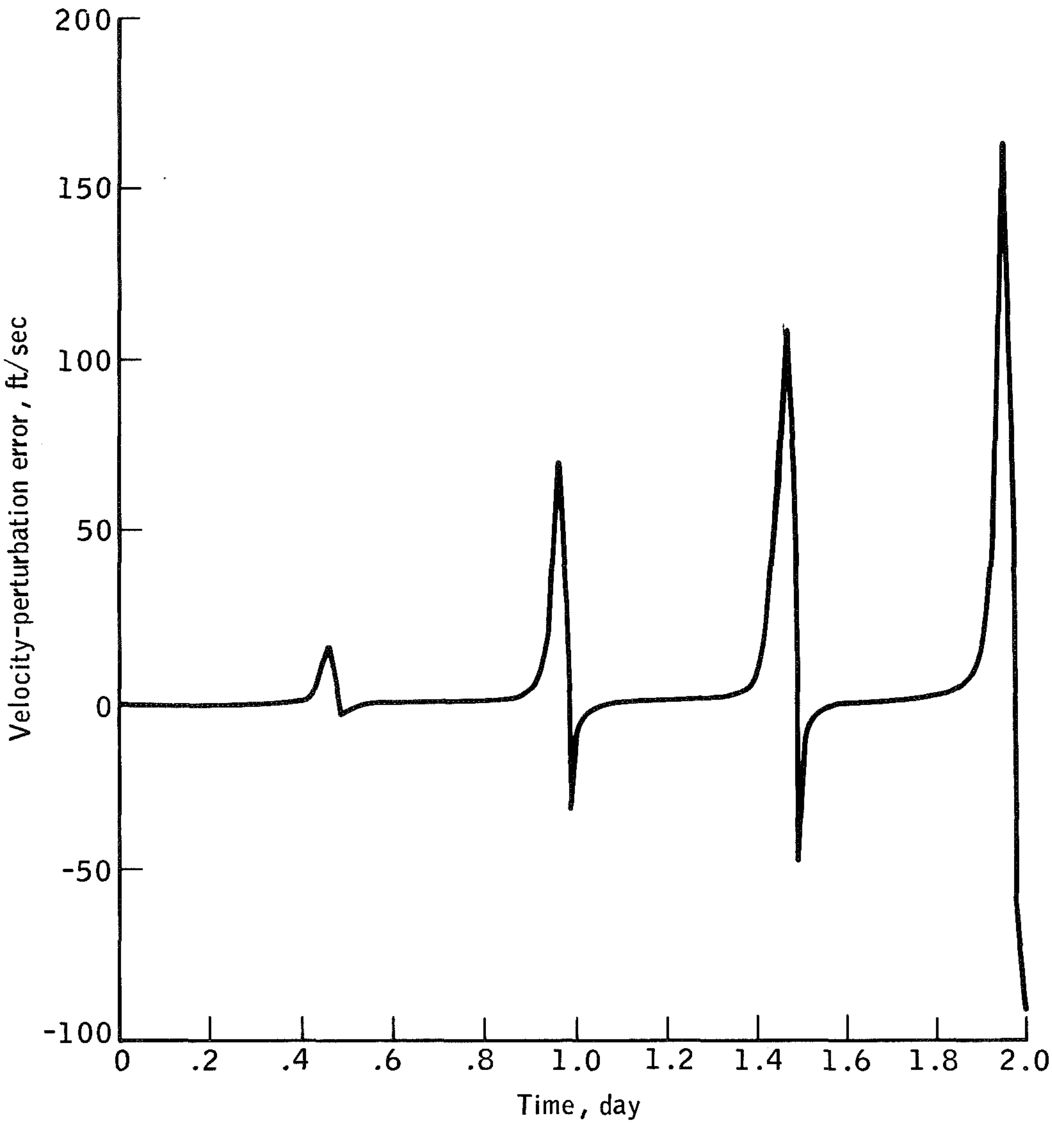

Figure 12. - Velocity-perturbation error for secular model, Mars orbiter.

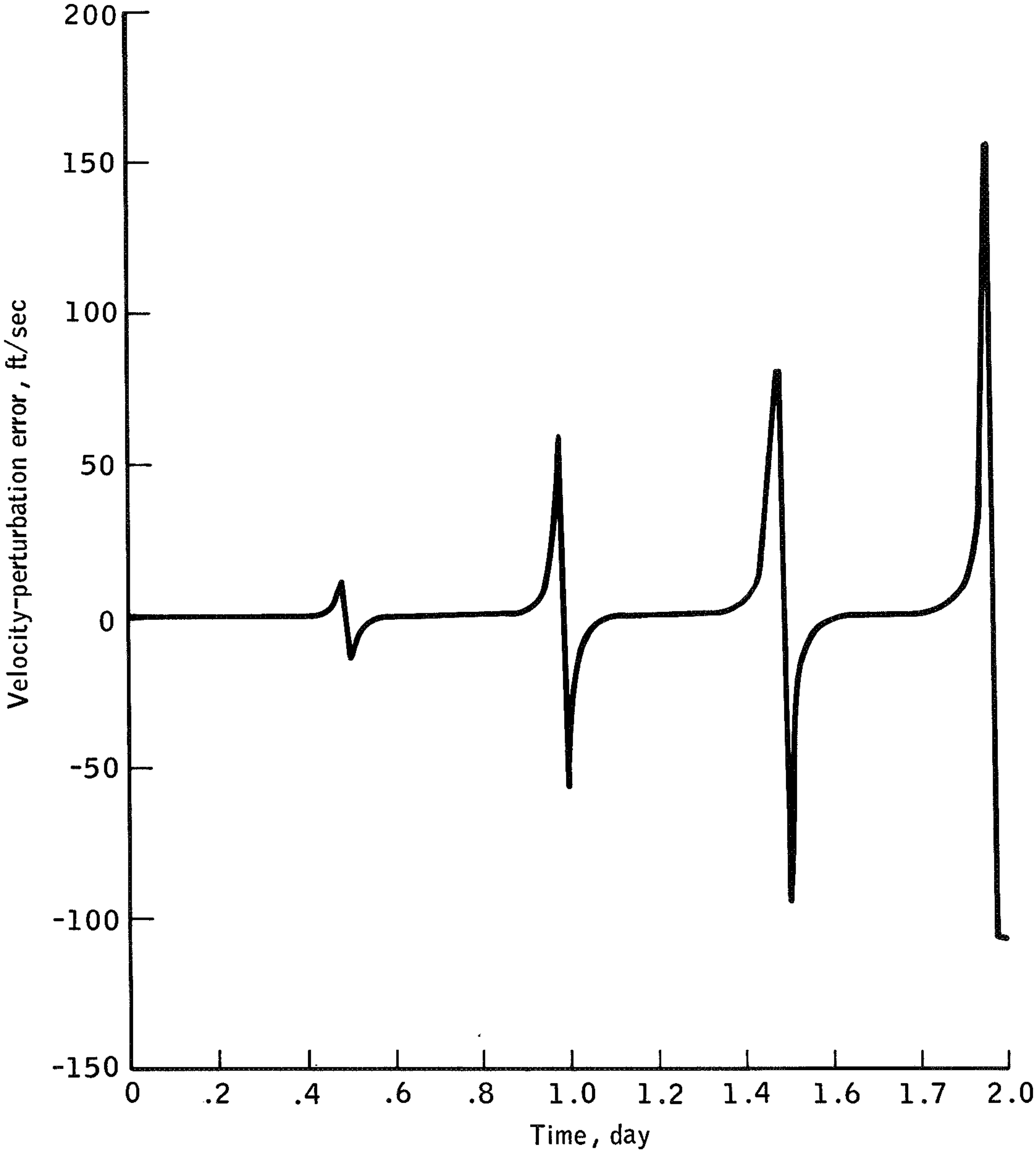

Figure 13. - Velocity-perturbation error for complete model, Mars orbiter.

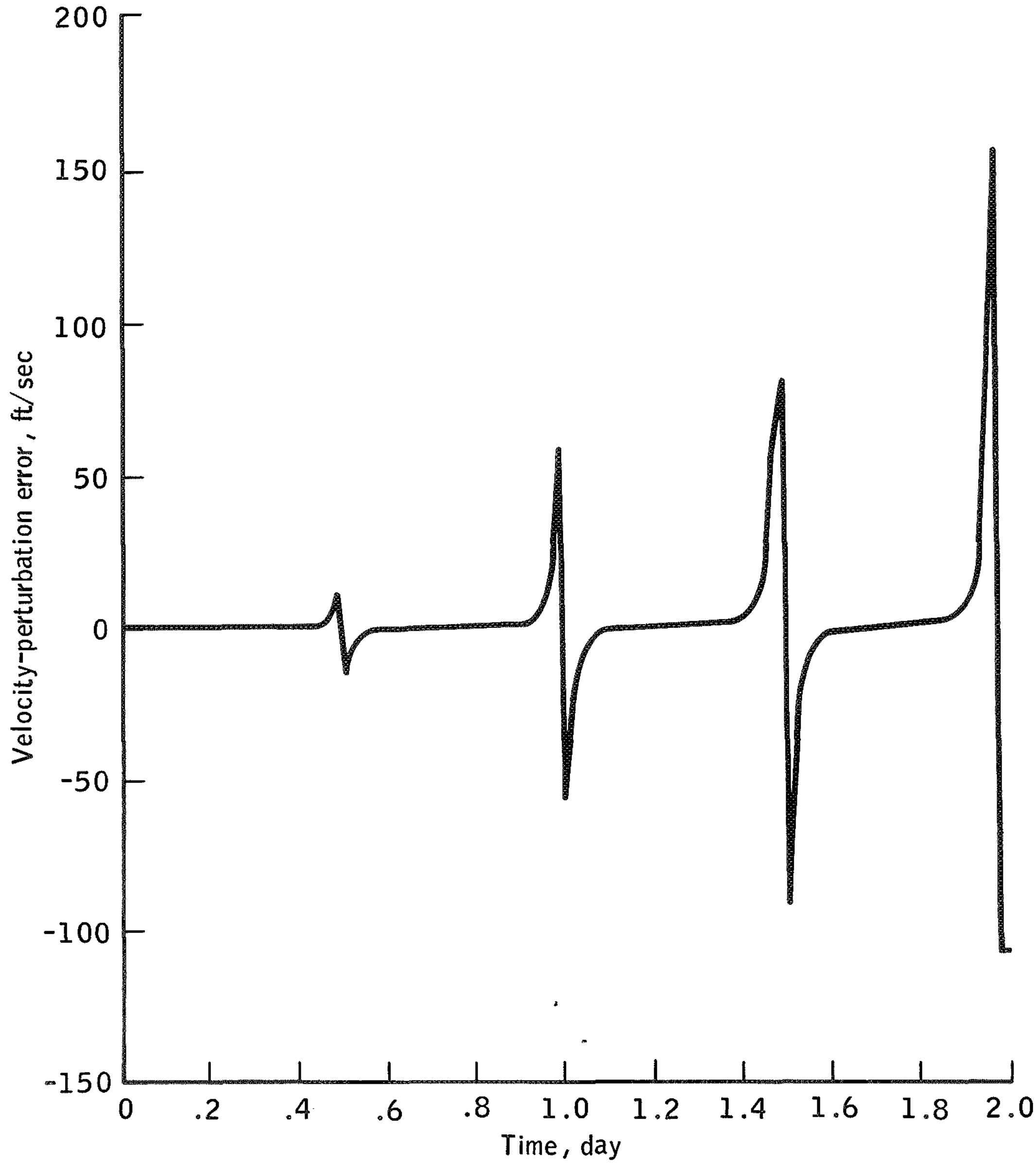

Figure 14. - Velocity-perturbation error for integrated model, Mars orbiter.

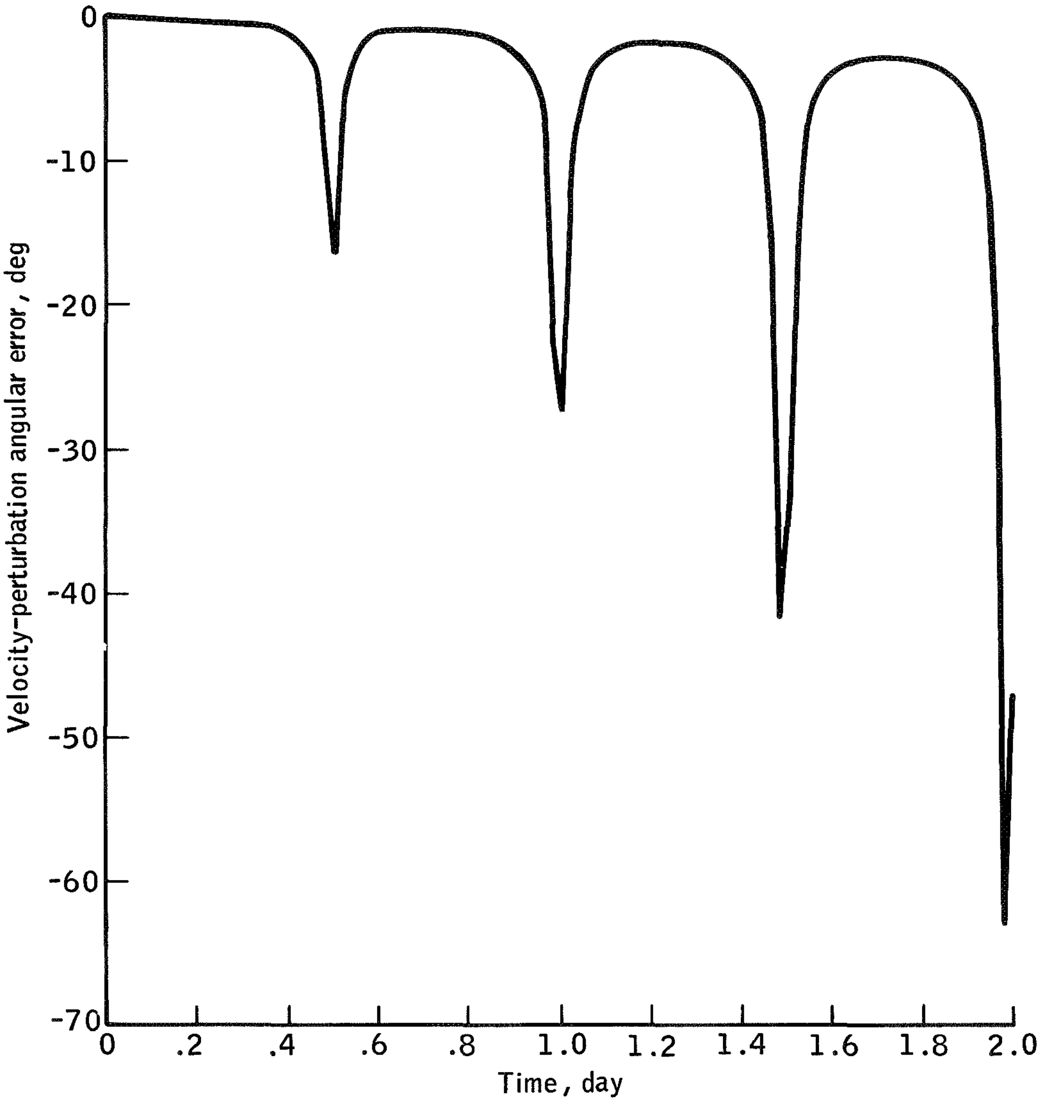

Figure 15. - Velocity-perturbation angular error for Keplerian model, Mars orbiter.

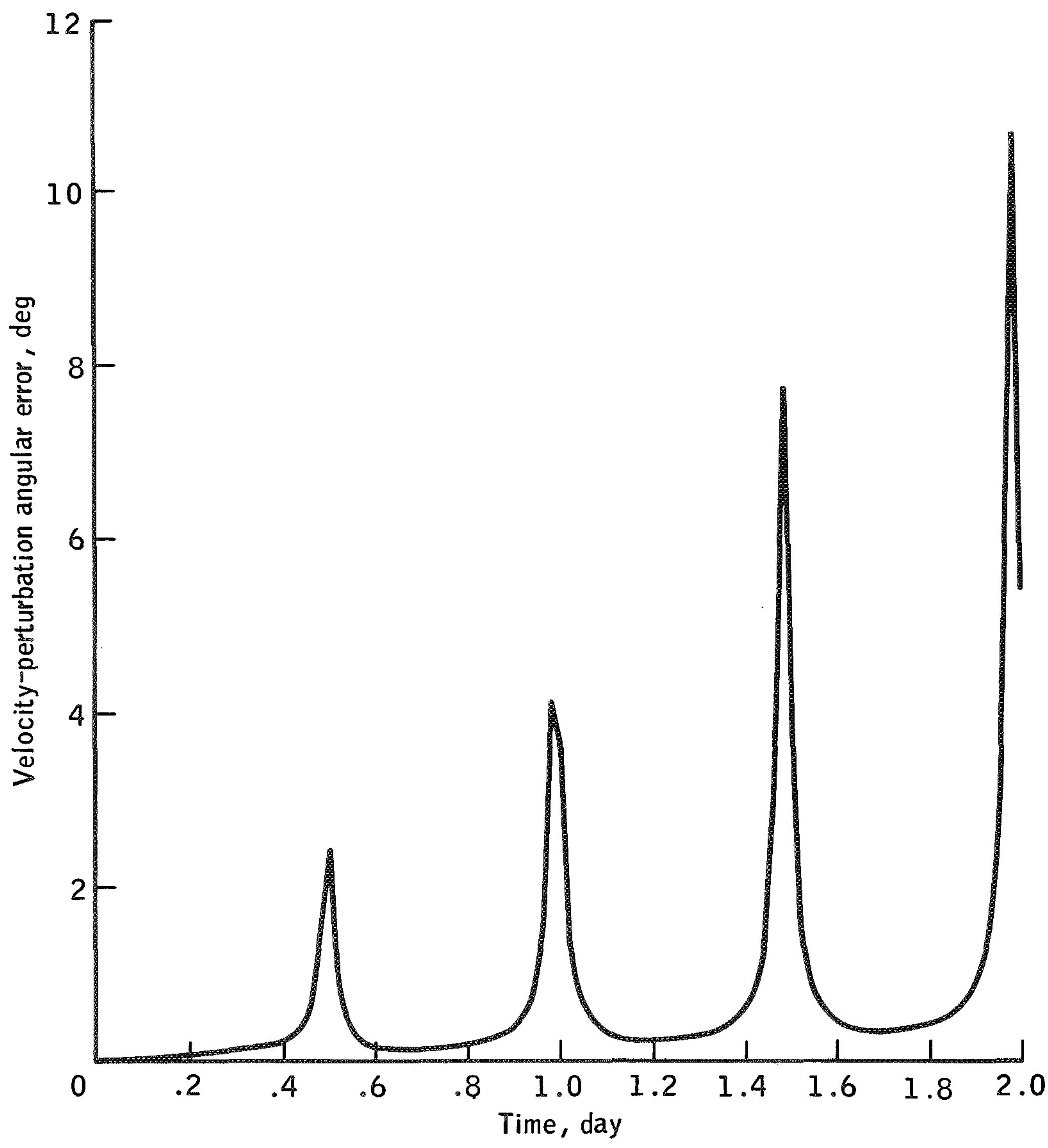

Figure 16. - Velocity-perturbation angular error for integrated model, Mars orbiter.

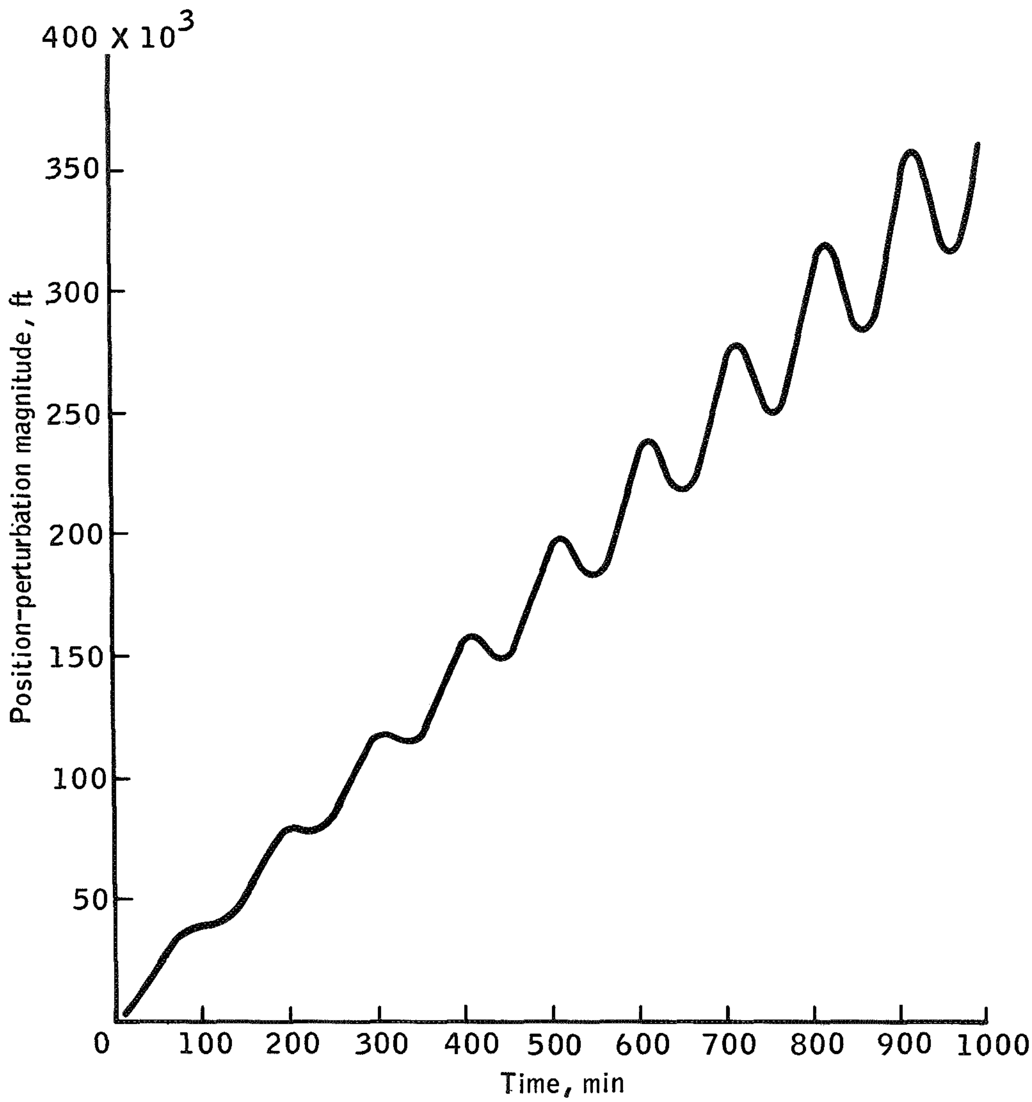

Figure 17. - Position-perturbation magnitude for Keplerian model, Earth orbiter.

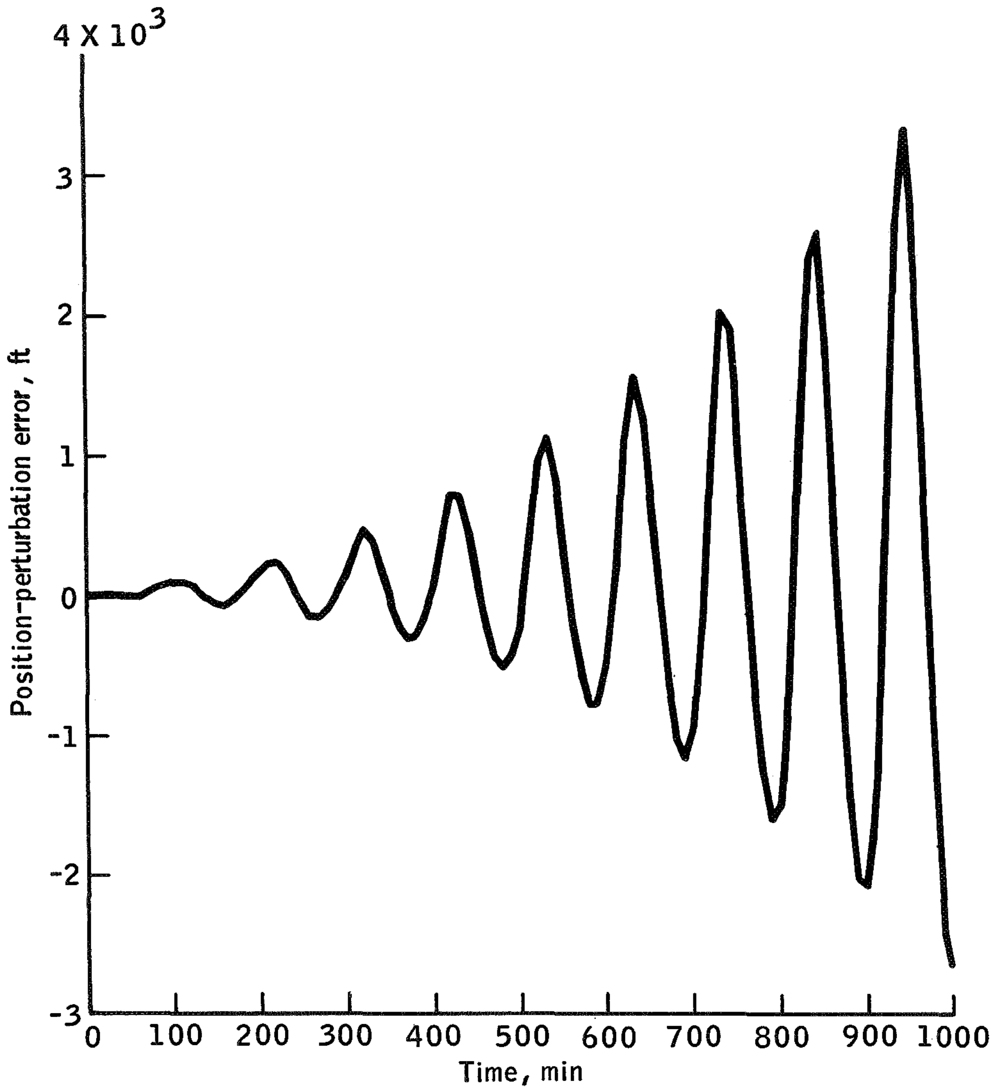

Figure 18. - Position-perturbation error for Keplerian model, Earth orbiter.

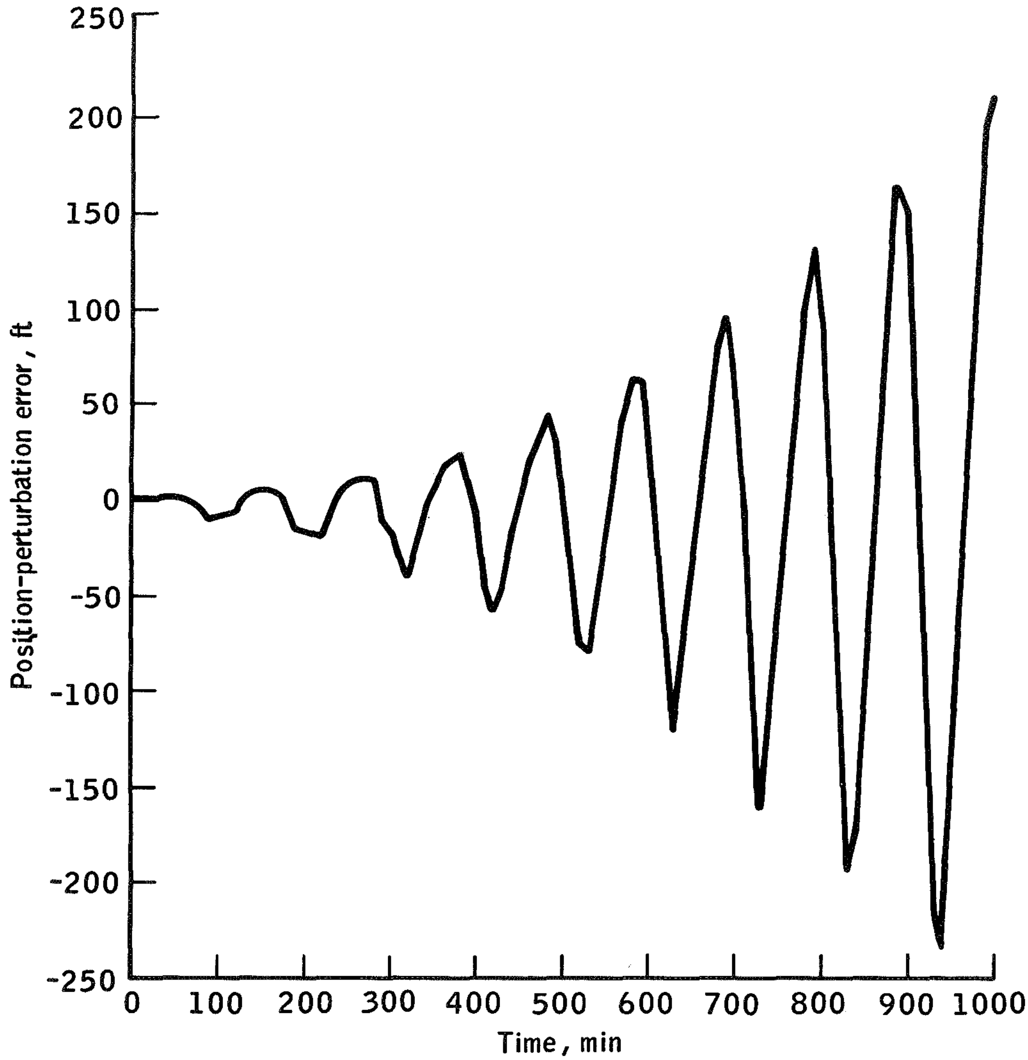

Figure 19. - Position-perturbation error for complete model, Earth orbiter.

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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