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A UNIVERSAL SOLUTION OF LAMBERT'S PROBLEM

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ABSTRACT

The problem of Lambert is reduced to the solution of a single equation which is valid for elliptic, parabolic, and hyperbolic motion. The equation has no singularity for 180° transfers, holds for any number of revolutions, and expresses a normalized time of flight as a function of one parameter and a universal two-body variable.

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I. INTRODUCTION

Consider a particle moving in a gravitational, inverse-square, central force field and let f and r be respectively the true anomaly and the distance of the particle from the center of attraction at time t. If subscripts 1 and 2 indicate values at times t_1 and t_2, Lambert's problem assumes that r_1, r_2, and $f_2 - f_1$ are known and seeks to find some parameter of the motion such as the semi-major axis a which will enable us to compute directly the velocity components at times t_1 and t_2, provided the plane of motion is known.

The problem of Lambert is reduced to the solution of two equations in two unknowns by Battin (1964) and Pitkin (1968). These equations are universally valid for the three types of conic motion. Battin (1964) reports the work of Deyst, wherein the problem is reduced to one universally valid equation but without provision for more than one revolution during the flight time $t_2 - t_1$. Pines (1965) has independently derived the equation of Deyst, with a modification which allows for any number of revolutions. The equation of Deyst and Pines expresses the time of flight as a function of two parameters and a universal variable.

In this paper Lambert's problem is reduced to a single universal equation which is simpler than the equation of Deyst and Pines. It expresses a normalized time of flight as a function of only one parameter and a universal variable.

We will first present separate equations for the elliptic and hyperbolic cases. We will then combine these equations into a single universal equation.

II. ELLIPTIC AND HYPERBOLIC CASES

For elliptic motion we have

$$r = a(1 - e\cos E), \tag{1}$$

$$(\mu/a^3)^{1/2}(t - t_p) = E - e\sin E, \tag{2}$$

$$r^{1/2}\sin\frac{1}{2}f = [a(1+e)]^{1/2}\sin\frac{1}{2}E\,, \tag{3}$$

$$r^{1/2}\cos\frac{1}{2}f = [a(1-e)]^{1/2}\cos\frac{1}{2}E\,, \tag{4}$$

where μ is a known constant, a is the semi-major axis, E is the eccentric anomaly, t_p is a time of periapsis passage, and e is the eccentricity of the orbit. We define

$$\alpha = \frac{1}{2}\left(E_2 - E_1\right) ,$$

$$\omega = e \cos \frac{1}{2} \left(E_1 + E_2\right) .$$

Place subscripts 1 and 2 on r and E in (1) and add the two resulting equations to obtain

$$\left(r_1 + r_2\right)/2a = 1 - \omega \cos \alpha . \tag{5}$$

Place subscripts 1 and 2 on t and E in (2) and subtract the resulting equations to obtain

$$\frac{1}{2} \left(\mu/a^3\right)^{1/2} \left(t_2 - t_1\right) = \alpha - \omega \sin \alpha . \tag{6}$$

Place subscripts 1 and 2 on r, f, and E in (3) and (4) and combine the resulting equations to obtain

$$\left(r_1 r_2\right)^{1/2} \cos \frac{1}{2} \left(f_2 - f_1\right) = a(\cos \alpha - \omega) . \tag{7}$$

If we define

$$T = (2\mu)^{1/2} \left(t_2 - t_1\right) / \left(r_1 + r_2\right)^{3/2} ,$$

$$q = \left[2 \left(r_1 r_2\right)^{1/2} / \left(r_1 + r_2\right)\right] \cos \frac{1}{2} \left(f_2 - f_1\right)$$

and eliminate a and ω from (5), (6), and (7), we get

$$T = (-1)^m (1 - q \cos \alpha)^{1/2} \; [q + (\alpha - \sin \alpha \cos \alpha) \, (1 - q \cos \alpha)/\sin^3 \alpha] \; , \qquad (8)$$

where m is the number of complete revolutions which occur during the time of flight. We note that

$$-1 \le q \le 1 \; .$$

An equation similar to (8) has been derived by Godal (1961) from formulas due to Gauss, but with the flight time expressed as a function of two parameters and the eccentric anomaly difference.

Eliminating ω from (5) and (7) gives

$$\left(r_1 + r_2\right) (1 - q \cos \alpha) = 2a \sin^2 \alpha \qquad (9)$$

for the calculation of a after α has been found from (8).

If q is near 1, it may be desirable to replace $1 - q \cos \alpha$ by $2 \sin^2 \frac{1}{2} \alpha + (1 - q) \cos \alpha$ in (8) and (9) for better numerical accuracy.

Kepler's equation for elliptic motion can be written in the form

$$\frac{1}{2} \left(\mu/a^3\right)^{1/2} \left(t_2 - t_1\right) = \alpha + \left[r_1 \dot{r}_1/(\mu a)^{1/2}\right] \sin^2 \alpha - \left(1 - r_1/a\right) \sin \alpha \cos \alpha, \qquad (10)$$

where $\dot{r}_1 = dr/dt$ at time t_1. Having found α and a, we can obtain $\dot{r}_1$ from (10).

In a similar manner we obtain for the case of hyperbolic motion

$$T = (1 - q \cosh \beta)^{1/2} \left[q - (\beta - \sinh \beta \cosh \beta)(1 - q \cosh \beta)/\sinh^3 \beta\right] , \qquad (11)$$

where q and T have the same definitions as before and

$$\beta = \frac{1}{2}\left(H_2 - H_1\right) ,$$

H being the usual hyperbolic parameter. The analogue of (9) is

$$\left(r_1 + r_2\right) (1 - q \cosh \beta) = -2a \sinh^2 \beta. \tag{12}$$

We note that we have adhered to the sign convention for the semi-major axis a, i.e., $a > 0$ for an ellipse and $a < 0$ for a hyperbola.

Kepler's equation for hyperbolic motion can be written in the form

$$\frac{1}{2}\left(\mu/-a^3\right)^{1/2}\left(t_2 - t_1\right) = -\beta + \left[\left(r_1\dot{r}_1\right/(-\mu a)^{1/2}\right]\sinh^2\beta + \left(1 - r_1/a\right)\sinh\beta\cosh\beta, \tag{13}$$

from which $\dot{r}_1$ can be obtained.

The following formulas for the eccentricity, semilatus rectum, and the component of velocity perpendicular to the t_1 radius vector are valid for all forms of two-body motion:

$$e^2 = \left(1 - r_1/a\right)^2 + \left(r_1\dot{r}_1\right)^2/\mu a, \tag{14}$$

$$p = 2r_1 - r_1^2/a - \left(r_1\dot{r}_1\right)^2/\mu, \tag{15}$$

$$u_1 = (\mu p)^{1/2}/r_1 . \tag{16}$$

III. THE UNIVERSAL FORM

We begin by defining two series

$$S(\gamma) = \sum_{i=0}^{\infty} (-1)^i\gamma^i/(2i + 3)!, \qquad C(\gamma) = \sum_{i=0}^{\infty} (-1)^i\gamma^i/(2i + 2)!, \tag{17}$$

convergent for all values of γ. With these series we can write

$$\sin\alpha = \alpha(1 - \gamma S), \qquad \cos\alpha = 1 - \gamma C \text{ if } \gamma = \alpha^2, \tag{18}$$

$$\sinh\beta = \beta(1 - \gamma S), \qquad \cosh\beta = 1 - \gamma C \text{ if } \gamma = -\beta^2. \tag{19}$$

Substituting these expressions into (8) and (11) gives

$$T = (-1)^m y^{1/2} \left[q + y\,(Cx + S)/x^3\right] \tag{20}$$

$$x = 1 - \gamma S, \quad y = 1 - q\,(1 - \gamma C) = \gamma C + (1 - q)(1 - \gamma C).$$

Equation (20) replaces (8) and (11) if we impose the convention that $\gamma > 0$ for elliptic motion and $\gamma < 0$ for hyperbolic motion. Equation (20) becomes truly universal if we agree that $\gamma = 0$ for parabolic motion. Of course $m = 0$ for hyperbolic and parabolic motions. γ is the universal two-body variable used in the universal solution of the two-body initial-value problem as described by Battin (1964).

Substituting (18) and (19) into (9) and (12) gives

$$\frac{1}{a} = 2\gamma x^2 z\,, \tag{21}$$

where

$$z = y/\left(r_1 + r_2\right).$$

Substituting (18) and (19) into (10) and (13) gives

$$r_1\dot{r}_1 = \mu\left(t_2 - t_1\right) z - (-1)^m\,(\mu/2z)^{1/2}\left[(Cx + S)/x^3 + 2r_1\,(1 - \gamma Cz)\right]. \tag{22}$$

The following identities are useful for computing the C and S functions for large values of γ:

$$2C(4\gamma) = [1 - \gamma S(\gamma)]^2,$$

$$4S(4\gamma) = S(\gamma) + C(\gamma)\,[1 - \gamma S(\gamma)]\,.$$

Since the C and S functions are evaluated so frequently, it is worthwhile to develop approximations that minimize the number of arithmetical operations required. Herron, et al. (1968) have developed several such approximations, based on theories of Chebyshev and Knuth, with bounds for the errors incurred when using them.

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