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# AN ALGORITHM FOR INTEGRATING SIMULTANEOUSLY THE RESTRICTED PROBLEM AND ITS FIRST VARIATIONS IN THIELE'S COORDINATES

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ABSTRACT

In the search for periodic orbits in the planar restricted problem of three-bodies, it is of interest to integrate simultaneously the orbital and variational equations in regularizing coordinates.[2] Estes and Lancaster[5] have presented an efficient algorithm for the integration of the equations of motion of the restricted problem in Thiele's variables and it is the purpose of this note to extend the algorithm to include the variational equations.

## AN ALGORITHM FOR INTEGRATING SIMULTANEOUSLY THE RESTRICTED PROBLEM AND ITS FIRST VARIATIONS IN THIELE'S COORDINATES

### EQUATIONS OF MOTION AND VARIATION

The Lagrangian for the planar restricted three-body problem in Thiele coordinates with the origin at the midpoint between the primaries and the x-axis passing through both primaries while rotating with the mean motion may be written

$$\mathcal{L} = \frac{1}{2}(u'^2 + v'^2) - \frac{1}{8}(u' \sinh 2v + v' \sin 2u) + \Omega \tag{1}$$

where

$$\Omega = \frac{1}{2}\left(\cosh v - [2\mu - 1] \cos u\right) + \frac{4t'}{8}\left(\frac{2\mu - 1}{2} \cos u \cosh v - J + \frac{1}{4}\left[\cosh^2 v + \cos^2 u\right]\right)$$

$$t' = \frac{1}{4}(\cosh^2 v - \cos^2 u) = r_1 r_2$$

$$r_1 = \frac{1}{2}(\cosh v - \cos u)$$

$$r_2 = \frac{1}{2}(\cosh v + \cos u)$$

$$u'^2 + v'^2 = 2\Omega = r_1 r_2 \left((1 - \mu) r_1^2 + \mu r_2^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - J\right)$$

and where $\mu$ is the ratio of the mass of the smaller primary (assumed to be to the left of the origin) to the sum of the two primary masses. Here prime accents indicate differentiation with respect to a regularizing pseudo-time $\tau$, $r_1$ and $r_2$ represent the distances between the primaries and the infinitesimal third body and $J$ represents the Jacobi constant of motion. The variables $u$ and $v$ are related to the Cartesian coordinates by the transformation

$$x = \frac{1}{2} \cos u \cosh v$$

$$y = -\frac{1}{2} \sin u \sinh v . \tag{2}$$

The Lagrangian equations of motion become

$$u'' - 2t'v' = \frac{\partial \Omega}{\partial u}$$

$$v'' + 2t'u' = \frac{\partial \Omega}{2 v} \tag{3}$$

where

$$\frac{\partial \Omega}{\partial u} = \frac{\sin u}{16} \left[8(2\mu - 1) - 4J \cos u + (2\mu - 1) \cosh v (3\cos^2 u - \cosh^2 v) + 2\cos^3 u\right]$$

$$\frac{\partial \Omega}{\partial v} = \frac{\sinh v}{16} \left[8 - 4J \cosh v + (2\mu - 1) \cos u (3\cosh^2 v - \cos^2 u) + 2\cosh^3 v\right].$$

The solution of Equations (3) depends upon initial conditions and the parameters $\mu$ and J. Thus

$$u = u(\tau;\ u_0,\ v_0,\ u_0',\ v_0',\ \mu,\ J)$$

$$v = v(\tau;\ u_0,\ v_o,\ u_0',\ v_0',\ \mu,\ J) .$$

)enoting $u_\epsilon = \frac{\partial u}{\partial \epsilon}$, $v_\epsilon = \frac{\partial v}{\partial \epsilon}$, etc., where $\epsilon$ represents one of the parameters or nitial conditions, the first variational equations of the system (3) become

$$!u_\epsilon'' - 4t' v_\epsilon' = u_\epsilon \left\{ \cos u \left[(2\mu - 1) - \frac{J}{2} \cos u + (2\mu - 1) \cosh v (3\cos^2 u - \cosh^2 v)\right.\right.$$

$$+ 2\cos^3 u + 2\sin u\, v' + \frac{3(2\mu - 1)}{4} \cosh v\,(\cos^2 u - 1)$$

$$+ \frac{3}{4}\cos^3 u\Big] + \frac{1}{2}J - \left(\frac{3 + 2J}{4}\right)\cos^2 u\Big\}$$

$$+ v_\epsilon \left\{\sinh v\left[2v' \cosh v - \frac{3(2\mu - 1)}{8}\sin u\,(\cosh^2 v - \cos^2 u)\right]\right\}$$

$$+ \mu_\epsilon \left\{2\sin u + \frac{\sin u \cosh v}{4}(3\cos^2 u - \cosh^2 v)\right\}$$

$$- \frac{1}{2}J_\epsilon \sin u \cos u$$

$$2v''_\epsilon + 4t'\, u'_\epsilon = v_\epsilon \Big\{\cosh v\Big[1 - \frac{\tau}{2}\cosh v + (2\mu - 1)\cos u\,(3\cosh^2 v - \cos^2 u)$$

$$+ 2\cosh^3 v - 2\sinh v\, u' + \frac{3(2\mu - 1)}{4}\cos u\,(\cosh^2 v - 1)$$

$$+ \frac{3}{4}\cosh^3 v\Big] + \frac{1}{2}J - \left(\frac{3 + 2J}{4}\right)\cosh^2 v\Big\}$$

$$+ u_\epsilon \left\{\sin u\left[- 2u' \cos u - \frac{3(2\mu - 1)}{8}\sinh v\,(\cosh^2 v - \cos^2 u)\right]\right\}$$

$$+ \mu_\epsilon \left\{\frac{\sin u \sinh v}{4}(3\cosh^2 v - \cos^2 u)\right\}$$

$$- \frac{1}{2}J_\epsilon \sinh v \cosh v \qquad (4)$$

## POWER SERIES SOLUTION

We now extend the notation introduced in Reference [4]. Let

$$k_1 = 2\mu - 1, \; k_2 = -\frac{J}{2}, \; k_3 = -\frac{k_1}{8}, \; k_4 = -3k_3 \tag{5}$$

$$a = \sin u, \; b = \cos u, \; c = \sinh v, \; d = \cosh v \tag{6}$$

$$p = b^2, \; q = d^2, \; w = q - p \tag{7}$$

$$r = k_4 d + b/4, \; s = k_4 b + d/4 \tag{8}$$

$$f = k_1 + k_2 b + rp + k_3 dq \tag{9}$$

$$g = 1 + k_2 d + sq + k_3 bp \tag{10}$$

$$u' = \alpha, \; v' = \beta, \; u'_\epsilon = \alpha_\epsilon, \; v'_\epsilon = \beta_\epsilon \tag{11}$$

$$\gamma = f + 2a\beta + \frac{3}{4} bp + 2k_4 d(p - 1) \tag{12}$$

$$\lambda = 2\beta d - k_4 aw \tag{13}$$

$$\rho = g - 2c\alpha + \frac{3}{4} dq + 2k_4 b(q - 1) \tag{14}$$

$$\sigma = -2b\alpha - k_4 cw \tag{15}$$

$$h = d(3p - q), \; z = b(3q - p) \tag{16}$$

$$\Gamma = -k_2 - \left(\frac{3}{4} - k_2\right) p + b\gamma \tag{17}$$

$$\Lambda = c\lambda, \; S = a\sigma \tag{18}$$

$$R = -k_2 - \left(\frac{3}{4} - k_2\right) q + d\rho \tag{19}$$

$$H = a\left(2 + \frac{h}{4}\right), \; Z = \frac{c}{4} z \tag{20}$$

Equations (3) and (4) then become

$$2\alpha' = a f + w\beta \tag{21}$$

$$2\beta' = c g - w\alpha \tag{22}$$

$$2\alpha_\epsilon' = u_\epsilon \Gamma + v_\epsilon \Lambda + w\beta_\epsilon + \mu_\epsilon H - \frac{1}{2} J_\epsilon a b \tag{23}$$

$$2\beta_\epsilon' = v_\epsilon R + u_\epsilon S - w\alpha_\epsilon + \mu_\epsilon Z - \frac{1}{2} J_\epsilon c d . \tag{24}$$

Assume the solutions of (7) - (24) can be represented by the power series

$$u = \sum_{i=0}^{\infty} u_i \tau^i, \quad v = \sum_{i=0}^{\infty} v_i \tau^i, \quad t = \sum_{i=0}^{\infty} t_i \tau^i$$

$$u_\epsilon = \sum_{i=0}^{\infty} u_{\epsilon_i} \tau^i, \quad v_\epsilon = \sum_{i=0}^{\infty} v_{\epsilon_i} \tau^i$$

and similar series for a, b, c, d, p, q, w, r, s, f, g, $\gamma$, $\lambda$, $\rho$, $\sigma$, h, z, $\Gamma$, $\Lambda$, R , S, H, and Z in a neighborhood of $\tau = 0$. Substituting these series into (7) - (24) and equating the coefficients of each power of $\tau$ gives, for $i \geq 0$

$$p_i = \sum_{j=0}^{i} b_j b_{i-j}, \quad q_i = \sum_{j=0}^{i} d_j d_{i-j}, \quad w_i = q_i - p_i \tag{25}$$

$$r_i = k_4 d_i + b_i/4, \quad s_i = k_4 b_i + d_i/4 \tag{26}$$

$$f_i = k_1 \delta_{i0} + k_2 b_i + \sum_{j=0}^{i} r_j p_{i-j} + k_3 \sum_{j=0}^{i} d_j q_{i-j} \tag{27}$$

$$g_i = \delta_{i0} + k_2 d_i + \sum_{j=0}^{i} s_j q_{i-j} + k_3 \sum_{j=0}^{i} b_j p_{i-j} \tag{28}$$

$$2(i+1)\alpha_{i+1} = \sum_{j=0}^{i} (a_j f_{i-j} + w_j \beta_{i-j}) \tag{29}$$

$$2(i+1)\beta_{i+1} = \sum_{j=0}^{i} (c_j g_{i-j} - w_j \alpha_{i-j}) \tag{30}$$

$$(i+2)u_{i+2} = \alpha_{i+1}, \quad (i+2)v_{i+2} = \beta_{i+1}, \quad 4(i+1)t_{i+1} = w_i \tag{31}$$

$$\gamma_i = f_i - 2k_4 d_i + 2\sum_{j=0}^{i} a_j \beta_{i-j} + \frac{3}{4}\sum_{j=0}^{i} b_j p_{i-j} + 2k_4 \sum_{j=0}^{i} d_j p_{i-j} \tag{32}$$

$$\rho_i = g_i - 2k_4 b_i - 2\sum_{j=0}^{i} b_j \alpha_{i-j} + \frac{3}{4}\sum_{j=0}^{i} d_j q_{i-j} + 2k_4 \sum_{j=0}^{i} b_j q_{i-j} \tag{33}$$

$$\lambda_i = 2\sum_{j=0}^{i} d_j \beta_{i-j} - k_4 \sum_{j=0}^{i} w_j a_{i-j}, \quad \sigma_i = -2\sum_{j=0}^{i} b_j \alpha_{i-j} - k_4 \sum_{j=0}^{i} w_j c_{i-j} \tag{34}$$

$$h_i = 3\sum_{j=0}^{i} d_j p_{i-j} - \sum_{j=0}^{i} d_j q_{i-j}, \quad z_i = 3\sum_{j=0}^{i} b_j q_{i-j} - \sum_{j=0}^{i} b_j p_{i-j} \tag{35}$$

$$\Gamma_i = -k_2 \delta_{i0} - \left(\frac{3}{4} - k_2\right) p_i + \sum_{j=0}^{i} b_j \gamma_{i-j} \tag{36}$$

$$R_i = -k_2 \delta_{i0} - \left(\frac{3}{4} - k_2\right) q_i + \sum_{j=0}^{i} d_j \rho_{i-j} \tag{37}$$

$$\Lambda_i = \sum_{j=0}^{i} c_j \lambda_{i-j}, \quad S_i = \sum_{j=0}^{i} a_j \sigma_{i-j} \tag{38}$$

$$H_i = 2a_i + \frac{1}{4}\sum_{j=0}^{i} a_j h_{i-j}, \quad Z_i = \frac{1}{4}\sum_{j=0}^{i} c_j z_{i-j} \tag{39}$$

$$2(i+1)\alpha_{\epsilon_{i+1}} = \mu_\epsilon H_i + \sum_{j=0}^{i}\left[\beta_{\epsilon_j} w_{i-j} + u_{\epsilon_j}\Gamma_{i-j} + v_{\epsilon_j}\Lambda_{i-j}\right] - \frac{1}{2}J_\epsilon \sum_{j=0}^{i} a_j b_{i-j} \tag{40}$$

$$2(i+1)\beta_{\epsilon_{i+1}} = \mu_\epsilon Z_i + \sum_{j=0}^{i}\left[v_{\epsilon_j} R_{i-j} - \alpha_{\epsilon_j} w_{i-j} + u_{\epsilon_j} S_{i-j}\right] - \frac{1}{2}J_\epsilon \sum_{j=0}^{i} c_j d_{i-j} \tag{41}$$

$$(i+2)\,u_{\epsilon_{i+2}} = \alpha_{\epsilon_{i+1}}, \quad (i+2)\,v_{\epsilon_{i+2}} = \beta_{\epsilon_{i+1}}. \tag{42}$$

$$(i+1)\,a_{i+1} = \sum_{j=0}^{i} b_j \alpha_{i-j}, \quad (i+1)\,b_{i+1} = -\sum_{j=0}^{i} a_j \alpha_{i-j} \tag{43}$$

$$(i+1)\,c_{i+1} = \sum_{j=0}^{i} d_j \beta_{i-j}, \quad (i+1)\,d_{i+1} = \sum_{j=0}^{i} c_j \beta_{i-j} \tag{44}$$

Here $\delta_{i0}$ is the Kronecker symbol which equals one if $i$ is zero, and zero otherwise. Given the initial conditions $u_0$, $v_0$, $\alpha_0 = u_1$, $\beta_0 = v_1$, $t_0$, $u_{\epsilon_0}$, $v_{\epsilon_0}$, $\alpha_{\epsilon_0} = u_{\epsilon_1}$ and $\beta_{\epsilon_0} = v_{\epsilon_1}$, (6) may be solved for $a_0$, $b_0$, $c_0$, $d_0$ and the algorithm (25) - (44) will then give all higher order coefficients in terms of the preceding coefficients, so that the solution may be extended optimally by analytic continuation using a variable step size and a variable number of terms in the power series expansion for each integration step.[4] It is to be noted that the equations of motion may be computed independently by deleting Equations (32) - (42) from the above algorithm.

As an example of applicability this algorithm may be used in conjunction with the well-known predictor-corrector technique for the numerical computation of natural families of periodic orbits.[1,2] Following this method we set $\mu_\epsilon = 0$ and the predictor variational equations would have $J_\epsilon = 1$, resulting in the expected nonhomogeneous system of linear equations. The iterative correcting procedure is isoenergetic so that $J_\epsilon = 0$ and the homogeneous system of variational equations yielding the corrected periodic orbit gives the characteristic stability exponents as eigenvalues of the numerically integrated matrizant at the end of the fundamental period.[3]

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