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LIAPUNOV FUNCTION FROM
AUXILIARY EXACT DIFFERENCE EQUATIONS
FOR NONLINEAR DIFFERENCE EQUATION
STABILITY ANALYSIS

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CONTENTS

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Abstract

A mathematical structure is presented to determine the stability of nonlinear difference equations by the second method of Liapunov using a direct extension of a procedure recently proposed for nonlinear differential equations. Definitions are given for the gradient of an implicit function of a discrete variable, a principal sum, a definite sum, and an exact difference equation, analogous to the corresponding quantities for an implicit function of a continuous variable. A theorem for exactness of a difference form is proved. It is then shown that Liapunov functions to determine the stability of nonlinear autonomous difference equations can be developed through the use of auxiliary exact difference equations.

LIAPUNOV FUNCTIONS

FROM AUXILIARY EXACT DIFFERENCE EQUATIONS FOR NONLINEAR DIFFERENCE EQUATION STABILITY ANALYSIS

I. INTRODUCTION

Stability characteristics of discrete-time dynamic systems described by nonlinear autonomous ordinary difference equations are defined to reflect the behavior of solutions about equilibrium states. The stability of these equilibrium state solutions can be determined by the second or direct method of Liapunov [1-4].

In general the application of the second method to difference equations is analogous to the application to differential equations. Specifically a sign definite scalar function is assumed in the region of space including the equilibrium state. The difference equation solutions projected onto the surface described by the scalar function are then examined. If the scalar function has a negative difference everywhere in the region of space about the equilibrium point when constrained to change according to the difference equation, any solution originating around the equilibrium state must converge to the equilibrium state. This equilibrium state is described as asymptotically stable. Scalar functions with the necessary properties to establish stability characteristics are called Liapunov functions [4,5].

In a recent paper, Liapunov functions for nonlinear autonomous differential equations of continuous-time dynamic systems were considered as solutions to auxiliary exact differential equations described in the phase space of the corresponding nonlinear equations [6]. It was shown that the problem of generating Liapunov functions from this point of view (1) leads to a variety of techniques that have been suggested in the literature, (2) has the advantage of reducing the freedom for selecting undetermined parameters for some of these techniques, and (3) suggests alternate and new methods for their development which are particularly useful for establishing finite regions of stability. The purpose of the following development is to present a mathematical structure which allows auxiliary exact difference equations to be used for constructing Liapunov functions for nonlinear difference equations describing discrete-time deterministic systems. Alternate methods are provided, consequently, from the trial and error selection usually suggested in the literature [3,5,7]. Other advantages correspond essentially to those stated for continuous-time system stability analysis.

The state space description of vector operations for difference equations are reviewed in the next section with the introduction of some new notations, definitions, and a theorem on exactness. Section III considers Liapunov functions for stability analysis as obtained from auxiliary exact difference equations. Illustrations are given in Section IV.

While this work is directed to the stability analysis of discrete-time dynamic systems, the details are only related to the describing difference equations. The definitions and procedures are applicable, therefore, to a broader class of convergence problems such as those arising in discrete solution techniques [5,8]. The development of difference equations for discrete-time stability analysis and questions of stability related to the continuous portions of hybrid discrete/continuous-time systems are not discussed.

II. DIFFERENCE EQUATIONS

Given a discrete-time system described by the difference equation

$$a_n y(k+n) + a_{n-1} y(k+n-1) + \cdots + a_1 y(k+1) + a_o y(k) + f[y(k)] = 0 . \quad (1)$$

This can conveniently be converted to a vector difference equation by using an operator notation.

Let $\quad Ey(k) \triangleq y(k+1)$

and $\quad \Delta y(k) \triangleq y(k+1) - y(k) . \quad (2)$

Then $\quad \Delta y(k) = (E-1)\, y(k) ,$

$$y(k+2) = Ey(k+1) = E^2 y(k) ,$$

and $\quad y(k+n) = E^n y(k)$.

As $\quad \Delta \Rightarrow E-1 \quad$ or $\quad E \Rightarrow \Delta+1$,

$$y(k+n) = E^n y(k) = (\Delta+1)^n y(k)$$

$$= \Delta^n y(k) + n\Delta^{n-1} y(k) + \frac{n(n-1)}{2!} \Delta^{n-2} y(k)$$

$$+ \cdots + n\Delta y(k) + y(k) .$$

Substituting into (1) and rearranging,

$$(b_n\Delta^n + b_{n-1}\Delta^{n-1} + \cdots + b_1\Delta + b_o) y(k) + f[y(k)] = 0 , \tag{3}$$

where the b_i are functions of the a_i in (1).

Define $\quad x_1(k) \triangleq y(k)$

$$x_2(k) \triangleq \Delta x_1(k) = \Delta x(k)$$

$$x_3(k) \triangleq \Delta x_2(k) = \Delta^2 x(k)$$

$$\vdots$$

$$x_n(k) \triangleq \Delta^{n-1} x(k) = \Delta x_{n-1}(k) .$$

Then $\Delta^n x(k) = \Delta x_n(k)$ and (3) can be written as

$$\Delta x_1(k) = x_2(k)$$

$$\Delta x_2(k) = x_3(k)$$

$$\vdots$$

$$\Delta x_{n-1}(k) = x_n(k) \tag{4}$$

$$\Delta x_n(k) = -\frac{1}{b_n}(b_{n-1}x_n(k) + b_{n-2}x_{n-1}(k) + \cdots + b_o x_1(k) + f[x_1(k)]) .$$

Substituting (2), this becomes

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + x_2(k) \\
x_2(k+1) &= x_2(k) + x_3(k) \\
&\vdots \\
x_{n-1}(k+1) &= x_{n-1}(k) + x_n(k) \\
x_n(k+1) &= -\frac{1}{b_n}[(-b_n+b_{n-1})x_n(k) + b_{n-2}x_{n-1}(k) \\
&\quad + \cdots + b_o x_1(k) + f(x_1(k))] \; .
\end{aligned}
\tag{5}
$$

Equation (4) is written in vector form as

$$\Delta \underline{x}(k) = \underline{f}(\underline{x}(k)) \; , \tag{6}$$

where $\underline{x}(k) \triangleq [x_1(k) \cdots x_n(k)]^t$ and $\underline{f} \triangleq [f_1 \cdots f_n]^t$, or equivalently corresponding to (5) as

$$\underline{x}(k+1) = \underline{f}'(\underline{x}(k)) \; , \tag{7}$$

where $\underline{f}' \triangleq \underline{f}(\underline{x}(k)) + \underline{x}(k)$.

Alternately [4], if $z_1(k) \triangleq y(k)$ and $z_{i-1}(k+1) \triangleq z_i(k)$, (1) becomes, instead of (5) and (7),

$$
\begin{aligned}
z_1(k+1) &= z_2(k) \\
z_2(k+1) &= z_3(k) \\
&\vdots \\
z_{n-1}(k+1) &= z_n(k) \\
z_n(k+1) &= -\frac{1}{a_n}[a_{n-1}z_n(k) + a_{n-2}z_{n-1}(k) \\
&\quad + \cdots + a_o z_o(k) + f(z_1(k))]
\end{aligned}
\tag{8}
$$

or simply $$\underline{z}(k+1) = \underline{g}(\underline{z}(k)) \ . \tag{9}$$

While (1) is nonlinear, equations (6), (7), and (9) are recognized as forms allowing more general nonlinear functions within the vector $\underline{f}$, $\underline{f}'$, or $\underline{g}$. For brevity only (6) is used in the subsequent development. Equivalent statements can often be made with respect to (7) and (9). k is a discrete variable defined conveniently on a one-dimensional space of all integers. Physically, k can be identified with a sequence of time instants or the sample times of a discrete-time system. $\underline{f}$ is defined in some domain Ω of n-dimensional Euclidean vector space E_n and is assumed to be real continuous functions of its arguments [1].

Example:

If $$y(k+2) + 5y(k+1) + 6y(k) + \alpha y^3(k) = 0 \ ,$$

then $$(E^2+5E+6)y(k) + \alpha y^3(k) = 0 \ ,$$

and $$(\Delta^2+7\Delta+12)y(k) + \alpha y^3(k) = 0 \ .$$

If $x_1(k) = y(k)$, etc., (6) becomes

$$\Delta\underline{x}(k) = \begin{bmatrix} 0 & 1 \\ (-12-\alpha x_1^2(k)) & -7 \end{bmatrix} \underline{x}(k) \ .$$

Using the alternate form (7),

$$\underline{x}(k+1) = \begin{bmatrix} 1 & 1 \\ (-12-\alpha x_1^2(k)) & -6 \end{bmatrix} \underline{x}(k) \ .$$

Corresponding to (9),

$$\underline{z}(k+1) = \begin{bmatrix} 0 & 1 \\ (-6-\alpha z_1^2(k)) & -5 \end{bmatrix} \underline{z}(k) \ .$$

Definition: Solution of a difference equation [5]

A function $\underline{X}(k;k_o,\underline{x}_o)$ is called a solution of the difference equation (6) with initial conditions k_o and x_o if

(a) $\underline{X}(k;k_o,\underline{x}_o)$ is defined for $k_o \leq k \leq k_o + p$, p some positive integer,

(b) $\underline{X}(k_o;k_o,\underline{x}_o) = \underline{x}_o$, the initial vector, and

(c) $\Delta\underline{X}(k;k_o,\underline{x}_o) = \underline{f}(\underline{X}(k;k_o,\underline{x}_o))$ for $k_o \leq k \leq k_o + p-1$, p some positive integer.

Definition: Equilibrium state [1,3]

A state $\underline{x}_e$ of the dynamic system (6) is an equilibrium state if $\underline{f}(\underline{x}_e) = \underline{0}$.

Definition: Stable equilibrium state [1-3]

An equilibrium state $\underline{x}_e$ of the dynamic system (6) is a stable equilibrium state if for any number $\varepsilon > 0$ there corresponds a number $\delta(\varepsilon,k_o) > 0$ such that if $||\underline{x}_o - \underline{x}_e|| \leq \delta$, then $||\underline{X}(k;\underline{x}_o,k_o) - \underline{x}_e|| \leq \varepsilon$ for all $k \geq k_o$*.

Further definitions of types of equilibrium points follow from direct analogies to continuous systems [1-3].

While the dependent variable $\underline{x}$ for (6) is defined over a continuous space, the discrete character of the independent variable k does not allow the use of the familiar definition of the gradient [9]. It is necessary, therefore, to redefine a suitable gradient. This is described initially for n = 2.

Consider a function of two dependent variables, each a function of an independent discrete variable,

---

*The Euclidean norm $||\underline{x}|| \triangleq (\Sigma x_i^2)^{1/2} = \langle\underline{x},\underline{x}\rangle = \underline{x}^t\underline{x}$.

$$F(x,y) = F(x(k),y(k)),\ k = \cdots -2,-1,0,1,2, \cdots .$$

A positive increment of these dependent variables is the change corresponding to a unit increase in k. For F, however, an increment is dependent on the increments of x and y which are not subject by limiting to arbitrarily small values. Therefore define the partial differences

$$\Delta_x F(x,y) \triangleq \frac{F(x+\Delta x,y) - F(x,y)}{\Delta x}$$

and

$$\Delta_y F(x,y) \triangleq \frac{F(x,y+\Delta y) - F(x,y)}{\Delta y} . \tag{10}$$

Then

$$\begin{aligned} \Delta F(x,y) &\triangleq F(x+\Delta x,y+\Delta y) - F(x,y) \\ &= F(x+\Delta x,y+\Delta y) - F(x,y+\Delta y) \\ &\quad + F(x,y+\Delta y) - F(x,y) \\ &= \Delta_x F(x,y+\Delta y)\ \Delta x + \Delta_y F(x,y)\ \Delta y , \end{aligned}$$

or

$$\begin{aligned} \Delta F(x,y) &= F(x+\Delta x,y+\Delta y) - F(x+\Delta x,y) \\ &\quad + F(x+\Delta x,y) - F(x,y) \\ &= \Delta_y F(x+\Delta x,y)\ \Delta y + \Delta_x F(x,y)\ \Delta x . \end{aligned}$$

It follows that

$$\begin{aligned} \Delta_x \Delta_y F(x,y) &= \frac{\Delta_y F(x+\Delta x,y) - \Delta_y F(x,y)}{\Delta x} \\ &= \frac{F(x+\Delta x,y+\Delta y) - F(x+\Delta x,y) - F(x,y+\Delta y) + F(x,y)}{\Delta x\ \Delta y} \\ &= \Delta_y \Delta_x F(x,y) . \end{aligned} \tag{11}$$

Thus a correspondence between implicit functions of continuous and discrete variables is made for $dF(x,y)$ and $\Delta F(x,y)$,

$\frac{\partial F(x,y)}{\partial x}$ and $\Delta_x F(x,y)$, and dx and Δx. The partial difference is not independent of the variation, however, so that $\Delta_x F(x,y)$ would accurately be represented, say, as $G(x,y,\Delta x)$. This function notation was rejected as too cumbersome in preference to $G(x,y)$, with the Δx dependency implied.

The extension of (10) to partial differences for a scalar function of n variables is direct and analogous to the definition from an implicit function of a continuous variable.

Definition: Gradient

The gradient of an implicit function of a discrete variable is

$$\nabla F(x_1,\ldots,x_n) \triangleq \begin{bmatrix} \Delta_{x_1} F(x_1,x_2+\Delta x_2,\ldots,x_n+\Delta x_n) \\ \Delta_{x_2} F(x_1,x_2,\ldots,x_n+\Delta x_n) \\ \vdots \\ \Delta_{x_n} F(x_1,x_2,\ldots,x_n) \end{bmatrix} . \qquad (12)$$

It follows that the difference of a scalar function is

$$\Delta F = \langle \nabla F, \Delta \underline{x} \rangle ,$$

where $\Delta \underline{x} \triangleq [\Delta x_1, \Delta x_2, \ldots, \Delta x_n]^t$ and ΔF and ∇F are functions of the components of both $\underline{x}$ and $\Delta \underline{x}$. It would also be possible to write

$$\Delta F(\underline{x}) = \sum_{i=1}^{n} \Delta x_i \cdot \Delta_{x_i} F(\underline{x}+\Delta \underline{x}^*) ,$$

where $\Delta x_j^* \triangleq \Delta x_j \cdot u(j-i)$.†

Principal Solution or Sum

If (6) is defined for 1 dimension only, i.e.

$$\Delta x = f(x) , \tag{13}$$

and if $X(k,k_o,x_o)$ is a solution, then

$$\Delta X = \Delta_x X(x) \cdot \Delta x = f(X) , \tag{14}$$

A principal sum and principal solution can be written for $f(x)$ and for (13) respectively $\mathcal{S} f(x)$ and

$$x = \mathcal{S} f(x) - C , \tag{15}$$

where C is a constant. Applying this notation to (14),

$$\mathcal{S} f(X) = \mathcal{S} \Delta X = \mathcal{S} \Delta_x X \cdot \Delta x = X + C ,$$

and it follows that the principal sum is simply the inverse of the difference operator. Using (14) and assuming existence, it is readily shown that (15) is a completely general form of the solution to (13). An alternate development appears in the literature [10].

Instead of (13), if in E_n

$$f(\underline{x}) = \Delta X(\underline{x}) = \langle \nabla X(\underline{x}) , \Delta \underline{x} \rangle ,$$

the principal sum is similarly defined and written as

†$u(\ )$ = 1 for a positive argument and zero otherwise.

$$\mathcal{S} f(\underline{x}) = \mathcal{S} \langle \nabla \underline{X} , \Delta \underline{x} \rangle . \tag{16}$$

Definite Sum

Given a function $F(x(k))$ defined and continuous in x. From the difference definitions (2) and (10),

$$\begin{aligned}
\Delta F(x(k)) &= F(x(k+1)) - F(x(k)) \\
&= F(x(k) + \Delta x(k)) - F(x(k)) \\
&= \Delta_x F(x(k)) \quad \Delta x(k) \\
&\triangleq G(x(k)) \ \Delta x(k) .
\end{aligned} \tag{17}$$

Thus $\quad F(x(k+n)) - F(x(k))$

$$= \sum_{i=1}^{n} G(x(k+i-1)) \quad \Delta x(k+i-1) .$$

This is defined as the definite sum of G and is written as

$$\mathcal{S}_{x(k)}^{x(k+n)} G(x) \ \Delta x \triangleq \sum_{i=1}^{n} G(x(k+i-1) \quad \Delta x(k+i-1)$$

$$= F(x(k+n)) - F(x(k)) , \tag{18}$$

using a notation suggested by Milne-Thompson [10].

The difference form $G(x)\ \Delta x$ of (17) is said to be an exact difference form if there exists a function $F(x)$ such that $\Delta_x F = G$. Then also

$$G(x) \quad \Delta x = 0$$

is called an exact difference equation with the solution $F(x)$. Similar statements can be made for an n-dimensional space, assuming the definition and continuity of the functions in Ω of E_n with respect to the components of $\underline{x}$. It is also assumed that $\underline{0} \subset \Omega$.

Definition: Exact Difference Form and Exact Difference Equation

The difference form $<\underline{G}(\underline{x}) \ , \ \Delta\underline{x}>$ is defined as an exact difference form if there exists a scalar function $F(\underline{x})$ such that $\nabla F(\underline{x}) = \underline{G}$. The difference equation

$$< \underline{G}(\underline{x}) \ , \ \Delta\underline{x} > = 0 \tag{19}$$

is said to be an exact difference equation with the solution $F(\underline{x})$ + constant.

Theorem: Exactness

The necessary and sufficient condition that $<\underline{G}(\underline{x}) \ , \ \Delta\underline{x}>$ be exact is that

$$\Delta_{x_j} G_i = \Delta_{x_i} G_j \ , \quad i,j = 1,2,\ldots,n, i \neq j \ , \tag{20}$$

i.e. the matrix

$$\underline{\Phi} = [\nabla G_1, \nabla G_2, \ldots, \nabla G_n]$$

is symmetric. Moreover, if (19) is exact, then the function $F(\underline{x})$ for which $\Delta_{x_j} F = G_j$ and $\Delta_{x_i} F = G_i$ is uniquely represented by the form

$$F(\underline{x}) = \mathop{\mathcal{S}}_{\underline{0}}^{\underline{x}} \Delta F(\underline{x}) = \mathop{\mathcal{S}}_{\underline{0}}^{\underline{x}} <\underline{G}(\underline{x}) \ , \ \Delta\underline{x}>$$

$$\triangleq \int_{0}^{x_1(x_2=x_3,\ldots=x_n=0)} G_1 \; \Delta x_1 + \int_{0}^{x_2(x_1=x_1,x_3=x_4=\ldots x_n=0)} G_2 \; \Delta x_2$$

$$+ \cdots + \int_{0}^{x_n(x_1=x_1,x_2=x_2,\ldots,x_{n-1}=x_{n-1})} G_n \; \Delta x_n \quad . \tag{21}$$

Proof: To prove necessity, assume that $<\underline{G}(\underline{x}) \; , \; \Delta\underline{x}>$ is exact. Then an $F(\underline{x})$ exists such that

$$\Delta F(\underline{x}) = <\nabla F(\underline{x}) \; , \; \Delta\underline{x}> = <\underline{G}(\underline{x}) \; , \; \Delta\underline{x}> ,$$

where

$$\nabla F(\underline{x}) = \begin{bmatrix} \Delta_{x_1} F(x_1, x_2+\Delta x_2, \ldots, x_n+\Delta x_n) \\ \Delta_{x_2} F(x_1, x_2, x_3+\Delta x_3, \ldots) \\ \vdots \\ \Delta_{x_n} F(x_1, x_2, \ldots, x_n) \end{bmatrix} .$$

Assuming $i<j$,

$$\Delta_{x_j}[\Delta_{x_i} F] = \Delta_{x_j} \frac{[F(x_1,\ldots,x_i+\Delta x_i,\ldots,x_j,\ldots,x_n) - F(\underline{x})]}{\Delta x_i}$$

$$= \frac{[\Delta_{x_j} F(x_1,\ldots,x_i+\Delta x_i,\ldots,x_j,\ldots,x_n) - \Delta_{x_j} F(\underline{x})]}{\Delta x_i}$$

$$= \frac{\begin{array}{c}[F(x_1,\ldots,x_i+\Delta x_i,\ldots,x_j+\Delta x_j,\ldots,x_n) \\ -F(x_1,\ldots,x_i+\Delta x_i,\ldots,x_j,\ldots,x_n) \\ -F(x_1,\ldots,x_i,\ldots,x_j+\Delta x_j,\ldots,x_n) + F(\underline{x})]\end{array}}{\Delta x_j \ \Delta x_i}$$

$$= \Delta_{x_i}[\Delta_{x_j}F] \ , \quad i,j = 1,2,\ldots,n, i \neq j \ .$$

As $\quad \Delta_{x_i}F = G_i(\underline{x}) \ ,$

then $\quad \Delta_{x_i}G_j = \Delta_{x_j}G_i \ .$

For sufficiency, it is required to show the existence of a function $F(\underline{x})$ such that $\Delta_{x_i}F(\underline{x}) = G_i$ and $\Delta_{x_j}F(\underline{x}) = G_j$ given that $\Delta_{x_i}G_j = \Delta_{x_j}G_i$, $i,j = 1,2,\ldots,n, i \neq j$. From (18) and (20), the first of the two conditions is satisfied iff

$$F(\underline{x}) = \mathcal{S}_{x_{io}}^{x_i} G_i(x_1,\ldots,t_i,\ldots,x_n)\Delta t_i + \phi_i(x_1,\ldots,x_n) \ , \tag{22}$$

where x_{io} is a constant and $\phi_i(x_1 \ldots x_n)$ is a function of $x_1,\ldots,x_n$ except the i^{th} component. The second condition will be satisfied iff

$$\Delta_{x_j}[\mathcal{S}_{x_{io}}^{x_i} G_i(x_1,\ldots,t_i,\ldots,x_n)\Delta t_i + \phi_i(x_1,\ldots,x_n)] = G_j \ . \tag{23}$$

Hence the function $F(\underline{x})$ given by (22) will satisfy both conditions if ϕ_i is determined so that (23) holds. The assumption on G_i and G_j require that

$$\Delta_{x_j} \mathop{S}_{x_{io}}^{x_i} G_i(x_1,\ldots,t_i,\ldots,x_n)\ \Delta t_i$$

$$= \mathop{S}_{x_{io}}^{x_i} \Delta_{x_j} G_i(x_1,\ldots,t_i,\ldots,x_n)\ \Delta t_i$$

$$= \mathop{S}_{x_{io}}^{x_i} \Delta_{t_i} G_j(x_1,\ldots,t_i,\ldots,x_n)\ \Delta t_i$$

$$= G_j(x_1,\ldots,x_i,\ldots,x_n) - G_j(x_1,\ldots,x_{io},\ldots,x_n)\ .$$

Therefore (23) holds iff

$$\Delta_{x_j}\phi_i(x_1,\ldots,x_n) = G_j(x_1,\ldots,x_{io},\ldots,x_n)\ ,$$

or

$$\phi_i(x_1,\ldots,x_n) = \mathop{S}_{x_{jo}}^{x_j} G_j(x_1,\ldots,x_{io},\ldots,t_j,\ldots,x_n)\ \Delta t_j + \psi_{ij}\ , \tag{24}$$

where ψ_{ij} is not a function of x_i or x_j. As (22) and (24) are valid $i,j = 1,2,\ldots,n, i\neq j$, (21) is obtained by letting $i = n$ in (22), $j = n - 1$ in (24) and continuing in sequence, and finally by setting $\underline{x}_o = \underline{0}$.

Both (16) and (21) are defined and require interpretation consistent with (12). If the gradient is defined alternately as

$$\nabla F = \begin{bmatrix} \Delta_{x_1} F(x_1,x_2,\ldots,x_n) \\ \Delta_{x_2} F(x_1+\Delta x_1,x_2,\ldots,x_n) \\ \cdot \\ \cdot \\ \cdot \\ \Delta_{x_n} F(x_1+\Delta x_1,x_2+\Delta x_2,\ldots,x_n) \end{bmatrix}, \tag{25}$$

then (16) and (21) would need corresponding interpretations.

The individual sums in (21) imply that when the sum is on x_i, then in the integrand, all $\Delta x_j=0$, $j\neq i$. This can be shown from (2) and (18).

Examples:

[1] For n = 2, consider

$$f(x_1,x_2) = x_1^2 + x_2 + 3x_1x_2 .$$

Then

$$\begin{aligned} \Delta f &= \Delta_{x_1} f(x_1,x_2+\Delta x_2)\Delta x_1 + \Delta_{x_2} f(x_1,x_2)\Delta x_2 \\ &= (2x_1+\Delta x_1+3x_2+3\Delta x_2)\Delta x_1 + (1+3x_1)\Delta x_2 \\ &= G_1(x_1,x_2)\Delta x_1 + G_2(x_1,x_2)\Delta x_2 . \end{aligned}$$

The exactness condition (20) is satisfied, i.e.

$$\Delta_{x_2} G_1 = 3 = \Delta_{x_1} G_2 .$$

Conversely, to find f from Δf, either (16) or (21) can be used. Considering (16) first, as

$$\Delta f = \Delta_{x_1} f(x_1,x_2+\Delta x_2)\Delta x_1 + \Delta_{x_2} f(x_1,x_2)\Delta x_2 ,$$

$$f(x_1,x_2) = \mathcal{S}\, G_2\, \Delta x_2 + F(x_1)\ .$$

Thus $f(x_1,x_2) = \mathcal{S}(1+3x_1)\Delta x_2 + F(x_1) = x_2 + 3x_1x_2 + F(x_1)$,

which can be verified by the reverse operation. Substituting this in

$$G_1 = \Delta_{x_1} f(x_1,x_2+\Delta x_2)\ ,$$

$$G_1 = 3(x_2+\Delta x_2) + \Delta_{x_1} F(x_1)\ .$$

But from Δf, $G_1 = 3(x_2+\Delta x_2) + 2x_1 + \Delta x_1$.

Thus $\Delta_{x_1} F(x_1) = 2x_1 + \Delta x_1$,

$$F(x_1) = \mathcal{S}(2x_1+\Delta x_1)\Delta x_1 + c = x_1^2 + c\ ,$$

and finally

$$f(x_1,x_2) = x_1^2 + x_2 + 3x_1x_2 + c\ ,$$

the original form within a constant.

To obtain f from (21),

$$f(x_1,x_2) = \mathcal{S}_{\underline{0}}^{\underline{x}} (G_1\Delta x_1 + G_2\Delta x_2)$$

$$= \mathcal{S}_0^{x_1} (2x_1+\Delta x_1)\Delta x_1 + \mathcal{S}_0^{x_2} (1+3x_1)\Delta x_2\ . \qquad (26)$$

Using (18) and (2) for the first definite sum,

$$\mathop{\mathcal{S}}_{0}^{x_1} (2x_1+\Delta x_1)\Delta x_1 = \sum_{i=1}^{n} [2x_1(k+i-1)+x(k+i)-x_1(k+i-1)][x_1(k+i)-x_1(k+i-1)]$$

$$= \sum_{i=1}^{n} [x_1^2(k+i) - x_1^2(k+i-1)]$$

$$= x_1^2(k+n) - x_1^2(k) ,$$

which is x_1^2 when the lower limit is set to zero. Similarly, for the second definite sum in (26),

$$\mathop{\mathcal{S}}_{0}^{x_2} (1+3x_1)\Delta x_2 = (1+3x_1) \mathop{\mathcal{S}}_{0}^{x_2} \Delta x_2 = (1+3x_1)x_2 ,$$

and

$$f(x_1,x_2) = x_1^2 + x_2 + 3x_1x_2 .$$

[2] For n = 3, consider

$$f(x_1,x_2,x_3) = -x_1 + x_1x_2^2 + x_1^2x_3^2 .$$

Then

$$\Delta f(x_1,x_2,x_3) = -(x_1+\Delta x_1)+(x_1+\Delta x_1)(x_2+\Delta x_2)^2$$
$$+(x_1+\Delta x_1)^2(x_3+\Delta x_3)^2-[-x_1+x_1x_2^2+x_1^2x_3^2] .$$

This does not provide unique terms for the gradient components, however. Therefore let

$$\Delta f = G_1 \Delta x_1 + G_2 \Delta x_2 + G_3 \Delta x_3 .$$

Using (25) for the definition of the gradient instead of (12),

$$G_1 = \Delta_{x_1} f(x_1,x_2,x_3)$$

$$= \frac{-(x_1+\Delta x_1)+(x_1+\Delta x_1)x_2^{\,2}+(x_1+\Delta x_1)^2 x_3^{\,2}-[-x_1+x_1x_2^{\,2}+x_1^{\,2}x_3^{\,2}]}{\Delta x_1}$$

$$= -1 + x_2^{\,2} + 2x_1x_3^{\,2} + x_3^{\,2}\Delta x_1 ,$$

$$G_2 = \Delta_{x_2} f(x_1+\Delta x_1 , x_2 , x_3)$$

$$= 2x_1x_2 + x_1 \Delta x_2 + 2x_2 \Delta x_1 + \Delta x_1 \Delta x_2 ,$$

and $G_3 = \Delta_{x_3} f(x_1+\Delta x_1, x_2+\Delta x_2, x_3)$

$$= 2x_3x_1^{\,2} + 4x_1x_3\Delta x_1 + 2x_3(\Delta x_1)^2 + x_1^{\,2} \Delta x_3$$

$$+2x_1\Delta x_1\Delta x_3+(\Delta x_1)^2\Delta x_3 .$$

Thus

$$\Delta_{x_1} G_2 = \frac{2(x_1+\Delta x_1)x_2 + (x_1+\Delta x_1)\Delta x_2 + 2x_2\Delta x_1 + \Delta x_1\Delta x_2}{\Delta x_1}$$

$$- \frac{[2x_1x_2 + x_1 \Delta x_2 + 2x_2 \Delta x_1 + \Delta x_1 \Delta x_2]}{\Delta x_1}$$

$$= 2x_2 + \Delta x_2 = \Delta_{x_2} G_1 .$$

Likewise

$$\Delta_{x_2} G_3(x_1+\Delta x_1, x_2, x_3) = 0 = \Delta_{x_3} G_2(x_1+\Delta x_1, x_2+\Delta x_2, x_3) ,$$

and

$$\Delta_{x_3} G_1 = 4x_1x_3 + 2x_1 \Delta x_3 + 6x_3 \Delta x_1 + 3 \Delta x_1 \Delta x_3$$

$$= \Delta_{x_1} G_3(x_1, x_2, x_3) ,$$

showing the exactness criteria to be satisfied.

Conversely, the function which satisfies the given difference form can be found from (16). Summing first on G_1

$$f(x_1,x_2,x_3) = \mathcal{S}[-1+x_2^2+2x_1x_3^2+x_3^2\Delta x_1]\Delta x_1 + F(x_2,x_3)$$

$$= -x_1 + x_1x_2^2 + x_1^2x_3^2 + F(x_2,x_3) .$$

Then

$$G_2 = \Delta_{x_2} f(x_1 + \Delta x_1, x_2, x_3)$$

$$= 2x_1x_2 + 2x_2\Delta x_1 + x_1\Delta x_2 + \Delta x_1\Delta x_2 + \Delta x_2 F(x_2,x_3) .$$

Compared to the given G_2,

$$\Delta_{x_2} F(x_2,x_3) = 0$$

and $\qquad F(x_2,x_3) = F_1(x_3) .$

Then $\qquad G_3 = \Delta_{x_3} f(x_1+\Delta x_1, x_2+\Delta x_2, x_3)$

$$= 2x_1^2x_3 + 4x_1x_3 \Delta x_1 + 2x_3(\Delta x_1)^2 + x_1^2 \Delta x_3$$

$$+ 2x_1 \Delta x_1 \Delta x_3 + (\Delta x_1)^2 \Delta x_3 + \Delta_{x_3} F_1(x_3) .$$

For G_3 as given, it is required that

$$\Delta_{x_3} F_1(x_3) = 0 \quad \text{or} \quad F_1(x_3) = C \; .$$

Therefore

$$f = -x_1 + x_1 x_2^2 + x_1^2 x_3^2 + C \; ,$$

the original form within a constant.

To find this function using the definite sum, (21),

$$f(x_1,x_2,x_3) = \mathop{\mathcal{S}}_0^{x_1} (-1)\Delta x_1 + \mathop{\mathcal{S}}_0^{x_2} (2x_1 x_2 + x_1 \Delta x_2)\Delta x_2$$

$$+ \mathop{\mathcal{S}}_0^{x_3} (2x_1^2 x_3 + x_1^2 \Delta x_3)\Delta x_3 \; .$$

From (2) and (18),

$$f = - \sum_{i=1}^{n} [x_1(k+i) - x_1(k+i-1)]$$

$$+ x_1 \sum_{i=1}^{n} [x_2^2(k+i) - x_1^2(k+i-1)]$$

$$+ x_1^2 \sum_{i=1}^{n} [x_3^2(k+i) - x_3^2(k+i-1)] \; ,$$

and with $\underline{x}(k) = \underline{0}$,

$$f(x_1,x_2,x_3) = - x_1 + x_1 x_2^2 + x_1^2 x_3^2 \; .$$

III. STABILITY AND LIAPUNOV FUNCTIONS FROM AUXILIARY EXACT DIFFERENCE EQUATIONS

For the following theorem and subsequent developments, it is assumed for (6) that $\underline{f}(\underline{0}) = \underline{0} \subset \Omega$ of E_n. If a system equilibrium state exists elsewhere, a transformation can be used to establish a new coordinate system with the equilibrium state at the origin.

Theorem: Stability of an Equilibrium Point [3,5]

For the autonomous difference equation (6), let Ω be a region of E_n such that $\underline{f}(\underline{0}) = 0 \subset \Omega$. If there exists a continuous scalar function $V(\underline{x})$ defined for all $\underline{x} \subset \Omega$ such that in Ω:

(a) $V(\underline{x})$ is positive definite,

(b) $\Delta_{x_i} V(\underline{x}) \cdot \Delta x_i$ is defined and exists for all x_i and Δ_{x_i} corresponding to the transition $k \rightarrow k+1$, for all k and $i=1,2,\ldots,n$,

(c) $\Delta V(\underline{x}) = \langle \nabla V(\underline{x}) , \underline{f}(\underline{x}) \rangle$ is negative semidefinite (definite),

the the equilibrium state $\underline{x}_e = \underline{0}$ of the system described by (6) is stable (asymptotically stable).

Assume for (6) that all f_i, $i=1,\ldots,n$, are defined and exist in Ω. Define

$$g_i \triangleq f_1 + f_2 + \cdots + f_{i-1} - f_{i+1} - \cdots - f_n . \tag{27}$$

Then
$$\sum_{i=1}^{n} g_i \; \Delta x_i \triangleq \langle \underline{g} , \Delta\underline{x} \rangle = 0 . \tag{28}$$

This is a difference equation representing the solution trajectories of (6) in phase space; solutions of (6) are solutions of (28). Note that (28) is only one possible difference equation that can be derived from linear and non-linear combinations of (6). All g_i are defined and exist in Ω.

Equation (28) is an exact difference equation if (20) is satisfied. If (28) is not exact, a function $\underline{h}(\underline{x})$ can be added to $\underline{g}(\underline{x})$ such that

$$< \underline{g} + \underline{h} \ , \ \Delta\underline{x} > = 0 \tag{29}$$

is an exact difference equation, i.e.

$$\Delta_{x_j}(g_i+h_i) = \Delta_{x_i}(g_j+h_j), \quad i,j = 1,2,\ldots,n, \ i \neq j \ . \tag{30}$$

The solution to (29), if $\underline{h}$ is selected to satisfy (30), is given by (16) or (21).

It may be possible to select $\underline{h}$ so that this solution has the characteristics of the Liapunov function $V(\underline{x})$ required by the stability theorem, with respect to the difference equation (6) [6]. Thus, given equations (6) and (28), if there exists an $\underline{h}$ such that

(a) $<\underline{g} + \underline{h} \ , \ \Delta\underline{x}> = \Delta V(\underline{x})$ is an exact difference form,

(b) $<\underline{h} \ , \ \underline{f}> = \Delta V(\underline{x})$ is at least negative semidefinite,

(c) the principal sum, $V(\underline{x})$, is positive definite,

then $V(\underline{x})$ is a Liapunov function with respect to (6). The corresponding difference equation is called an auxiliary exact difference equation.

The necessary and sufficient condition for exactness given by (30) is equivalent to the condition that

(a') the matrix

$$[\Delta_{x_j}(g_i+h_i)] = [\nabla(g_1+h_1), \ \nabla(g_2+h_2), \ldots, \nabla(g_n+h_n)]$$

be symmetric.

It is readily shown that the existence of an $\underline{h}$ satisfying the three conditions (a) or (a')-(c) is both necessary and sufficient for the existence of a Liapunov function for (6) [6].

Consequently the problem of developing a Liapunov function for (6) in terms of determining components of a vector $\underline{h}$ for an auxiliary exact difference equation is analogous to the corresponding problem for continuous-time systems [6]. The various techniques for selecting $\underline{h}$ are those proposed for continuous-time systems, including methods based on a principal or first sum, a variable gradient, a quadratic plus sum or Lure' form, or as proposed by Zubov [6,9]. A series expansion procedure recently proposed can also be used with minor modifications [11].

Extensions of the statements in this section for other stabilities and instability follow directly.

IV. ILLUSTRATIONS

Example 1. Consider the nonlinear difference equation

$$y(k+2) + \alpha y^n(k) = 0 ,$$

where n is any positive real number ≥ 1 and α is any real constant.

Corresponding to (3) and to (6), if $x_1(k) \triangleq y(k)$,

$$\Delta^2 y(k) + 2\Delta y(k) + y(k) + \alpha y^n(k) = 0$$

and

$$\Delta x_1(k) = x_2$$

$$\Delta x_2(k) = -x_1 - 2x_2 - \alpha x_1{}^n .$$

From (27)

$$g_1 = -f_2 = x_1 + 2x_2 + \alpha x_1{}^n$$

$$g_2 = f_1 = x_2 \ .$$

If $$h_1 = 3x_1 + 2\Delta x_1 - \alpha x_1{}^n$$

$$h_2 = 2(x_1+\Delta x_1) + x_2 + \Delta x_2 \ ,$$

then $$\Delta_{x_1}(g_2+h_2) = 2 = \Delta_{x_2}(g_1+h_1)$$

and $$\Delta V = \langle \underline{g} + \underline{h} \ , \ \underline{f} \rangle$$

is exact. Expanding ΔV and substituting the system equation,

$$\Delta V = x_1{}^2(\alpha^2 x_1{}^{2n-2}-1) \ .$$

From (21)

$$V = \mathcal{S}_{\underline{0}}^{\underline{x}} \langle \underline{g} + \underline{h} \ , \ \Delta\underline{x} \rangle$$

$$= \mathcal{S}_0^{x_1} (4x_1+2\Delta x_1) \ \Delta x_1 + \mathcal{S}_0^{x_2} (2x_1+2x_2+\Delta x_2) \ \Delta x_2$$

$$= 2x_1{}^2 + (2x_1x_2+x_2{}^2) \ ,$$

as readily verified by the inverse operation, and

$$V(\underline{x}) = x_1{}^2 + (x_1+x_2)^2 \ .$$

For the region $\alpha^2 x_1{}^{2n-2}(k)<1$, $k=0,1$, ΔV is negative semi-definite, V is positive definite, and the equilibrium state $\underline{0}$ is stable. Asymptotic stability can also be shown by further argument.

Example 2. To examine the stability characteristic of

$$\Delta x_1 = 0.01x_2 \qquad = f_1$$

$$\Delta x_2 = -0.000001x_1-0.00985x_2-0.0001x_1^2-0.01x_1^3 = f_2 ,$$

about the origin, let

$$g_1 = -f_2$$

$$g_2 = f_1$$

and (28) is not satisfied. If, however,

$$h_1 = \frac{1}{2}(4x_1^3+6x_1^2\Delta x_1+4x_1\Delta x_1^2+\Delta x_1^3) + \frac{1}{2}(2x_1+\Delta x_1) + x_2 + \Delta x_2$$

$$h_2 = (2x_2+\Delta x_2) + x_1 ,$$

(30) is satisfied and $<\underline{g} + \underline{h} , \Delta\underline{x}>$ is an exact difference form. Then

$$<\underline{h} , \underline{f}> = \{\frac{1}{2}(4x_1^3+6x_1^2\Delta x_1+4x_1\Delta x_1^2+\Delta x_1^3) + \frac{1}{2}(2x_1+\Delta x_1) + x_2 + \Delta x_2\} \Delta x_1 + \{(2x_2+\Delta x_2) + x_1\} \Delta x_2 .$$

Substituting the original equations for $\Delta\underline{x}$, after some manipulation,

$$10^4 \langle \underline{h} , \underline{f} \rangle = x_1^2[x_1^4+0.02x_1^3-100x_1^2-x_1-0.01]$$

$$+ x_2^2[-96.51+0.00005x_2^2+3x_1^2+0.02x_1x_2$$

$$+ (0.97\frac{x_1^3}{x_2} - 1.9903\frac{x_1^2}{x_2} + 1.48\frac{x_1}{x_2})]$$

$$\triangleq \alpha x_1^2 + \beta x_2^2 .$$

Using numerical methods, it is readily shown that all roots of $\alpha(x_1) = 0$ are outside $-1 \leq x_1 \leq 1$. Consequently

$$\alpha < 0 \quad , \quad |x_1| \leq 1 .$$

Also

$$\beta \leq [-93 + \frac{x_1}{x_2}(0.97x_1^2-1.9903x_1+1.48)]$$

for $\quad |x_1| \leq 1 \; , \; |x_2| \leq 1 ,$

or $\beta < 0$ in this region if

$$\frac{x_1}{x_2} < \frac{93}{0.97x_1^2-1.9903x_1+1.48} .$$

As the denominator for $|x_1| \leq 1,$

$$0 < 0.97x_1^2 - 1.9903x_1 + 1.48 \leq 4.4403 ,$$

$\beta < 0$ if $\frac{x_1}{x_2} < \frac{93}{4.4403}$ or, more conservatively, if $\frac{x_1}{x_2} < 10.$

For the remaining range, $\frac{x_1}{x_2} \geq 10$, $|x_1| \leq 1$ and $|x_2| \leq 1$, consider equivalently

$$x_2 = \frac{\eta x_1}{10} \quad , \quad 0 \leq \eta \leq 1 \ , \quad |x_1| \leq 1 \ .$$

Then

$$10^4 \langle \underline{h} \ , \ \underline{f} \rangle \leq x_1^2 [x_1^4 + 0.02x_1^3 - 100x_1^2 - x_1 - 0.01$$
$$-0.93\eta^2 + \eta(0.097x_1^2 - 0.19903x_1 + 0.148)]$$
$$= -x_1^2 [x_1^2(100 - 0.097\eta - x_1^2)$$
$$+ x_1(-0.02x_1^2 + 0.19903\eta + 1)$$
$$+ (0.93\eta^2 - 0.148\eta + 0.01)] \ .$$

But $\quad 100 - 0.097\eta - x_1^2 \geq 98.9$

and $\quad 0.93\eta^2 - 0.148\eta + 0.01 > 0.004$

for $\quad 0 \leq \eta_1 \leq 1 \ , \quad |x_1| \leq 1 \ .$

If $\quad \theta \triangleq (-0.02x_1^2 + 0.19903\eta + 1) \ ,$

then $\quad 0.98 \leq \theta < 1.2 \quad$ in this region and

$$10^4 \langle \underline{h} \ , \ \underline{f} \rangle \leq -x_1^2 [98x_1^2 + \theta(x_1, \eta)x_1 + 0.004] \ .$$

Finally the quantity in the brackets is

$$98\left(x_1 + \frac{0}{196}\right)^2 + 0.000327 > 0$$

and $\quad 10^4\langle \underline{h}\ ,\ \underline{f}\rangle < 0 \quad$ for $\quad |x_1| \leq 1\ ,\ |x_2| \leq 1\ ,$

excepting the origin.

For the principal sum

$$V = \mathop{\mathcal{S}}_{0}^{x_1(x_2=0)} (g_1+h_1)\ \Delta x_1 + \mathop{\mathcal{S}}_{0}^{x_2(x_1=x_1)} (g_2+h_2)\ \Delta x_2$$

$$= \frac{1}{2}\,x_1^{\,4} + \frac{1}{2}\,x_1^{\,2} + x_2^{\,2} + x_1x_2\ ,$$

which can be verified readily by the inverse operation. V is positive definite and the difference equation is locally asymptotically stable in the region $|x_1| \leq 1\ ,\ |x_2| \leq 1.$

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