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PARAMETRIC SENSITIVITY ANALYSIS FOR STAGED PROPULSIVE VEHICLES

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ABSTRACT

A mathematical method from random variable theory was used to analyze simultaneous variation of engine performance and stage inert weight for some typical 2-stage rocket vehicles. This represents an extension of the conventional sensitivity analysis which is often performed for conceptual designs. The study was incomplete in that too few examples were analyzed for conclusions to be drawn regarding the usefulness of the approach, but mathematical consistency was shown.

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ADVANCED SYSTEMS OFFICE
RESEARCH AND DEVELOPMENT OPERATIONS

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DEFINITION OF SYMBOLS

NOMENCLATURE

g .... Gravitational acceleration ($9.805\ m/sec^2$)

I .... Propulsive specific impulse, seconds

M .... Mass, kg

$p(x)$ ...... Density function for x

$P(x)$ ...... Distribution function for x

v .... Velocity, m/sec

y .... Dummy variable

z .... Dummy variable

λ .... Stage propellant fraction

μ .... Vehicle mass ratio, (mass fully loaded)/(mass with impulse propellant expended)

ζ .... Mass ratio, M_T/M_L

SUBSCRIPTS

L .... Payload

T .... Total

1 .... First stage

2 .... Second stage

Where appended to functional relationships (e.g., g_1, g_2, etc.), numerical subscripts do not refer to stages.

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SUMMARY

A mathematical method from random variable theory was used to analyze simultaneous variation of engine performance and stage inert weight for some typical 2-stage rocket vehicles. This represents an extension of the conventional sensitivity analysis which is often performed for conceptual designs. The study was incomplete in that too few examples were analyzed for conclusions to be drawn regarding the usefulness of the approach, but mathematical consistency was shown.

BACKGROUND AND INTRODUCTION

During preliminary and conceptual design phases of studies of flight systems, it is common practice to perform sensitivity analyses to determine the effect of variation of system parameters that cannot be precisely predicted in the course of such early studies. These sensitivity data are commonly in the form of derivatives such as change in vehicle gross weight as a function of change in propulsion I_{sp} of one of the stages, and change in system gross weight versus the sub-parameters of interest. In a plotted form one can readily see, for example, how much the system gross weight will be affected by a 1% variation in second stage inert weight. It is of interest to investigate the combined effect of simultaneous variation in more than one parameter. For example: What is the combined effect of uncertainty in delivered I_{sp} and inert weights in a 1- or 2-stage vehicle?

A method that can be used to simulate simultaneous variation in critical sub-parameters is the treatment of these sub-parameters as random variables. This method can be readily employed whenever it is possible to express the desired vehicle performance parameters in mathematical form. Gross weight-to-payload ratio may be expressed as a mathematical function of the various sub-parameters such that the pertinent partial derivatives may be analytically evaluated. This may be done with a simple ideal velocity model of rocket vehicle performance. Even this simple model will provide valuable insight into the likelihood of achieving design goals and into the selection of certain parameters, such as staging velocity, to minimize the development risk.

The study reported herein is incomplete in that too few examples were analyzed to achieve a good evaluation of the usefulness of the technique. However, a mathematical approach was formulated using the random variable theory, and enough calculations were made on a digital computer to show that the approach is at least mathematically consistent.

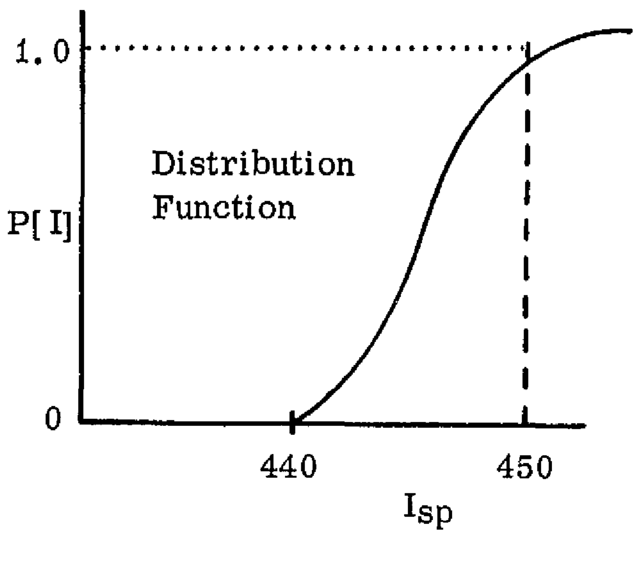
PURPOSE

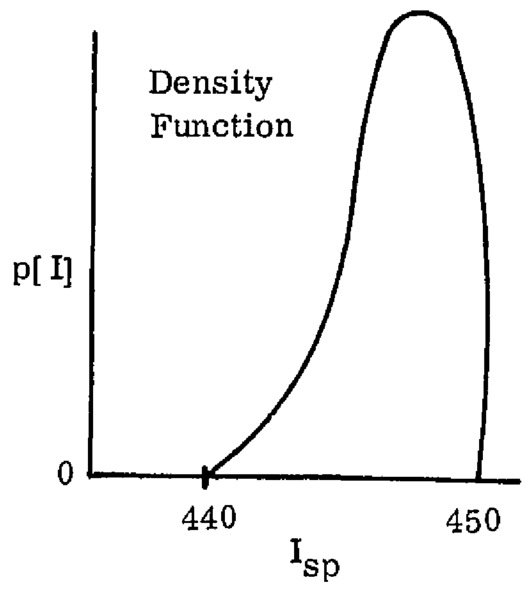
The purpose of this internal note is to report on an exploratory investigation of a method of analysis for treatment of vehicle performance by random variable techniques and to give example results of the analysis for representative rocket propelled vehicles.

Application of the random variable theory as suggested above requires some information about the variables considered to be random. This information is normally in the form of a distribution density function. Such a function must be either known or assumed.

As a simple example, suppose that it is desired to determine the density function for required mass ratio (initial weight/cutoff weight), knowing the density function for engine performance in terms of I_{sp}. The density function of a random variable is defined as the derivative of the distribution function of the variable, which is the probability of the variable being smaller than a given value. This relation is sketched graphically below, where

$$p\,[I] = \frac{\partial P[I]}{\partial I_{sp}} \tag{1}$$





Where an assumed function p [I] is provided, with an arbitrary ordinate, it must be normalized by the relation that follows from the definition of the distribution function P [I] and equation (1):

$$\int_0^{\infty} p\,[I]\;dI = P\,[\infty] = 1 \tag{2}$$

There is a distribution function for the required mass ratio, μ, that corresponds to the distribution function for I. For a given value I' there is a corresponding value μ'. The probability that μ is greater than μ' is equal to the probability that I is smaller than I', i. e.,

$$P\,[\mu'] = 1 - P\,[I'] \tag{3}$$

Usually it is the density function that is desired.

$$p\,[\mu] = \frac{\partial P\,[\mu]}{\partial \mu} \tag{4}$$

Substituting from (3),

$$\begin{aligned} p\,[\mu] &= \frac{-\partial\, P[I]}{\partial \mu} \\ &= \frac{-\partial P[I]}{\partial I}\;\frac{\partial I}{\partial \mu} \\ &= p\,[I]\left(\frac{\partial \mu}{\partial I}\right)^{-1} \end{aligned} \tag{5}$$

The relation between μ and I is

$$\mu = \exp \frac{\Delta V}{gI} \tag{6}$$

where ΔV is the ideal ΔV (disregarding gravity and drag losses).

$$\frac{\partial \mu}{\partial I} = \frac{-\Delta V}{gI^2}\;\exp \frac{\Delta V}{gI} \tag{7}$$

Substituting (7) and (5) allows determination of p $[\mu]$.

Analysis of a vehicle could proceed using I as a random variable instead of μ. However, it may be interesting to know the density function of μ. It will be assumed in the following analysis that a preliminary step has determined p $[\mu]$.

The equation for total mass of the second stage of a 2-stage vehicle or the total mass of a single stage vehicle is

$$\frac{M_T}{M_L} = \frac{\lambda\mu}{1-(1-\lambda)\mu} \tag{8}$$

M_T/M_L is a function of two random variables, λ and μ. The density function for M_T/M_L may be found as follows:

A dummy variable is introduced: $y = \mu$

Inverse relations are constructed: $\mu = g_1\left(y, \frac{M_T}{M_L}\right)$

$$\lambda = g_2\left(y, \frac{M_T}{M_L}\right) = \frac{\frac{M_T}{M_L}(1-y)}{y\left(1-\frac{M_T}{M_L}\right)} \tag{10}$$

The joint density function $p_{\mu\lambda}$ $[\mu, \lambda]$ is known:

$$p_{\mu\lambda}\,[\mu,\lambda] = p\,[\mu]\;\; p\,[\lambda] \tag{11}$$

This follows from the assumption that p $[\mu]$ and p $[\lambda]$ are independent. For convenience, letting $\zeta = M_T/M_L$, the joint density function $P_{\lambda,\,\zeta}\,[\lambda]$ if then found by:

$$p_{y,\,\zeta}\,(y,\zeta) = p_{\mu,\lambda}[g_1(y,\zeta), g_2(y,\zeta)] \begin{vmatrix} \frac{\partial g_1}{\partial\mu} & \frac{\partial g_2}{\partial\mu} \\ \frac{\partial g_1}{\partial\zeta} & \frac{\partial g_2}{\partial\zeta} \end{vmatrix} \tag{12}$$

$$p_{\lambda\zeta}\ (y,\zeta) = p_{\mu}\ (\mu)\ p_{\lambda}\left(\frac{\zeta\ (1-y)}{y\ (1-\zeta)}\right)\begin{vmatrix} \frac{\partial y}{\partial \mu} & \frac{\partial \lambda}{\partial \mu} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial \lambda}{\partial \zeta} \end{vmatrix} \tag{12a}$$

$\frac{\partial \lambda}{\partial \mu}$ and $\frac{\partial y}{\partial \zeta}$ are both zero.

$$\frac{\partial y}{\partial \mu} = 1 \tag{13}$$

$$\frac{\partial \lambda}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left[\frac{\zeta\ (1-y)}{y\ (1-\zeta)}\right] = \frac{1-y}{y\ (1-\zeta)^2}$$

(12a) becomes

$$p[y,\zeta] = p_{\mu}[\mu]\quad p_{\lambda}\left[\frac{\zeta\ (1-y)}{y\ (1-\zeta)}\right]\begin{vmatrix} 1 & 0 \\ 0 & \frac{1-y}{y\ (1-\zeta)^2} \end{vmatrix} \tag{14}$$

It is assumed that $p\ [y,\zeta] = p\ [y]\ \ p\ [\zeta]$.

Therefore, $p\ [\zeta]$ may be found by integrating over y:

$$p\ (\zeta) = \int_{-\infty}^{\infty} p\ (y,\zeta)\ dy \tag{15}$$

Since $y = \mu$ this is equivalent to

$$p\ (\zeta) = \int_{\mu\ min.}^{\mu\ max.} p_{y,\ \zeta}\ (\mu\ ,\ \zeta)\quad d\mu \tag{15a}$$

This integral may be performed for a number of values in the range of ζ. If it is also desired to get the distribution function for ζ, this may be found by

$$p\ (\zeta) = \int_{\zeta min.}^{\zeta} p\ (\zeta')\ d\zeta_1 \tag{16}$$

For a 2-stage vehicle, ζ in equation (16) becomes ζ_2 and $\zeta = \zeta_1\ \zeta_2$, since the second stage is the payload of the first stage. ζ_2 is now a random variable defined by equation (16). Analagous to equation (8),

$$\zeta = \frac{\zeta_2 \ \lambda_2 \ \mu_2}{1 - (1-\lambda_1)\mu_1} \tag{17}$$

Two dummy variables are required:

$$y_1 = \mu_1$$

$$z_1 = \zeta_2 \tag{18}$$

The same type of inverse relations hold:

$$\mu_1 = g_1 \ (y_1, z_1, \zeta) = y_1$$

$$\zeta_2 = g_2 \ (y_1, z_1, \zeta) = z_1$$

$$\lambda_1 = g_3 \ (y_1, z_1, \zeta) = \frac{\zeta \ (1-y_1)}{y_1 \ (1-\zeta)} \tag{19}$$

The joint density function is now:

$$p \ [\mu, \lambda_1, \zeta_2] = p \ [\mu_1] \ p \ [\lambda_1] \ p \ [\zeta_2] \tag{20}$$

and the joint density function $p_{y,z,\zeta} \ [y_1, z_1, \zeta]$

is then founded by:

$$p_{y,z,\zeta} \ [y_1, z_1, \zeta] = P\mu\lambda\zeta_2 \ [g_1, g_2, g_3] \begin{vmatrix} \frac{\partial g_1}{\partial \mu_1} & \frac{\partial g_2}{\partial \mu_1} & \frac{\partial g_3}{\partial \mu_1} \\ \frac{\partial g_1}{\partial z_1} & \frac{\partial g_2}{\partial z_1} & \frac{\partial g_3}{\partial z_1} \\ \frac{\partial g_1}{\partial \zeta} & \frac{\partial g_2}{\partial \zeta} & \frac{\partial g_3}{\partial \zeta} \end{vmatrix} \tag{21}$$

All of the partial derivatives are zero except

$$\frac{\partial g_1}{\partial \mu_1} = \frac{\partial g_2}{\partial z_1} = 1 \ \text{and} \ \frac{\partial g_3}{\partial \zeta} = \frac{1-y_1}{y_1 \ (1-\zeta)^2} \tag{22}$$

$p \ [\zeta]$ may be found by integrating y_1 and z_1

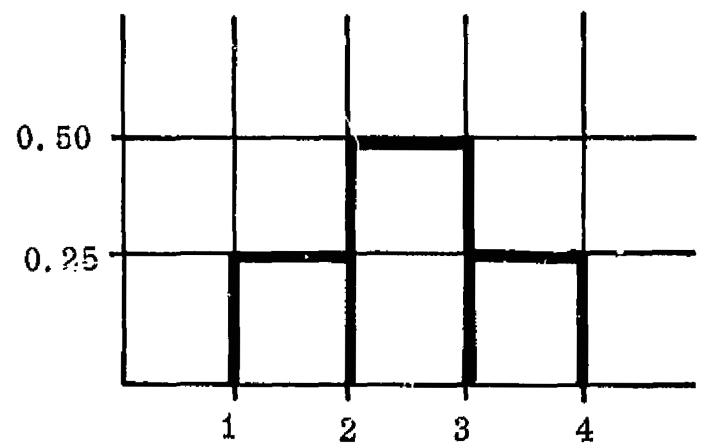
$$p \ [\zeta] = \int_{\zeta_2 \text{ min.}}^{\zeta_2 \text{ max.}} \int_{\mu_1 \text{ min.}}^{\mu_1 \text{ max.}} p \ (\mu_1, \zeta_2, \zeta) \ d\mu, \ d\zeta_2 \tag{23}$$

The distribution function P [ζ] is again found by equation (16).

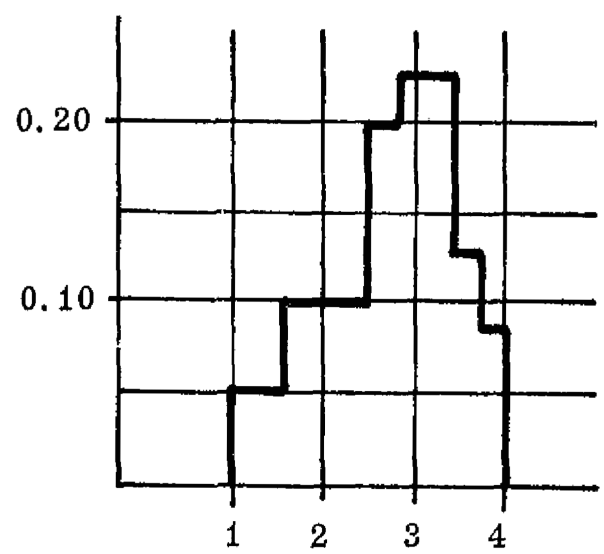
RESULTS

Explanation of Terminology

For the non-technical reader, an explanation is in order regarding the format of data to be shown. First, what is a density function? The density function represents something that is not precisely known and about which there is some uncertainty; it does, however, provide information on probability. If we have a function whose exact value is not known, and we know that there is a 25% probability that its value is between 3 and 4, we can draw a histogram, as shown below.



If we have more information about this function, that is, if we know the probability of its being between 1 and $1\frac{1}{2}$ and 2, etc., we can draw a histogram showing more detail, as in the following sketch.



If we divide the region into arbitrarily small elements instead of a stepped curve, then a continuous curve results. This is a density function. If the density function is normalized, any point on the curve represents the probability that the value of the function will fall within a unit range about the point. For example, consider a density function plotted versus a specific impulse obtainable from a rocket engine; a point plotted at a value such as 400 seconds represents the probability that the I_{sp} obtained will lie between 399.5 and 400.5 seconds. The importance of normalization lies in the fact that the integral from zero to infinity of a normalized density function is exactly equal to 1.

This integral then forms what is known as the distribution function. The distribution function is either constant or monotonically increasing and represents the probability that a random variable will be less than the abscissa value against which the distribution is plotted. If at an I_{sp} value of 400 the distribution function is equal to 0.4, then the probability is 40% that the I_{sp} will be less than 400, and 60% that it will be greater than 400.

It is convenient for the analyst using this random variable technique to make an "eyeball" estimate of the uncertainties in his sub-parameters in the form of density function. These may then be employed to show the effects of variations in sub-parameters on overall system performance parameters. It is convenient that the analyst should not have to produce a normalized density function. An ordinate scale of 0 to 5 is typically convenient. Normalization can be readily carried out by the computer, prior to its using the input density functions in the calculations.

Example Calculations

Computation of p $[\xi]$ and P $[\xi]$ was performed by a digital computer routine for certain 2-stage vehicle examples. Inputs were assumed density functions for first and second stage I_{sp} and propellant fractions, payload mass, and overall impulsive velocity increment for the vehicle, and staging velocity. Input density functions were given an arbitrary scale of 0 to 5 and were normalized by the computer routine. The following examples were given: a lunar direct flight and return vehicle, a Saturn-type 2-stage to orbit expendable vehicle, and a reusable space transport vehicle with rocket propulsion. Results from the calculations were automatically plotted on the SC-4020 plotter.

For the lunar direct flight vehicle a single-stage lander and single-stage return vehicle staging necessarily occurs at the lunar surface for the case shown. The given impulsive velocity increments are not typical of those currently used for lunar mission analysis, but they serve to illustrate typical results of this analysis. Both stages were assumed to have the same engine performance. The landing stage is assumed to have a better propellant fraction because it is larger and because it is not required to be well insulated for extended storage of cryogenic propellants. Figures 1 through 5 give the results for this vehicle. Irregularities in Figure 2 result from the finite difference nature of the numerical integrations employed.

The next two examples, where the staging velocity is not fixed by mission characteristics, were calculated for several staging velocities. This allows selection of a staging velocity that will give reasonable all-up weights with minimum sensitivity to uncertainty in inert weights and engine performance. Accumulated errors in numerical integrations may be noted by the fact that distribution function curves for these cases do not terminate exactly at 1. 0 as they should.

The random variable approach is not limited to the relatively simple cases shown. It could be extended to more than two stages, and variation in nominal propellant fraction with propellant loading could be incorporated without much difficulty.

Lunar Direct Flight Vehicle

One-stage Cryogenic Lander
Return Payload Mass = 5670 Kg
Return Ideal Delta V = 2.75Km/Sec
Landing Ideal Delta V = 3.697 Km/Sec

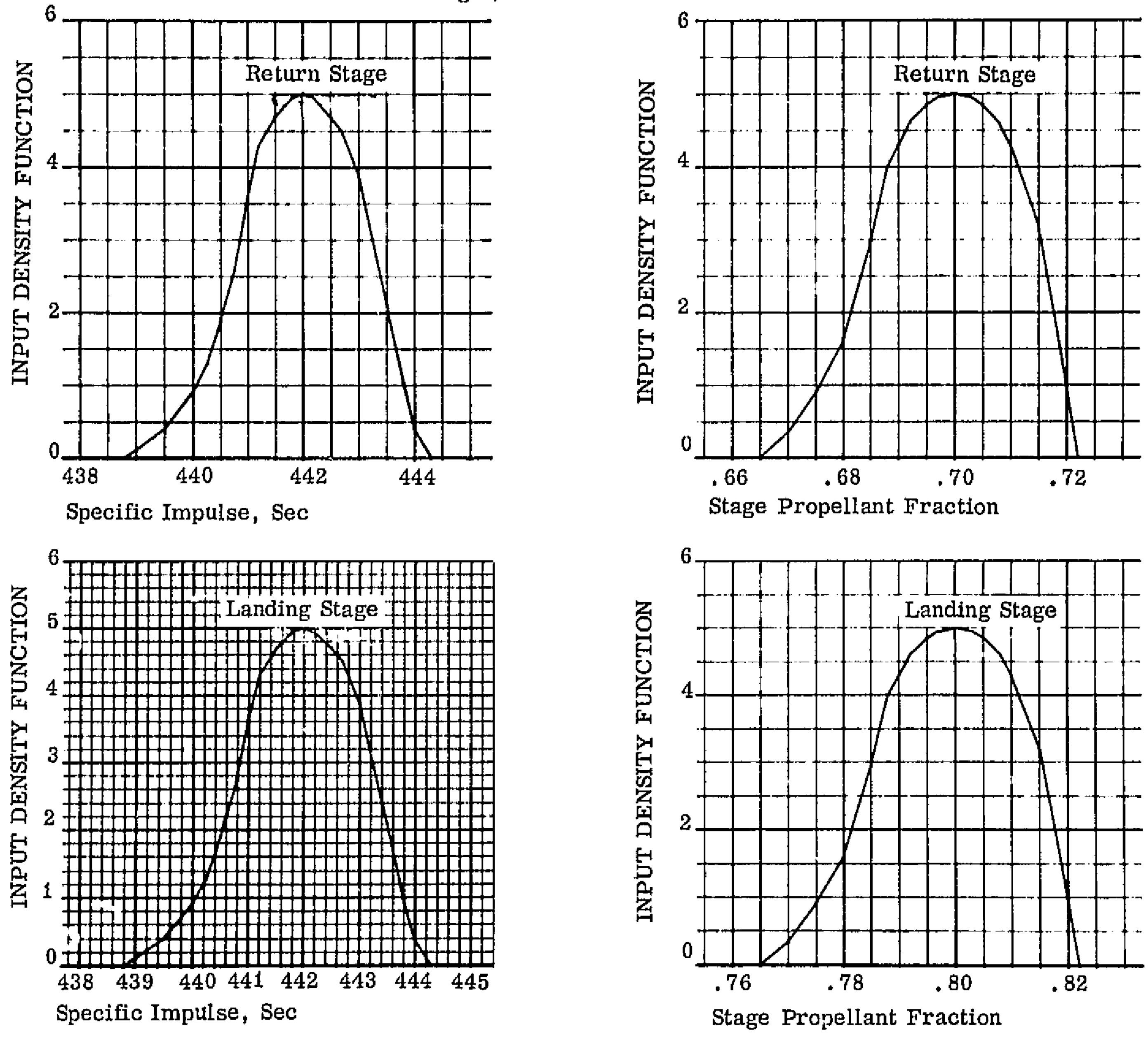


FIGURE 1

Lunar Direct Flight Vehicle
One-stage Cryogenic Lander
Return Payload Mass = 5670 Kg
Return Ideal Delta V = 2.75Km/Sec
Landing Ideal Delta V = 3.697 Km/Sec

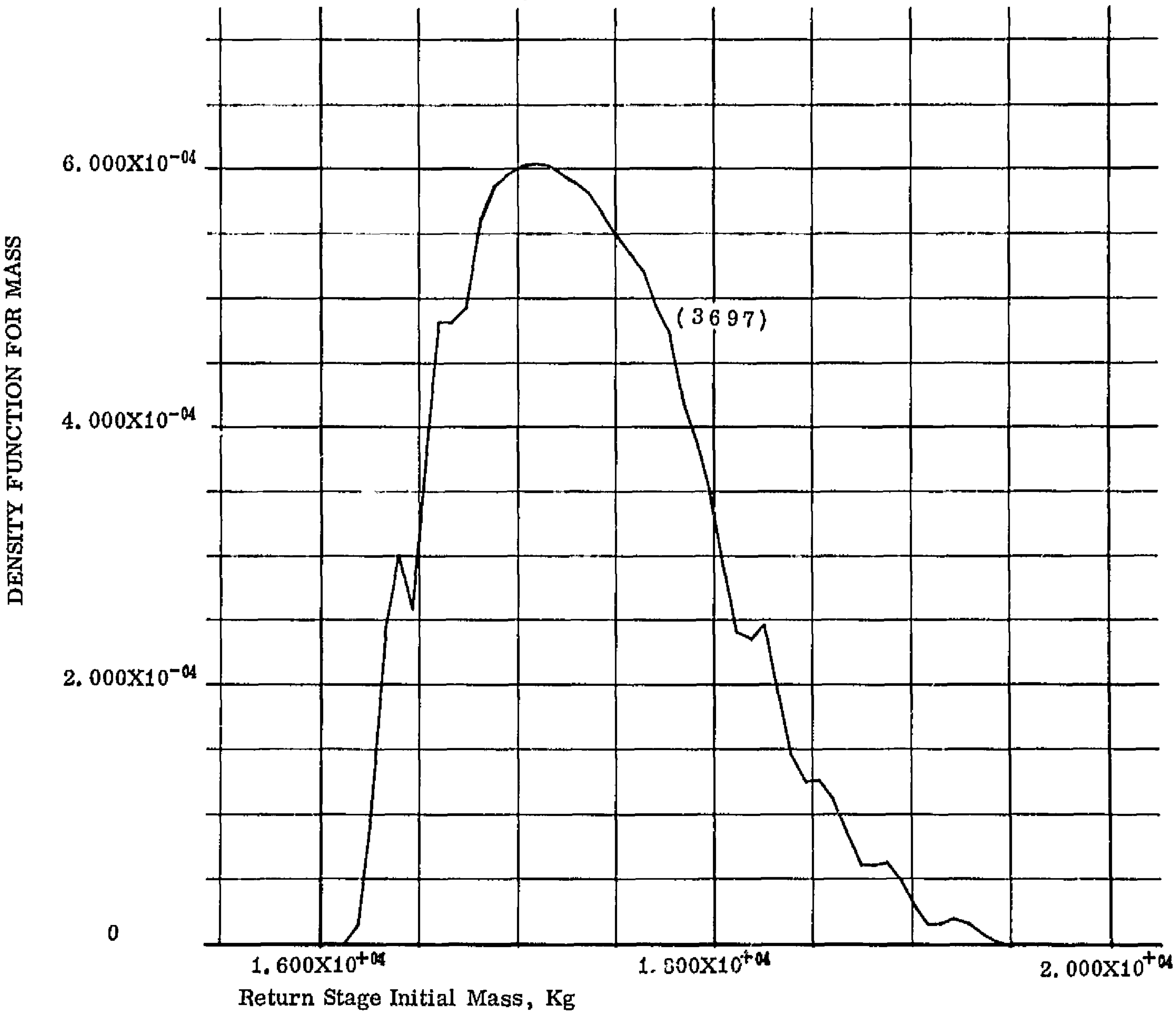
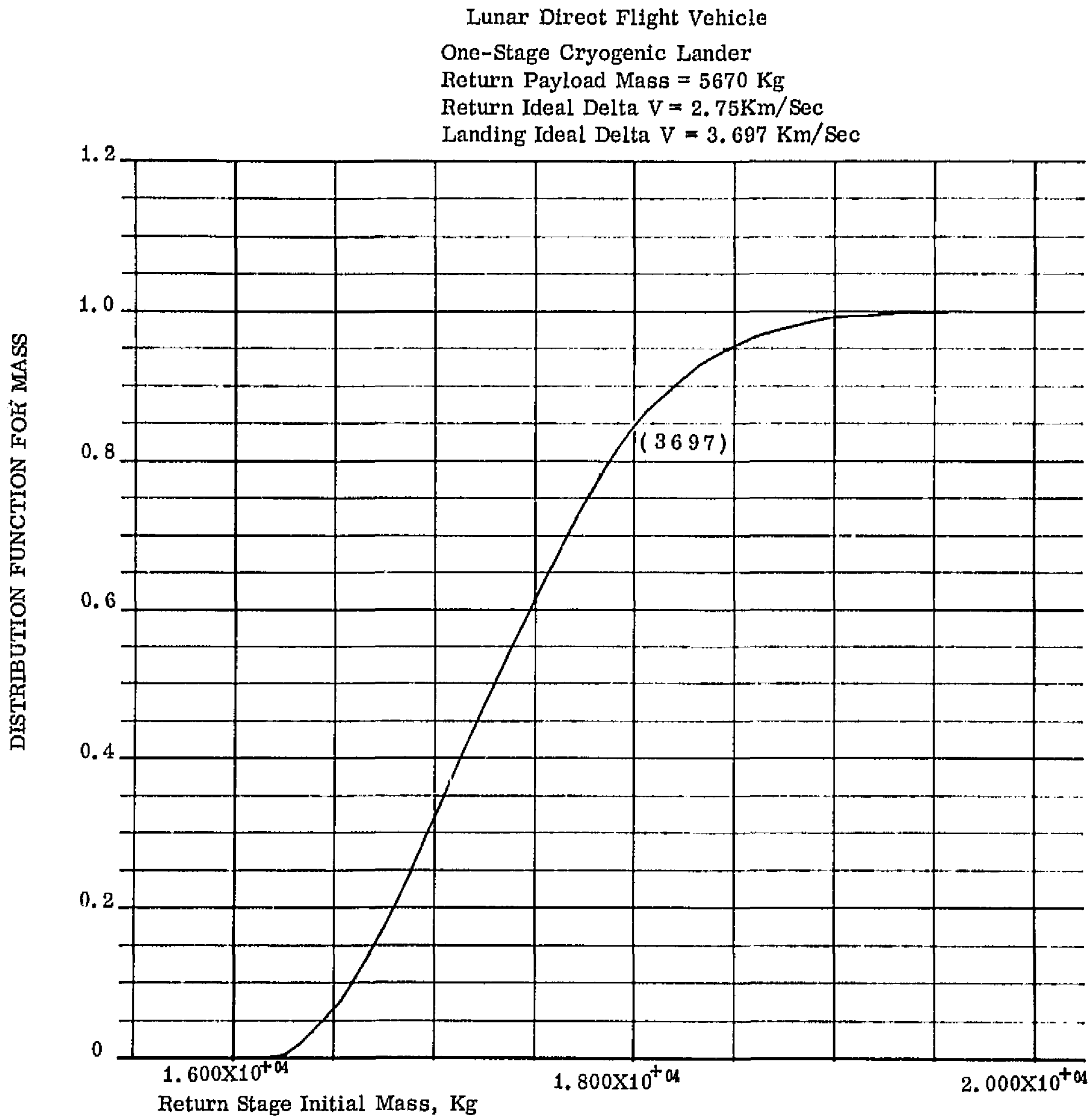


FIGURE 2



Lunar Direct Flight Vehicle
One-Stage Cryogenic Lander
Return Payload Mass = 5670 Kg
Return Ideal Delta V = 2.75Km/Sec
Landing Ideal Delta V = 3.697 Km/Sec
DISTRIBUTION FUNCTION FOR MASS
1.2
1.0
0.8
0.6
0.4
0.2
0
(3697)
1.600X10+04
1.800X10+04
2.000X10+04
Return Stage Initial Mass, Kg

FIGURE 3

Lunar Direct Flight Vehicle

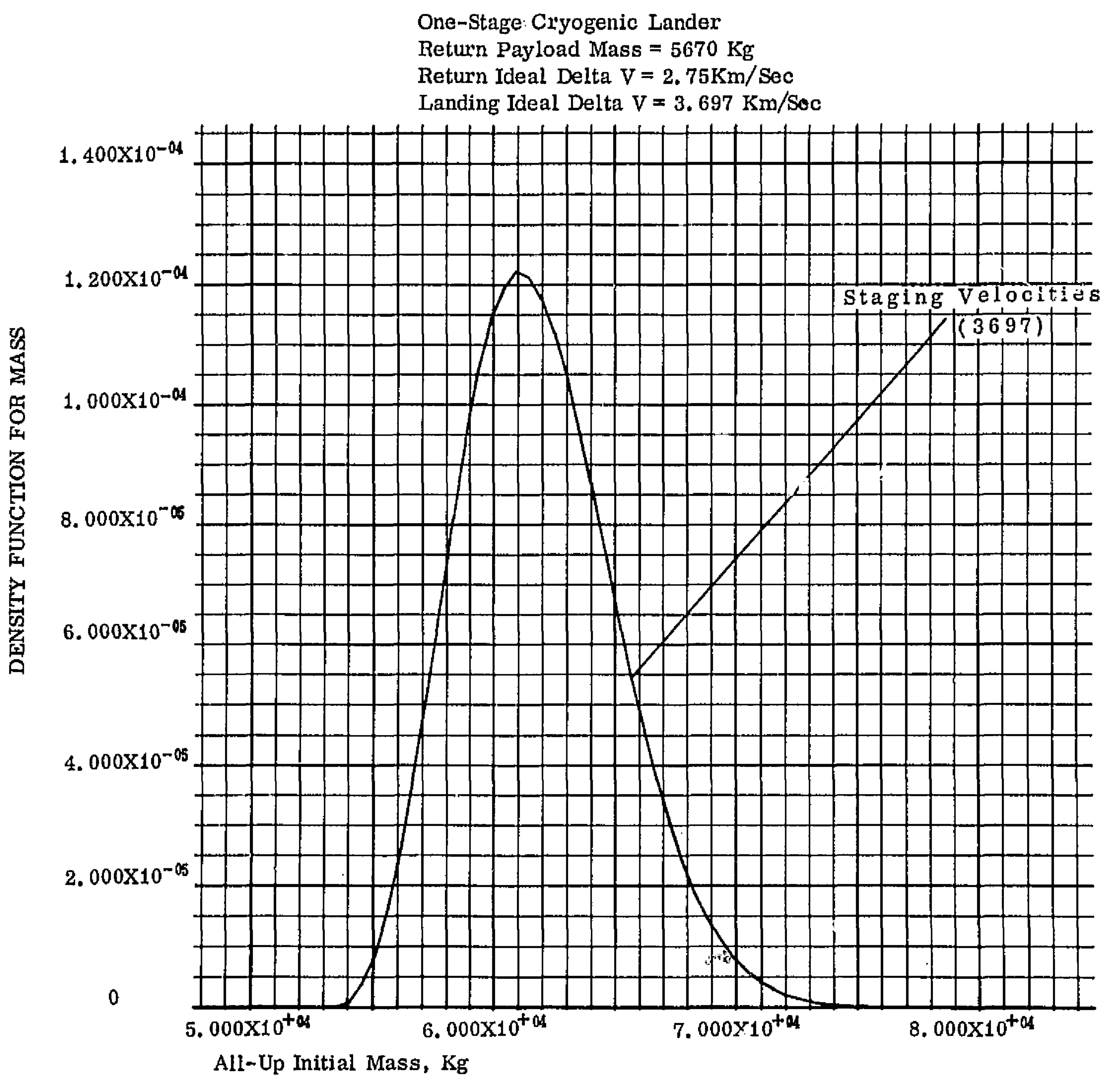


FIGURE 4

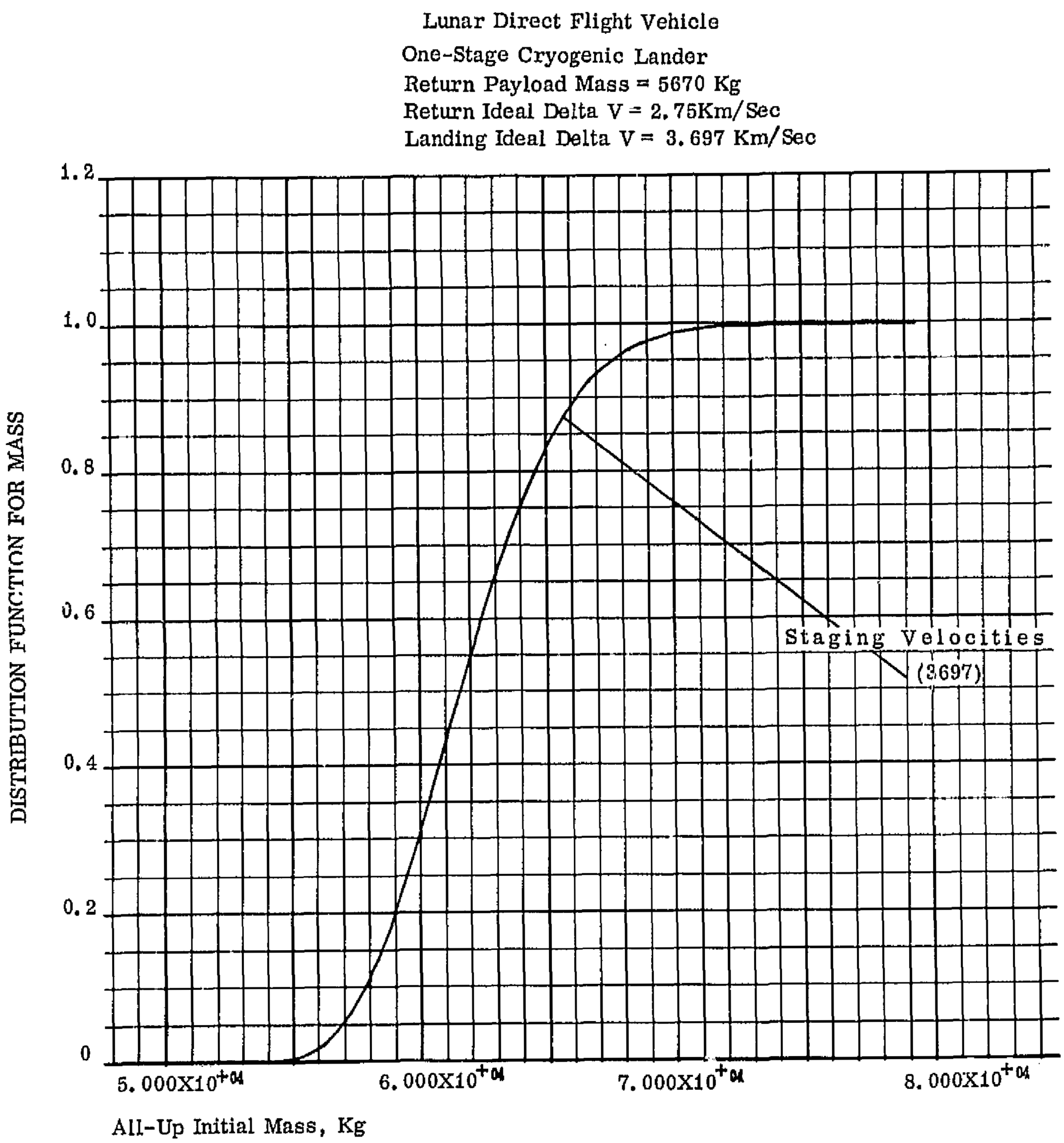

Lunar Direct Flight Vehicle
One-Stage Cryogenic Lander
Return Payload Mass = 5670 Kg
Return Ideal Delta V = 2.75Km/Sec
Landing Ideal Delta V = 3.697 Km/Sec
1.2
1.0
0.8
0.6
0.4
0.2
0
DISTRIBUTION FUNCTION FOR MASS
Staging Velocities
(3697)
5.000X10+04
6.000X10+04
7.000X10+04
8.000X10+04
All-Up Initial Mass, Kg

FIGURE 5

Saturn-Type Two Stage to Orbit Launch Vehicle

Payload Mass = 125 Metric Tons
Total Ideal Velocity Increment = 9150 M/Sec

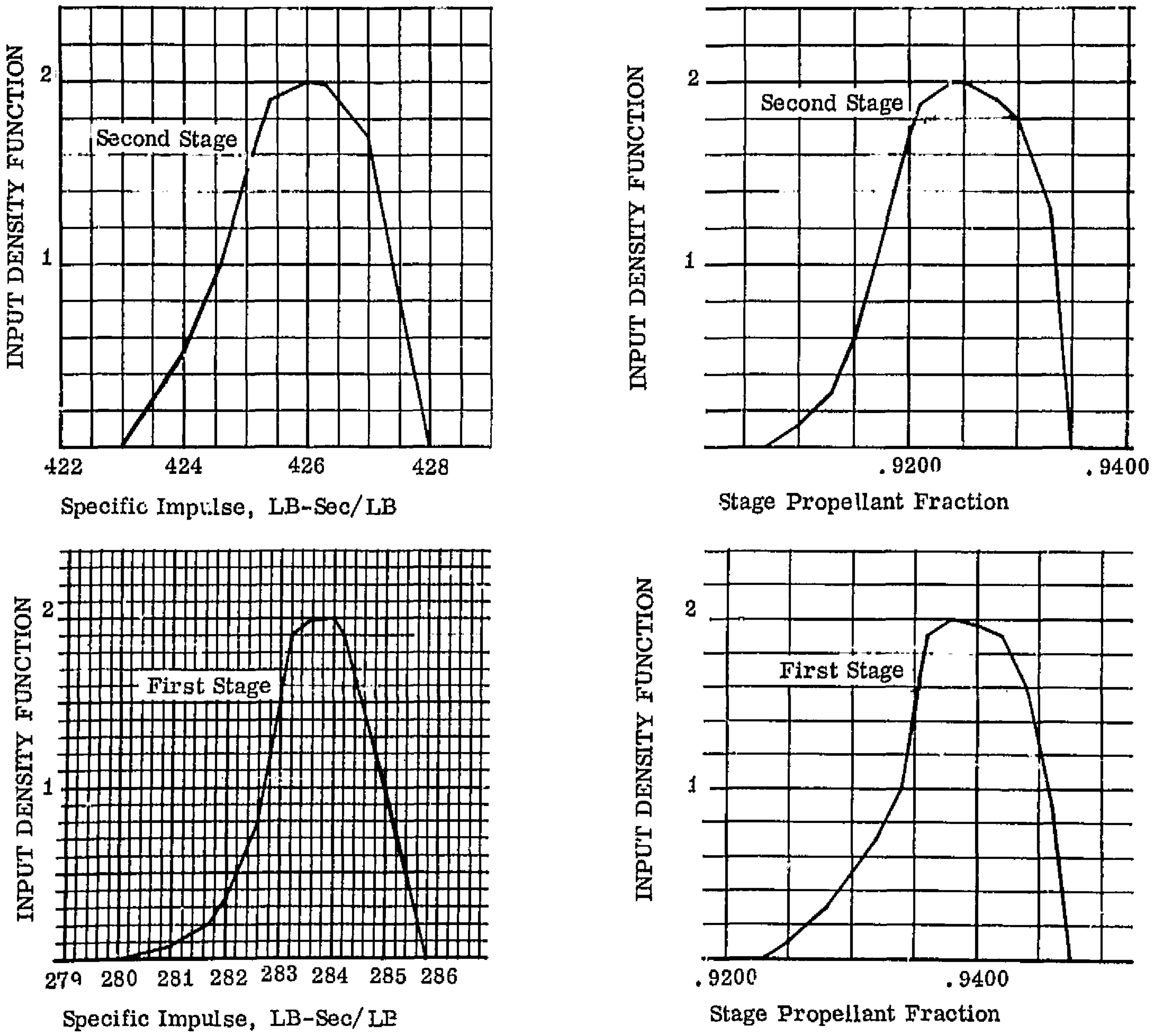


FIGURE 6

Saturn-Type Two-Stage-to-Orbit Launch Vehicle

Payload Mass = 125 Metric Tons
Total Ideal Velocity Increment = 9150 M/Sec

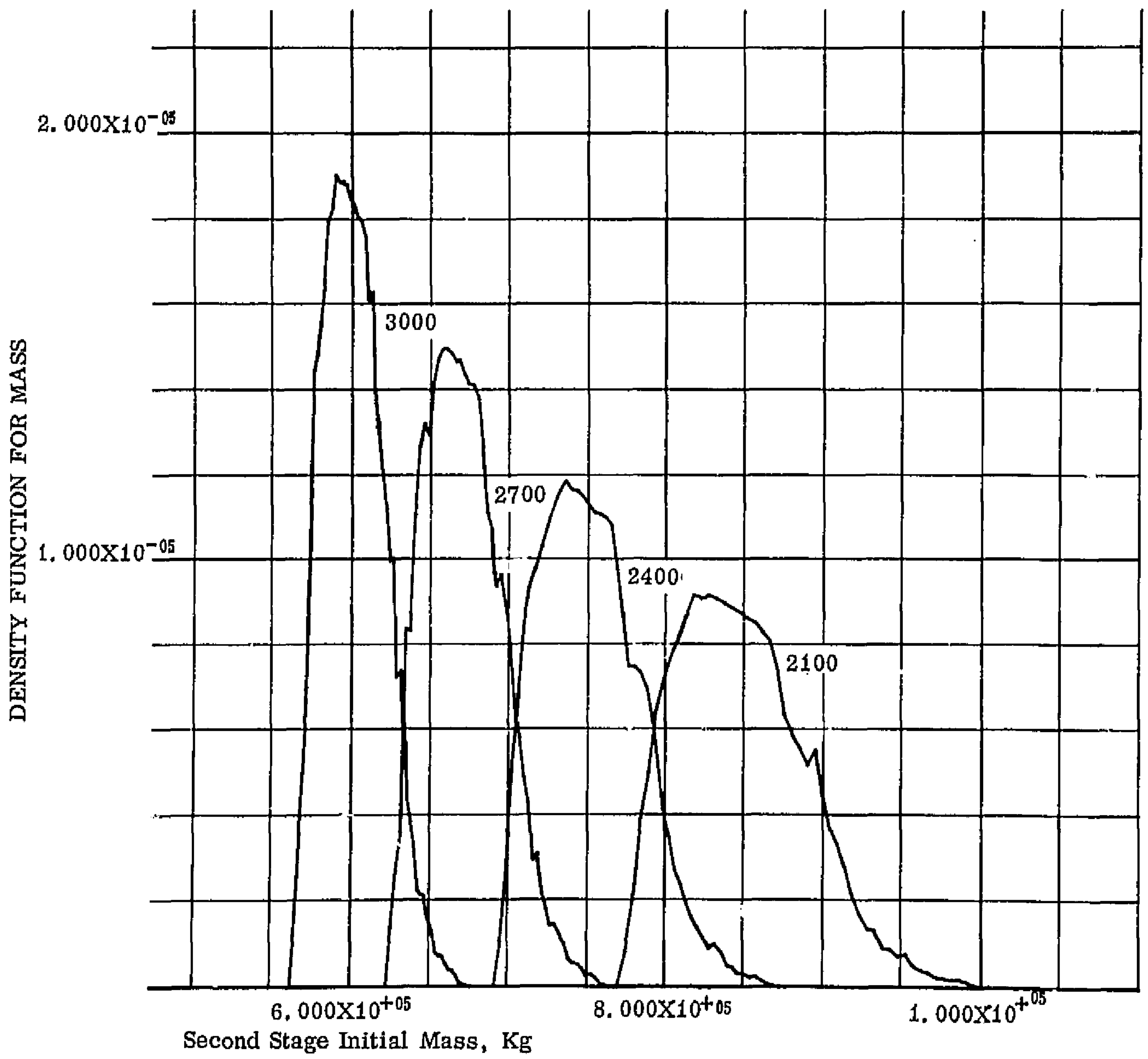


FIGURE 7

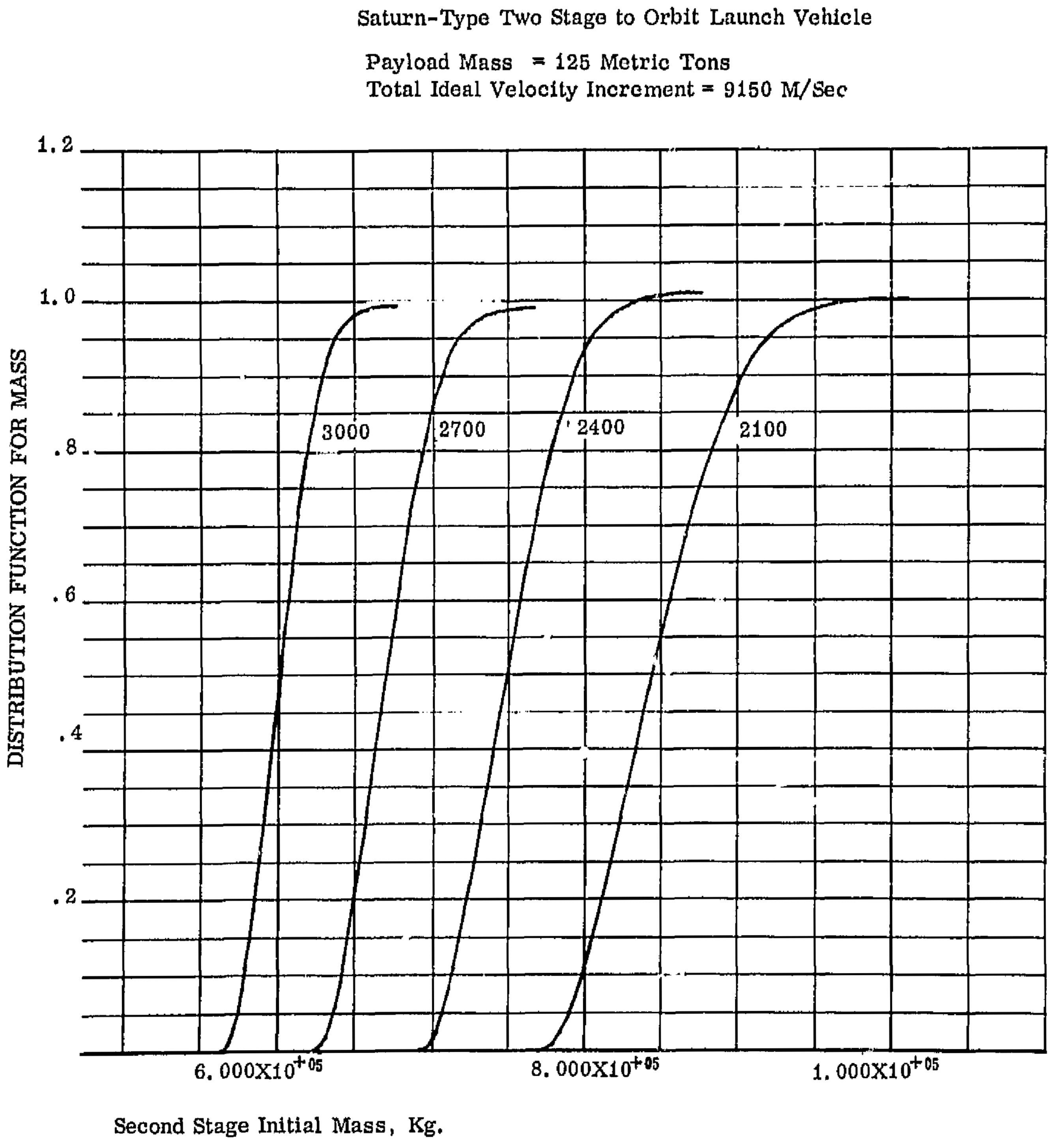

Saturn-Type Two Stage to Orbit Launch Vehicle
Payload Mass = 125 Metric Tons
Total Ideal Velocity Increment = 9150 M/Sec
1.2
1.0
.8
.6
.4
.2
DISTRIBUTION FUNCTION FOR MASS
3000
2700
2400
2100
6.000X10+05
8.000X10+05
1.000X10+05
Second Stage Initial Mass, Kg.

FIGURE 8

Saturn-Type Two Stage to Orbit Launch Vehicle
Payload Mass = 125 Metric Tons
Total Ideal Velocity Increment = 9150 M/Sec

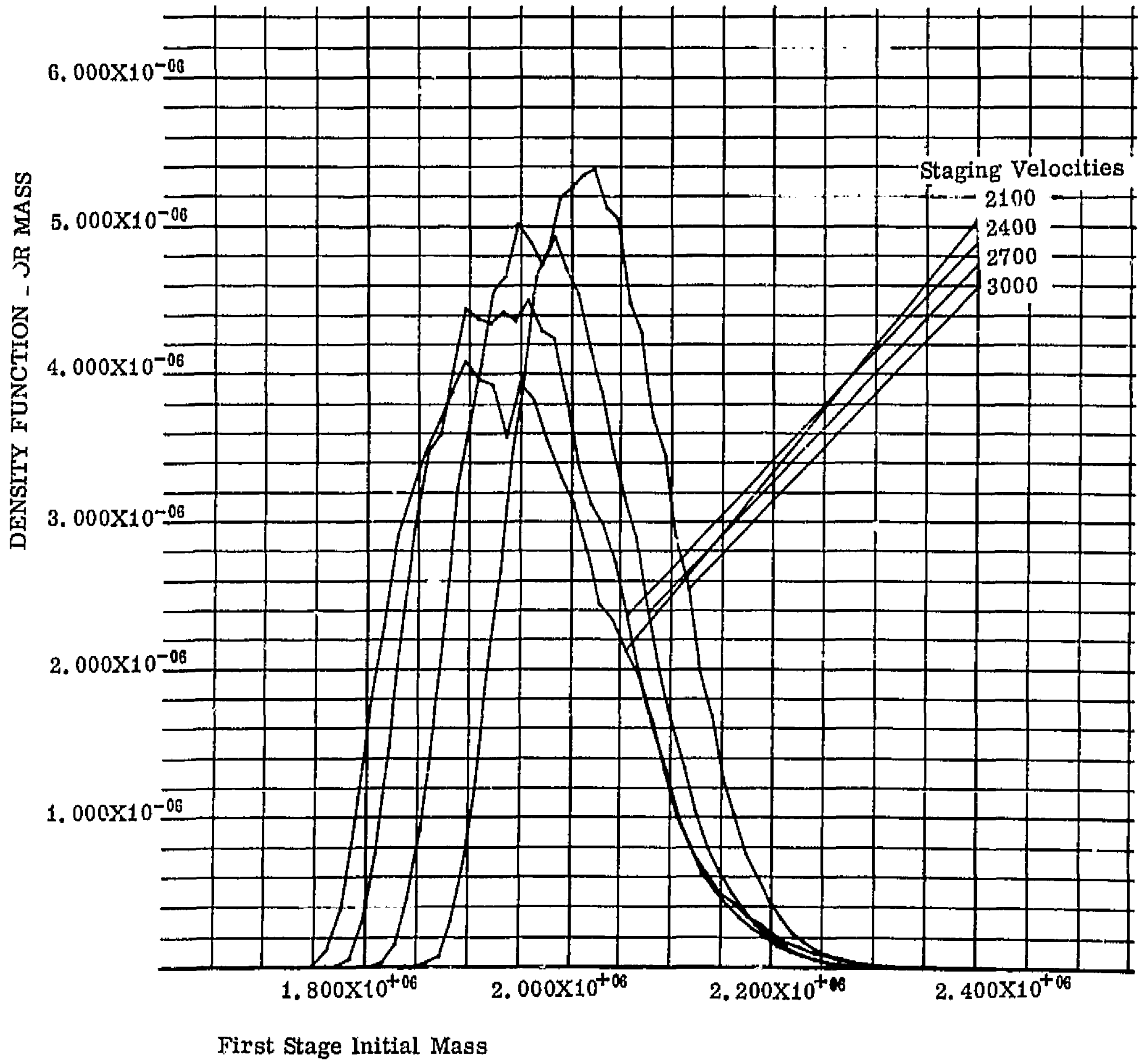


FIGURE 9

Saturn-Type Two Stage to Orbit Launch Vehicle

Payload Mass = 125 Metric Tons
Total Ideal Velocity Increment = 9150 M/Sec

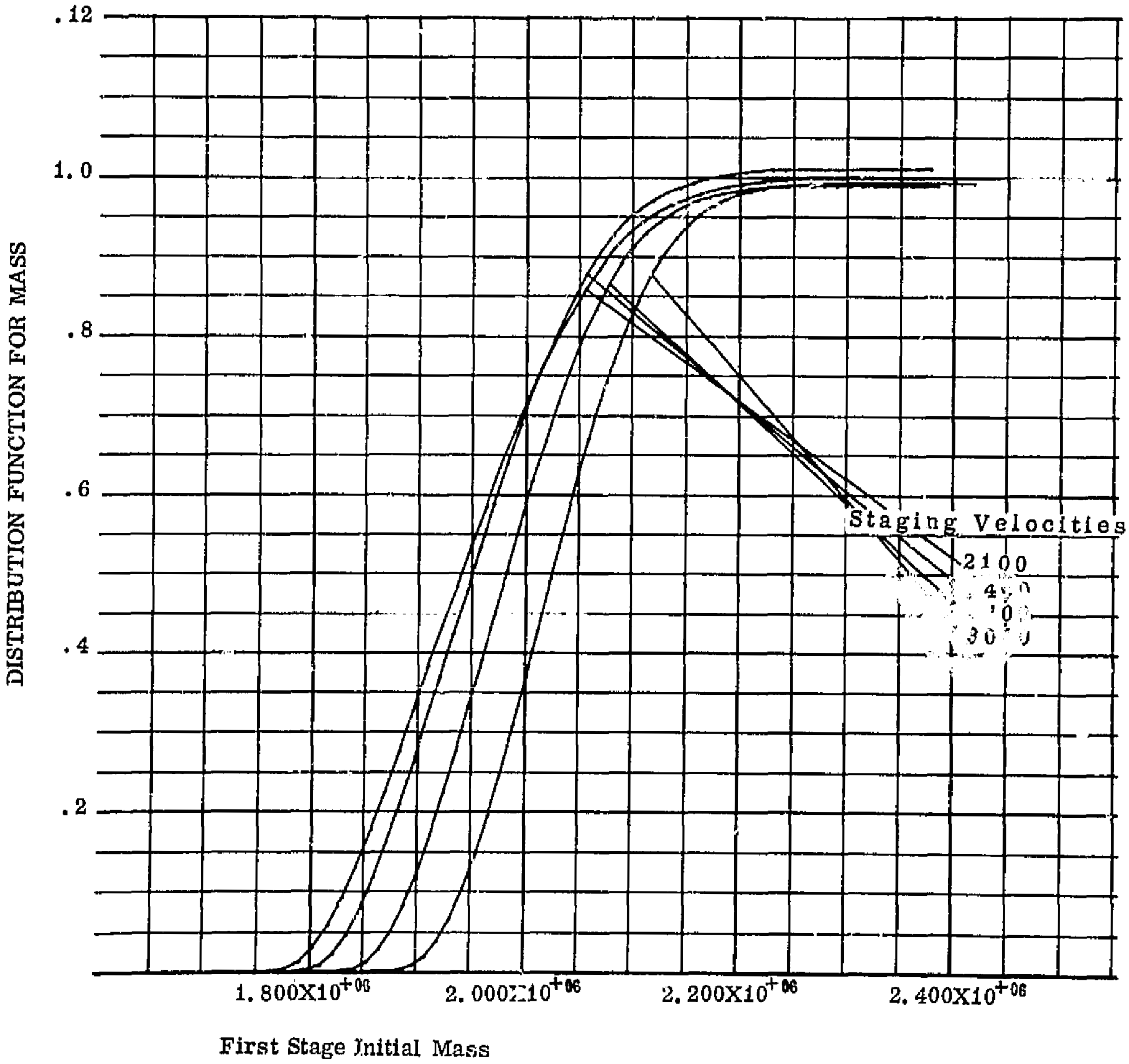


FIGURE 10

Reusable Space Transport, 2 Stage
F-1 First Stage, Hg-3 Second Stage
Payload Mass = 11.35 Metric Tons
Total Ideal Velocity Increment = 8.85 Km/Sec

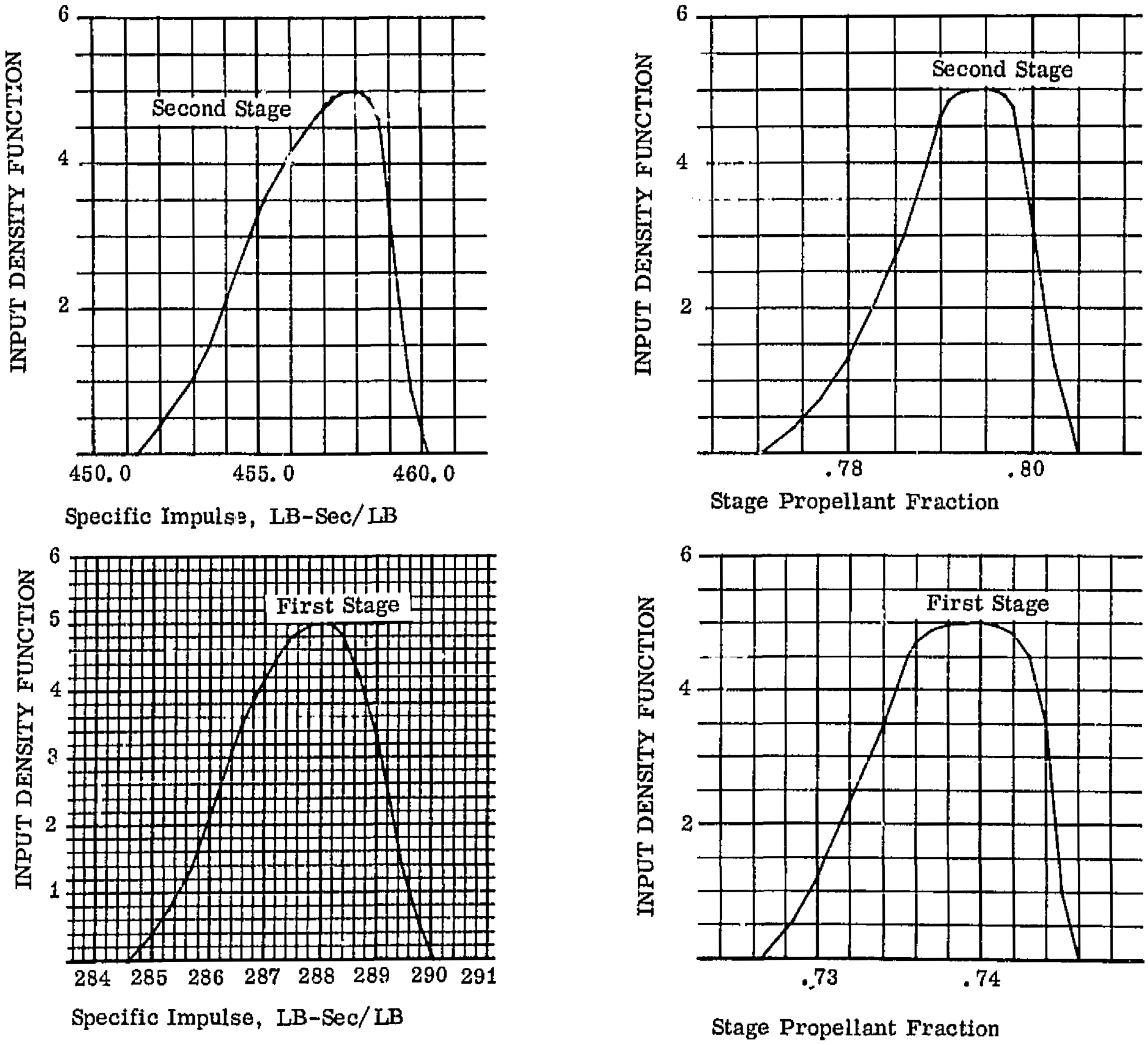


FIGURE 11

Reusable Space Transport, 2 Stage

F-1 First Stage, Hg-3 Second Stage
Payload Mass = 11.33 Metric Tons
Total Ideal Velocity Increment = 8.85 Km/Sec

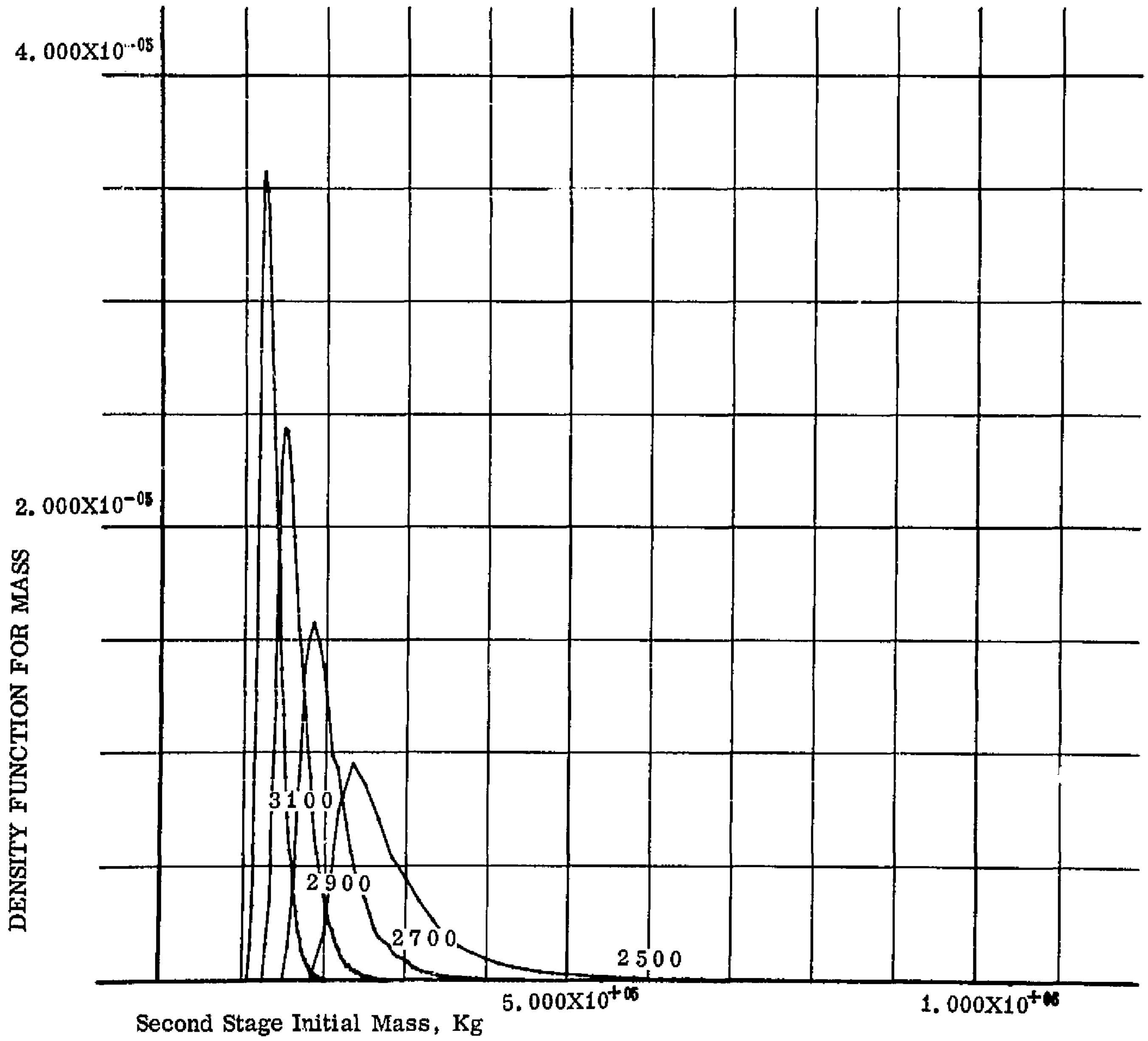


FIGURE 12

Reusable Space Transport, 2 Stage

F-1 First Stage, Hg-3 Second Stage
Payload Mass = 11.35 Metric Tons
Total Ideal Velocity Increment = 8.85 Km/Sec

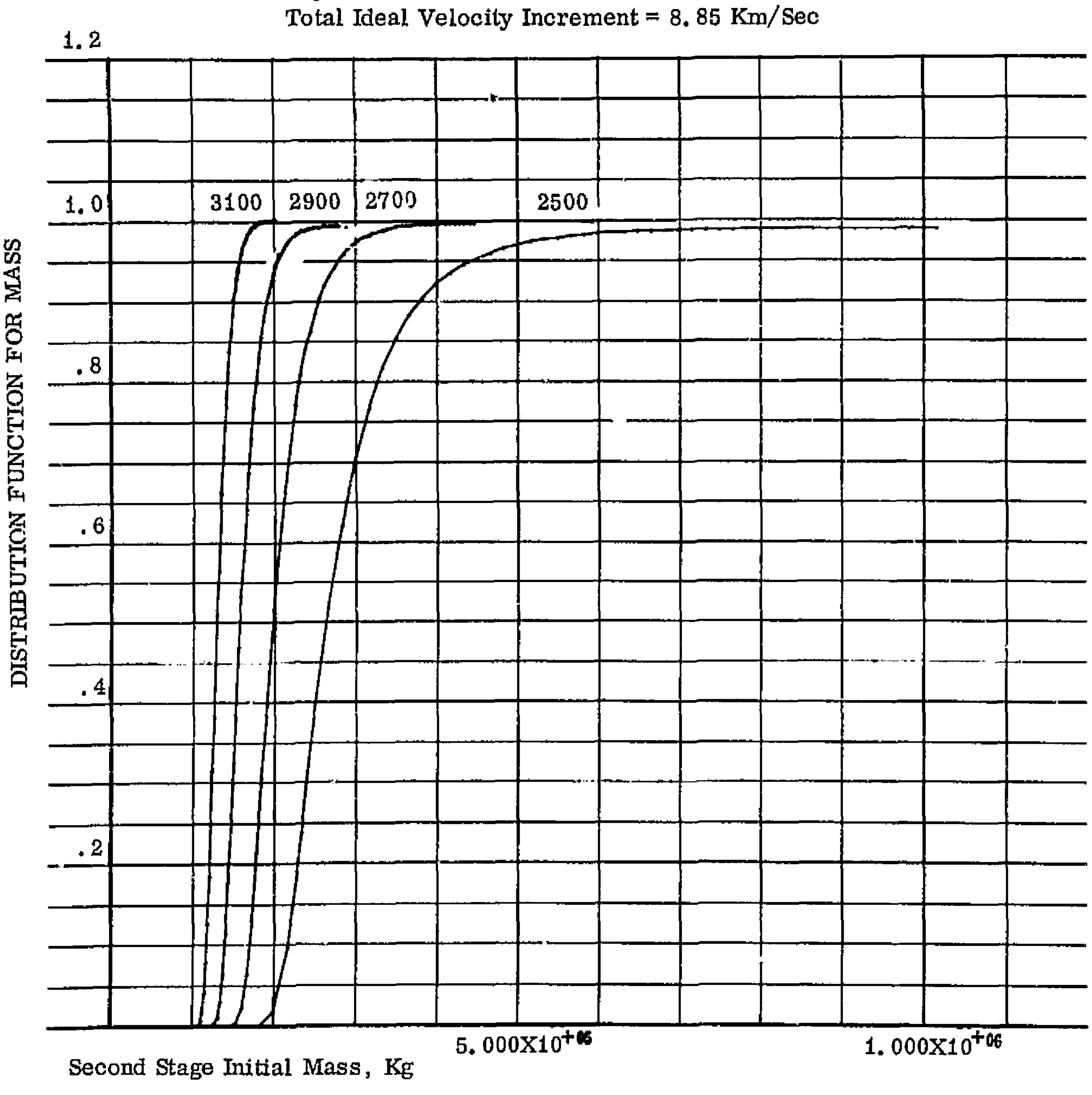


FIGURE 13

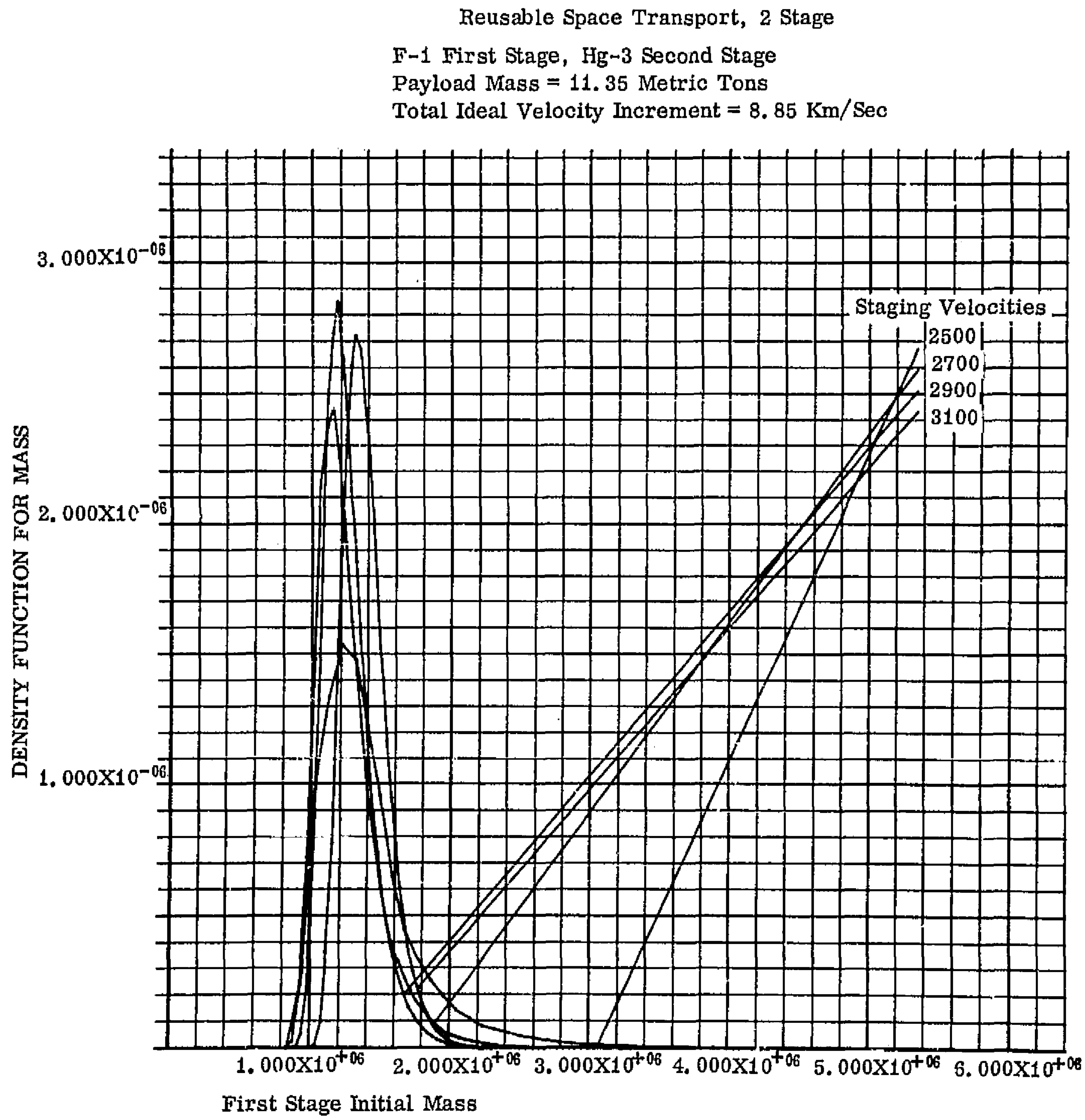

Reusable Space Transport, 2 Stage
F-1 First Stage, Hg-3 Second Stage
Payload Mass = 11.35 Metric Tons
Total Ideal Velocity Increment = 8.85 Km/Sec
DENSITY FUNCTION FOR MASS
3.000X10-06
2.000X10-06
1.000X10-06
Staging Velocities
2500
2700
2900
3100
1.000X10+06
2.000X10+06
3.000X10+06
4.000X10+06
5.000X10+06
6.000X10+06
First Stage Initial Mass

FIGURE 14

Reusable Space Transport, 2 Stage
F-1 First Stage, Hg-3 Second Stage
Payload Mass = 11.35 Metric Tons
Total Ideal Velocity Increment = 8.85 Km/Sec

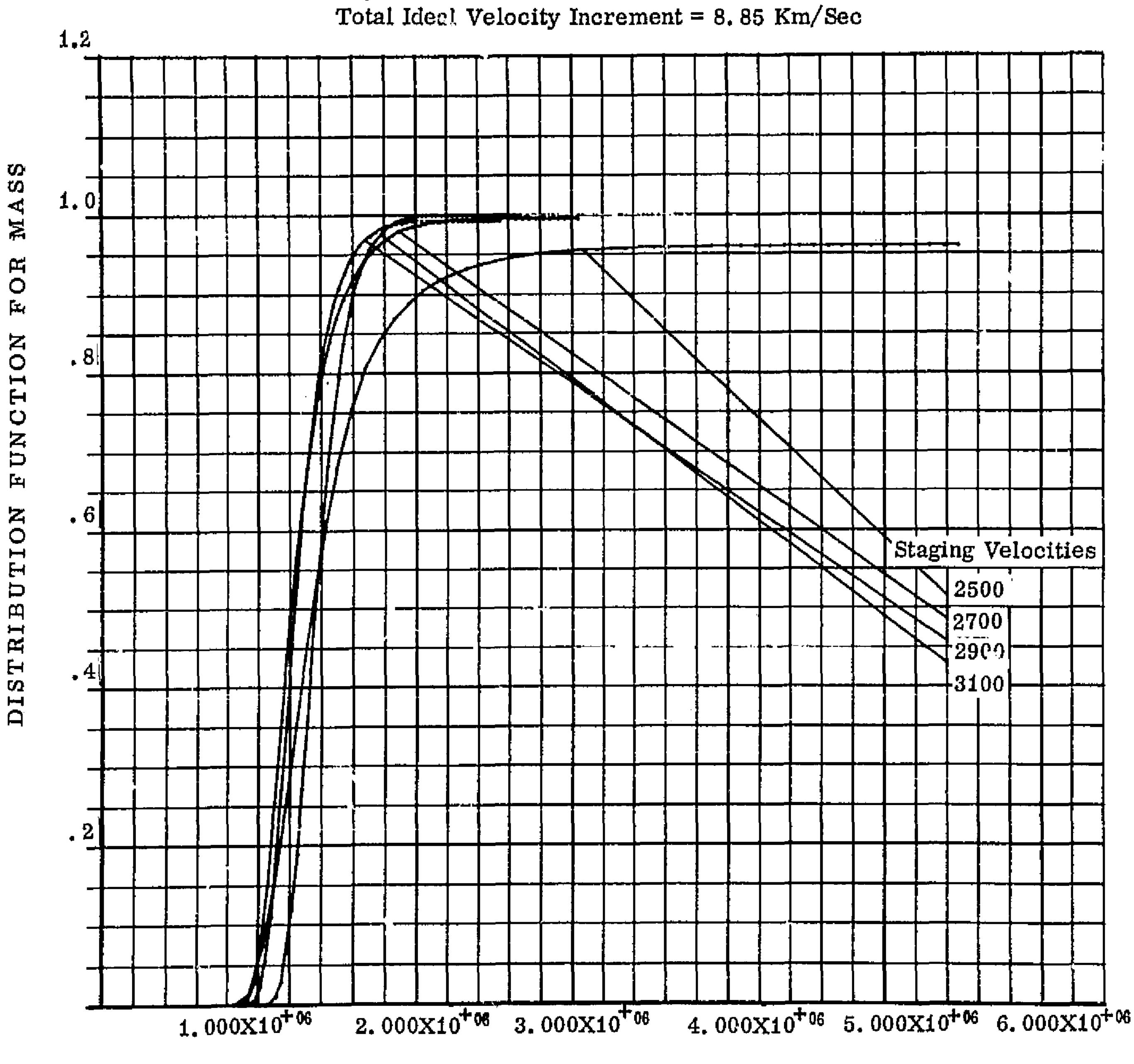


FIGURE 15

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APPROVAL

PARAMETRIC SENSITIVITY ANALYSIS FOR STAGED PROPULSIVE VEHICLES

By G. R. Woodcock

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

L. T. Spears

L. T. SPEARS
Chief, Voyager Definition Task Group