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45

PHYSICS OF THE CLOCK EXPERIMENT

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SUMMARY

A simple treatment of the relativistic clock experiment has been given in terms of observed frequency shifts of a standard oscillator or clock. No attempt has been made to introduce the details of atmospheric disturbances, measurement errors, or electronic changes in the basic clock frequency to provide for more convenient transmission, reception, or signal comparisons.

I. INTRODUCTION

Any device which changes with time can be used as a clock, whether it is the amount of water in a bucket with a hole in it, the amount of sand in an hour-glass, or the frequency of an atomic oscillator. Ordinarily, some periodic device, such as a pendulum or an electronic oscillator, is a convenient clock because we can simply count the number of oscillations to define an interval of time. It is a basic assumption of general relativity that an ideal clock next to an observer always runs at the same rate (i. e., always gives the proper time). It may happen, however, that two observers at different places, but with identical clocks, may find that their clocks disagree (i. e., that their clocks appear to run at different speeds).

II. THE DOPPLER SHIFT

The Doppler shift for electromagnetic radiation is a function of the relative motion between the transmitter and receiver. If we define an inertial coordinate system with the origin at the transmitter and locate the receiver by a position vector $\overline{r}$, the following relationship may be expressed as

$$\gamma \nu = \nu_0 \left(1 - \frac{\overline{v} \cdot \hat{r}}{c}\right), \tag{1}$$

where γ is the time dilatation factor of special relativity, ν is the frequency received, ν_0 is the frequency transmitted, c is the speed of signal propagation, and $\overline{v}$, $\hat{r}$, and γ are defined by the equations,

$$\overline{v} = d\overline{r}/dt, \tag{2}$$

$$\hat{r} = \overline{r}/|\overline{r}|, \tag{3}$$

$$\gamma = 1/(1 - \overline{v} \cdot \overline{v}/c^2)^{\frac{1}{2}}. \tag{4}$$

If $\overline{v}$ and $\hat{r}$ are perpendicular, Equation (1) becomes

$$\gamma \nu = \nu_0, \tag{5}$$

which is the basis for the relativity statement that moving clocks (or oscillators) appear to run more slowly than identical clocks that are not moving.

If the term $\overline{v} \cdot \overline{v}/c^2$ of Equation (4) is defined as

$$\overline{v} \cdot \overline{v}/c^2 \equiv \beta^2, \tag{6}$$

and

$$\overline{v} \cdot \hat{r}/c \equiv \beta \cos \alpha, \tag{7}$$

where α is the angle $(\overline{v}, \hat{r})$, Equation (1) can be restated as

$$\nu = \nu_0 (1 - \beta^2)^{\frac{1}{2}} (1 - \beta \cos \alpha). \tag{8}$$

If β is considered to be a small quantity this expression can be expanded in the form,

$$\nu = \nu_0 (1 - \beta \cos \alpha - \tfrac{1}{2} \beta^2 + \ldots). \tag{9}$$

The Doppler shift Δ_D can then be written as

$$\Delta_D = \nu - \nu_0 = - \beta \cos \alpha - \tfrac{1}{2} \beta^2, \qquad (10)$$

where the first term on the right is the classical Doppler shift and the second term (always negative) arises from the time dilatation of special relativity.

III. THE GRAVITATIONAL SHIFT

The gravitational shift can be explained most simply by thinking of the rf signal as a collection of photons having a transmission frequency of ν_0 . The energy E_0 of each photon is defined as

$$E_0 = h\nu_0 , \qquad (11)$$

and the equivalent mass m is defined as

$$m = E_0/c^2 = h\nu_0 /c^2, \qquad (12)$$

where h is the Planck constant. Whenever a mass m moves in a conservative force field $\bar{F}(\bar{r})$ from a position $\bar{r}_1$ to a position $\bar{r}_2$, its potential energy V is changed by an amount ΔV,

$$\Delta V = - \int_{\bar{r}_1}^{\bar{r}_2} \bar{F} \cdot d\bar{r}, \qquad (13)$$

where $\bar{F}$ is the force acting on the mass m, and $\bar{r}$ is the position vector locating the mass. Let us imagine that a photon is constrained to move radially outward from a position $\bar{r}_1$ on the rotating Earth to a position $\bar{r}_2$ stationary above some particular point on the Earth. The change in potential energy is given by the expression,

$$\Delta V = - \int_{\bar{r}_1}^{\bar{r}_2} \left(- \frac{GMm}{r^2} + m\,\omega^2 r\right) dr \equiv m\Delta\phi, \tag{14}$$

in which G is the gravitational constant, M is the mass of the Earth, ω is the angular rotational velocity of the Earth, r is the distance between the centers of M and m, the first term of the integrand represents the attractive gravitational force, the second term is the centrifugal force, and ϕ is the "potential" function associated with the rotating gravitational field. This "potential" function is defined by the equation,

$$\phi = - \frac{GM}{r} - \frac{\omega^2 r^2}{2}, \tag{15}$$

so that we can compute

$$\Delta\phi = \phi(\bar{r}_2) - \phi(\bar{r}_1). \tag{16}$$

Assuming that the total energy H of a rising photon is conserved, we can write

$$H \equiv E + V = \text{constant}, \tag{17}$$

or

$$\Delta E + \Delta V = 0. \tag{18}$$

Therefore, from Equations (11), (12), (14), and (18), we have

$$\Delta E = \Delta(h\nu) = h\Delta\nu, \tag{19}$$

$$\Delta V = m\,\Delta\phi = (h\nu_0/c^2)\,\Delta\phi, \tag{20}$$

$$h\Delta\nu = -(h\nu_0/c^2)\,\Delta\phi. \tag{21}$$

We define

$$\Delta_G \equiv |\Delta\nu| = |\nu_0\,(\Delta\phi/c^2)|, \tag{22}$$

where Δ_G is the absolute value of the gravitational frequency shift. It should be noted that the gravitational frequency shift is less than zero for a rising photon and greater than zero for a falling photon.

IV. DISCUSSION

Let us suppose that we have two observers, one on the Earth (ground) and one on a satellite. The ground observer has a clock of frequency ν_0, two receivers to monitor broadcasts from the satellite, and a transmitter to transmit his clock frequency ν_0 to the satellite. The satellite observer (which can be a circuit and not a man) has a receiver-transmitter to receive signals from the ground and retransmit them to the ground. He can also transmit a signal ν_0 from a clock identical to the one on the ground.

The ground transmitter will transmit a signal of frequency ν_0. The frequency ν_1 received on the satellite will be

$$\nu_1 = \nu_0 - \Delta_G + \Delta_D , \tag{23}$$

where Δ_G is the absolute gravitational shift and Δ_D is the Doppler shift. If the frequency ν_1 is compared to the satellite frequency ν_0, the frequency difference $\Delta\nu_1$ is found to be

$$\Delta\nu_1 \equiv \nu_1 - \nu_0 = -\Delta_G + \Delta_D . \tag{24}$$

If the frequency ν_1 is rebroadcast to the ground, it is received as a frequency ν_2,

$$\nu_2 = \nu_1 + \Delta_G + \Delta_D = \nu_0 + 2\Delta_D , \tag{25}$$

so that we find

$$\Delta\nu_2 \equiv \nu_2 - \nu_0 = 2\Delta_D . \tag{26}$$

It is interesting to note here that the round trip has eliminated the gravitational shift and has doubled the Doppler shift.

If the satellite broadcasts a frequency ν_0 toward the ground, the received frequency ν_3 is given by the equation,

$$\nu_3 = \nu_0 + \Delta_G + \Delta_D, \tag{27}$$

and the frequency shift, compared to the ν_0 frequency standard on the ground, is

$$\Delta \nu_3 \equiv \Delta_G + \Delta_D. \tag{28}$$

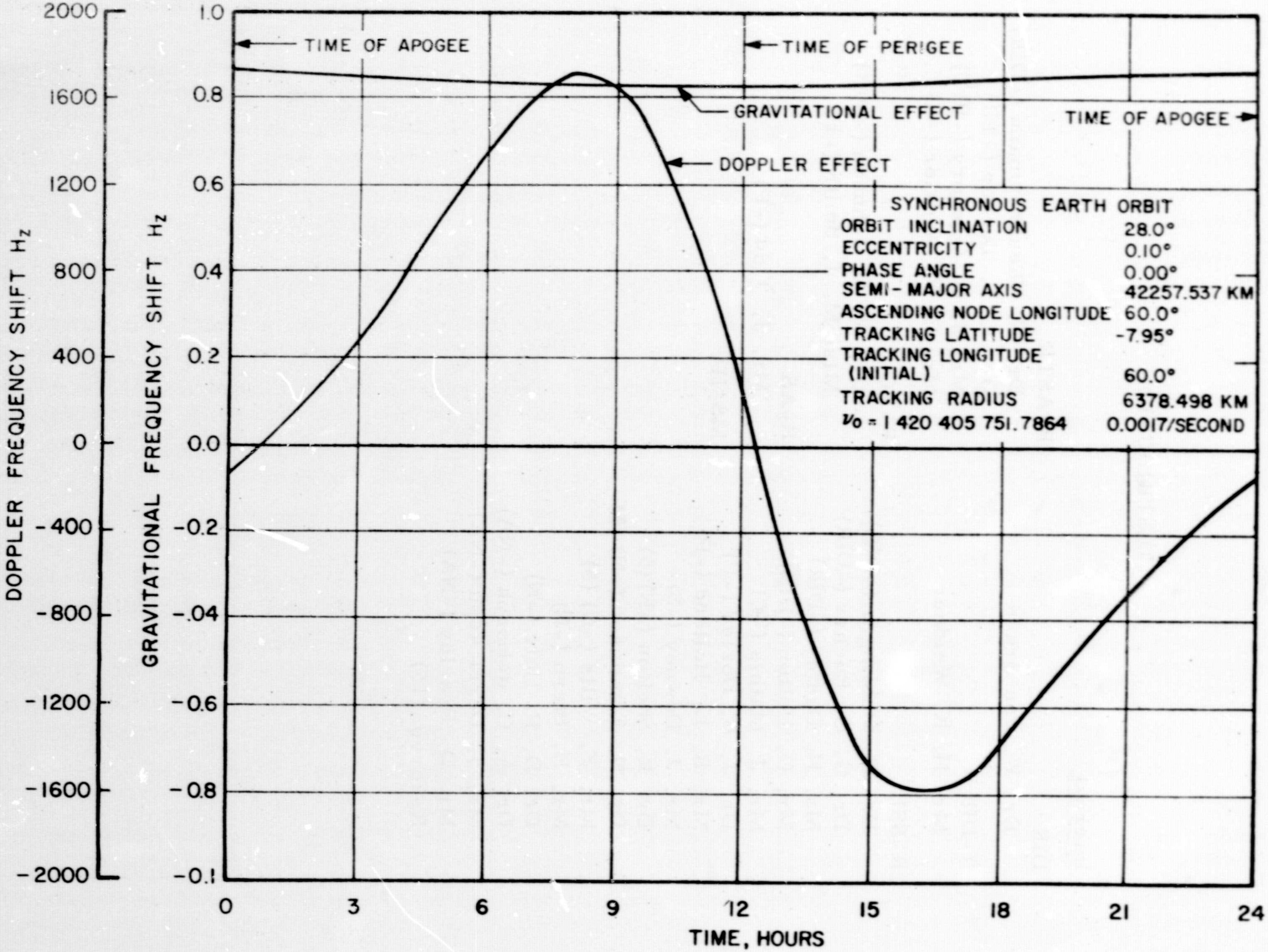
The frequency shifts $\Delta \nu_2$ and $\Delta \nu_3$ given by Equations (26) and (28), respectively, are measured on the ground, and simple algebraic operations yield Δ_D and Δ_G to be

$$\Delta_D = \Delta \nu_2 / 2, \tag{29}$$

$$\Delta_G = \Delta\nu_3 - \Delta \nu_2 / 2. \tag{30}$$

The example stated in this discussion is illustrated in the following graph, which shows predicted values of the absolute frequency shifts.

The data for the curves were calculated on the basis of a 24-hour satellite orbit having an inclination of 28° and an eccentricity of 0.1. The ground transmitter and receiver are considered to be on Ascension Island. The effects of Earth rotation are not included in the calculations; if included, they would lower the value of the gravitational shift approximately 10 percent.



TIME OF APOGEE
TIME OF PERIGEE
GRAVITATIONAL EFFECT
TIME OF APOGEE
DOPPLER EFFECT
SYNCHRONOUS EARTH ORBIT
ORBIT INCLINATION 28.0°
ECCENTRICITY 0.10°
PHASE ANGLE 0.00°
SEMI-MAJOR AXIS 42257.537 KM
ASCENDING NODE LONGITUDE 60.0°
TRACKING LATITUDE -7.95°
TRACKING LONGITUDE (INITIAL) 60.0°
TRACKING RADIUS 6378.498 KM
ν_o = 1 420 405 751.7864 0.0017/SECOND
DOPPLER FREQUENCY SHIFT Hz
2000
1600
1200
800
400
0
-400
-800
-1200
-1600
-2000
GRAVITATIONAL FREQUENCY SHIFT Hz
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-.04
-0.6
-0.8
-0.1
0
3
6
9
12
15
18
21
24
TIME, HOURS

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