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USE OF LOW CONSTANT THRUST TO ACCOMPLISH ORBITAL TRANSFER

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ABSTRACT

A fuel-minimized trajectory is generated, satisfying initial and final boundary conditions for the problem of a constantly thrusting space vehicle which is transferred from an initial elliptical orbit around Mars to a circular orbit of lower energy. The constant-thrust solution is amenable to electric ion-propelled spacecraft having an initial elliptical or circular planetocentric orbit which must then be powered to a circular orbit of desired altitude.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

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SPACE SCIENCES LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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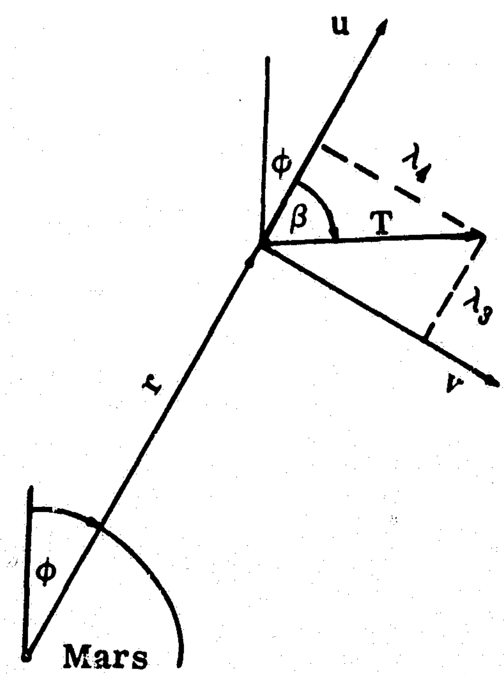
DEFINITION OF SYMBOLS

Symbol	Definition
ϕ	range angle (deg)
r	radial distance of spacecraft to center of mass of planet (km)
u	radial velocity (km/sec)
v	velocity perpendicular to radial velocity (km/sec)
T	thrust force (newton)
β	steering control angle (deg)
a	radius of Mars (km)
g_e	gravitational acceleration at surface of Earth (9.80665×10^{-3} km/sec^2)
g_m	gravitational acceleration at surface of Mars (3.837×10^{-3} km/sec^2)
μ	Mars gravitational constant (km^3/sec^2) $= g_m a^2$
I_{sp}	engine specific impulse (sec)
m_0	initial mass (kg)
$\dot{m}$	constant time rate of change of mass (kg/sec) $= \dfrac{T}{g_e \cdot I_{sp}}$
t	time (sec)



DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
m	mass at time t (kg) $= m_0 - \dot{m}(t - t_0)$
λ_1	first Lagrangian variable (sec)
λ_2	second Lagrangian variable (sec/km)
λ_3	third Lagrangian variable (sec^2/km)
λ_4	fourth Lagrangian variable (sec^2/km)
A	semimajor axis of starting ellipse (km)
J_2	coefficient of the second harmonic of Mar's potential function $J_2 = 0.002001$

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SUMMARY

A fuel-minimized trajectory is generated, satisfying initial and final boundary conditions for the problem of a constantly thrusting space vehicle which is transferred from an initial elliptical orbit around Mars to a circular orbit of lower energy. The constant-thrust solution is amenable to electric ion-propelled spacecraft having an initial elliptical or circular planetocentric orbit which must then be powered to a circular orbit of desired altitude.

INTRODUCTION

The Advanced Reconnaissance Electric Spacecraft (ARES) is a mission designed principally to map the neighboring planets of Venus, Mars, and Jupiter. To map the planets in a minimum length of time, high data transmission rates (on the order of 2 megabits/sec) are required, which, in turn, necessitate large quantities of electric power. The on-board power source is already required to accomplish the mission mapping objective once an orbit is established; hence, it appears practicable to use the available electric power source for main propulsion to achieve the desired orbit. For cartographic purposes, the most desirable orbits are circular, as opposed to elliptic, because the problem of changing altitude (and, hence, resolution) as a function of time is eliminated. In the ARES mission, once the planet is mapped from a high circular orbit, the altitude is decreased to a lower circular orbit so that a higher- resolution map may be made for easy reference to the previous map. The optimal solution of the equations of motion for a minimum transfer time from one circular orbit to another is desired which includes low continuous thrust and the precessional effects of planetary oblateness. The results of such a solution for a typical problem are presented in this report. However, for purposes of generality the initial orbit is chosen as an ellipse, as if the spacecraft has been captured in an initial elliptical orbit and then powered to a circular orbit.

DISCUSSION

The Differential Equation System

The second order Newton differential equations of motion in plane polar coordinates are used. Included are the acceleration components from continuous thrust (T) along and perpendicular to the radius vector:

$$\ddot{r} = r\dot{\phi}^2 + \frac{T}{m}\cos\beta - \mu/r^2 \left[1 - \frac{3}{2} J_2\left(\frac{a}{r}\right)^2 (3\sin^2\phi - 1)\right] \tag{1a}$$

$$\frac{d(r\dot{\phi})}{dt} = -\dot{r}\dot{\phi} + \frac{T}{m}\sin\beta - \mu/r^2 \left[3J_2\left(\frac{a}{r}\right)^2 \sin\phi\cos\phi\right] \tag{1b}$$

where the terms involving J_2 within the brackets of (1) account for the oblateness of Mars [1].

Introduction of the variables $u = \dot{r}$ and $v = r\dot{\phi}$ into (1) yields the following first order differential equation system:

$$\dot{\phi} = \frac{v}{r} \tag{2a}$$

$$\dot{r} = u \tag{2b}$$

$$\dot{u} = \frac{v^2}{r} + \frac{T}{m}\cos\beta - \frac{\mu}{r^2}\left[1 - \frac{3}{2} J_2\left(\frac{a}{r}\right)^2 (3\sin\phi - 1)\right] \tag{2c}$$

$$\dot{v} = \frac{-uv}{r} + \frac{T}{m}\sin\beta - \frac{\mu}{r^2}\left[3J_2\left(\frac{a}{r}\right)^2 \sin\phi\cos\phi\right] \tag{2d}$$

The variable β of set (2) is not associated with any particular differential equation so it is removed by the definitions:

$$\tan\beta = \frac{\lambda_4}{\lambda_3}\ , \quad \cos\beta = \frac{\lambda_3}{\sqrt{\lambda_3^2+\lambda_4^2}}\quad , \ \sin\beta = \frac{\lambda_4}{\sqrt{\lambda_3^2+\lambda_4^2}}$$

where λ_3, λ_4 are functions of time. Recourse is made to the calculus of variations to supply the Euler-Lagrange (E-L) differential equations for λ_3 and λ_4 thus linking the steering angle β to the concept of minimal time. In this example, the solution satisfying the boundary conditions and the E-L equations yields a relative minimum. The E-L equations furnish the necessary criteria for a minimum to exist but are not sufficient to guarantee an absolute minimum. Another trajectory history might exist that uses less time than this example.

The Appendix constitutes a brief exposition of the isoperimetric problem solved here. The integral to be extremalized specializes to the total time of flight: $I = \int_{t_0}^{t_1} dt$, $f = 1$ via equation (A1). The constraining differential equations are:

$$G_1 = \dot{\phi} - \frac{v}{r} = 0, \ G_3 = \dot{u} - \frac{v^2}{r} - \frac{T}{m}\frac{\lambda_3}{\sqrt{\lambda_3^2+\lambda_4^2}}$$

$$+ \frac{\mu}{r^2}\left[1 - \frac{3}{2} J_2\left(\frac{a}{r}\right)^2 (3\sin^2\phi - 1)\right] = 0$$

$$G_2 = \dot{r} - u = 0, \ G_4 = \dot{v} + \frac{uv}{r} - \frac{T}{m}\frac{\lambda_4}{\sqrt{\lambda_3^2+\lambda_4^2}}$$

$$+ \frac{\mu}{r^2}\left[3 J_2\left(\frac{a}{r}\right)^2 \sin\phi\cos\phi\right] = 0 \quad .$$

The isoperimetric integrand function, equation (A5), becomes

$$F = 1 + \lambda_1\left(\dot{\phi} - \frac{v}{r}\right) + \lambda_2(\dot{r} - u) + \lambda_3\left\{\dot{u} - \frac{v^2}{r} - \frac{T}{m}\frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}}\right.$$

$$\left. + \frac{\mu}{r^2}\left[1 - \frac{3}{2}J_2\left(\frac{a}{r}\right)^2(3\sin^2\phi - 1)\right]\right\}$$

$$+ \lambda_4\left\{\dot{v} + \frac{uv}{r} - \frac{T}{m}\frac{\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}} + \frac{\mu}{r^2}\left[3J_2\left(\frac{a}{r}\right)^2\sin\phi\cos\phi\right]\right\} .$$

Application of the Euler-Lagrange equations (A-9) gives

$$\frac{\partial F}{\partial \phi} = -9\lambda_3\frac{J_2\mu a^2}{r^4}\sin\phi\cos\phi + 3\lambda_4\frac{J_2\mu a^2}{r^4}(\cos^2\phi - \sin^2\phi)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\phi}}\right) = \dot{\lambda}_1$$

$$\frac{\partial F}{\partial r} = \lambda_1\frac{v}{r^2} + \lambda_3\left[\frac{v^2}{r^2} - \frac{2\mu}{r^3} + 6J_2\frac{\mu a^2}{r^5}(3\sin^2\phi - 1)\right]$$

$$- \lambda_4\left[\frac{uv}{r^2} + \frac{12J_2\mu a^2}{r^5}\sin\phi\cos\phi\right]$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{r}}\right) = \dot{\lambda}_2$$

$$\frac{\partial F}{\partial u} = -\lambda_2 + \lambda_4\frac{v}{r}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) = \dot{\lambda}_3$$

$$\frac{\partial F}{\partial v} = -\lambda_1 \frac{1}{r} - 2\lambda_3 \frac{v}{r} + \lambda_4 \frac{u}{r}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right) = \dot{\lambda}_4$$

The complete differential equation system is:

$$\dot{\phi} = \frac{v}{r} \tag{3a}$$

$$\dot{r} = u \tag{3b}$$

$$\dot{u} = \frac{v^2}{r} + \frac{T}{m} \frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}} - \frac{\mu}{r^2}\left[1 - \frac{3}{2} J_2\left(\frac{a}{r}\right)^2 (3 \sin^2\phi - 1)\right] \tag{3c}$$

$$\dot{v} = -\frac{uv}{r} + \frac{T}{m} \frac{\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}} - \frac{\mu}{r^2}\left[3 J_2\left(\frac{a}{r}\right)^2 \sin\phi \cos\phi\right] \tag{3d}$$

$$\dot{\lambda}_1 = -9\lambda_3 \frac{J_2 \mu a^2}{r^4} \sin\phi \cos\phi + 3\lambda_4 \frac{J_2 \mu a^2}{r^4} (\cos^2\phi - \sin^2\phi) \tag{3e}$$

$$\dot{\lambda}_2 = \lambda_1 \frac{v}{r^2} + \lambda_3\left[\frac{v^2}{r^2} - \frac{2\mu}{r^3} + 6 \frac{J_2 \mu a^2}{r^5} (3 \sin^2\phi - 1)\right]$$

$$- \lambda_4 \left[\frac{uv}{r^2} + 12 \frac{J_2 \mu a^2}{r^5} \sin\phi \cos\phi\right] \tag{3f}$$

$$\dot{\lambda}_3 = -\lambda_2 + \lambda_4 \frac{v}{r} \tag{3g}$$

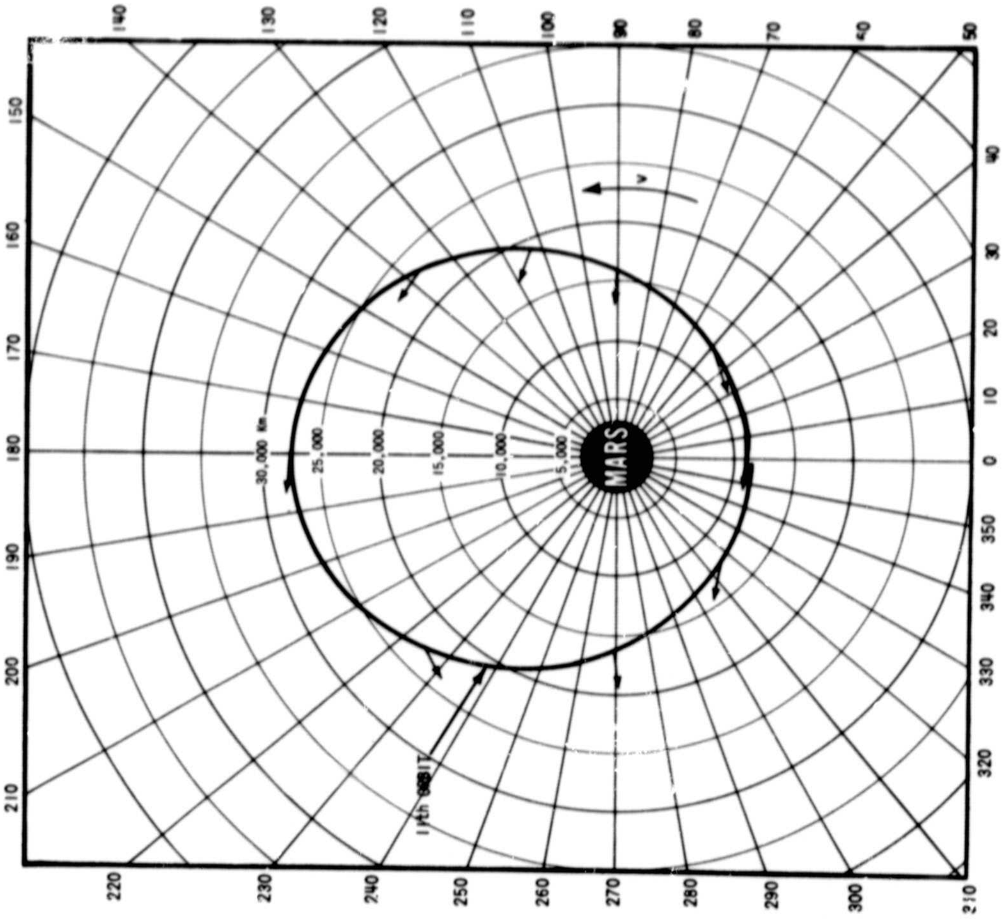
$$\dot{\lambda}_4 = -\lambda_1 \frac{1}{r} - 2\lambda_3 \frac{v}{r} + \lambda_4 \frac{u}{r} \tag{3h}$$

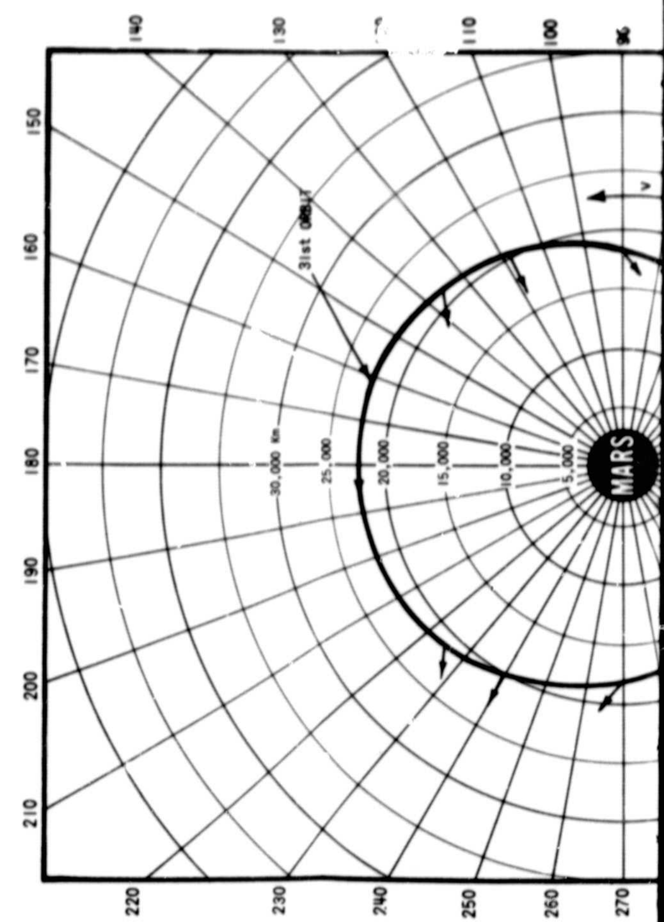
Boundary Conditions

The curves of Figure 1 illustrate to scale the elliptic starting orbit and the final circular orbit. Mars is located at a focus of the ellipse and the center of the circle. The radius of Mars is 3332.15 km. The semimajor and semi-minor axes of the ellipse are 18 500 km and 16 000 km. The eccentricity is 0.502; the perihelion distance is 9212.91 km; and the aphelion distance is 27 787.09 km. The radius of the circle is 13 332.15 km, or at 10 000 km altitude above the surface of Mars. The initial spacecraft mass is 2950 kg, and the engine I_{sp} is 3500 sec. The ionic thrust (T) is 0.4 newton. The constant time rate of change of mass ($\dot{m}$) is 1.165×10^{-5} kg/sec. Initial and final conditions and derived quantities are given in Table I.

TABLE I. INITIAL AND FINAL CONDITIONS AND DERIVED QUANTITIES

Conditions / Variables	Initial Conditions and Derived Quantities	Final Conditions and Derived Quantities
ϕ	0	78 orbits + 195°
r	9212.91	13 332.15
u	0	0
v	2.6355	1.7876
t	0	64 days 4.6 hours
m	2950.0	2885.37
g	501.9×10^{-6}	239.7×10^{-6}





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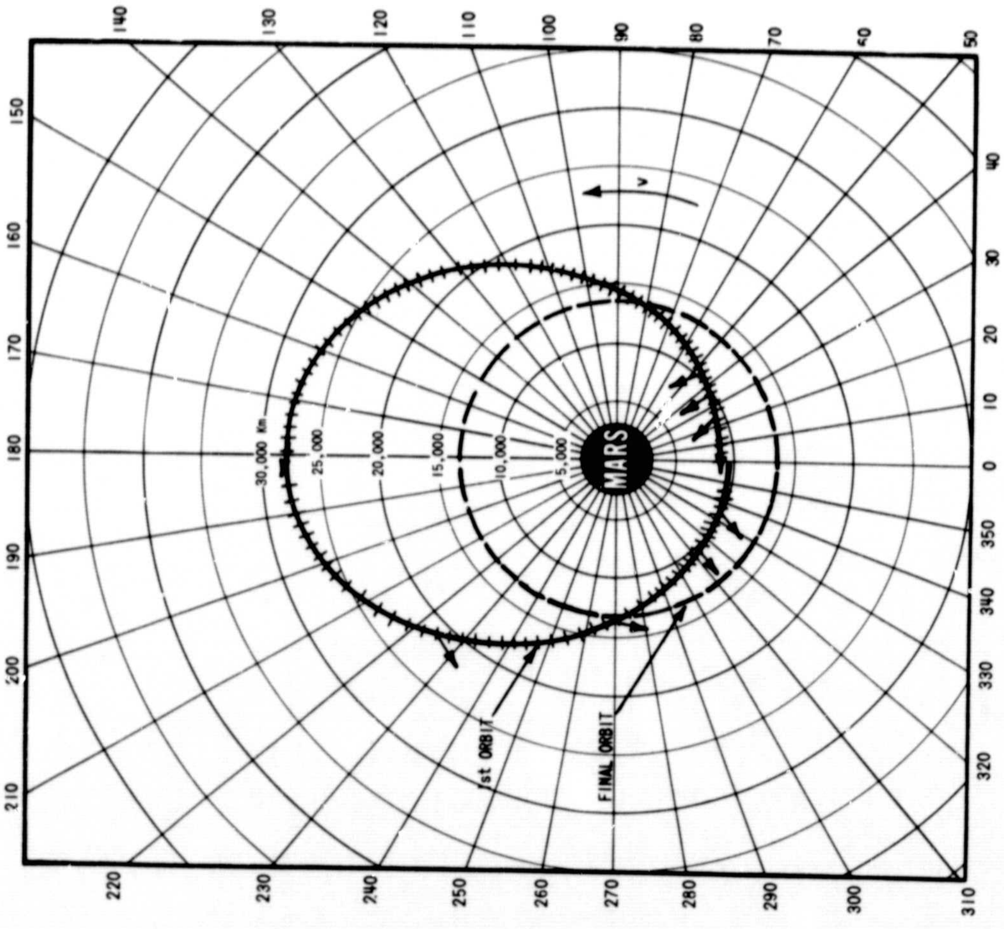
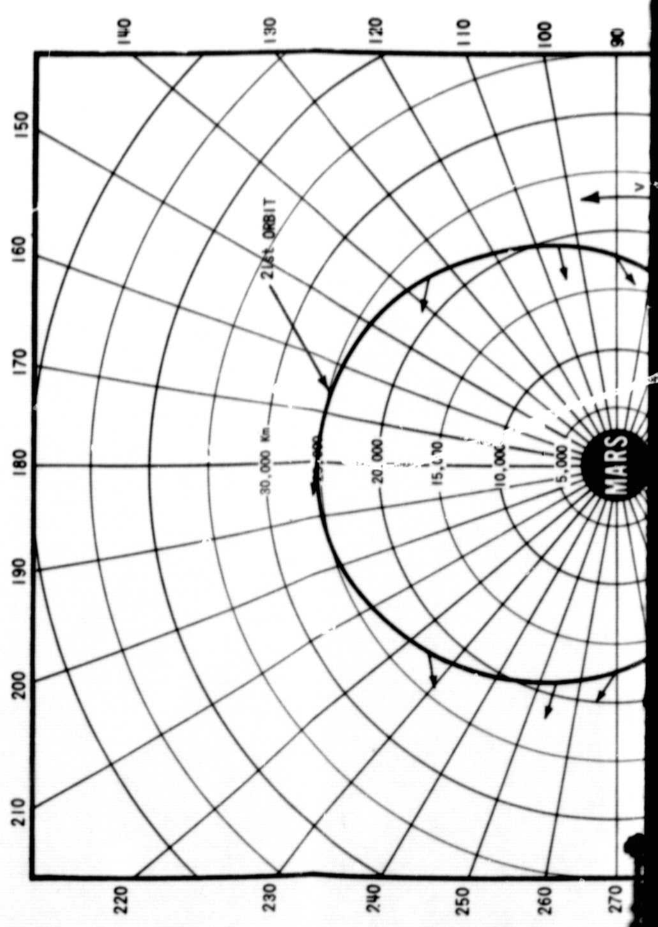
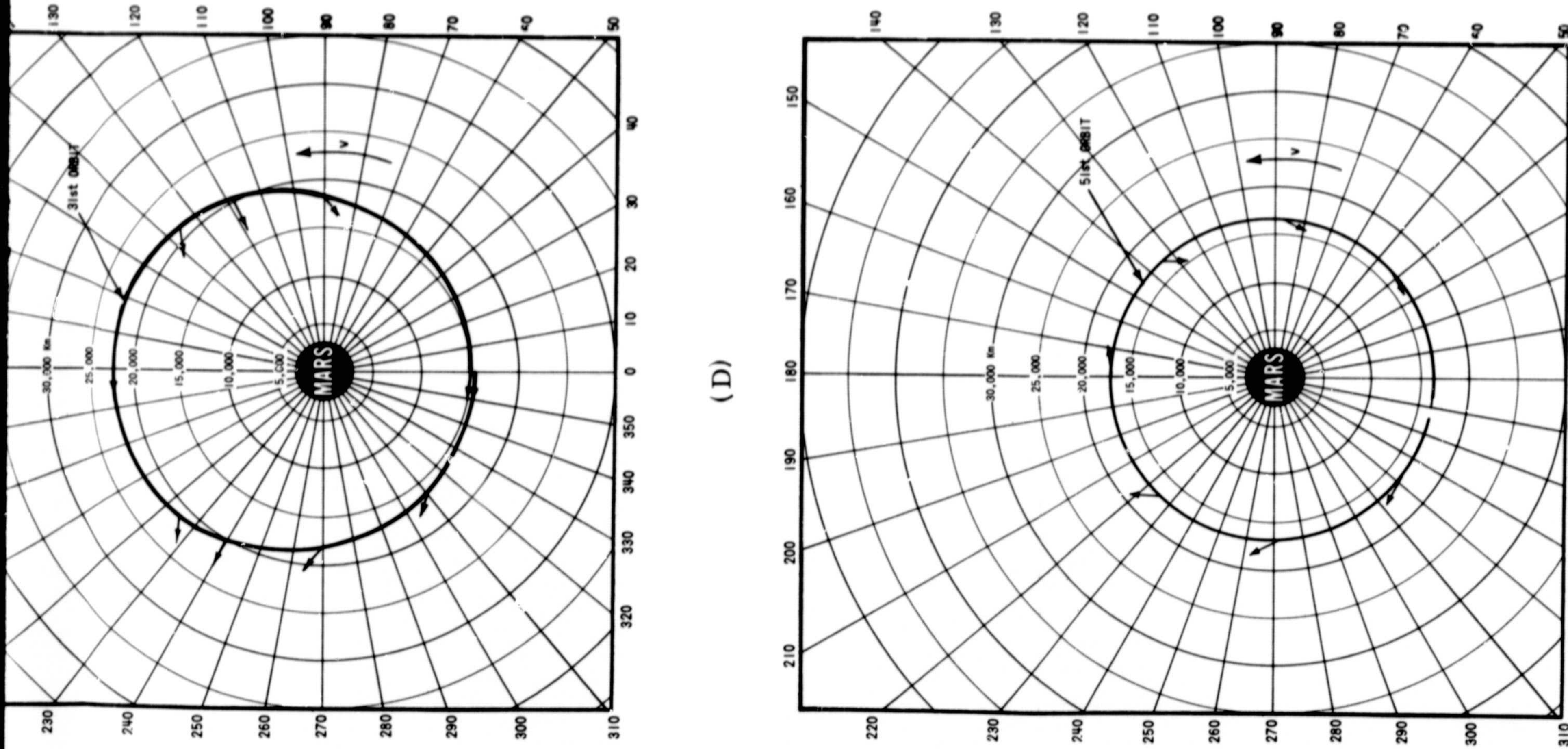




(A)

FOLDOUT FRAME 1



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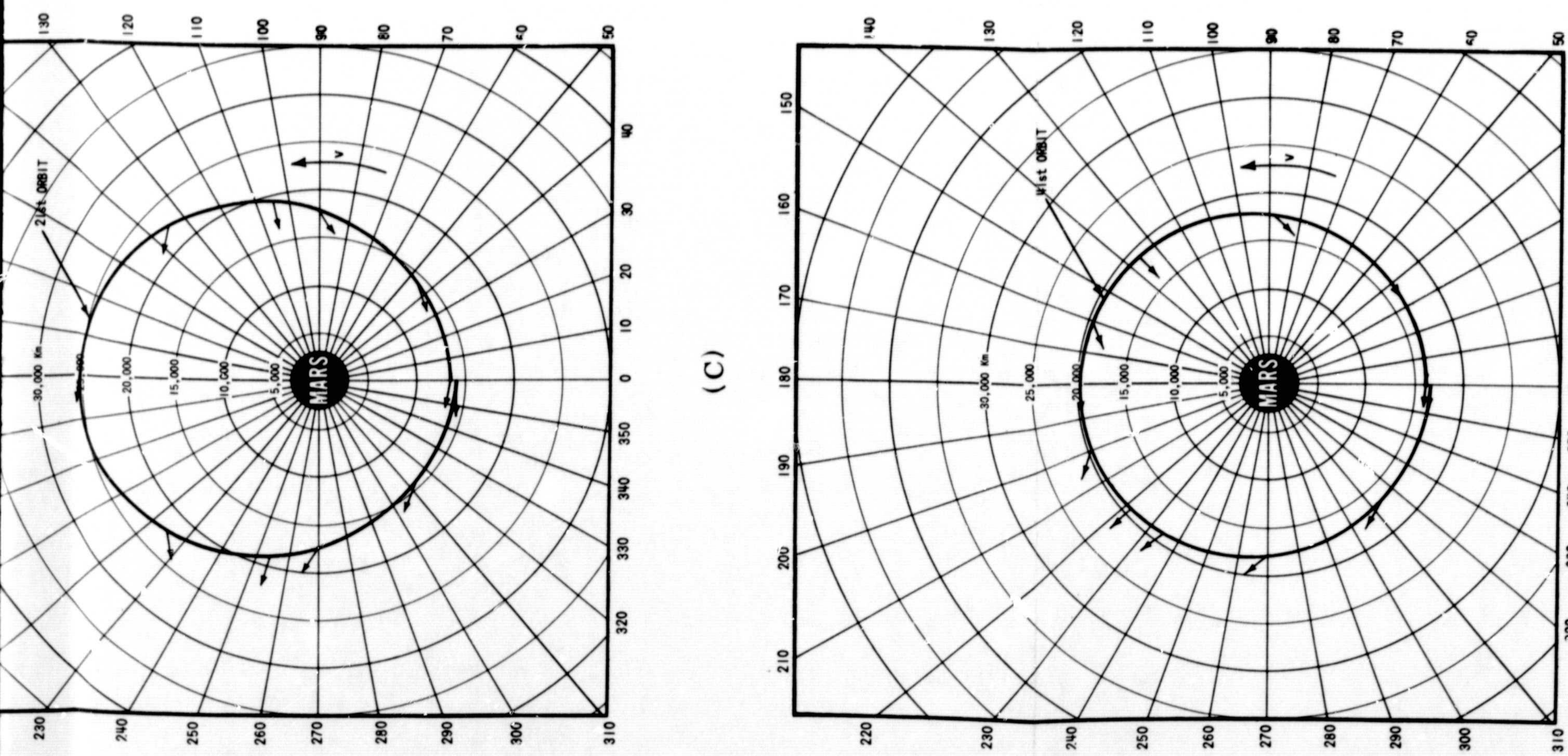
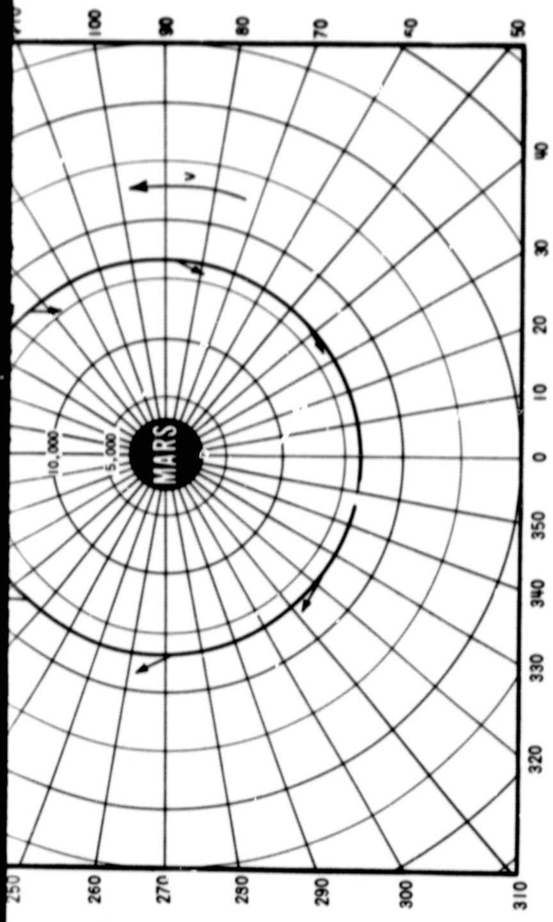
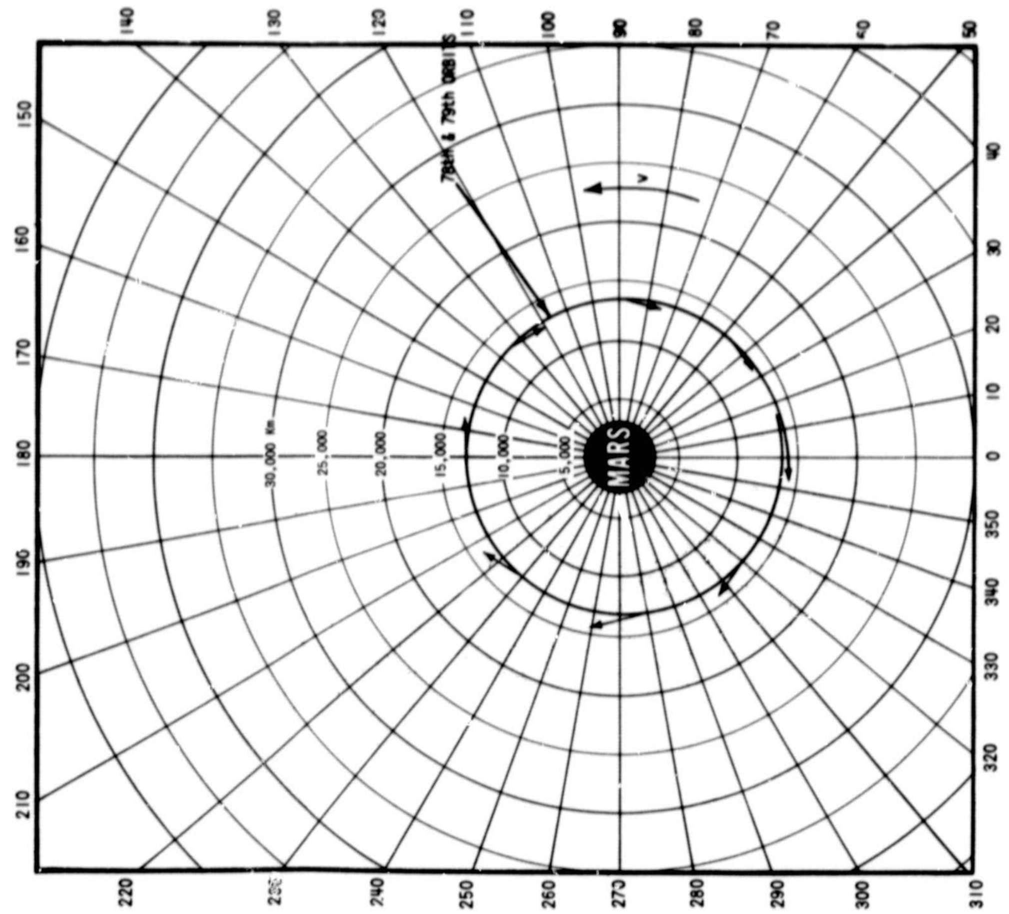


(C)

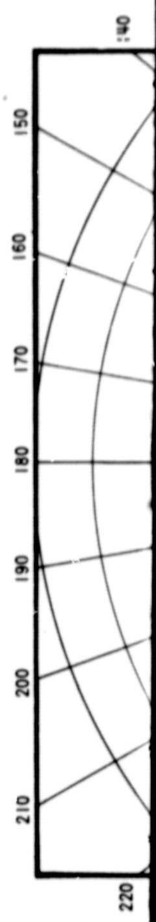
FOLDOUT FRAME 2

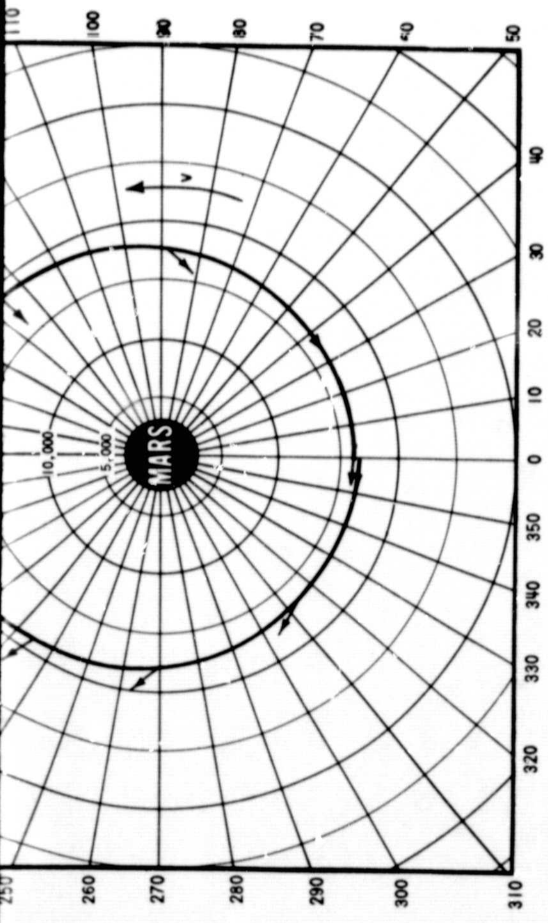


(F)

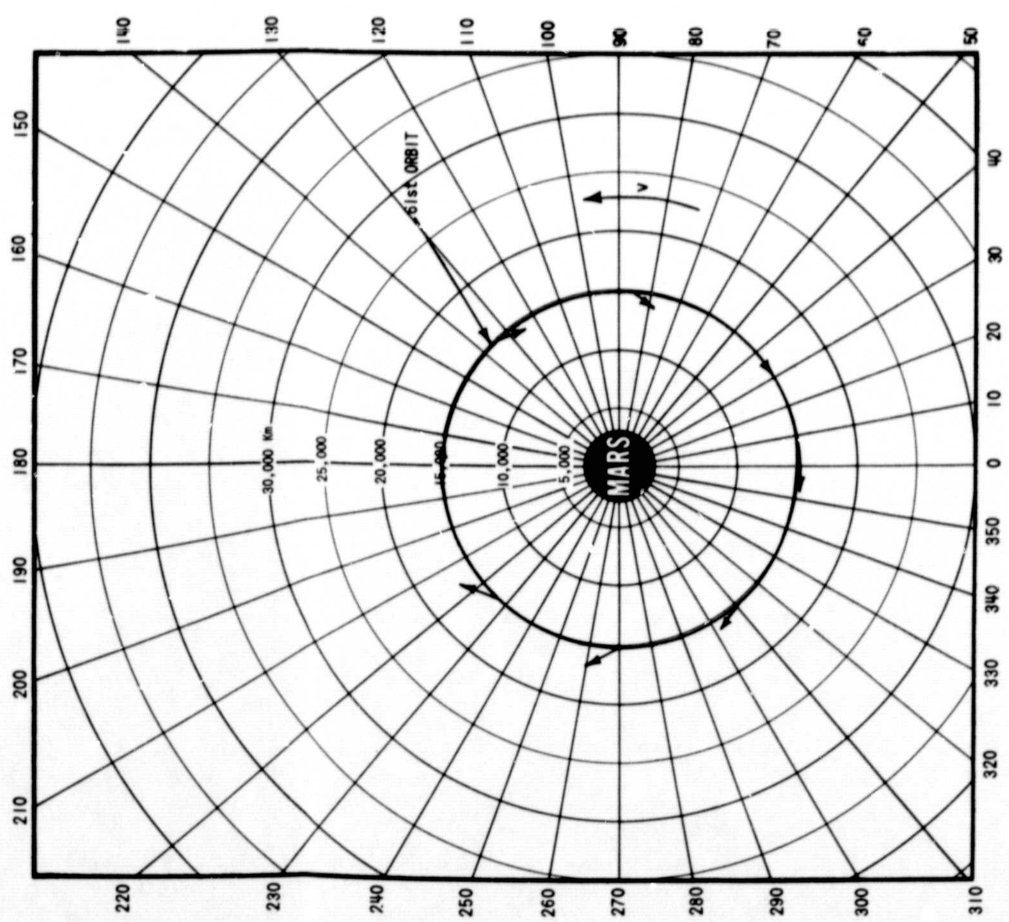


(H)





(E)



(G)

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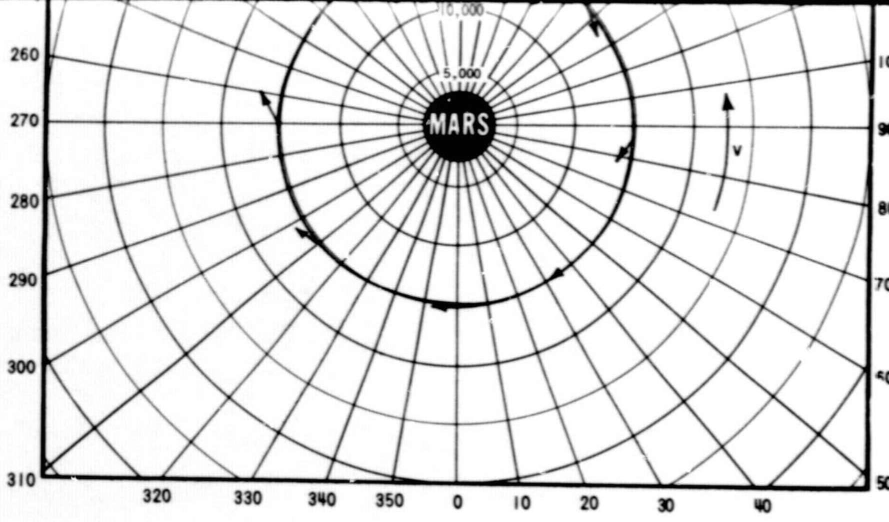
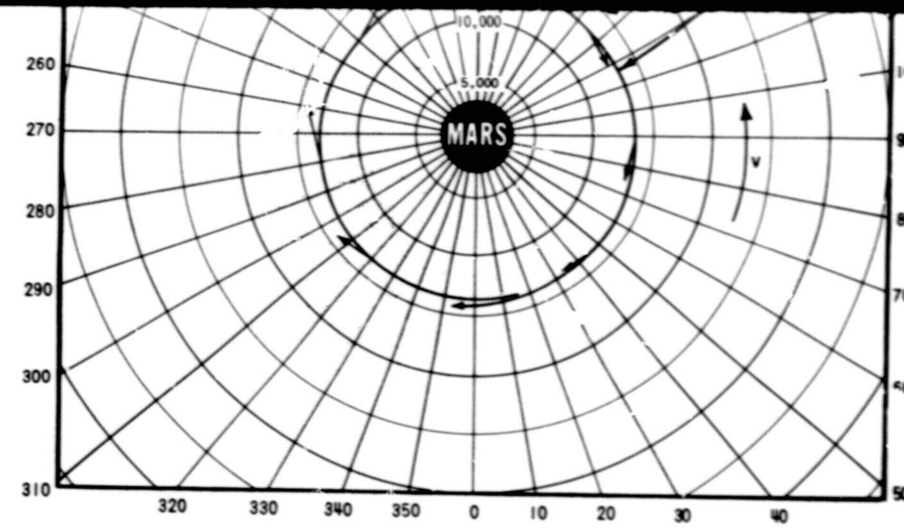
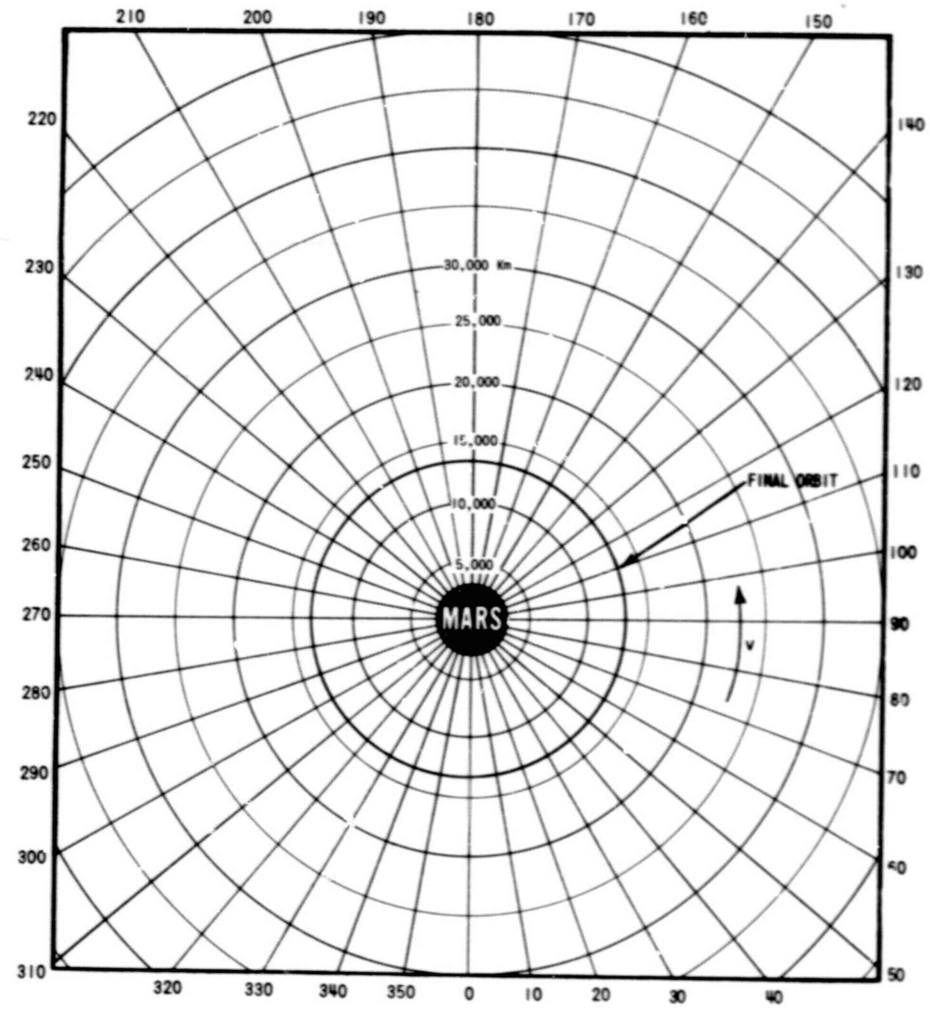


(G)



(H)

FIGURE 1. TYPICAL ORBIT GEOMETRIES



(I)

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Numerical Integration

The procedure used was a powerful seventh order Runge-Kutta-type method with automatic step size control developed by Dr. Erwin Fehlberg of the MSFC Computation Laboratory. A tolerance assuring eight-place accuracy throughout the history for all variables allowed a minimum step size of 227 sec and a maximum of 1355 sec. The first orbit required 115 time steps, or about 78 000 sec (21.7 hr); the intervals are shown in Figure 1. The 78th orbit required 45 time steps each of 1085 sec, or about 47 000 sec (13.1 hr). The IBM 7094 computer was used with double precision (16-digit word length), and machine time was 12 minutes for the trajectory history.

Remarks

Figures 1 depicts the trajectory history with motion in a counterclockwise direction from the ellipse to the final circle. The first 50 orbits change the initial ellipse to a circular orbit about 14 700 km above the surface of Mars (Figs. 1A through 1F). The thrust vectors (small arrows) of Figures 1 A through 1E never create more than a 90° angle with the longitudinal axis of a nonrotating spacecraft until the 51st orbit is reached. Orbits 51 through 78 shrink the radius of the circular orbit to the desired altitude of 10 000 km above the surface of Mars. The thrust vectors of Figures 1F through 1H show continuous braking action, and they are nearly aligned with the tangent of the trajectory curve. The final boundary conditions are met about halfway through the 79th orbit. There is no evidence of precession as the ellipse is turned into the circle.

Figures 2A through 2D give radius and steering angle as functions of time. Figure 2C shows the continuous braking effect of the thrust vector that oscillates about the angle $\beta = 270°$.

The effects of oblateness were not significant for this example. The total time difference between two comparison runs for $J_2 = 0$ and $J_2 = 0.002001$ was slightly over 1000 sec, with about a one degree change in the range angle ϕ.

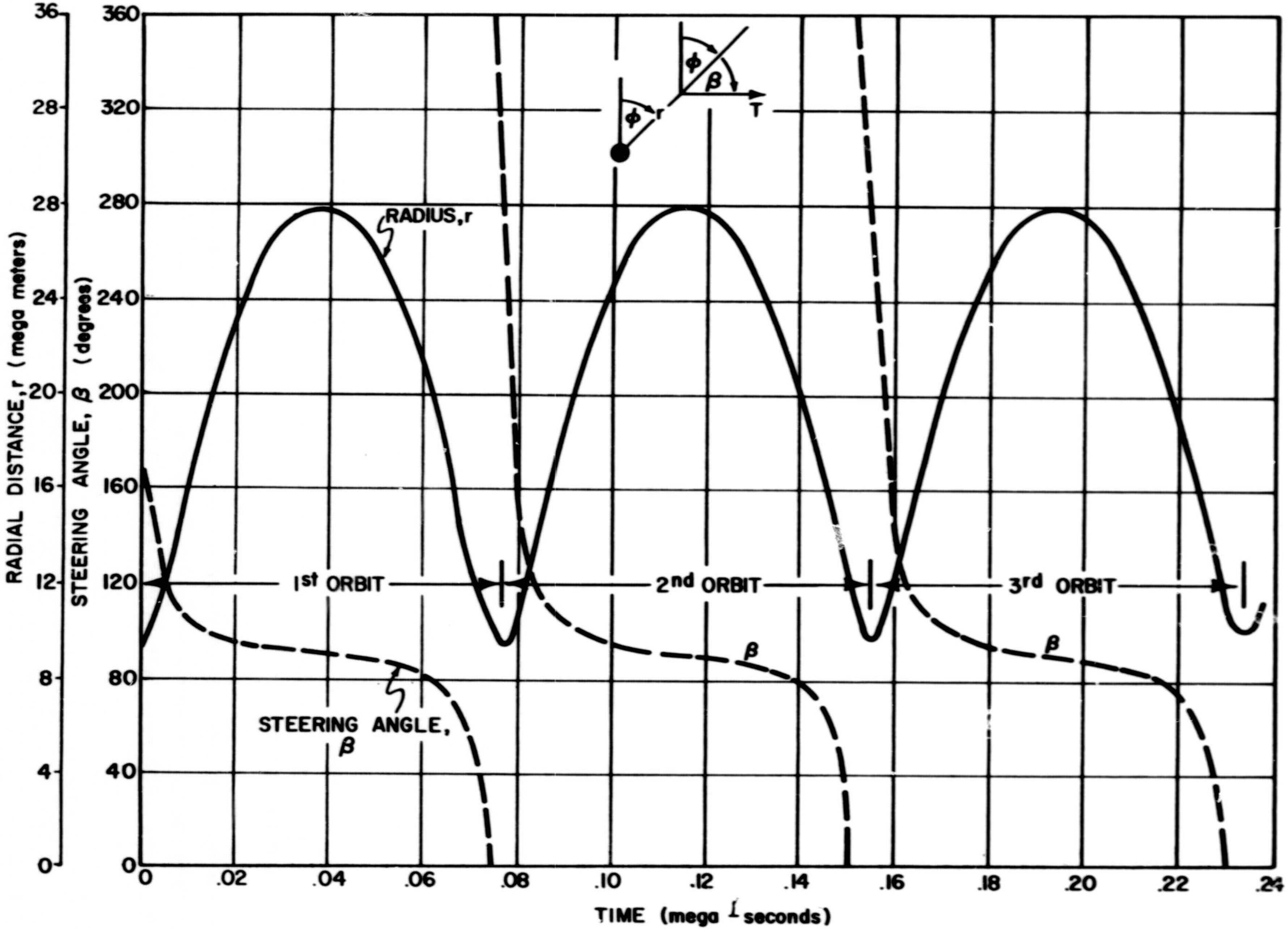

FIGURE 2A. RADIAL DISTANCE AND STEERING ANGLE β AS A FUNCTION OF TIME

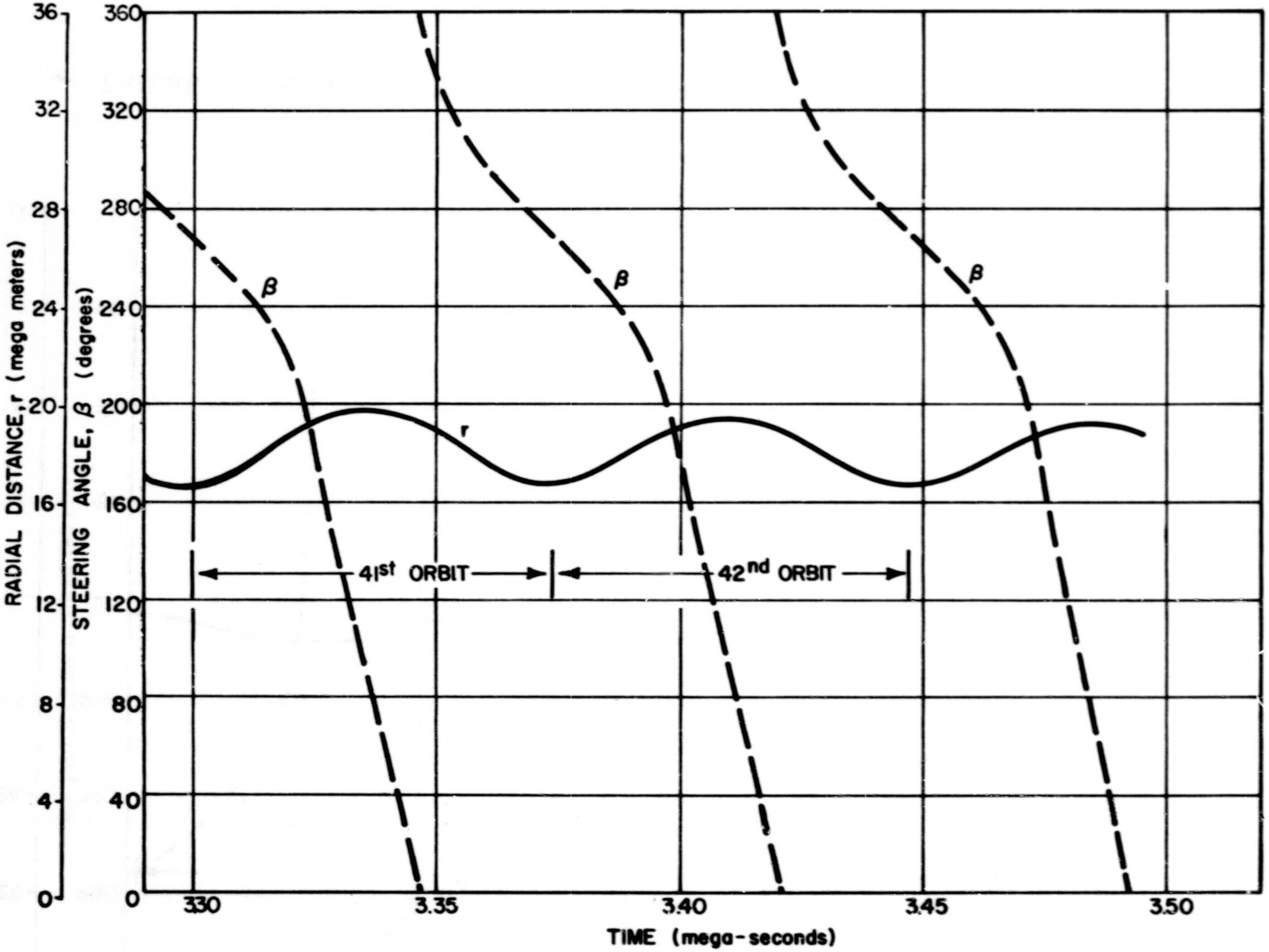

FIGURE 2B. RADIAL DISTANCE AND STEERING ANGLE β AS A FUNCTION OF TIME

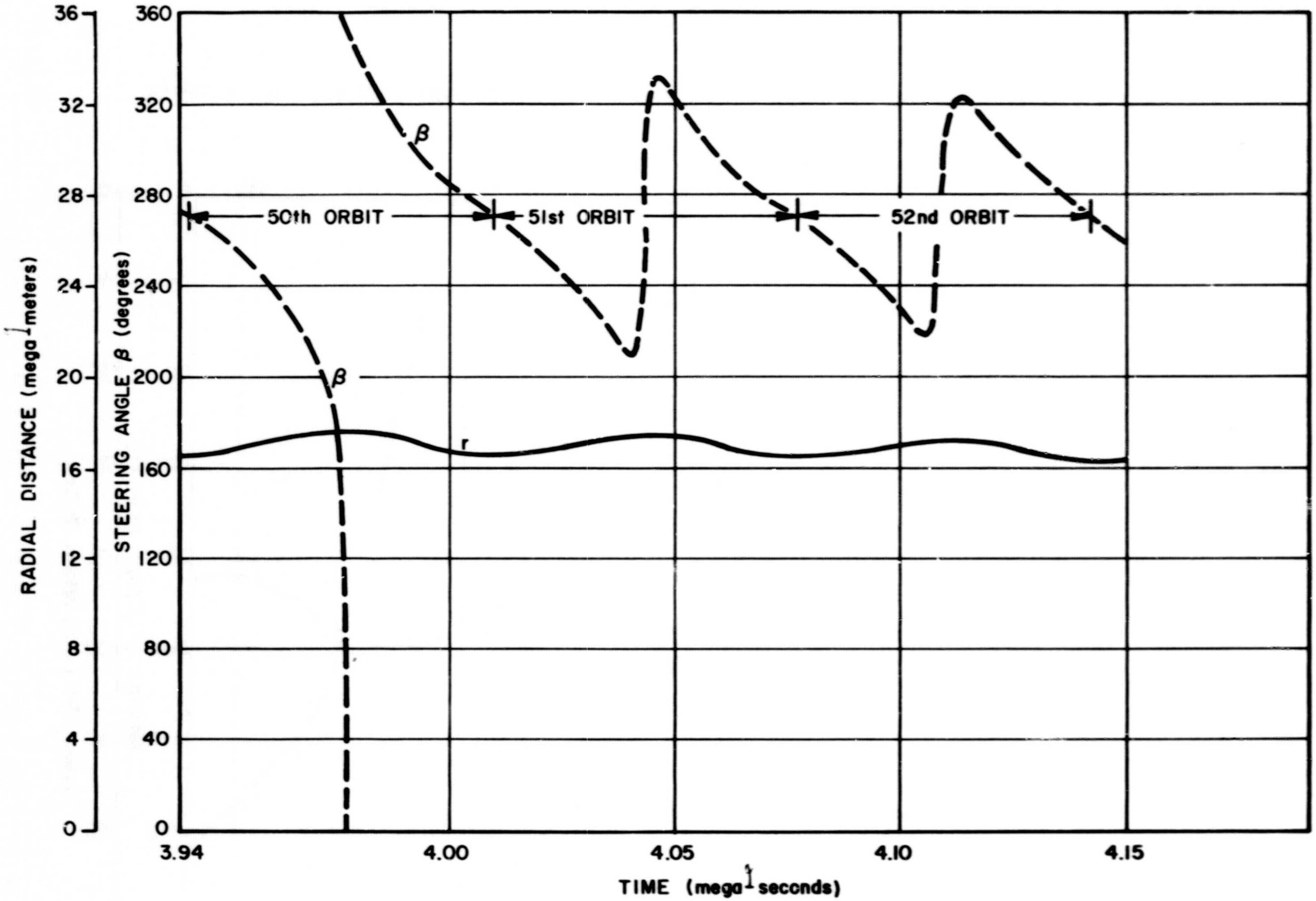

FIGURE 2C. RADIAL DISTANCE AND STEERING ANGLE β AS A FUNCTION OF TIME

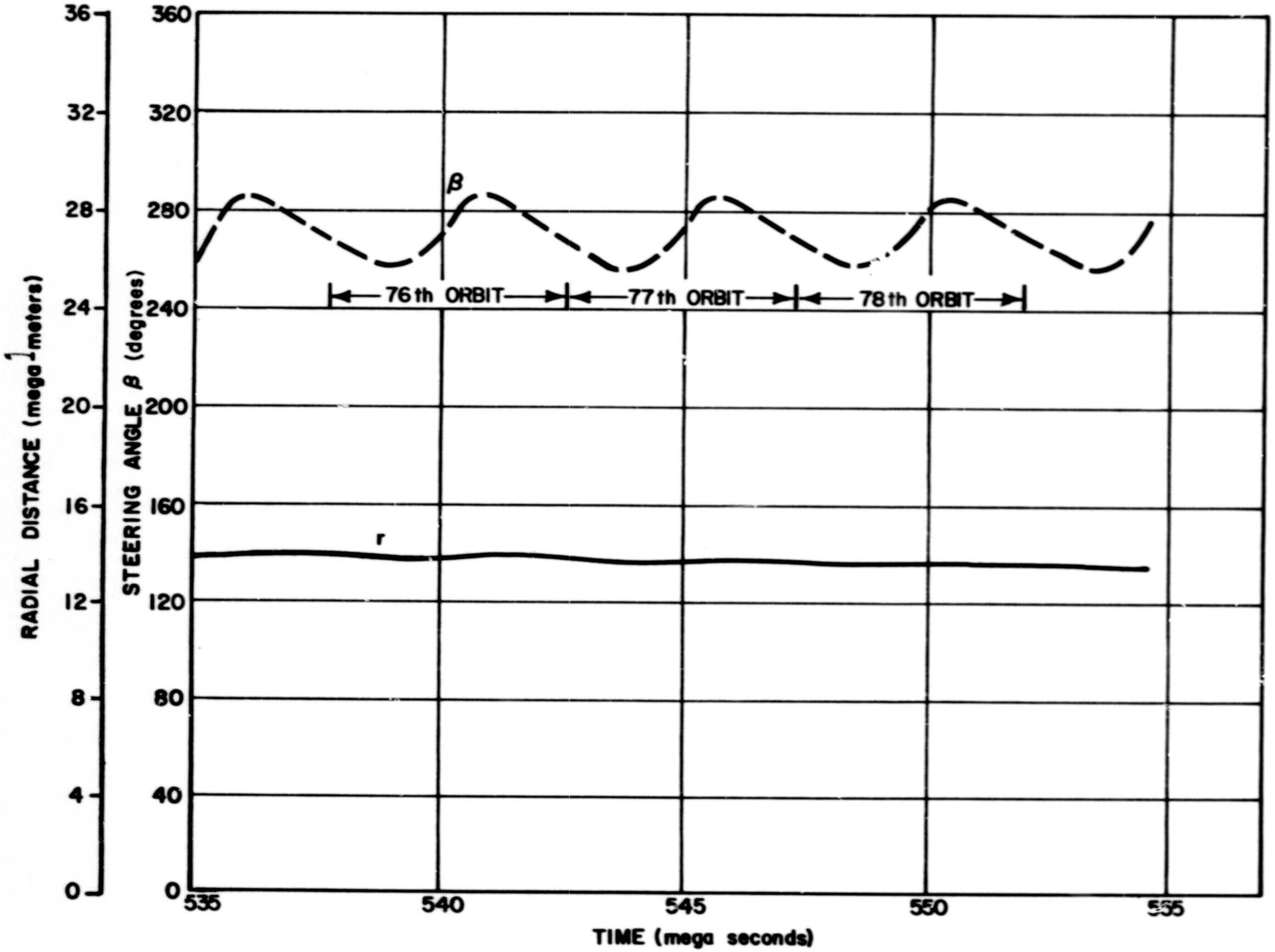

FIGURE 2D. RADIAL DISTANCE AND STEERING ANGLE β AS A FUNCTION OF TIME

CONCLUSIONS

The problem of changing the planetocentric orbit of a long-term, constantly low-thrusting spacecraft from an initial elliptic to a final circular orbit, so as to obtain a trajectory which requires a minimum of fuel is shown to be tractable. The effect of the oblateness of the planet Mars, which contributes perturbations to the equations of motion in the form of harmonics to the gravitational field representation, was shown to be negligible for the case considered.

REFERENCES

1. Menzel, D. H.: Mathematical Physics. Dover Publications, Inc., New York, 1961.

2. Weinstock, Robert: Calculus of Variations. McGraw-Hill Book Co., New York, 1952.

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APPENDIX

The integral

$$I = \int_{t_0}^{t_1} f(x_1, x_2, \ldots x_K \dot{x}_1, \dot{x}_2, \ldots \dot{x}_K, t)\, dt \quad [2] \tag{A-1}$$

is to be extremalized where the functions eligible for the extremalization must obey $N \leq K$ differential equations as side conditions. The problem is defined as isoperimetric.

A set of comparison functions $X_i = x_i + \epsilon \eta_i(t)$ $i = 1, 2, \ldots K$ is selected with ϵ an arbitrary parametric variable and $\eta_i(t)$ $i = 1, 2, \ldots K$ arbitrary functions of t subject to the restrictions $\eta_i(t_0) = \eta_i(t_1) = 0$, $i = 1, 2, \ldots K$.

The integral becomes

$$I(\epsilon) = \int_{t_0}^{t_1} f(X_1, X_2, \ldots X_K, \dot{X}_1, \dot{X}_2, \ldots \dot{X}_K, t)\, dt,$$

and for $I(\epsilon)$ to be an extremum for $\epsilon = 0$, it is necessary that $I'(0) = 0$. Differentiation under the integral sign with respect to ϵ produces

$$I'(\epsilon) = \int_{t_0}^{t_1} \left[\frac{\partial f}{\partial X_1} \frac{\partial X_1}{\partial \epsilon} + \ldots + \frac{\partial f}{\partial X_K} \frac{\partial X_K}{\partial \epsilon} + \frac{\partial f}{\partial \dot{X}_1} \frac{\partial \dot{X}_1}{\partial \epsilon} + \ldots + \frac{\partial f}{\partial \dot{X}_K} \frac{\partial \dot{X}_K}{\partial \epsilon} \right] dt$$

and for $\epsilon = 0$ this becomes

$$I'(0) = \int_{t_0}^{t_1} \left[\frac{\partial f}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial f}{\partial x_K} \eta_K(t) + \frac{\partial f}{\partial \dot{x}_1} \dot{\eta}_1(t) + \ldots + \frac{\partial f}{\partial \dot{x}_K} \dot{\eta}_K(t) \right] dt$$

$$= 0 . \tag{A-2}$$

N of the arbitrary functions $\eta_i(t)$ $i = 1, 2, \ldots K$ are dependent because of the N differential equations of constraint

$$G_j(X_1, X_2, \ldots X_K, \dot{X}_1, \dot{X}_2, \ldots \dot{X}_K, t) = 0 \qquad j = 1, 2, \ldots N \quad .$$

Each of these N equations is differentiated with respect to ϵ and multiplied by a Lagrangian variable to give

$$\lambda_j(t)\left[\frac{\partial G_j}{\partial X_1}\frac{\partial X_1}{\partial \epsilon} + \ldots + \frac{\partial G_j}{\partial X_K}\frac{\partial X_K}{\partial \epsilon} + \frac{\partial G_j}{\partial \dot{X}_1}\frac{\partial \dot{X}_1}{\partial \epsilon} + \ldots + \frac{\partial G_j}{\partial \dot{X}_K}\frac{\partial \dot{X}_K}{\partial \epsilon}\right] = 0$$

$$j = 1, 2, \ldots N$$

and, for $\epsilon = 0$, this becomes

$$\lambda_j(t)\left[\frac{\partial G_j}{\partial x_1}\eta_1(t) + \ldots + \frac{\partial G_j}{\partial x_K}\eta_K(t) + \frac{\partial G_j}{\partial \dot{x}_1}\dot{\eta}_K(t) + \ldots + \frac{\partial G_j}{\partial \dot{x}_K}\dot{\eta}_K(t)\right] = 0$$

$$j = 1, 2, \ldots N \tag{A-3}$$

The equations of (A-3) do not disturb the numerical value of the integrand of (A-2), so they may be added to it to form

$$I'(0) = \int_{t_0}^{t_1}\left\{\left[\frac{\partial f}{\partial x_1} + \sum_{j=1}^{N}\lambda_j(t)\frac{\partial G_j}{\partial x_1}\right]\eta_1(t) + \ldots + \left[\frac{\partial f}{\partial x_K} + \sum_{j=1}^{N}\lambda_j(t)\frac{\partial G_j}{\partial x_K}\right]\eta_K(t)\right.$$

$$+\left[\frac{\partial f}{\partial \dot{x}_1} + \sum_{j=1}^{N}\lambda_j(t)\frac{\partial G_j}{\partial \dot{x}_1}\right]\dot{\eta}_1(t) + \ldots$$

$$\left. + \left[\frac{\partial f}{\partial \dot{x}_K} + \sum_{j=1}^{N}\lambda_j(t)\frac{\partial G_j}{\partial \dot{x}_K}\right]\dot{\eta}_K(t)\right\} dt = 0 \tag{A-4}$$

A new integrand function is defined to be

$$F = f + \sum_{j=1}^{N} \lambda_j(t)\, G_j \tag{A-5}$$

which when differentiated partially with respect to x_i and $\dot{x}_i$ $i = 1, 2, \ldots K$ gives

$$\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial x_i}, \quad \frac{\partial F}{\partial \dot{x}_i} = \frac{\partial f}{\partial \dot{x}_i} + \sum_{j=1}^{N} \lambda_j(t) \frac{\partial G_j}{\partial \dot{x}_i}$$

$$i = 1, 2, \ldots K \qquad j = 1, 2, \ldots N$$

The right side of these equations is identical to the bracketed terms of (A-4) so that

$$I'(0) = \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x_1} \eta_1(t) + \ldots + \frac{\partial F}{\partial x_K} \eta_K(t) \right.$$

$$\left. + \frac{\partial F}{\partial \dot{x}_1} \dot{\eta}_1(t) + \ldots + \frac{\partial F}{\partial \dot{x}_K} \dot{\eta}_K(t) \right) dt = 0 \tag{A-6}$$

Integration by parts on the right half of the integrand of (A-6) is performed using the formula

$$\int_{t_0}^{t_1} u_i\, dv_i = u_i v_i \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} v_i\, du_i$$

$$\left.\begin{array}{ll} u_i = \dfrac{\partial F}{\partial \dot{x}_i} & dv_i = \dot{\eta}_i(t)\ dt \\[2ex] du_i = d\left(\dfrac{\partial F}{\partial \dot{x}_i}\right) & v_i = \eta_i(t) \end{array}\right\} i = 1, 2, \ldots K$$

$$u_i v_i \Bigg|_{t_0}^{t_1} = \frac{\partial F}{\partial \dot{x}_i}\ \eta_i(t) \Bigg|_{t_0}^{t_1} \quad , \quad \int_{t_0}^{t_1} v_i\, du_i = \int_{t_0}^{t_1} \eta_i(t)\ \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) dt \ .$$

Equation (A-6) becomes

$$I'(0) = \left[\frac{\partial F}{\partial \dot{x}_1}\ \eta_1(t) + \ldots + \frac{\partial F}{\partial \dot{x}_K}\ \eta_K(t)\right]_{t_0}^{t_1}$$

$$+ \int_{t_0}^{t_1} \left\{ \left[\frac{\partial F}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_1}\right)\right] \eta_1(t) + \ldots \right.$$

$$\left. + \left[\frac{\partial F}{\partial x_K} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_K}\right)\right] \eta_K(t) \right\} dt = 0 \ . \qquad \text{(A-7)}$$

The first term of (A-7) is zero because of the restrictions $\eta_i(t_0) = \eta_i(t_1) = 0$ $i = 1, 2, \ldots K$ so that (A-7) becomes

$$I'(0) = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial F}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_1}\right)\right] \eta_1(t) + \ldots + \left[\frac{\partial F}{\partial x_K} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_K}\right)\right] \eta_K(t) \right\} dt = 0$$

(A-8)

The Lagrangian variables $\lambda_j(t)$ $j = 1, 2, \ldots N$ can be algebraically assigned to insure that the coefficients of the first N terms of (A-8) will vanish. (The K - N terms can be unconditionally brought to zero because $\eta_i(t)$ $i = N + 1, N + 2, \ldots K$ are independent and arbitrary.) The Euler-Lagrange differential equations of (A-8)

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = 0 \quad i = 1, 2, \ldots N \tag{A-9}$$

provide the differential equations for the Lagrangian variables.