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SMALL OSCILLATIONS OF A VISCOUS ISOTHERMAL ATMOSPHERE

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SMALL OSCILLATIONS OF A VISCOUS ISOTHERMAL ATMOSPHERE

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PREFACE

The study of the earth's atmosphere has attracted the talents of many investigators. The author hopes that this research is not an unworthy contribution to this field of scientific investigation.

During the conduct of this research the author has received considerable aid from several individuals. Since many numerical calculations were performed, it was necessary to develop a large number of computer programs. During this phase of the research, Melvin J. Ardlt and E. O. Smith of the Lockheed Electronics Company were most helpful and the author wishes to express his appreciation.

The author has been very fortunate since an unusual set of ci.cumstances* has lead to the completion of this research. The author wishes to express his gratitude to Henry P. Decell† of the University of Houston and Jay M. Lewallen of the Manned Spacecraft Center for providing the opportunity to complete this research. In addition, Dr. Decell and Dr. Lewallen have made many helpful

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†Formerly Chief of the Theory and Analysis Office of the Manned Spacecraft Center.

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The author wishes to express his gratitude to Dr. Michael Yanowitch of Adelphi University who served as advisor. His patience, advice, and intuitive understanding have made this undertaking worthwhile.

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ABSTRACT

The problem of small oscillations of an isothermal atmosphere is investigated. The assumptions of small, two-dimensional waves in a viscous, compressible, stratified fluid with a constant dynamic viscosity coefficient μ leads to a linear system of two second-order, ordinary differential equations in the vertical z coordinate.

In solving the viscous problem for small $\mu > 0$ it is found that there is an inviscid region in which the solution behaves like a linear combination of inviscid ($\mu = 0$) solutions. Several interesting cases develop depending on the values of the frequency σ and horizontal wave number k. The most interesting case concerns the viscous solution for those values of σ and k which lead to the development of inviscid solutions which are wavelike in z. For this case the viscous solution does not satisfy the radiation condition in the inviscid region since viscosity reflects waves in addition to damping wave motion for large z. Thus, the correct solution of the inviscid problem consists of an incident and a reflected wave. As $\mu \to 0$ the ratio of the amplitudes of the incident and reflected waves approaches a limiting value for each fixed z in the inviscid region.

However, the viscous solution does not approach a limiting value since the reflecting layer shifts to infinity as $\mu \to 0$ and thus alters the phase of the reflected wave.

The remaining cases are investigated and several numerically computed solutions are determined. On the basis of the computations, the validity of the linearization is also discussed.

TABLE OF CONTENTS

TABLE OF CONTENTS (Concluded)

LIST OF TABLES

LIST OF FIGURES

PARTIAL LIST OF SYMBOLS

The symbols are listed in order of their appearance in the text, starting with Section 2 and ending with Appendix B. Each symbol is briefly defined, and the page number where the symbol is first introduced is stated.

Symbol	Definition	Page
t	time	9
u, w	horizontal and vertical velocity components	9
x, z	horizontal and vertical coordinates.	9
ρ	density.	9
γ	ratio of specific heats.	9
μ	dynamic viscosity coefficient.	9
p	pressure	9
$\vec{v} = \begin{bmatrix} u \\ w \end{bmatrix}$	two-dimensional velocity vector. . . .	9
g	acceleration of gravity.	9
c	speed of sound	10
ρ_g	unperturbed density at $z = 0$.	10

Symbol	Definition	Page
T_0	temperature.	10
R	gas constant	10
H	density scale height	10
N	Brunt-Väisälä frequency.	10
k	horizontal wave number	11
σ	frequency.	11
U	complex horizontal velocity amplitude.	11
i	$=\sqrt{-1}$	11
W	complex vertical velocity amplitude.	11
ε	$=\frac{\mu}{\gamma\rho_g H^2}\sqrt{\frac{H}{g}}$; small, dimensionless parameter	11
α	$=\frac{\sigma^2}{\gamma} - k^2 + \frac{\gamma - 1}{\gamma}\frac{k^2}{\sigma^2}$; dimensionless parameter.	15
λ_1, λ_2	roots of the dispersion relation . .	16

Symbol	Definition	Page
β	dimensionless vertical wave number. .	18
ξ	$=\frac{e^{-z}}{i\varepsilon\sigma}$; dimensionless independent variable.	21
$\underline{y}$	four-dimensional vector solution. . .	22
I	4×4 identity matrix	22
K, R, D	4×4 matrices.	22
$\underline{DC}_1$, $\underline{DC}_2$	four-dimensional solutions which satisfy the DC.	26
$\overrightarrow{dc}_1$, $\overrightarrow{dc}_2$	two-dimensional solutions which satisfy the DC.	26
$\underline{INV}_1$, $\underline{INV}_2$, $\underline{BLSOL}$, $\underline{TLSOL}$	four-dimensional vector solutions	29
$\hat{\underline{y}}_{j,L}(\xi)$	formal truncated asymptotic expansions defined for $j = 1,2,3$ and 4 and $L = 0,1,2,3,\cdots$	37
$\{\underline{y}_j(\xi)\}$	family of solutions asymptotic to $\hat{\underline{y}}_{j,\infty}(\xi)$ and defined for $j = 1,2,3$ and 4	38
$\underline{y}_j(\xi;\xi_0)$	a member of $\{\underline{y}_j(\xi)\}$ which is in canonical form at $\xi = \xi_0$ and defined for $j = 1,2,$ and 3 . . .	43,46

Symbol	Definition	Page
$\underline{y}_T(\xi;\xi_0)$	temporary canonical form.	45
ξ_I	initial ξ value at which canonical numerical integration begins.	49
τ	$=\sqrt{\xi}$	49
$\underline{\varepsilon}_0$	vector error.	53
$\underline{y}_{je}(\xi)$	a solution with an initial vector error.	53
$E_{je}(\xi;\xi_0)$	the absolute error of $\underline{y}_{je}(\xi)$ with respect to the family $\{\underline{y}_j(\xi)\}$	53
$f_{j1}(\xi)$, $f_{j2}(\xi)$	minimum norms of members of the family $\{\underline{y}_j(\xi)\}$ subject to constraints	59
$A_{je}(\xi;\xi_0)$	relative accuracy of $\underline{y}_{je}(\xi)$ with respect to the family $\{\underline{y}_j(\xi)\}$	61
$R_{je}(\xi;\xi_0)$	relative error of $\underline{y}_{je}(\xi)$ with respect to the family $\{\underline{y}_j(\xi)\}$	61
$\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, $\underline{c}_{1,3}$	lead vectors in the formal asymptotic expansions $\underline{y}_{i,\infty}(\xi)$	67
κ_R	reflection coefficient.	102

Symbol	Definition	Page
δ	$\ell n \left(\frac{1}{\varepsilon}\right)$	102
κ_{RA}, κ_{RG}	acoustic and gravity reflection coefficients.	109
$z_{t.\ell.}$	lower boundary of the transition layer.	114
$z_{b.\ell.}$	upper boundary of the boundary layer	114
S	$\sqrt{\lvert\xi\rvert}$.	125
$\Phi(\xi)$	fundamental matrix solution	137
S_m	constant 4×4 matrices defined for $m = 0,1,2,\cdots$	137
J	Jordan canonical form of R	137
$\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, $\underline{e}_4$,	eigenvectors and generalized eigenvectors of R	138,139
$\hat{\Phi}(\xi)$	$S_0\xi^J$	145
$\hat{\phi}_j(\xi)$	*jth* column of $\hat{\Phi}(\xi)$.	146
$\phi_j(\xi)$	*jth* column of $\Phi(\xi)$.	146
$\phi_j^{(n)}(\xi)$	*nth* Picard iterant defined for $j = 3$ or 4 and $n = 0,1,2,3,\cdots$ . .	148

Symbol	Definition	Page
A_n	constant 4×4 matrices defined for $n = 0,1,2,3,\cdots$	158
E	constant nonsingular matrix	160
$\tilde{\underline{y}}$	$=E^{-1}\underline{y}$	160
$\tilde{A}_n$	$=E^{-1}A_nE$	161
$\tilde{\tilde{\underline{y}}}$	$=\text{diagonal}(1, 1, 1, \tau)\tilde{\underline{y}}$	162
$\tilde{\tilde{A}}_n$	constant 4×4 matrices defined for $n = 0,1,2,3,\cdots$	162
F	constant nonsingular matrix	165
$\tilde{\tilde{\tilde{\underline{y}}}}$	$=F^{-1}\tilde{\tilde{\underline{y}}}$	165
$\tilde{\tilde{\tilde{A}}}_n$	$=F^{-1}\tilde{\tilde{A}}_nF$	166
$P(\tau)$	Sibuya transformation	168
$\underline{X}$	$\tilde{\tilde{\tilde{\underline{y}}}} = P(\tau)\underline{X}$	168
P_0, P_1	$P(\tau) = P_0 + \frac{P_1}{\tau}$	169
C_n	constant 4×4 matrices defined for $n = 0,1,2,3,\cdots$	169
$\tilde{P}(\tau)$	combination of a Sibuya and similarity transformation	170

Symbol	Definition	Page
$\underline{Z}$	$\underline{X} = \tilde{P}(\tau)\underline{Z}$	170
$\tilde{P}_0$, $\tilde{P}_1$	$\tilde{P}(\tau) = \tilde{P}_0 + \frac{\tilde{P}_1}{\tau}$	170
B_n	constant 4×4 matrices defined for $n = 0,1,2,3,\cdots$	172
$\hat{\underline{Z}}_{i,L}(\tau)$	formal truncated asymptotic solutions of a transformed differential equation which are defined for $i = 1,2,3$ and 4 and $L = 0,1,2,3,\cdots$	173
$\underline{p}_n$, $\underline{q}_n$, $\underline{r}_n$, $\underline{s}_n$	constant vector coefficients in the formal asymptotic expansions $\hat{\underline{Z}}_{i,\infty}(\tau)$ and defined for $n = 0,1,2,\cdots$	173
$\hat{\hat{\underline{y}}}_{i,L}(\xi)$	formal truncated asymptotic solutions of the original differential equation and defined for $i = 1$ or 2 and $L = 0,1,2,\cdots$	189
$\underline{a}_{n,i}$	constant vector coefficients in the formal asymptotic expansions $\hat{\hat{\underline{y}}}_{i,L}(\xi)$ and defined for $n = 0,1,2,3,\cdots$	189

1. INTRODUCTION

In the study of atmospheric waves, a model which is frequently encountered is one of waves in an ideal compressible fluid in a half-space, with a specified vertical temperature profile (see e.g. [1]). For the linearized isothermal problem the variables are separable and a system of two first-order, ordinary differential equations in the vertical z-coordinate is obtained.

There is little difficulty in determining a fundamental set of solutions since the coefficient matrix is constant. However, the velocity components of both solutions are not uniformly bounded for all z and, hence, violate the assumptions underlying the linearization.

For the problem of forced oscillations two conditions must be imposed to specify a unique solution. One condition is obtained for the vertical velocity at the ground by assuming that the fluid maintains contact with the lower boundary $z = 0$ and another condition must be obtained.

For certain values of the frequency σ and horizontal wave number k, one inviscid solution has finite kinetic energy in an infinite column of fluid of finite cross section, and the other solution has infinite kinetic energy. For this case it is reasonable to select the solution with finite kinetic energy. Thus, a unique solution of the inviscid problem can be obtained by imposing this additional requirement. The solution for this case is of the form

$$\begin{bmatrix} u(x,z,t) \\ w(x,z,t) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} e^{\ell z + i(kx - \sigma t)}$$

where $k > 0$, $\sigma > 0$ and ℓ is real, $u(x,z,t)$ is the horizontal velocity component, $w(x,z,t)$ is the vertical velocity component, and A and B are constants. The solution thus propagates in a horizontal x-direction.

If other values of σ and k are considered it is possible to find two solutions with oblique lines of constant phase:

$$\begin{bmatrix} u_\pm(x,z,t) \\ w_\pm(x,z,t) \end{bmatrix} = \begin{bmatrix} A_\pm \\ B_\pm \end{bmatrix} e^{\ell z + i(kx \pm \beta z - \sigma t)}$$

where ℓ, k, β and σ are all positive. Both solutions have infinite kinetic energy. Since one of the solutions has upward energy flux and the other has downward energy propagation, it is possible to determine a unique solution to the inviscid problem by neglecting the solution with downward energy propagation. This assumption seems reasonable since all the energy is being supplied by the lower boundary. However, it must be assumed that reflection of the wave with upward energy flux is negligible.

This assumption (no reflection) will be called the radiation condition.

Recently Yanowitch [2] considered a similar problem: two-dimensional waves in an isothermal, incompressible fluid occupying the upper half-space with a density distribution which decreased exponentially with increasing altitude. By examining the viscous problem for a constant dynamic viscosity coefficient μ , Yanowitch was able to introduce a reasonable requirement on the dissipation of energy due to viscosity, which he called the "dissipation condition." This condition requires the solution of the viscous problem to dissipate only a finite amount of energy in a column per period of oscillation of the lower boundary. The dissipation condition, the no-slip condition ($u = 0$ at $z = 0$), and the boundary condition on w at $z = 0$ are sufficient to prescribe a unique solution for every small $\mu > 0$. Yanowitch then analyzed the viscous problem as μ tended to zero. He was able to show that the inviscid solution with finite kinetic energy is approached uniformly, on an interval $0 < A \leq z \leq B < \infty$ as $\mu \to 0$; if the parameters σ and k yield one inviscid solution with finite kinetic energy and another with infinite kinetic energy. For other values of σ and k which yield two inviscid solutions with oblique lines of phase propagation and both having infinite kinetic energy, he was able to show that the reflection of waves is not always negligible. More precisely, he was able to show that the reflection coefficient, κ_R (ratio of the amplitudes of the incident and reflected waves), satisfied

$$|\kappa_R| = e^{-\pi\beta} ,$$

where $\beta = \frac{2\pi H}{\ell}$, H is the density scale height and ℓ is the vertical wavelength.[1] The reflection in Yanowitch's problem was due to a relatively thin layer which receded to infinity as $\mu \to 0$. Thus, κ_R did not approach a limiting value, i.e., arg κ_R varied as $\mu \to 0$. For an incompressible isothermal model of the atmosphere it appears that the inclusion of viscosity not only damps the wave motion for large z but is also capable of causing reflection. Since the wave motion is damped for large z, Yanowitch was also able to determine reasonable estimates for the amplitude of the oscillation of the lower boundary such that the viscous solution remained small for all z. Thus, viscosity also provided a justification of the linearization.

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[1]Similar results were obtained by Yanowitch [3] for vertical oscillations of an isothermal, compressible atmosphere. In addition, Lindzen [4] obtained a similar result for tidal waves in an isothermal atmosphere subject to Newtonian cooling.

Clearly, compressibility should be included in a model of the earth's atmosphere. Thus, a natural extension of Yanowitch's approach would be to investigate two-dimensional, linearized wave motion in a viscous, isothermal, compressible, stratified fluid occupying the upper half-space $z > 0$.

The incompressible and compressible models differ in several respects. The most notable difference is that the incompressible model has only a low frequency (gravity region) range of values for σ such that the inviscid solutions are wavelike in z , whereas the compressible model has both a high frequency (acoustic region) and a low frequency (gravity region) range separated by an excluded region. In addition, the compressible model has inviscid solutions, for certain values of σ and k , which are eigensolutions, or free oscillations. These solutions are the so-called Lamb waves and the incompressible model has no such solutions. Thus, for certain values of σ and k a resonant situation will develop for the compressible model, and no resonant case is possible for the incompressible model.

Many of the results obtained by Yanowitch for the incompressible fluid model are also obtained in this thesis for a compressible fluid model. There is one significant exception: the magnitude of the reflection coefficient for the case of inviscid solutions which are wavelike in z depends on the horizontal wave number k. It was found that for the range of parameters considered, the reflection coefficient κ_R satisfies

$$|\kappa_R| \leq e^{-\pi\beta} ,$$

where $\beta = \frac{2\pi H}{\ell}$, H is the density scale height, and ℓ is the vertical wavelength. In general, the magnitude of the gravity reflection coefficient more nearly equals $e^{-\pi\beta}$ than does the acoustic reflection coefficient. If an error of 15 percent is tolerated then the magnitude of the gravity reflection coefficient can be considered approximately equal to $e^{-\pi\beta}$ for all k in the range of computations. The acoustic reflection coefficient has rather peculiar properties for k values which correspond to horizontal wavelengths of 7H to 30H. For example, if a horizontal wavelength of 13H (about 90 km) is considered, then the reflection coefficient is no longer monotonically decreasing as β increases. If the horizontal wavelength is greater than 30H or less than 7H, then $e^{-\pi\beta}$ provides a reasonable approximation of the magnitude for both the acoustic and gravity reflection coefficients. A summary of the results for the case of inviscid solutions which are wavelike in z is provided in figures 2 through 7 in Section 5.2.

In Section 2 the differential equations for linearized wave motion in a stratified compressible isothermal fluid are developed. A system of two second-order, ordinary differential equations in z (the altitude) are obtained if the fluid is assumed viscous but thermally nonconducting. The inviscid equations are also developed in Section 2 and a fundamental set of inviscid solutions are found.

The remainder of this paper is concerned with an analysis of the viscous problem as the dynamic viscosity coefficient tends to zero. The most difficult mathematical problem encountered is the so-called asymptotic continuation problem (Section 4). The viscous problem is not completely analytically tractable[2] and thus a numerical integration of the viscous ordinary differential equation was performed. Since the asymptotic continuation problem is inherently unstable (Section 4), a numerical integration is not a trivial procedure. A modification of

[2]For the incompressible problem, which Yanowitch considered [2], the corresponding differential equation is simple enough to make possible the determination of an integral representation of the solution. The asymptotic relations are then obtained from the integral representation. A similar procedure was attempted for the compressible problem but it was unsuccessful.

an algorithm developed by Conte [5] is contained in Section 4. In addition, an analysis of the error is performed in Section 4 which is not included in Conte's paper. In fact, the definition of error is rather novel. A summary of all the computations and some concluding remarks are in Section 5.

2. FORMULATION

The complete set of equations governing two-dimensional flow in a viscous but thermally nonconducting fluid[3] are:

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial x} = \mu\,\Delta u + \frac{\mu}{3}\,\frac{\partial}{\partial x}\,(\mathrm{div}\,\vec{v}) \quad , \qquad (1a)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + w\,\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z} + \rho g = \mu\,\Delta w + \frac{\mu}{3}\,\frac{\partial}{\partial z}\,(\mathrm{div}\,\vec{v}) \quad , \qquad (1b)$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\vec{v}) = 0 \quad , \qquad (1c)$$

and

$$\frac{d}{dt}\,(p\rho^{-\gamma}) = 0 \quad , \qquad (1d)$$

where ρ is the density, u is the horizontal component of the velocity, w is the vertical component of the velocity, $\vec{v} = \begin{bmatrix} u \\ w \end{bmatrix}$, and p is the pressure; also, γ is the ratio of the specific heats, t is time, x is the horizontal space coordinate, z is the vertical

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[3]For a gas, viscosity and thermal conductivity are related [6, pp. 47-50] and, thus, it is inconsistent to assume a viscous but thermally nonconducting fluid. The model is very simple, but it still provides a qualitative description of reflection.

space coordinate, g is the acceleration of gravity,[4] μ is the dynamic viscosity coefficient,[5] and Δ is the two-dimensional Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. Equations (1a) and (1b) are the Navier-Stokes equations, equation (1c) is derived by assuming conservation of mass, and equation (1d) is the adiabatic law which neglects second-order dissipative terms.

For small deviations from equilibrium the equations (1) can be approximated by the linearized equations:

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \mu \Delta u + \frac{\mu}{3} \frac{\partial}{\partial x} (\mathrm{div}\ \vec{v}) \quad , \tag{2a}$$

$$\rho_0 \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} + g\rho = \mu \Delta w + \frac{\mu}{3} \frac{\partial}{\partial z} (\mathrm{div}\ \vec{v}) \quad , \tag{2b}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 (\mathrm{div}\ \vec{v}) + w \frac{\partial \rho_0}{\partial z} = 0 \quad , \tag{2c}$$

and

$$\frac{\partial p}{\partial t} + c^2 \rho_0 (\mathrm{div}\ \vec{v}) - g\rho_0 w = 0 \quad , \tag{2d}$$

where the variables without subscript are the perturbed values and the variables with the subscript zero are the

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[4]The acceleration of gravity g is assumed constant and, also, it is assumed to be the only body force acting on the fluid.

[5]The dynamic viscosity coefficient μ is assumed small but constant.

equilibrium values of these variables. The equilibrium values can be obtained from the following equations:

$$\frac{dp_0}{dz} + g\rho_0 = 0 \ , \tag{3a}$$

$$p_0 = R\rho_0 T_0 \tag{3b}$$

and

$$c^2 = \gamma R T_0 \ , \tag{3c}$$

where R is the gas constant and the temperature T_0 is assumed constant (isothermal).

If a scale height H is defined by

$$H = \frac{RT_0}{g} \tag{4}$$

then

$$\frac{1}{\rho_0}\frac{d\rho_0}{dz} = -\frac{1}{H} \ , \tag{5a}$$

or

$$\rho_0(z) = \rho_g e^{-z/H} \tag{5b}$$

and

$$c^2 = \gamma g H \tag{5c}$$

If the Brunt-Väisälä frequency is defined by

$$\frac{N^2}{g} = -\left(\frac{g}{c^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\right) \tag{6}$$

then

$$\rho_0\left\{\frac{\partial^2 u}{\partial t^2} + g\frac{\partial w}{\partial x} - c^2\frac{\partial}{\partial x}(\text{div }\vec{v})\right\} = \mu\frac{\partial}{\partial t}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}(\text{div }\vec{v})\right] \tag{7a}$$

and

$$\rho_0\left\{\frac{\partial^2 w}{\partial t^2} + g\frac{\partial w}{\partial z} - c^2\frac{\partial}{\partial z}(\text{div }\vec{v}) + \frac{N^2c^2}{g}(\text{div }\vec{v})\right\}$$

$$= \mu\frac{\partial}{\partial t}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}(\text{div }\vec{v})\right] \tag{7b}$$

If the following variables are introduced

$$u(x,z,t) = U(z)e^{i(kx-\sigma t)} , \tag{8a}$$

$$w(x,z,t) = iW(z)e^{i(kx-\sigma t)} , \tag{8b}$$

$$\tilde{\rho}(z) = \rho_0(z)/\rho_g , \tag{8c}$$

$$\tilde{z} = z/H , \tag{8d}$$

$$\tilde{k} = kH , \tag{8e}$$

$$\tilde{\sigma} = \sigma\sqrt{\frac{H}{g}} \tag{8f}$$

and

$$\varepsilon = \frac{\mu}{\gamma\rho_g H^2}\sqrt{\frac{H}{g}} \tag{8g}$$

into the differential equation (7) then a system of second-order differential equations is obtained:

$$\begin{bmatrix} i\varepsilon\sigma & 0 \\ 0 & \rho - i\frac{4}{3}\varepsilon\sigma \end{bmatrix} \begin{bmatrix} U(z) \\ W(z) \end{bmatrix}'' + \begin{bmatrix} 0 & k\left(\rho - \frac{i\varepsilon\sigma}{3}\right) \\ k\left(\rho - \frac{i\varepsilon\sigma}{3}\right) & -\rho \end{bmatrix} \begin{bmatrix} U(z) \\ W(z) \end{bmatrix}'$$

$$+ \begin{bmatrix} \rho\left(k^2 - \frac{\sigma^2}{\gamma}\right) - i\varepsilon\sigma\frac{4}{3}k^2 & -\frac{k\rho}{\gamma} \\ -k\frac{\gamma - 1}{\gamma}\rho & \frac{\rho\sigma^2}{\gamma} + i\varepsilon\sigma k^2 \end{bmatrix} \begin{bmatrix} U(z) \\ W(z) \end{bmatrix} = \vec{0} \, , \tag{9}$$

where $' = \frac{d}{dz}$ and $\rho = e^{-z}$. The tilde has been omitted from the variables ρ, z, k and σ in equation (9). No confusion should arise in what follows, since only the dimensionless variables will be considered.

Equation (9) might arise if the periodic solution is desired when the lower boundary is forced to oscillate with frequency σ . The fluid maintains contact with the lower boundary and, hence, must satisfy the no-slip condition, $U(0) = 0$. Without loss in generality, the oscillation of the lower boundary can be normalized so that $W(0) = 1$.

The problem of forced oscillations is, of course, artificial and the boundary conditions

$$U(0) = 0 \tag{10a}$$

and $$W(0) = 1 \qquad (10b)$$

are only a part of the complete mathematical formulation. Only the effects of the "upper boundary" are going to be assessed, not the effects of the oscillating lower boundary. For this problem it is necessary only to investigate a plane wave incident on the upper boundary; it is not necessary to examine a realistic mechanism for the development of such a wave. Thus, the boundary conditions (10) only provide a normalization of the solution of the viscous problem and are otherwise physically meaningless.

In addition to the boundary conditions (10), other requirements must be imposed in order to insure uniqueness of the solution of (9). The lower boundary is capable of performing only a finite amount of work per period of oscillation per unit area. Thus, it seems reasonable to require that only a finite amount of energy be dissipated in an infinite column of fluid of finite cross section. This condition will be referred to as the dissipation condition and denoted by DC.

The local dissipation of energy depends on the dynamic viscosity μ and the squares of the space derivatives of u and w. Thus the DC is equivalent to requiring

$$\int_0^\infty \left\{ |U'(z)|^2 + |W'(z)|^2 + |U(z)|^2 + |W(z)|^2 \right\} dz < \infty \tag{11}$$

where k is assumed to be nonzero.

Thus, the complete mathematical formulation of the viscous problem consists of a system of two second-order, linear, ordinary differential equations (9), boundary conditions (10) and the DC (11).

For ρ large and ε small it is expected that the solution of the viscous problem can be approximately obtained by considering $\varepsilon = 0$ or $\mu = 0$. If μ is set equal to zero in (2) and relations (3) through (6) and (8) are used then

$$\begin{bmatrix} U(z) \\ W(z) \end{bmatrix}' = \begin{bmatrix} \dfrac{\gamma - 1}{\gamma} & \dfrac{1}{k}\left(\dfrac{\gamma - 1}{\gamma}\dfrac{k^2}{\sigma^2} - k^2\right) \\ \dfrac{1}{k}\left(\dfrac{\sigma^2}{\gamma} - k^2\right) & \dfrac{1}{\gamma} \end{bmatrix} \begin{bmatrix} U(z) \\ W(z) \end{bmatrix} \tag{12}$$

is obtained.

The solutions of (12) are easily determined since the coefficient matrix is constant. The first step is to solve for the eigenvalues of the coefficient matrix. This leads to the dispersion relation

$$\lambda^2 - \lambda + \alpha = 0 \tag{13a}$$

and $$\alpha = \frac{\sigma^2}{\gamma} - k^2 + \frac{\gamma - 1}{\gamma}\frac{k^2}{\sigma^2} \quad , \tag{13b}$$

where λ is an eigenvalue of the constant coefficient matrix in (12). The roots of the dispersion relation (13) are

$$\lambda_1 = \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha} \tag{14a}$$

and $$\lambda_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \alpha} \tag{14b}$$

If $\alpha < \frac{1}{4}$ in (13), then λ_1 and λ_2 are real and $\lambda_2 < \frac{1}{2} < \lambda_1$. The solutions of the inviscid equation (12) are of the form

$$\begin{bmatrix} U_j(z) \\ W_j(z) \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\lambda_j - \frac{1}{\gamma}\right)} \end{bmatrix} e^{\lambda_j z} \quad , \qquad j = 1 \text{ or } 2 \quad , \tag{15}$$

for $\sigma^2/\gamma - k^2 \neq 0$ and

$$\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix} = \begin{bmatrix} \dfrac{-\left(\lambda_1^2 - \lambda_1 + \frac{\sigma^2}{\gamma}\right)}{k\left(\lambda_1 + \frac{1}{\gamma} - 1\right)} \\ 1 \end{bmatrix} e^{\lambda_1 z} \tag{16a}$$

and $$\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda_2 z} \tag{16b}$$

for $\frac{\sigma^2}{\gamma} - k^2 = 0$.

The solutions with the subscript 2 in relations (15) and (16) have finite kinetic energy in an infinite column of finite cross section, since

$$\left\{|U_j(z)|^2 + |W_j(z)|^2\right\}\rho(z) = \text{constant} \times e^{(2\lambda_j - 1)z} \tag{17}$$

and $\lambda_2 < \frac{1}{2}$. The solutions corresponding to the subscript 1 have infinite kinetic energy since $\lambda_1 > \frac{1}{2}$. For the case $\frac{\sigma^2}{\gamma} - k^2 = 0$ the solution with finite kinetic energy, the so-called Lamb wave, is a free oscillation. Hence, for this case a resonant situation can be expected to develop.

If $\alpha = \frac{1}{4}$ then the dispersion relation (13) yields only one root. For this case the solutions of (12) have the form

$$\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\frac{1}{2} - \frac{1}{\gamma}\right)} \end{bmatrix} e^{z/2} \tag{18a}$$

and

$$\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{(2+\gamma)(\sigma^2/\gamma - k^2)}{(2-\gamma)k\left(\frac{1}{2} - \frac{1}{\gamma}\right)} \end{bmatrix} e^{z/2} + z \begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix} . \tag{18b}$$

Both solutions have infinite kinetic energy in an infinite column of fluid of finite cross section.

The most interesting case occurs when the roots of the dispersion relation (13) are complex. This occurs when

$$\alpha = \frac{\sigma^2}{\gamma} - k^2 + \frac{\gamma - 1}{\gamma}\frac{k^2}{\sigma^2} > \frac{1}{4} \tag{19}$$

The roots of (13a) are then of the form

$$\lambda_1 = \frac{1}{2} + i\beta \quad , \tag{20a}$$

$$\lambda_2 = \frac{1}{2} - i\beta \quad , \tag{20b}$$

where
$$\beta = \sqrt{\alpha - \frac{1}{4}} \quad . \tag{20c}$$

Thus β is a dimensionless vertical wave number.

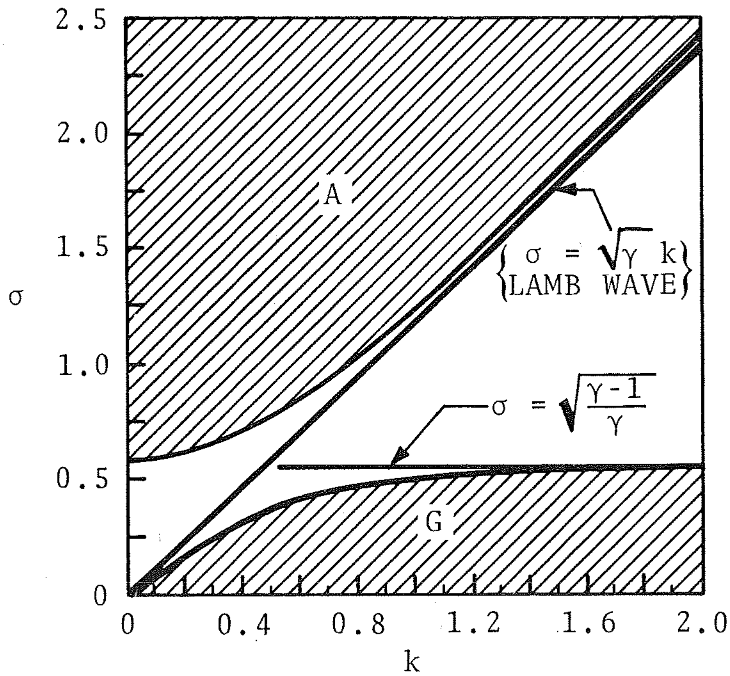

Figure 1. — Inviscid dispersion relation for $\gamma = 1.4$.

The solutions of the inviscid differential equation (12) are wavelike in z if the dimensionless parameters σ and k have values which lie in the shaded regions of figure 1. The upper shaded region A in figure 1 is referred to as the acoustic region and the lower shaded region G is called the gravity region. Both solutions of (12) for $\alpha > \frac{1}{4}$ are of the form (15). Both solutions have constant kinetic energy for all z and, hence, have infinite kinetic energy in a column of fluid.

If the group velocities of solutions (15) for complex λ_j are investigated, then it can be shown that $\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix}$ has an upward energy flux for $\sigma^2/\gamma - k^2 > 0$ and a downward energy flux for $\sigma^2/\gamma - k^2 < 0$ (see [7]). Similarly, $\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix}$ can be shown to have a downward energy flux for $\sigma^2/\gamma - k^2 > 0$ and an upward energy flux for $\sigma^2/\gamma - k^2 < 0$. Thus, in the acoustic region the vertical phase propagation and the energy flux are in the same direction for solutions (15), and for the gravity region the vertical phase propagation and the energy flux are in opposite directions.

It is worth noting that in some publications prior to Yanowitch's paper [2] the upward- and downward-going waves are handled separately, that is, linear combinations of the two solutions are not considered. It appears that the tacit assumption regarding this wave motion is that reflection does not occur.

In the following sections the differential equation (9) will be investigated subject to the boundary conditions (10) and the DC (11). The solution of the viscous problem will be investigated in the limit as $\varepsilon \to 0$ (or $\mu \to 0$) and also for a value $\varepsilon = 10^{-11}$, which is comparable to the value in the earth's atmosphere. The object is to obtain some insight into the correct formulation of the inviscid problem ($\mu \equiv 0$).

3. REFORMULATION AND DISCUSSION OF THE VISCOUS PROBLEM

3.1 REFORMULATION

It will be convenient to introduce a new independent dimensionless variable ξ, defined by

$$\xi = \frac{e^{-z}}{i\varepsilon\sigma} \tag{21}$$

The differential equation (9) then takes the form

$$\begin{bmatrix} 1 & 0 \\ 0 & \xi - \frac{4}{3} \end{bmatrix} \Theta^2 \begin{bmatrix} U(\xi) \\ W(\xi) \end{bmatrix} - \begin{bmatrix} 0 & k\left(\xi - \frac{1}{3}\right) \\ k\left(\xi - \frac{1}{3}\right) & -\xi \end{bmatrix} \Theta \begin{bmatrix} U(\xi) \\ W(\xi) \end{bmatrix}$$
$$+ \begin{bmatrix} \left(k^2 - \frac{\sigma^2}{\gamma}\right)\xi - \frac{4k^2}{3} & -\frac{k}{\gamma}\xi \\ -k\frac{\gamma - 1}{\gamma}\xi & \frac{\sigma^2}{\gamma}\xi + k^2 \end{bmatrix} \begin{bmatrix} U(\xi) \\ W(\xi) \end{bmatrix} = \vec{0}\,, \tag{22}$$

where $\Theta = \xi\frac{d}{d\xi} = -\frac{d}{dz}$. The boundary conditions (10) become

$$\begin{bmatrix} U(\xi_1) \\ W(\xi_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{23a}$$

where

$$\xi_1 = \frac{1}{i\varepsilon\sigma} \tag{23b}$$

Thus, as $\varepsilon \to 0$ through positive values $\xi_1 \to \infty$ along the ray $\arg \xi = -\frac{\pi}{2}$. Any bounded region, $0 < A \le z \le B < \infty$, is restricted to a line segment on the ray $\arg \xi = -\frac{\pi}{2}$ and as $\varepsilon \to 0$ the line segment shifts to infinity. Hence, an investigation of (9) as $\varepsilon \to 0$ on $0 < A \le z \le B < \infty$ is equivalent to an investigation of (22) for large ξ.

If
$$\underline{y}(\xi) = \begin{bmatrix} U(\xi) \\ \Theta U(\xi) \\ W(\xi) \\ \Theta W(\xi) \end{bmatrix}, \tag{24a}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{24b}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3/4 \end{bmatrix}, \tag{24c}$$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{4k^2}{3} & 0 & 0 & -\frac{k}{3} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k}{4} & \frac{3k^2}{4} & 0 \end{bmatrix} \tag{24d}$$

and

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\sigma^2}{\gamma} - k^2 & 0 & \frac{k}{\gamma} & k \\ 0 & 0 & 0 & 0 \\ \frac{-3(\gamma - 1)k}{4\gamma} & \frac{-3k}{4} & \frac{3\sigma^2}{4\gamma} & \frac{3}{4} \end{bmatrix} \tag{24e}$$

then

$$(I + \xi K) \frac{d}{d\xi} \underline{y}(\xi) = \frac{1}{\xi} (R + \xi D)\underline{y}(\xi) \cdot \tag{25}$$

Equation (25) has regular singularities at $\xi = 0$ and $\xi = \frac{4}{3}$ and an irregular singularity at $\xi = \infty$. The viscous problem, formulated in Section 2, can be restated in terms of the new ξ-variable. The problem is to investigate solutions of equation (25) on the ray $\arg \xi = -\frac{\pi}{2}$, which satisfy the boundary condition (23) and the DC (11) as $\varepsilon \to 0$.

Near $\xi = 0$ there exist solutions of equation (25) which exhibit the scalar growths ξ^k, $(\ln \xi)\xi^k$, ξ^{-k}, and $(\ln \xi)\xi^{-k}$ as $\xi \to 0$ (see Appendix A). Only the solutions which grow like ξ^k and $(\ln \xi)\xi^k$ satisfy the DC. Hence, the solution of the viscous problem is merely a linear combination of the two solutions which satisfy the DC.

For large ξ a fundamental set of asymptotic solutions of equation (25) can be found (see Appendix B.1) which exhibit the scalar growths $\xi^{-\lambda_1}$, $\xi^{-\lambda_2}$, $\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$ and $\xi^{\frac{1}{4}}e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}$, where λ_1 and λ_2 are the roots of the dispersion relation (13). Due to the different scalar rates of growth the asymptotic solutions are significant in different regions. In particular, due to the boundary condition (23) and the rapid growth of $\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$, as $|\xi|$ increases along $\arg \xi = -\frac{\pi}{2}$, the solution of equation (25), which exhibits this approximate scalar growth, is significant only near $z = 0$ or $\xi = \xi_1$.[6]

Thus, if $\varepsilon > 0$ is small, the two solutions which satisfy the DC must combine so as to eliminate approximately the rapidly growing solution over most of the interval $0 < |\xi| < |\xi_1|$ and $\arg \xi = -\frac{\pi}{2}$. Except for a scaling constant the viscous problem can be approximately solved over most of the ξ-interval by determining a linear

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[6]This is discussed more fully in Section 3.2.

combination of the solutions satisfying the DC, which is asymptotic to a linear combination of the solutions exhibiting the scalar growths $\xi^{-\lambda_1}$, $\xi^{-\lambda_2}$, and $\xi^{\frac{1}{4}}e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}$.

In solving the viscous problem four distinct regions develop. Near $\xi = 0$ the viscous forces dominate and the solution decays like ξ^k and $(\ln \xi)\xi^k$ and, hence, this region is called the viscous region. For large ξ the solution of the viscous problem is approximately a linear combination of inviscid solutions and, hence, this region is called the inviscid region. Connecting the viscous region and the inviscid region is a transition region where ξ varies from large to small values or equivalently the kinematic viscosity varies from small to large values. In addition, near $\xi_1 = \frac{1}{i\varepsilon\sigma}$ a boundary layer develops. The object is to investigate the viscous problem in the inviscid region as $\varepsilon \to 0$.

What is required mathematically is a means of connecting the viscous behavior to the inviscid behavior through the transition region, or a method to connect the expansions about the regular singularity $\xi = 0$ to the asymptotic expansions about the irregular singularity $\xi = \infty$. Due to the very complicated three-term recursion relation for the expansions about $\xi = 0$, it was not possible to attack this problem in the same manner in which Yanowitch solved the incompressible problem. However, it is possible to solve the viscous problem numerically (See Section 4).

3.2 HEURISTIC DISCUSSION OF THE VISCOUS PROBLEM

In this section a heuristic discussion of the viscous problem, formulated in Section 2, is carried out. Unless certain pathological situations develop, it is shown that it is reasonable to expect existence and uniqueness of the solution of the viscous problem for every sufficiently small value of $\varepsilon > 0$. For the particular values of the parameters used in the calculations it appears that none of the pathological situations developed. In addition, the four regions (viscous, transition, inviscid, and boundary layer) are discussed.

In Appendix A it is shown that there exist precisely two linearly independent solutions of the differential equation (25) which satisfy the DC and two linearly independent solutions which violate the DC. The two solutions which satisfy the DC will be denoted by $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$. No singularities exist on the ray $\arg \xi = -\frac{\pi}{2}$ for $0 < |\xi| < \infty$; hence it is possible to analytically continue $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ to the whole ray.

THEOREM 1: Let the two-dimensional vectors $\vec{dc}_1(\xi)$ and $\vec{dc}_2(\xi)$ consist of the first and third components of the four-dimensional vectors $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$, respectively. If $\xi_1 = \frac{1}{i\varepsilon\sigma}$ and $\vec{dc}_1(\xi_1)$ and $\vec{dc}_2(\xi_1)$ are linearly independent, then the viscous problem has one and only one solution.

NOTE: It is already known that $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ are linearly independent since these solutions of equation (25) are linearly independent for small values of $\xi \neq 0$. For linear differential equations two or more solutions are linearly independent on an interval which excludes singularities if and only if they are linearly independent at a single point in the interval [8, Chapter 3].

It is not unreasonable to assume that $\vec{dc}_1(\xi_1)$ and $\vec{dc}_2(\xi_1)$ are linearly independent. See theorems B3 and B4 in Appendix B.2 for a weaker hypothesis in establishing the existence and uniqueness of the solution of the viscous problem.

PROOF: The viscous problem consists of a differential equation (25), boundary condition (23), and the DC (11). Any solution of equation (25) which satisfies the DC (11) must be a linear combination of $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ (see Appendix A). The only question is whether there is one and only one linear combination of $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ which satisfies the boundary condition (23) at $\xi = \xi_1$. The boundary condition (23) imposes a requirement only on the first and third components of $\underline{DC}_1(\xi_1)$ and $\underline{DC}_2(\xi_1)$. Since $\vec{dc}_1(\xi_1)$ and $\vec{dc}_2(\xi_1)$ are linearly independent it is clear that there is one and only one solution of

$$c_1 \times \vec{dc}_1(\xi_1) + c_2 \times \vec{dc}_2(\xi_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{Q.E.D.}$$

It is important to note that it is reasonable to expect the existence of a unique solution to the viscous problem formulated in Section 1 only if there are precisely two linearly independent solutions of the differential equation (25) which satisfy the DC (11). If only one solution of equation (25) satisfied the DC, there would be little hope of also satisfying the boundary condition (23). If more than two linearly independent solutions of (25) satisfied the DC, then there would probably be infinitely many solutions to the viscous problem.

Theorem 1 is not very useful for obtaining additional qualitative information for ξ large. The region $|\xi| > 1$ is important since it corresponds to the physical region $0 \leq z < \ell n \left(\frac{1}{\varepsilon\sigma}\right)$. The most important analysis from a physical standpoint concerns the region $|\xi| > 1$.

In order to obtain additional qualitative information for ξ large it is necessary to make use of the asymptotic expansions about the irregular singularity $\xi = \infty$. In Appendix B a fundamental set of formal asymptotic expansions is developed about the irregular singularity $\xi = \infty$. The principal results are the following:

a. The lead terms in two of the asymptotic expansions correspond to the two inviscid solutions, that is, two of the asymptotic expansions are asymptotic to the inviscid solutions.

b. There are two remaining asymptotic solutions which exhibit the scalar growths $\xi^{\frac{1}{4}} e^{\pm 2\sigma \sqrt{\frac{\xi}{\gamma}}}$.

The solutions of equation (25) which are asymptotic to the inviscid solutions will be denoted by $\underline{INV}_1(\xi)$ and $\underline{INV}_2(\xi)$. The solution of equation (25) which is exponentially increasing in $\sqrt{\xi}$ as $|\xi|$ increases will be denoted by $\underline{BLSOL}(\xi)$. The exponentially decreasing solution will be denoted by $\underline{TLSOL}(\xi)$.

If $\underline{y}_{VP}(\xi)$ is the solution of the viscous problem for some small $\varepsilon > 0$, then $\underline{y}_{VP}(\xi)$ satisfies

$$\underline{y}_{VP}(\xi) = c_1 \times \underline{DC}_1(\xi) + c_2 \times \underline{DC}_2(\xi) \qquad (26)$$

for some constants c_1 and c_2 . Since $\underline{INV}_1(\xi)$, $\underline{INV}_2(\xi)$, $\underline{TLSOL}(\xi)$, and $\underline{BLSOL}(\xi)$ grow at different asymptotic rates, they are linearly independent for $0 < |\xi| < \infty$ and

$\arg \xi = -\frac{\pi}{2}$. Hence, there exist constants d_1, d_2, d_3, and d_4 such that

$$\underline{y}_{VP}(\xi) = d_1 \times \underline{INV}_1(\xi) + d_2 \times \underline{INV}_2(\xi) + d_3 \times \underline{TLSOL}(\xi) + d_4 \times \underline{BLSOL}(\xi) . \tag{27}$$

The boundary condition (23) restricts the norm[7] of $\underline{y}_{VP}(\xi)$ near $z = 0$ as $\varepsilon \to 0$, provided it is assumed that there exists a unique solution to the viscous problem for every $\varepsilon > 0$ sufficiently small. This additional hypothesis is reasonable since $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ are linearly independent for all ξ such that $0 < |\xi| < \infty$ and $\arg \xi = -\frac{\pi}{2}$ (see theorems B3 and B4 in Appendix B, Section B.2).

If the first and third components of $\underline{y}_{VP}(\xi)$ are $0(1)$ near $z = 0$ for each $\varepsilon > 0$ sufficiently small, then

$$d_1 = 0\left(|\xi_1|^{\mathrm{Re}\lambda_1}\right) \quad \text{since} \quad ||\underline{INV}_1(\xi_1)|| = 0\left(|\xi_1|^{-\mathrm{Re}\lambda_1}\right) , \tag{28a}$$

$$d_2 = 0\left(|\xi_1|^{\mathrm{Re}\lambda_2}\right) \quad \text{since} \quad ||\underline{INV}_2(\xi_1)|| = 0\left(|\xi_1|^{-\mathrm{Re}\lambda_2}\right) , \tag{28b}$$

[7]The only exception is the resonant Lamb-wave case. The details for this case are found in Appendix B, case 4, and in Section 5.

$$d_3 = 0\left(e^{Re 2\sigma\sqrt{\frac{\xi_1}{\gamma}}}\right) \qquad (28c)$$

since $$||\underline{TLSOL}(\xi_1)|| > 0\left(e^{-Re 2\sigma\sqrt{\frac{\xi_1}{\gamma}}}\right),$$

and $$d_4 = 0\left(e^{-Re 2\sigma\sqrt{\frac{\xi_1}{\gamma}}}\right) \qquad (28d)$$

since $$||\underline{BLSOL}(\xi_1)|| > 0\left(e^{Re 2\sigma\sqrt{\frac{\xi_1}{\gamma}}}\right),$$

where $\xi_1 = \frac{1}{i\varepsilon\sigma}$ and λ_1 and λ_2 are the roots of the dispersion relation (see Appendix B).

As $|\xi|$ is decreased (z increased) from the value $|\xi_1| = \frac{1}{\varepsilon\sigma}$ ($z = 0$), the term $d_4 \times \underline{BLSOL}(\xi)$ decreases very rapidly since

$$||d_4 \times \underline{BLSOL}(\xi)|| \approx \left(\frac{\xi}{\xi_1}\right)^{\frac{1}{4}} ||d_4 \times \underline{BLSOL}(\xi_1)|| \times e^{-\frac{2\sigma}{\sqrt{\gamma}}(Re\sqrt{\xi_1} - Re\sqrt{\xi})} \cdot \qquad (29)$$

Since $Re\sqrt{\xi_1} > Re\sqrt{\xi}$ if $|\xi_1| > |\xi|$ and $\arg \xi = -\frac{\pi}{2}$, thus, the term $||d_4 \times \underline{BLSOL}(\xi)||$ decreases exponentially in $\sqrt{|\xi|}$ as $|\xi|$ is decreased or equivalently as z is increased. The rapid decrease of $d_4 \times \underline{BLSOL}(\xi)$ is even more remarkable when it is related to changes in z. If $\Delta\mathrm{Re}\sqrt{\xi} = \mathrm{Re}\sqrt{\xi_1} - \mathrm{Re}\sqrt{\xi}$, then

$$\Delta\mathrm{Re}\sqrt{\xi} = \frac{|\sqrt{\xi_1} - \sqrt{\xi}|}{\sqrt{2}} = \frac{1 - e^{\Delta z/2}}{\sqrt{2\varepsilon\sigma}} \tag{30a}$$

and

$$e^{-\frac{2\sigma}{\sqrt{\gamma}}\Delta\mathrm{Re}\sqrt{\xi}} = \mathrm{EXP}\left\{-\sqrt{\frac{2\sigma}{\gamma\varepsilon}}\,(1 - e^{-\Delta z/2})\right\} \approx e^{-\sqrt{\frac{\sigma}{2\gamma\varepsilon}}\Delta z}. \tag{30b}$$

Thus, for an increase in z of about $\sqrt{\frac{2\gamma\varepsilon}{\sigma}}$ the term $d_4 \times$ <u>BLSOL</u>(ξ) is multiplied by a factor of about $\frac{1}{e}$. For small positive ε the solution <u>BLSOL</u>(ξ) is only important near the boundary $z = 0$, that is, the term $d_4 \times$ <u>BLSOL</u>(ξ) in equation (27) decreases so rapidly that it is insignificant and can be neglected outside a thin boundary layer. The boundary layer thickness is $0\left(\sqrt{\frac{\varepsilon}{\sigma}}\right)$ as $\varepsilon \to 0$. The solution <u>BLSOL</u>(ξ) is called a boundary layer solution.

Suppose the constants c_1 and c_2 in equation (26) are not known, but the vectors <u>DC</u>$_1(\xi)$, <u>DC</u>$_2(\xi)$, <u>INV</u>$_1(\xi)$, <u>INV</u>$_2(\xi)$, <u>TLSOL</u>(ξ), and <u>BLSOL</u>(ξ) are all determined at $\xi = \frac{1}{i}$. For small positive values of ε the term $d_4 \times$ <u>BLSOL</u>(ξ) is negligible at $\xi = \frac{1}{i}$ and, thus, the constants c_1 and c_2 in equation (26) satisfy

$$c_1 \times \underline{DC}_1\left(\frac{1}{i}\right) + c_2 \times \underline{DC}_2\left(\frac{1}{i}\right) \approx d_1 \times \underline{INV}_1\left(\frac{1}{i}\right) + d_2 \times INV_2\left(\frac{1}{i}\right) + d_3 \times \underline{TLSOL}\left(\frac{1}{i}\right) . \qquad (31)$$

For a value of ε [8] like 10^{-11} the equation (31) is correct to hundreds of significant figures and, hence, is certainly consistent with the accuracy of the mathematical model which has been developed.

If $d_2 \neq 0$ in (31), then the viscous problem can be solved in two steps. First solve for the constants e_1, e_2, e_3, and e_4 such that

$$e_1 \times \underline{DC}_1\left(\frac{1}{i}\right) + e_2 \times \underline{DC}_2\left(\frac{1}{i}\right) + e_3 \times \underline{INV}_1\left(\frac{1}{i}\right) + e_4 \times \underline{TLSOL}\left(\frac{1}{i}\right) = \underline{INV}_2\left(\frac{1}{i}\right) , \qquad (32)$$

and then solve for the constants d_2 and d_4 such that the vector $\underline{y}_{VP}(\xi)$ is given by

$$\underline{y}_{VP}(\xi_1) = d_2(\underline{INV}_2(\xi_1) - e_3\underline{INV}_1(\xi_1)) + d_4 \times \underline{BLSOL}(\xi_1) , \qquad (33)$$

that is, satisfies the boundary condition (23) at $\xi_1 = \frac{1}{i\varepsilon\sigma}$ or the boundary condition (10) at $z = 0$. The solution, $\underline{TLSOL}(\xi)$, is neglected in equation (33) since it

[8] For the earth's atmosphere, ε is comparable to 10^{-11} [6, Appendix 1].

decays exponentially fast. For $\varepsilon \approx 10^{-11}$ equation (33) is correct to hundreds of significant figures.

Thus, for large ξ the vector $\underline{y}_{VP}(\xi)$ is simply a linear combination of the solutions $\underline{INV}_1(\xi)$ and $\underline{INV}_2(\xi)$ which are asymptotic to the solutions of the inviscid differential equation (12). This linear combination of solutions is modified in a region very close to the boundary $z = 0$. The region where the inviscid solutions accurately approximate $\underline{INV}_1(\xi)$ and $\underline{INV}_2(\xi)$ and the boundary layer solution is negligible is the inviscid region mentioned in Section 3.1.

For values of ξ such that $|\xi| < 1$ the solutions $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ can be obtained from the expansions about the regular singularity $\xi = 0$ (see Appendix A, Section A.1). Due to viscosity the solutions decay as does ξ^k , $(\ell n\ \xi)\xi^k$ as $\xi \to 0$ or e^{-kz} and ze^{-kz} as $z \to \infty$. The region $0 < |\xi| < 1$ is the viscous region since viscous forces dominate and force the decay of the solution of the viscous problem.

Connecting the viscous region to the inviscid region is the transition region. In the transition region $\underline{TLSOL}(\xi)$ will be significant and for this reason it will be called the transition layer solution.

The viscous problem is solved for small $\varepsilon > 0$ when equations (32) and (33) are solved. The equation (33) is easily solved since the solutions $\underline{INV}_1(\xi)$, $\underline{INV}_2(\xi)$, and $\underline{BLSOL}(\xi)$ are easily computed from the asymptotic expansions (see Appendix B, Section B.1). However, equation (32) must be solved prior to equation (33) since the scalar e_3 in equation (32) is required in equation (33). Accurate values for $\underline{INV}_1(\xi)$, $\underline{INV}_2(\xi)$, and $\underline{TLSOL}(\xi)$ are obtained from the asymptotic expansions (see Section B.1) for large ξ. Thus, in order to solve equation (32) it is necessary to continue the accurate values of the asymptotic expansions to small values of ξ, that is, to $\xi = \frac{1}{i}$. This problem of continuing the accurate values of the asymptotic solutions will be called the asymptotic continuation problem. It will be investigated in Section 4.

4. INVESTIGATION OF THE ASYMPTOTIC CONTINUATION PROBLEM

This section was developed as a series of modifications of a paper by Conte [5]. A numerical algorithm is developed which is specifically aimed at solving the asymptotic continuation problem.

4.1 NUMERICAL ALGORITHM

A concept which arises frequently in numerical analysis is stability. Often a numerical algorithm is considered either stable or unstable. However, certain problems have "inherent instability", quite independently of the particular numerical algorithm used [9]. Due to the different asymptotic growths of solutions of (25), the asymptotic continuation problem is inherently unstable. Only the solution which grows most rapidly is easily computed with small relative error.

It is expected that the relative error for a standard numerical integration scheme will grow exponentially fast for the computation of any subdominant solution in the asymptotic continuation problem. However, if a certain definition of error (Section 4.2) is introduced, then it is found that the relative error grows only algebraically fast. In the remainder of this section a numerical algorithm is developed. An analysis of the algorithm follows in Section 4.2.

In Appendix B four distinct formal asymptotic solutions of (25) about the irregular singularity $\xi = \infty$ are developed. The formal truncated asymptotic expansions are denoted by $\hat{\underline{y}}_{j,L}(\xi)$ for $j = 1,2,3,$ or 4 and $L = 0,1,2,\cdots$. It is shown in Section B.2 that $\hat{\underline{y}}_{1,0}(\xi)$ and $\hat{\underline{y}}_{2,0}(\xi)$ are multiples of inviscid solutions. The vector $\hat{\underline{y}}_{3,L}(\xi)$ exhibits the scalar growth $\xi^{\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$ as $\xi \to \infty$ and $\hat{\underline{y}}_{4,L}(\xi)$ exhibits the scalar growth $\xi^{\frac{1}{4}} e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}$.

In order to solve the asymptotic continuation problem it is necessary to determine the solutions of (25) at $\xi = \frac{1}{i}$,[9] which exhibit the distinct asymptotic growths. Thus, it is necessary to determine vectors $\underline{y}_j(\xi)$ at $\xi = \frac{1}{i}$ such that

$$\underline{y}_j(\xi) \sim \hat{\underline{y}}_{j,\infty}(\xi) \quad \text{as} \quad \xi \to \infty , \quad \arg \xi = -\frac{\pi}{2} , \tag{34}$$

for $j = 1,2,3,$ and 4 . Relation (34) does not uniquely specify all of the vectors $\underline{y}_j(\xi)$. Since any set of four solutions of (25) which satisfy (34) can be used in the asymptotic continuation problem, it is natural to consider families of solutions of (25) rather than uniquely

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[9]The value $\xi = \frac{1}{i}$ was found to be a reasonable value at which to determine the asymptotic connection relations for most of the computations.

defined solutions. The families are chosen so that each member has the same asymptotic expansion on the ray[10] $\arg \xi = -\frac{\pi}{2}$. The family of solutions of the differential equation (25) which are asymptotic to $\hat{\underline{y}}_{j,\infty}(\xi)$, is denoted by $\{\underline{y}_j(\xi)\}$.

THEOREM 2: If $\underline{y}_j(\xi)$ is a member of $\{\underline{y}_j(\xi)\}$ for $j = 1,2,3,$ and 4, respectively, then $\underline{Y}_j(\xi)$ is also a member of $\{\underline{y}_j(\xi)\}$ if

$$\underline{Y}_4(\xi) \equiv \underline{y}_4(\xi) \quad , \tag{35a}$$

$$\underline{Y}_1(\xi) = \underline{y}_1(\xi) + c\underline{y}_4(\xi) \quad , \tag{35b}$$

$$\underline{Y}_2(\xi) = \underline{y}_2(\xi) + d\underline{y}_4(\xi) \tag{35c}$$

and

$$\underline{Y}_3(\xi) = \underline{y}_3(\xi) + c_1\underline{y}_1(\xi) + c_2\underline{y}_2(\xi) + c_4\underline{y}_4(\xi) \quad , \tag{35d}$$

where c, d, c_1, c_2, and c_4 are arbitrary complex scalars.

PROOF: The vectors $\underline{y}_1(\xi)$ and $\underline{y}_2(\xi)$ exponentially dominate $\underline{y}_4(\xi)$ as $\xi \to \infty$. Hence, any multiple of $\underline{y}_4(\xi)$ can be added to $\underline{y}_1(\xi)$ or $\underline{y}_2(\xi)$ without altering the asymptotic properties of $\underline{y}_1(\xi)$ or $\underline{y}_2(\xi)$.

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[10] The families are easily extended to the sector $|\arg \xi| < \pi$. However, this extension is unnecessary for the immediate problem of interest.

Similarly, $\underline{y}_3(\xi)$ exponentially dominates $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, and $\underline{y}_4(\xi)$ as $\xi \to \infty$. Hence, arbitrary multiples of $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, and $\underline{y}_4(\xi)$ can be added to $\underline{y}_3(\xi)$ without altering the asymptotic properties of $\underline{y}_3(\xi)$. Q.E.D.

THEOREM 3: If $\underline{y}_j(\xi)$ and $\underline{Y}_j(\xi)$ are members of $\{\underline{y}_j(\xi)\}$ for $j = 1,2,3,$ and 4, respectively, then

$$\underline{Y}_4(\xi) \equiv \underline{y}_4(\xi) \quad , \tag{36a}$$

$$\underline{Y}_1(\xi) = \underline{y}_1(\xi) + c\underline{y}_4(\xi) \quad , \tag{36b}$$

$$\underline{Y}_2(\xi) = \underline{y}_2(\xi) + d\underline{y}_4(\xi) \tag{36c}$$

and

$$\underline{Y}_3(\xi) = \underline{y}_3(\xi) + c_1\underline{y}_1(\xi) + c_2\underline{y}_2(\xi) + c_4\underline{y}_4(\xi) \quad ; \tag{36d}$$

for some scalars c, d, c_1, c_2, and c_4.

PROOF: The vectors $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, $\underline{y}_3(\xi)$, and $\underline{y}_4(\xi)$ are linearly independent since they have different asymptotic growths. Hence, any solution $\underline{y}(\xi)$ of the differential equation (25) can be represented in the form

$$\underline{y}(\xi) = \sum_{j=1}^{4} b_j \underline{y}_j(\xi) \quad .$$

In particular, there exist constants c_{jk} such that

$$\underline{Y}_j(\xi) = \sum_{k=1}^{4} c_{jk}\underline{y}_k(\xi) \quad . \tag{37}$$

However,

$$||\underline{y}_j(\xi) - \hat{\underline{y}}_{j,L}(\xi)|| = 0\left(\frac{1}{|\xi|^{\frac{L+1}{2}}}\right)\hat{\underline{y}}_{j,0}(\xi)$$

$$||\underline{Y}_j(\xi) - \hat{\underline{y}}_{j,L}(\xi)|| = 0\left(\frac{1}{|\xi|^{\frac{L+1}{2}}}\right)\hat{\underline{y}}_{j,0}(\xi)$$

and

$$||\underline{y}_j(\xi) - \underline{Y}_j(\xi)|| \leq ||\underline{y}_j(\xi) - \hat{\underline{y}}_{j,L}(\xi)|| + ||Y_j(\xi) - \hat{\underline{y}}_{j,L}(\xi)|| \quad .$$

Thus

$$||\underline{y}_j(\xi) - \underline{Y}_j(\xi)|| = 0\left(\frac{1}{|\xi|^{\frac{L+1}{2}}}\right)\hat{\underline{y}}_{j,0}(\xi) \quad , \tag{38}$$

for $j = 1,2,3$ and 4 , where L is an arbitrarily large integer. The theorem is an immediate consequence of relations (37) and (38). Q.E.D.

Suppose a numerical integration of the differential equation (25) in the direction of decreasing $|\xi|$ along the ray $\arg \xi = -\frac{\pi}{2}$ is performed. The solution $\underline{y}_4(\xi)$ increases exponentially fast (in $\sqrt{|\xi|}$), $\underline{y}_1(\xi)$ and $\underline{y}_2(\xi)$

exhibit algebraic growth, and $\underline{y}_3(\xi)$ decreases exponentially fast (in $\sqrt{|\xi|}$) as $|\xi|$ decreases. Hence, it appears that for a numerical integration in the direction of decreasing $|\xi|$ that the transition layer solution ($\underline{y}_4(\xi)$) is exponentially dominant. There should be little difficulty in obtaining an accurate value of the transition layer solution at, say, $\xi = \frac{1}{i}$. However, it is anticipated that the remaining solutions will be more difficult to determine accurately.

If the initial vectors for a numerical integration of the differential equation (25) are determined from the truncated asymptotic expansions $\hat{\underline{y}}_{j,L}(\xi)$, for $j = 1,2,3,$ and 4 , then an initial relative error $O\left(\frac{1}{|\xi|^{\frac{L+1}{2}}}\right)$ is introduced. The initial error can be made to approach zero algebraically fast as the initial $\xi \to \infty$. Due to the inherent instability of the continuation problem, an initially small error is expected to be magnified. The initial error may grow exponentially fast. Hence, it may happen that a simple numerical integration to $\xi = \frac{1}{i}$ will yield no significant figures in the determination of $\underline{y}_j(\xi)$, even though subsequent errors due to numerical integration can be made arbitrarily small by carrying sufficient precision in the calculations.

In order to solve the asymptotic continuation problem, it is necessary only to obtain an accurate member in each of the families $\{\underline{y}_j(\xi)\}$, for $j = 1,2,3,$ and 4 . An accurate determination of a member $\{\underline{y}_3(\xi)\}$ will serve as an error check. Thus, an attempt will be made to determine a member of $\{\underline{y}_3(\xi)\}$, although it is not required for the solution of equation (32).

One additional requirement must be imposed. Linear independence of the particular members of the families $\{\underline{y}_j(\xi)\}$ must be required. It will be shown in the next section that it is possible to obtain *accurate* members of the individual families which are essentially linearly dependent (ill-conditioned) if only finitely many significant figures are retained. In order to ensure linear independence of the particular members of the different families a canonical form will be introduced. To some extent the canonical form is arbitrary. The two main properties of the canonical form are that it

a. Ensures linear independence.

b. Singles out unique members in each family of solutions.

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In addition, as a bonus the canonical form provides an effective error control which overcomes the inherent instability of the continuation problem in many cases.

The transition layer solution ($\underline{y}_4(\xi)$) is uniquely defined and, thus, for any value of ξ it is possible to determine which component is greatest in modulus. This component will be called the maximum component of $\underline{y}_4(\xi)$. If more than one component achieves the maximum, then the component with the lowest index is called the maximum component. Thus, for a specified value of ξ there will correspond a unique maximum component of $\underline{y}_4(\xi)$.

If $\underline{y}_1(\xi)$ is a member of $\{\underline{y}_1(\xi)\}$, then consider

$$\underline{Y}_1(\xi) = \underline{y}_1(\xi) + c\underline{y}_4(\xi) \quad . \tag{39}$$

Due to theorem 2, $\underline{Y}_1(\xi)$ is also a member of $\{\underline{y}_1(\xi)\}$. The vector $\underline{Y}_1(\xi)$ is said to be in canonical form at ξ_0 if $\underline{Y}_1(\xi_0)$ has a zero component corresponding to the maximum component of $\underline{y}_4(\xi_0)$. A member of $\{\underline{y}_1(\xi)\}$, which is in canonical form at ξ_0 , is denoted by $\underline{y}_1(\xi;\xi_0)$. Similarly, a canonical form for the family $\{\underline{y}_2(\xi)\}$ is introduced.

The vector $\underline{y}_2(\xi;\xi_0)$ is a solution of (25), such that $\underline{y}_2(\xi_0;\xi_0)$ is in canonical form, that is, $\underline{y}_2(\xi_0;\xi_0)$

has a zero component corresponding to the maximum component of $\underline{y}_4(\xi_0)$.

THEOREM 4: For each value of ξ_0 such that $0 < |\xi_0| < \infty$ and $\arg \xi_0 = -\frac{\pi}{2}$, the canonical vector, $\underline{y}_j(\xi;\xi_0)$, defines a unique member of $\{\underline{y}_j(\xi)\}$ for $j = 1$ or 2.

PROOF: Existence of at least one solution, $\underline{y}_j(\xi;\xi_0)$, is assured since any member of $\{\underline{y}_j(\xi)\}$ ($j = 1$ or 2) can be reduced to canonical form at any ξ_0 on the ray $\arg \xi_0 = -\frac{\pi}{2}$. Each member of $\{\underline{y}_j(\xi)\}$ can be analytically continued to the entire ray, $\arg \xi = -\frac{\pi}{2}$ and $0 < |\xi| < \infty$, since there are no singularities of the differential equation (25) on this ray.

All that remains is to show that any two members of $\{\underline{y}_j(\xi)\}$ ($j = 1$ or 2), which are simultaneously in canonical form at ξ_0, must be identically equal.

Suppose $\underline{y}_j(\xi;\xi\ )$ and $\underline{Y}_j(\xi;\xi\ )$ are two members of $\{\underline{y}_j(\xi)\}$, which are in canonical form at $\xi = \xi_0$. The vector $\underline{y}_j(\xi;\xi_0)-\underline{Y}_j(\xi;\xi_0)$ is equal to a multiple of $\underline{y}_4(\xi)$ due to theorem 3. Thus,

$$\underline{y}_j(\xi;\xi_0) - \underline{Y}_j(\xi;\xi_0) = c\underline{y}_4(\xi) \quad . \tag{40}$$

The constant c in (40) must be zero since $\underline{y}_j(\xi_0;\xi_0)-\underline{Y}_j(\xi_0;\xi_0)$ has a zero component corresponding to the maximum component of $\underline{y}_4(\xi)$ at $\xi = \xi_0$. Hence,

$$\underline{y}_j(\xi;\xi_0) - \underline{Y}_j(\xi;\xi_0) \equiv \underline{0} . \qquad \text{Q.E.D.}$$

The canonical form of a particular member of $\{\underline{y}_j(\xi)\}$ at $\xi = \xi_0$ (j = 1 or 2) can be considered the canonical form of the family of solutions at $\xi = \xi_0$, since any two members of $\{\underline{y}_j(\xi)\}$ yield the same canonical vector at $\xi = \xi_0$. Hence, the canonical form is a property of the family and not merely a property of the particular member which is reduced to canonical form.

The vectors $\underline{y}_1(\xi;\xi_0)$ and $\underline{y}_2(\xi;\xi_0)$ are uniquely defined members of $\{\underline{y}_1(\xi)\}$ and $\{\underline{y}_2(\xi)\}$, respectively. For every ξ_0 it is possible to determine which component of $\underline{y}_1(\xi_0;\xi_0)$ is greatest in modulus. This component will be called the maximum component of $\underline{y}_1(\xi_0;\xi_0)$. If more than one component achieves the maximum, then the component with lowest index is called the maximum component.

Consider

$$\underline{y}_T(\xi;\xi_0) = \underline{y}_2(\xi;\xi_0) + c\underline{y}_1(\xi;\xi_0) . \tag{41}$$

The vector $\underline{y}_T(\xi;\xi_0)$ is said to be in temporary canonical form[11] at $\xi = \xi_0$ if c is chosen so that

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[11]For computational purposes the vector $\underline{y}_T(\xi_0;\xi_0)$ will be used and then discarded. It is in this sense that the vector $\underline{y}_T(\xi_0;\xi_0)$ is of *temporary* value.

$\underline{y}_T(\xi_0;\xi_0)$ has a zero component corresponding to the maximum component of $\underline{y}_1(\xi_0;\xi_0)$. Since $\underline{y}_1(\xi;\xi_0)$ and $\underline{y}_2(\xi;\xi_0)$ are unique linearly independent solutions of (25), the vector $\underline{y}_T(\xi_0;\xi_0)$ is uniquely defined and nonzero for each ξ_0 such that $0 < |\xi\ | < \infty$ and $\arg \xi_0 = -\frac{\pi}{2}$. The vector $\underline{y}_T(\xi_0;\xi_0)$ has a component which is greatest in modulus. The first component to achieve the maximum is called the maximum component.

Now it is possible to introduce a canonical form for $\{\underline{y}_3(\xi)\}$, the boundary layer family of solutions. Suppose $\underline{y}_3(\xi)$ is a particular member of $\{\underline{y}_3(\xi)\}$ and consider

$$\underline{y}_3(\xi;\xi_0) = y_3(\xi) + a\underline{y}_1(\xi;\xi_0) + b\underline{y}_T(\xi;\xi_0) + c\underline{y}_4(\xi) , \qquad (42)$$

where a, b, and c are scalars which are chosen such that $\underline{y}_3(\xi_0;\xi_0)$ has three components which are zero. The three zero components correspond to the distinct maximum components of $\underline{y}_4(\xi_0)$, $\underline{y}_1(\xi_0;\xi_0)$, and $\underline{y}_T(\xi_0;\xi_0)$. The scalars a, b, and c can be determined in essentially a back substitution process by first solving for c, then a, and finally b.

THEOREM 5: For each value of ξ_0 such that $0 < |\xi_0| < \infty$ and $\arg \xi_0 = -\frac{\pi}{2}$, the canonical form (42) defines a unique member, $\underline{y}_3(\xi;\xi_0)$, of $\{\underline{y}_3(\xi)\}$.

PROOF: Any member of $\{\underline{y}_3(\xi)\}$ can be reduced to canonical form (42) for any ξ_0 such that $\arg \xi_0 = -\frac{\pi}{2}$ and $0 < |\xi_0| < \infty$. The only question is whether or not $\underline{y}_3(\xi;\xi_0)$ is uniquely defined.

Let $\underline{y}_3(\xi;\xi_0)$ and $\underline{Y}_3(\xi;\xi_0)$ be any two vectors which are both members of $\{\underline{y}_3(\xi)\}$ and in canonical form at $\xi = \xi_0$. From theorem 3

$$\underline{Y}_3(\xi;\xi_0) - \underline{y}_3(\xi;\xi_0) = c_1\underline{y}_1(\xi;\xi_0) + c_2\underline{y}_2(\xi;\xi_0) + c_4\underline{y}_4(\xi) \quad , \tag{43}$$

where c_1, c_2, and c_4 are uniquely determined, or

$$\underline{Y}_3(\xi;\xi_0) - \underline{y}_3(\xi;\xi_0) = a\underline{y}_1(\xi;\xi_0) + b\underline{y}_T(\xi;\xi_0) + c_4\underline{y}_4(\xi) \quad , \tag{44}$$

where a and b in (44) are uniquely determined from c_1 and c_2 in (43). But $\underline{Y}_3(\xi_0;\xi_0)$ and $\underline{y}_3(\xi_0;\xi_0)$ have zero components corresponding to the maximum components of $\underline{y}_4(\xi_0)$, $\underline{y}_1(\xi_0;\xi_0)$, and $\underline{y}_T(\xi_0;\xi_0)$. Thus, the constants a, b, and c_4 in (44) are all zero and this implies $\underline{Y}_3(\xi_0;\xi_0) - \underline{y}_3(\xi_0;\xi_0) = \underline{0}$.

Thus,

$$\underline{Y}_3(\xi;\xi_0) - \underline{y}_3(\xi;\xi_0) \equiv \underline{0} \,. \qquad \text{Q.E.D.}$$

In order to solve the asymptotic continuation problem it is necessary to determine members of the families $\{\underline{y}_j(\xi)\}$ for $j = 1,2,3,$ and 4 at $\xi = \frac{1}{i}$. Due to theorems 2 and 3 it is permissible to use the canonical vectors $\underline{y}_j(\xi_0;\xi_0)$ at $\xi_0 = \frac{1}{i}$ to solve equation (32).

The asymptotic continuation problem is inherently unstable. However, theorems 2 and 3 imply that the solutions of the differential equation (25), which exponentially dominate $\underline{y}_j(\xi)$ for decreasing ξ , do not destroy the calculations. For example, consider $\underline{y}_1(\xi)$, for decreasing $|\xi|$ the only solution which exponentially dominates $\underline{y}_1(\xi)$, is $\underline{y}_4(\xi)$. If an error is initiated at some ξ_0 , which introduces a small multiple of $\underline{y}_4(\xi)$, then as $|\xi|$ is decreased the growth of $\underline{y}_4(\xi)$ may swamp the calculation. However, theorems 2 and 3 imply that arbitrary multiples of $\underline{y}_4(\xi)$ are acceptable, that is, do not affect the asymptotic continuation problem. The only difficulty which $\underline{y}_4(\xi)$ creates in the computation of $\underline{y}_1(\xi)$ is that the multiple of $\underline{y}_4(\xi)$, introduced via an error, may grow to such proportions that $\underline{y}_1(\xi)$ is masked by the multiple of $\underline{y}_4(\xi)$, that is, several of the significant figures of the numerical approximation of $\underline{y}_1(\xi)$ may reflect the useless multiple of $\underline{y}_4(\xi)$, which was introduced via an error. In order to avoid this situation it is necessary to control (not eliminate) the multiple of $\underline{y}_4(\xi)$ present in the numerical approximation of $\underline{y}_1(\xi)$. This can be effectively accomplished by reducing the numerical approximation of $\underline{y}_1(\xi)$ to canonical form several times over the interval of numerical integration.

There are essentially five steps in solving the continuation problem. They are the following:

a. Initial vectors are determined from the truncated asymptotic expansions $\hat{\underline{y}}_{j,L}(\xi)$ at an appropriate initial value of $\xi = \xi_I$ (see Appendix B, Section B.2).

b. The interval from ξ_I to $\xi = \frac{2}{i}$ is divided into N subintervals.[12] If the variable τ is defined by $\tau = \sqrt{\xi}$, then the N subintervals are chosen to have equal length in τ. The integer N is chosen sufficiently large so that the transition layer solution does not grow in norm by more than a factor of 10.

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[12]For the case $k = 0.005$, the gravity wave and Lamb-wave numerical integrations were terminated at $\xi = -2 \times 10^4 i$. The DC solutions were continued from $\xi = \frac{1}{i}$ to $\xi = -2 \times 10^4 i$ in order to determine the asymptotic connection relations.

c. A numerical integration[13] is performed over the first subinterval starting at $\xi = \xi_I$ with the initial vector $\underset{\sim}{y}_{j,L}(\xi_I)$ for $j = 1,2,3,$ and 4. At the end of the subinterval the numerical solutions are reduced to the appropriate canonical form. Step c is repeated for each successive subinterval. The canonical vectors obtained at the end of the k*th* subinterval are the initial vectors for the k+1*st* subinterval.

d. A numerical integration[14] from $\xi = \frac{2}{i}$ to $\xi = \frac{1}{i}$ is performed. The numerical solutions of (25) are once again reduced to canonical form.

e. The boundary layer solution ($\underset{\sim}{y}_3(\xi)$) obtained at $\xi = \frac{1}{i}$ is numerically integrated from $\xi = \frac{1}{i}$ to $\xi = \xi_I$. The vector so obtained at ξ_I is reduced to the boundary layer canonical form, $\underset{\sim}{y}_3(\xi_I;\xi_I)$, by adding the proper

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[13] The numerical step in ξ, which is used in the numerical integration, is uniform in $\sqrt{\xi}$; that is, if H is the uniform step in $\sqrt{\xi}$ and h the step in ξ, then $h = (\sqrt{\xi} + H)^2 - \xi = 2\sqrt{\xi}\,H + H^2$.

[14] The numerical step in ξ which is used for this portion of the numerical integration is uniform in ξ.

multiples of the vectors $\hat{\underline{y}}_{j,L}(\xi_I)$ for $j = 1,2,$ and 4. The canonical form of the vector $\hat{\underline{y}}_{3,L}(\xi_I)$ is then compared with the canonical form of the vector which has been integrated from $\xi = \frac{1}{i}$ to $\xi = \xi_I$. The agreement, or lack of agreement, between these two vectors is a good error indication. In addition, the approximation of $\vec{y}_3\left(\xi_I;\frac{1}{i}\right)$ was compared with $\hat{\vec{y}}_{3,L}(\xi_I)$.

Since the numerical solutions, which approximate $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, and $\underline{y}_4(\xi)$, must be accurate in order to reduce $\underline{y}_3(\xi)$ accurately to canonical form, it follows that step e is an error check on all of the solutions. Since $\underline{y}_3(\xi)$ dominates $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, and $\underline{y}_4(\xi)$ for increasing $|\xi|$, there should be little difficulty in numerically continuing $\underline{y}_3(\xi)$ from $\xi = \frac{1}{i}$ to $\xi = \xi_I$. Thus, the error check at $\xi = \xi_I$ is a good indication of the errors present at $\xi = \frac{1}{i}$. For several problems where a closed-form solution was available, the algorithm (steps a through e) yielded accurate canonical vectors for small values of the independent variable. In addition, step e provided a one-significant-figure estimate of the maximum relative error present in the canonical vectors. This partially justifies the claim that step e is an error check for all the numerical solutions.

4.2 NUMERICAL ANALYSIS

In this section the algorithm developed in Section 4.1 is investigated. The approximate numerical value of $\underline{y}_j(\xi)$, so obtained at a fixed value[15] of ξ , is shown to approach a vector associated with a member of $\{\underline{y}_j(\xi)\}$. It will be tacitly assumed that it is possible to compute numerical solutions of the differential equation (25) which are arbitrarily close to actual solutions of (25). The continuation problem, thus limited, is simply a question of whether the truncated asymptotic expansions, $\hat{\underline{y}}_{j,L}(\xi_I)$, yield sufficient accuracy to determine the vectors $\underline{y}_j(\xi)$ at $\xi = \frac{1}{i}$ in the limit as $\xi_I \to \infty$. Since the initial error in $\hat{\underline{y}}_{j,L}(\xi_I)$ decreases algebraically fast as $\xi_I \to \infty$ and the solutions of the differential equation (25) grow at different exponential rates, it is not obvious that the continuation problem can be solved in this manner.

Of course, the more difficult problem of accumulated error due to numerical integration cannot be entirely ignored. The fifth step of the numerical

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[15]For convenience the fixed value of ξ is chosen to be $\xi = \frac{1}{i}$.

algorithm, developed in Section 4.1, provides a reasonable check of the accumulated errors.

Consider

$$\underline{y}_{je}(\xi_0) = \underline{y}_j(\xi_0) + \underline{\varepsilon}_0 \quad , \tag{45}$$

where $\underline{y}_{je}(\xi)$ is a solution of the differential equation (25) for all ξ . It will be assumed in all that follows that $\arg \xi = -\frac{\pi}{2}$.

For an arbitrary $\underline{\varepsilon}_0$ in equation (45) the vector $\underline{y}_{je}(\xi)$ will not necessarily be a member of $\{\underline{y}_j(\xi)\}$ even though $\underline{y}_j(\xi)$ is a member of $\{\underline{y}_j(\xi)\}$. It is necessary for the analysis to quantify or measure in some way the disparity (error) of $\underline{y}_{je}(\xi)$ with members of $\{\underline{y}_j(\xi)\}$. Of course, it is possible to call $||\underline{\varepsilon}_0||$ the error in $\underline{y}_{je}(\xi)$ at $\xi = \xi_0$.[16] However, this concept of error does not lead to a unique value[17] since there are infinitely many members in the families $\{\underline{y}_j(\xi)\}$, for $j = 1,2,$ and 3 .

THEOREM 6: If

$$E_{je}(\xi;\xi_0) = \min_{\substack{\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\} \\ \xi \text{ fixed}}} ||\underline{y}_{je}(\xi) - \underline{y}_j(\xi)|| \quad ,$$

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[16]The maximum norm is used throughout this paper.

[17]If $j = 4$ then $||\underline{\varepsilon}_0||$ uniquely defines the error at $\xi = \xi_0$.

where $\underline{y}_{je}(\xi)$ and ξ_0 are defined in equation (45), then the function $E_{je}(\xi;\xi_0)$ exists, is continuous in ξ, and nonnegative for $0 < |\xi| < \infty$.

PROOF: If $j = 4$, then theorem 6 is trivial since $\{\underline{y}_4(\xi)\}$ has only one member, $\underline{y}_4(\xi)$. Therefore, the limit process, implicit in the definition of $E_{4e}(\xi;\xi_0)$, is trivial. Continuity of $E_{4e}(\xi;\xi_0)$ follows since $\underline{y}_{4e}(\xi)$ and $\underline{y}_4(\xi)$ are continuous in ξ.

In order to show that $E_{je}(\xi;\xi_0)$ exists for $j = 1,2,$ and 3, it is necessary to show that there exists a member, $\underline{y}_j(\xi)$ of $\{\underline{y}_j(\xi)\}$, corresponding to each value of ξ such that $||\underline{y}_{je}(\xi)-\underline{y}_j(\xi)||$ achieves a minimum.

Let $S_j(\xi)$ be a set of nonnegative numbers corresponding to each value of ξ. A number s is included in $S_j(\xi)$ if and only if

$$s = ||\underline{y}_{je}(\xi) - \underline{y}_j(\xi)||$$

for some $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$. Thus, $S_j(\xi)$ is defined for all ξ such that $0 < |\xi| < \infty$ since $\underline{y}_{je}(\xi)$ and $\{\underline{y}_j(\xi)\}$ exist for $0 < |\xi| < \infty$. In addition, $S_j(\xi)$ is bounded below by zero for each value of ξ. Hence, $S_j(\xi)$ has a greatest lower bound (g.l.b.) for each value of ξ. The function $E_{je}(\xi;\xi_0)$ exists if the g.l.b. is achieved for a member of $\{\underline{y}_j(\xi)\}$.

Let

$$s_j(\xi) \quad = \quad \text{g.l.b. } S_j(\xi) \quad .$$

The function $E_{je}(\xi;\xi_0)$ exists if $s_j(\xi) \varepsilon S_j(\xi)$. Since $s_j(\xi)$ is the g.l.b. of $S_j(\xi)$, there exists a sequence of functions $\underline{y}_{jn}(\xi)$ such that for a prescribed ξ $\underline{y}_{jn}(\xi) \varepsilon \{\underline{y}_j(\xi)\}$ for $n = 1,2,3,\cdots$ and $\lim_{n\to\infty} ||\underline{y}_{je}(\xi) - \underline{y}_{jn}(\xi)|| = s_j(\xi)$.

The sequence $\underline{y}_{jn}(\xi)$ must remain bounded for fixed ξ , since for arbitrary $\varepsilon > 0$ prescribed $||\underline{y}_{je}(\xi) - \underline{y}_{jn}(\xi)|| \leq s_j(\xi) + \varepsilon$ for n sufficiently large. Hence, $||\underline{y}_{jn}(\xi)|| \leq ||\underline{y}_{je}(\xi)|| + s_j(\xi) + \varepsilon$. Due to theorems 2 and 3 the vector $\underline{y}_{jn}(\xi)$ can be expressed in the form

$$\underline{y}_{jn}(\xi) \quad = \quad \underline{y}_j(\xi) + \sum_k c_{kn}\underline{y}_k(\xi) \quad , \tag{46}$$

where $\underline{y}_j(\xi)$ and $\underline{y}_k(\xi)$ are particular members of $\{\underline{y}_j(\xi)\}$ and $\{\underline{y}_k(\xi)\}$, respectively; that is, $\underline{y}_j(\xi)$ and $\underline{y}_k(\xi)$ do not vary with the subscript n .

NOTE: If $j = 1$ or 2 in equation (46), then the sum over k includes $k = 4$ only; that is, $c_{1n} = c_{2n} = c_{3n} = 0$. If $j = 3$ in equation (46), then the sum over k includes $k = 1,2,$ and 4 . If $j = 4$ in equation (46), then $c_{1n} = c_{2n} = c_{3n} = c_{4n} = 0$.

The scalars c_{kn} in equation (46) are uniformly bounded in n since the vectors $\underline{y}_{jn}(\xi)$ are uniformly bounded in n and $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, $\underline{y}_3(\xi)$, and $\underline{y}_4(\xi)$ are linearly independent. Hence, there exists a convergent subsequence of the scalars c_{kn}, call it c_{kn_ℓ}. Thus, $\underline{y}_{kn_\ell}$ is a convergent sequence of vectors and this sequence converges to a vector which is a member of $\{\underline{y}_j(\xi)\}$. Denote the limit vector by $\underline{y}_{j0}(\xi)$, then $||\underline{y}_{je}(\xi) - \underline{y}_{j0}(\xi)|| = s_j(\xi)$; thus, $E_{je}(\xi;\xi_0)$ exists and is clearly nonnegative.

Clearly, $||\underline{y}_j(\xi)||$ is continuous in ξ for a particular member of $\{\underline{y}_j(\xi)\}$ since $\underline{y}_j(\xi)$ is continuous. Suppose at $\xi = \xi_1$, $\underline{y}_{j1}(\xi)$ is selected so that $E_{je}(\xi;\xi_0) = ||\underline{y}_{je}(\xi_1) - \underline{y}_{j1}(\xi_1)||$ and $\underline{y}_{j1}(\xi)\varepsilon\{\underline{y}_j(\xi)\}$. The vector $\underline{y}_{j1}(\xi)$ exists due to the analysis which established the existence of $E_{je}(\xi;\xi_0)$ for $0 < |\xi| < \infty$. For any ξ_2

$$E_{je}(\xi_2;\xi_0) \le ||\underline{y}_{je}(\xi_2) - \underline{y}_{j1}(\xi_2)|| \qquad (47)$$

since $\underline{y}_{j1}(\xi_2)$ is not necessarily a vector which minimizes $||\underline{y}_{je}(\xi_2) - \underline{y}_j(\xi_2)||$. Consider an arbitrary sequence v_n of ξ values such that $v_n \to \xi_1$ as $n \to \infty$. The corresponding sequence of nonnegative numbers $E_{je}(v_n;\xi_0)$ is bounded due to (47), and hence, there exists a convergent subsequence $E_{je}\left(v_{n_k};\xi_0\right)$. Inequality (47) implies

$$\lim_{n_k\to\infty} E_{je}\left(v_{n_k};\xi_0\right) \leq E_{je}(\xi_1;\xi_0) \quad . \qquad (48)$$

Relation (48) follows if $\xi_2 = v_{n_k}$ in (47).

If only equality in (48) can hold, then theorem 6 is established since the implication is that every convergent subsequence $E_{je}\left(v_{n_k};\xi_0\right)$ converges to the same limit, $E_{je}(\xi_1;\xi_0)$. Thus, it would follow that

$$\lim_{\xi_2\to\xi_1} E_{je}(\xi_2;\xi_0) = E_{je}(\xi_1;\xi_0)$$

or $E_{je}(\xi;\xi_0)$ is continuous in the variable ξ .

Suppose

$$\lim_{n_k\to\infty} E_{je}\left(v_{n_k};\xi_0\right) < E_{je}(\xi_1;\xi_0) \qquad (49)$$

and let

$$\varepsilon = E_{je}(\xi_1;\xi_0) - \lim_{n_k\to\infty} E_{je}\left(v_{n_k};\xi_0\right) \quad .$$

Now consider the sequence of vectors $\underline{y}_{jn_k}(\xi)$ such that

$$E_{je}\left(v_{n_k};\xi_0\right) = ||\underline{y}_{je}\left(v_{n_k}\right) - \underline{y}_{jn_k}\left(v_{n_k}\right)||$$

and $\underline{y}_{jn_k}\varepsilon\{\underline{y}_j(\xi)\}$. There exists such vectors $\underline{y}_{jn_k}(\xi)$ due to the analysis which establish the existence of $E_{je}(\xi;\xi_0)$.

If the differential equation (25) is written in the form

$$\frac{d\underline{y}(\xi)}{d\xi} = A(\xi)\underline{y}(\xi) \qquad (50)$$

and

$$||A(\xi)|| = \sup_{\substack{||\underline{v}||=1 \\ \xi \text{ fixed}}} ||A(\xi)\underline{v}|| \ , \qquad (51)$$

then it is readily seen that $||A(\xi)||$ is bounded on $0 < \delta \le |\xi| < \infty$ $\left(\text{recall} \ \arg \xi = -\frac{\pi}{2}\right)$. Relations (50) and (51) imply

$$||\underline{y}(\xi)|| \le ||\underline{y}(\xi_0)|| + \int_{\xi_0}^{\xi} ||A(S)|| \ ||\underline{y}(S)|| \ |dS| \ , \qquad (52)$$

where $\underline{y}(\xi)$ is a solution of (50). An immediate consequence of (52) is

$$||\underline{y}(\xi_1)|| \le ||\underline{y}(\xi_2)|| e^{\int_{\xi_2}^{\xi_1} ||A(S)|| |dS|}$$

Thus,

$$||\underline{y}_{je}(\xi_1) - \underline{y}_{n_k}(\xi_1)|| \le ||\underline{y}_{je}\left(v_{n_k}\right) - \underline{y}_{n_k}\left(v_{n_k}\right)|| \times e^{\int_{v_{n_k}}^{\xi_1} ||A(S)|| |dS|}$$

However, as $n_k \to \infty$

$$||\underline{y}_{je}(v_{n_k}) - \underline{y}_{n_k}(v_{n_k})|| \to E_{je}(\xi_1;\xi_0) - \varepsilon$$

and $e^{\int_{v_{n_k}}^{\xi_1} ||A(S)|| |ds|} \to 1$ since $v_{n_k} \to \xi_1$ and $||A(S)||$ is bounded. Thus, for sufficiently large n_k

$$||\underline{y}_{je}(\xi_1) - \underline{y}_{n_k}(\xi_1)|| < E_{je}(\xi_1;\xi_0) \quad . \tag{53}$$

However, (53) contradicts the definition of $E_{je}(\xi;\xi_0)$. Since (53) follows from relation (49), it follows that (49) is false. Hence, only equality can hold in (48). Therefore, $E_{je}(\xi;\xi_0)$ is continuous in ξ . Q.E.D.

DEFINITION 1: The function $E_{je}(\xi;\xi_0)$ is called the absolute error of $\underline{y}_{je}(\xi)$ with respect to the family $\{\underline{y}_j(\xi)\}$.

It is often more useful to deal with relative errors rather than absolute errors. This is generally true when the solutions being investigated are capable of growing or decaying exponentially fast. Before a useful definition of relative error is developed, some preliminary analysis will be performed.

LEMMA 1: If $f_{j1}(\xi) = \min||\underline{y}_{j1}(\xi)||$, subject to the constraints

a. $\underline{y}_{j1}(\xi)\varepsilon\{\underline{y}_j(\xi)\}$,

b. ξ fixed, and

c. $E_{je}(\xi;\xi_0) = ||\underline{y}_{je}(\xi) - \underline{y}_{j1}(\xi)||$,

then the function $f_{j1}(\xi)$ exists for $0 < |\xi| < \infty$. If $f_{j2}(\xi) = \min||\underline{y}_{j2}(\xi)||$ subject to the constraints

a. $\underline{y}_{j2}(\xi)\varepsilon\{\underline{y}_j(\xi)\}$ and

b. ξ fixed,

then the function $f_{j2}(\xi)$ exists and is continuous in ξ for $0 < |\xi| < \infty$.

PROOF: Due to theorem 6, there is at least one vector, $\underline{y}_{j1}(\xi)\varepsilon\{\underline{y}_j(\xi)\}$, corresponding to each ξ such that

$$E_{je}(\xi;\xi_0) = ||\underline{y}_{je}(\xi) - y_{j1}(\xi)|| \quad . \tag{54}$$

If there are only finitely many such vectors, then $f_{j1}(\xi)$ is trivially constructed. If infinitely many vectors $\underline{y}_{j1}(\xi)$ satisfy (54), then the analysis in the proof of theorem 6 can be repeated to establish that $f_{j1}(\xi)$ exists. Similarly, $f_{j2}(\xi)$ exists. In addition, the analysis in the proof of theorem 6 can be repeated to establish the continuity of $f_{j2}(\xi)$.

Now return to the development of a definition of relative error. In general, relative error is defined by a ratio of the form

$$R = \frac{||\underset{\sim}{y}_{APPROXIMATE} - \underset{\sim}{y}_{EXACT}||}{||\underset{\sim}{y}_{EXACT}||} .$$

The absolute error $||\underset{\sim}{y}_{APPROXIMATE} - \underset{\sim}{y}_{EXACT}||$ is defined in definition 1. The problem is how to choose $||\underset{\sim}{y}_{EXACT}||$ from the family $\{\underset{\sim}{y}_{EXACT}\}$. The following two definitions are possible ways of defining a unique relative error. To distinguish these two concepts one will be called the relative accuracy and the other will be called the relative error.

DEFINITION 2: The relative accuracy of $\underset{\sim}{y}_{je}(\xi)$ with respect to the family $\{\underset{\sim}{y}_j(\xi)\}$ is denoted by $A_{je}(\xi;\xi_0)$ and is defined by

$$A_{je}(\xi;\xi_0) = \frac{E_{je}(\xi;\xi_0)}{f_{j1}(\xi)} , \qquad (55)$$

where $\underset{\sim}{y}_{je}(\xi)$ is defined in (45), $E_{je}(\xi;\xi_0)$ is defined in theorem 6 and $f_{j1}(\xi)$ is defined in lemma 1.

DEFINITION 3: The relative error of $\underset{\sim}{y}_{je}(\xi)$, with respect to the family $\{\underset{\sim}{y}_j(\xi)\}$, is denoted by $R_{je}(\xi;\xi_0)$ and is defined by

$$R_{je}(\xi;\xi_0) = \frac{E_{je}(\xi;\xi_0)}{f_{j2}(\xi)} , \tag{56}$$

where $\underline{y}_{je}(\xi)$ is defined in (45), $E_{je}(\xi;\xi_0)$ is defined in theorem 6, and $f_{j2}(\xi)$ is defined in lemma 1.

Clearly,

$$A_{je}(\xi;\xi_0) \leq R_{je}(\xi;\xi_0) \tag{57a}$$

since

$$f_{j1}(\xi) \geq f_{j2}(\xi) . \tag{57b}$$

The relative accuracy $A_{je}(\xi;\xi_0)$ has peculiar properties which render it unsuitable as a criterion for a numerical analysis. It is possible for $A_{je}(\xi;\xi_0)$ to be extremely small even though $\underline{y}_{je}(\xi)$ may be useless for solving the continuation problem.

EXAMPLE: Suppose $\underline{y}_4(\xi_0) = \begin{bmatrix} -4 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and

$\underline{y}_1(\xi_0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c\underline{y}_4(\xi_0)$. If $c = 10^{30}$ and $\underline{\varepsilon}_0 = 10^{10}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

in (45), then $A_{1e}(\xi_0;\xi_0) \approx 10^{-20}$ and $R_{1e}(\xi_0;\xi_0) \approx 10^{10}$.

For the above example the relative accuracy is small since $\underline{y}_{1e}(\xi_0)$ equals $\underline{y}_1(\xi_0)+\underline{\varepsilon}_0$ and $\underline{y}_{1e}(\xi_0)$ agrees with a member of $\{\underline{y}_1(\xi)\}$ to 20 significant figures.

The relative accuracy $A_{1e}(\xi_0;\xi_0)$ is roughly a measure of the number of significant figures of agreement between $\underline{y}_{1e}(\xi_0)$ and some member of $\{\underline{y}_1(\xi)\}$ at ξ_0 . On the other hand, the relative error $R_{1e}(\xi;\xi_0)$ is not influenced by the apparent agreement between $\underline{y}_{1e}(\xi_0)$ and some member of $\{\underline{y}_1(\xi)\}$. The relative accuracy of $\underline{y}_{1e}(\xi)$ can be made arbitrarily small at any fixed finite value of ξ by simply adding a sufficiently large multiple of $\underline{y}_4(\xi)$ to $\underline{y}_{1e}(\xi)$. The absolute error $E_{1e}(\xi;\xi_0)$ is not necessarily small if $A_{1e}(\xi;\xi_0)$ is small. However, $R_{1e}(\xi;\xi_0)$ can only be made small at a fixed finite value of ξ by making $E_{1e}(\xi;\xi_0)$ small. The asymptotic continuation problem will be considered solved if the relative error $R_{je}(\xi;\xi_0)$ can be made t approach zero at $\xi = \frac{1}{i}$, for $j = 1,2,3,$ and 4 .

Suppose $\underline{y}_j(\xi)$ are particular members of $\{\underline{y}_j(\xi)\}$, respectively, for $j = 1,2,3,$ and 4 ; then the vector error, $\underline{\varepsilon}_0$ in (45), can be expressed in the form

$$\underline{\varepsilon}_0 = \sum_{j=1}^{4} \varepsilon_k \underline{y}_k(\xi_0) \quad , \tag{58}$$

where the scalars ε_k are uniquely determined.

THEOREM 7: The relative error $R_{je}(\xi;\xi_0)$ satisfies

$$R_{1e}(\xi;\xi_0) \le \frac{M_1}{f_{12}(\xi_0)} \left(|\varepsilon_1| \, ||\underline{y}_1(\xi_0)|| + |\varepsilon_2| \, ||\underline{y}_2(\xi_0)|| \times \left(\frac{|\xi_0|}{|\xi|}\right)^{c+|Re\lambda_2 - Re\lambda_1|} + |\varepsilon_3| \, ||\underline{y}_3(\xi_0)|| \right), \tag{59a}$$

$$R_{2e}(\xi;\xi_0) \le \frac{M_2}{f_{22}(\xi_0)} \left(|\varepsilon_1| \, ||\underline{y}_1(\xi_0)|| \left(\frac{|\xi_0|}{|\xi|}\right)^{|Re\lambda_1 - Re\lambda_2|+c} + |\varepsilon_2| \, ||\underline{y}_2(\xi_0)|| + |\varepsilon_3| \, ||\underline{y}_3(\xi_0)|| \right), \tag{59b}$$

$$R_{3e}(\xi;\xi_0) \le \frac{M_3 |\varepsilon_3| \, ||\underline{y}_3(\xi_0)|| \sqrt{|\xi_0|}}{f_{32}(\xi_0)} \tag{59c}$$

and

$$R_{4e}(\xi;\xi_0) \le \frac{M_4}{f_{42}(\xi_0)} \left(|\varepsilon_1| \, ||\underline{y}_1(\xi_0)|| + |\varepsilon_2| \, ||\underline{y}_2(\xi_0)|| + |\varepsilon_3| \, ||\underline{y}_3(\xi_0)| + |\varepsilon_4| \, ||\underline{y}_4(\xi_0)|| \right), \tag{59d}$$

for $1 \le |\xi| \le |\xi_0|$, where λ_1 and λ_2 are the roots of the dispersion relation (13), the scalars ε_1, ε_2, ε_3, and ε_4 are defined in (58), the scalar c in relations (59a) and (59b) can be set equal to zero if $\lambda_1 \neq \lambda_2$ and if $\lambda_1 = \lambda_2$, then the scalar c can be chosen arbitrarily small but positive. The constants M_1, M_2, M_3 and M_4 are finite and independent of ξ_0. The functions $f_{j2}(\xi)$ are defined in lemma 1.

PROOF: The proof of this theorem involves a good deal of heavy analysis. For this reason the main elements in the proof of theorem 7 will be outlined first.

By far the most important element in establishing theorem 7 is the definition of the absolute error $E_{je}(\xi;\xi_0)$. The definition of $E_{je}(\xi;\xi_0)$ implies

$$E_{1e}(\xi;\xi_0) \leq |\varepsilon_1| \, ||\underline{y}_1(\xi)|| + |\varepsilon_2| \, ||\underline{y}_2(\xi)|| + |\varepsilon_3| \, ||\underline{y}_3(\xi)|| \quad ,$$

$$E_{2e}(\xi;\xi_0) \leq |\varepsilon_1| \, ||\underline{y}_1(\xi)|| + |\varepsilon_2| \, ||\underline{y}_2(\xi)|| + |\varepsilon_3| \, ||\underline{y}_3(\xi)|| \quad ,$$

$$E_{3e}(\xi;\xi_0) \leq |\varepsilon_3| \, ||\underline{y}_3(\xi)||$$

and

$$E_{4e}(\xi;\xi_0) \leq |\varepsilon_1| \, ||\underline{y}_1(\xi)|| + |\varepsilon_2| \, ||\underline{y}_2(\xi)|| + |\varepsilon_3| \, ||\underline{y}_3(\xi)|| + |\varepsilon_4| \, ||\underline{y}_4(\xi)|| \quad .$$

Theorem 7 relates the error at ξ to the initial error at ξ_0 and this is where the heavy analysis arises. If $|\xi_0|$ were bounded above by some finite constant, then the theorem would follow trivially since the solutions $\underline{y}_j(\xi)$ have only a bounded growth on a bounded interval which excludes $\xi = 0$. However, $|\xi_0|$ can be chosen arbitrarily large in theorem 7.

For particular vectors $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$ it is possible to consider two cases:

a. $1 \leq |\xi_0| \leq M$

b. $|\xi_0| \geq M$

The first case is easily handled for any value of $M < \infty$. The second case can be handled by the asymptotic expansions if M is sufficiently large.

LEMMA 2. If $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$ for $j = 1,2,3,$ and 4 and the roots, λ_1 and λ_2, of the dispersion relation (13) are distinct, then

$$\frac{||\underline{y}_j(\xi)||}{f_{j2}(\xi)} \leq M_j \quad , \tag{60a}$$

for $j = 1,2,$ and 4 and $1 \leq |\xi| < \infty$ and

$$\frac{||\underline{y}_3(\xi)||}{f_{32}(\xi)} \leq M_3\sqrt{|\xi|} \quad , \tag{60b}$$

for $\qquad 1 \leq |\xi| < \infty \quad .$ (60c)

PROOF: Since $\underline{y}_j(\xi)$ and $f_{j2}(\xi)$ are continuous for $1 \le |\xi| < \infty$ and $f_{j2}(\xi) > 0$, it follows that $\frac{||\underline{y}_j(\xi)||}{f_{j2}(\xi)}$ is uniformly bounded for $j = 1,2,3,$ and 4 and $1 \le |\xi| \le M_0 < \infty$ (M_0 held fixed). Thus, it is only necessary to examine $||\underline{y}_j(\xi)||/f_{j2}(\xi)$ for ξ large. But for ξ large (see Appendix B.2)

$$\underline{y}_j(\xi) \sim (\underline{b}_{0,j} + \cdots)\xi^{-\lambda_j}, \quad (j = 1 \text{ or } 2) \tag{61}$$

$$\underline{y}_3(\xi) \sim \left(\underline{c}_{0,3} + \underline{c}_{1,3}/\sqrt{\xi} + \cdots\right)\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}, \tag{62}$$

and

$$\underline{y}_4(\xi) \sim \left(-\underline{c}_{0,3} + \underline{c}_{1,3}/\sqrt{\xi} + \cdots\right)\xi^{\frac{1}{4}}e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}, \tag{63}$$

where $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, and $\underline{c}_{1,3}$ are linearly independent.

It will be convenient for the analysis to consider the following vectors

$$\underline{V}_j(\xi) = \xi^{\lambda_j}\underline{y}_j(\xi), \quad \text{for } j = 1 \text{ or } 2$$

$$\underline{V}_3(\xi) = \frac{1}{2}\left\{\xi^{-\frac{1}{4}}e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}\sqrt{\xi}\;\underline{y}_3(\xi) + \xi^{-\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}\sqrt{\xi}\;\underline{y}_4(\xi)\right\}$$

and $$\underline{V}_4(\xi) = \xi^{-\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \underline{y}_4(\xi) \quad .$$

Due to the relations (61), (62), and (63) it follows that

$$\underline{V}_j(\xi) \to \underline{b}_{0,j} \quad \text{as} \quad \xi \to \infty \quad , \quad (j = 1 \text{ or } 2) \quad ,$$

$$\underline{V}_3(\xi) \to \underline{c}_{1,3} \quad \text{as} \quad \xi \to \infty \quad ,$$

and $$\underline{V}_4(\xi) \to -\underline{c}_{0,3} \quad \text{as} \quad \xi \to \infty \quad .$$

Therefore, $\underline{V}_1(\xi)$, $\underline{V}_2(\xi)$, $\underline{V}_3(\xi)$, and $\underline{V}_4(\xi)$ are uniformly bounded and linearly independent for $1 \leq |\xi| < \infty$.

CASE 1:

$$\frac{||\underline{y}_4(\xi)||}{f_{42}(\xi)} \equiv 1$$

since $\{\underline{y}_4(\xi)\}$ consists of a single unique member $\underline{y}_4(\xi)$.

CASE 2: Consider $||\underline{y}_1(\xi)||/f_{12}(\xi)$. Lemma 1, which established the existence of $f_{12}(\xi)$, implies the existence of a function $c_{min}(\xi)$, such that

$$|\xi^{\lambda_1}| f_{12}(\xi) = ||\underline{V}_1(\xi) + c_{min}(\xi)\underline{V}_4(\xi)|| \quad .$$

Clearly,

$$\frac{||\underline{y}_1(\xi)||}{f_{12}(\xi)} = \frac{||\underline{V}_1(\xi)||}{||\underline{V}_1(\xi) + c_{min}(\xi)\underline{V}_4(\xi)||} \quad .$$

It is necessary only to show that $||\underline{V}_1(\xi)+c_{min}(\xi)\underline{V}_4(\xi)||$ is uniformly bounded away from zero since $||\underline{V}_1(\xi)||$ is uniformly bounded. The first step is to show that $c_{min}(\xi)$ is uniformly bounded. Consider

$$|c_{min}(\xi)| \; ||\underline{V}_4(\xi)|| - ||\underline{V}_1(\xi)|| \leq ||\underline{V}_1(\xi) + c_{min}(\xi)\underline{V}_4(\xi)||$$

and $$||V_1(\xi) + c_{min}(\xi)\underline{V}_4(\xi)|| \leq ||\underline{V}_1(\xi)|| .$$

Thus, $$|c_{min}(\xi)| \leq 2\frac{||\underline{V}_1(\xi)||}{||\underline{V}_4(\xi)||} ,$$

but $||\underline{V}_1(\xi)||$ and $||\underline{V}_4(\xi)||$ are uniformly bounded, continuous, and for all finite $\xi \neq 0$, $||\underline{V}_4(\xi)|| > 0$. As $\xi \to \infty$, $||\underline{V}_4(\xi)|| \to ||\underline{c}_{0,3}||$ and hence, $||\underline{V}_4(\xi)||$ is uniformly bounded away from zero. Thus, $c_{min}(\xi)$ is uniformly bounded.

There exists a constant, c_∞ , such that

$$||\underline{b}_{0,1} - c_\infty \underline{c}_{0,3}|| = \text{minimum} > 0 \tag{64}$$

since $\underline{b}_{0,1}$ and $\underline{c}_{0,3}$ are linearly independent vectors. Now consider $||\underline{V}_1(\xi)+c_{min}(\xi)\underline{V}_4(\xi)||$ for large ξ ,

$$||\underline{V}_1(\xi) + c_{min}(\xi)\underline{V}_4(\xi)|| = ||\underline{V}_1(\xi) - \underline{b}_{0,1} + c_{min}(\xi) \times (\underline{V}_4(\xi) + \underline{c}_{0,3}) + (\underline{b}_{0,1} - c_{min}(\xi)\underline{c}_{0,3})|| \quad . \tag{65}$$

Relation (65) implies

$$||\underline{V}_1(\xi) + c_{min}(\xi)\underline{V}_4(\xi)|| \geq ||\underline{b}_{0,1} - c_{min}(\xi)\underline{c}_{0,3}|| - \left\{||\underline{V}_1(\xi) - \underline{b}_{0,1}|| + |c_{min}(\xi)| \, ||\underline{V}_4(\xi) + \underline{c}_{0,3}||\right\} . \tag{66}$$

The first term on the right side of (66) is bounded away from zero as $\xi \to \infty$ due to (64), and the remaining two terms tend to zero as $\xi \to \infty$ since $\underline{V}_1(\xi) \to \underline{b}_{0,1}$, $\underline{V}_4(\xi) \to -\underline{c}_{0,3}$, and $|c_{min}(\xi)|$ is uniformly bounded. Hence, relation (66) implies $||\underline{V}_1(\xi)+c_{min}(\xi)\underline{V}_4(\xi)||$ is uniformly bounded away from zero, for sufficiently large ξ and it is already known to be uniformly bounded away from zero for $1 \leq |\xi| \leq M_0$ (M_0 arbitrarily large).

CASE 3: Consider $||\underline{y}_2(\xi)||/f_{22}(\xi)$. Replace subscript 1 with 2 in case 2. No new analysis is required.

CASE 4: Consider $||\underline{y}_3(\xi)||/f_{32}(\xi)$. Lemma 1 implies that there exist functions, $c_{1min}(\xi)$, $c_{2min}(\xi)$ and $c_{4min}(\xi)$, such that

$$\left|\sqrt{\xi} \times \xi^{-\frac{1}{4}} e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}\right| f_{32}(\xi) = ||\underline{V}_3(\xi) + c_{1min}(\xi)\underline{V}_1(\xi) + c_{2min}(\xi)\underline{V}_2(\xi) + c_{4min}(\xi)\underline{V}_4(\xi)|| \quad .$$

Thus,

$$\frac{||\underline{y}_3(\xi)||}{\sqrt{|\xi|}\, f_{32}(\xi)} = \frac{\left|\xi^{-\frac{1}{4}} e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}\right| \, ||\underline{y}_3(\xi)||}{||\underline{V}_3(\xi)+c_{1min}(\xi)\underline{V}_1(\xi)+c_{2min}(\xi)\underline{V}_2(\xi)+c_{4min}(\xi)\underline{V}_4(\xi)||} \quad . \tag{67}$$

It is necessary only to show that $||\underline{V}_3(\xi)+c_{1min}(\xi)\underline{V}_1(\xi)+c_{2min}(\xi)\underline{V}_2(\xi)+c_{4min}(\xi)\underline{V}_4(\xi)||$ is uniformly bounded away from zero on $1 \leq |\xi| < \infty$ since $\left|\xi^{-\frac{1}{4}} e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}\right| \, ||\underline{y}_3(\xi)||$ is uniformly bounded. In addition, it is necessary to consider only large ξ since the lemma is valid for $1 \leq |\xi| \leq M_0$ trivially.

The first step in establishing that $||\underline{V}_3(\xi)+c_{1min}(\xi)\underline{V}_1(\xi)+\cdots||$ is uniformly bounded away from zero is to establish that $c_{1min}(\xi)$, $c_{2min}(\xi)$, and $c_{4min}(\xi)$ are uniformly bounded functions. Clearly

$$||\underline{V}_3(\xi) + c_{1min}(\xi)\underline{V}_1(\xi) + \cdots|| \leq ||\underline{V}_3(\xi)||$$

and

$$||c_{1min}(\xi)\underline{V}_1(\xi) + c_{2min}(\xi)\underline{V}_2(\xi) + c_{4min}(\xi)\underline{V}_4(\xi)|| - ||\underline{V}_3(\xi)|| \leq ||\underline{V}_3(\xi) + c_{1min}(\xi)\underline{V}_1(\xi) + \cdots|| \quad .$$

Therefore,

$$||c_{1min}(\xi)\underline{V}_1(\xi) + c_{2min}(\xi)\underline{V}_2(\xi) + c_{4min}(\xi)\underline{V}_4(\xi)|| \leq 2||\underline{V}_3(\xi)|| \qquad (68)$$

and $||\underline{V}_3(\xi)||$ is uniformly bounded on $1 \leq |\xi| < \infty$. This implies that the functions $c_{jmin}(\xi)$ $(j = 1,2,4)$ are uniformly bounded. If the functions $c_{jmin}(\xi)$ were not uniformly bounded, then there would exist a sequence $\xi_n \to \infty$ as $n \to \infty$ such that

$$|c_{jmin}(\xi_n)| \to \infty \ , \qquad (69)$$

for some $j = 1,2,$ or 4 .

NOTE: $c_{jmin}(\xi)$ cannot become unbounded for finite ξ since $\underline{V}_1(\xi)$, $\underline{V}_2(\xi)$, and $\underline{V}_4(\xi)$ are linearly independent for all finite ξ . Hence, relation (68) implies that $c_{jmin}(\xi)$ are bounded for all finite ξ .

If more than one subscript j satisfies (69), then consider the subscript j for which

$$|c_{jmin}(\xi_n)| \geq |c_{kmin}(\xi_n)| \tag{70}$$

for infinitely many subscripts n , where $k = 1,2,$ and 4 . If more than one subscript j satisfies (70), then choose the first subscript j which satisfies (70). Now consider for this subscript j a subsequence n_ℓ such that

$$\left|c_{jmin}\left(\xi_{n_\ell}\right)\right| \geq \left|c_{kmin}\left(\xi_{n_\ell}\right)\right| \quad , \tag{71}$$

for $k = 1,2,$ and 4 . Let

$$d_k\left(\xi_{n_\ell}\right) = \frac{c_{kmin}\left(\xi_{n_\ell}\right)}{c_{jmin}\left(\xi_{n_\ell}\right)} \quad , \tag{72}$$

then (71) and (72) imply $\left|d_k\left(\xi_{n_\ell}\right)\right| < 1$. In addition, there exist constants c_k such that

$$\left|\left|\underline{V}_j(\infty) + \sum_{\substack{k\neq j\\k=1,2,4}} c_k\underline{V}_k(\infty)\right|\right| = \text{minimum} > 0 \quad , \tag{73}$$

where $$\underline{V}_k(\infty) \equiv \lim_{\xi\to\infty}\underline{V}_k(\xi) \quad . \tag{74}$$

The limit in (74) exists, due to the asymptotic properties of $\underline{V}_j(\xi)$, for $j = 1,2,3,$ and 4 . The vectors, $\underline{V}_1(\infty)$, $\underline{V}_2(\infty)$, $\underline{V}_3(\infty)$, and $\underline{V}_4(\infty)$ are linearly independent; thus, (73) follows.

$$||\underline{V}_j\left(\xi_{n_\ell}\right) + \sum_{\substack{k\neq j\\ k=1,2,4}} d_k\left(\xi_{n_\ell}\right)\underline{V}_k\left(\xi_{n_\ell}\right)||$$

$$\leq ||\underline{V}_j(\infty) + \sum_{\substack{k\neq j\\ k=1,2,4}} c_k\underline{V}_k(\infty)|| - \left\{||\underline{V}_j\left(\xi_{n_\ell}\right) - V_j(\infty)|| + \sum_{\substack{k\neq j\\ k=1,2,4}} |d_k| \, ||\underline{V}_k\left(\xi_{n_\ell}\right) - \underline{V}_k(\infty)||\right\} \quad (75)$$

However, relation (73) implies that the right side of (75) is positive for sufficiently large ξ ; that is, it is bounded away from zero for sufficiently large ξ since $||\underline{V}_k\left(\xi_{n_\ell}\right) - \underline{V}_k(\infty)|| \to 0$ as $n_\ell \to \infty$ and $|d_k| \leq 1$. However, this implies

$$||\underline{V}_j\left(\xi_{n_\ell}\right) + \sum_{\substack{k\neq j\\ k=1,2,4}} d_k\left(\xi_{n_\ell}\right)\underline{V}_k\left(\xi_{n_\ell}\right)||$$

is uniformly bounded away from zero, or

$$||c_{jmin}\left(\xi_{n_\ell}\right)\underline{V}_j\left(\xi_{n_\ell}\right) + \sum_{\substack{k\neq j\\ k=1,2,4}} c_{kmin}\left(\xi_{n_\ell}\right)\underline{V}_k\left(\xi_{n_\ell}\right)|| \to \infty$$

as $n_\ell \to \infty$. This contradicts (68). Hence, relation (69) is false, that is $c_{jmin}(\xi)$ are uniformly bounded for $j = 1,2,$ and 4 and $1 \le |\xi| < \infty$.

If the $c_{jmin}(\xi)$ are bounded, then consider

$$||\underline{V}_3(\xi) + c_{1min}(\xi)\underline{V}_1(\xi) + c_{2min}(\xi)\underline{V}_2(\xi) + c_{4min}(\xi)\underline{V}_4(\xi)||$$

$$\begin{aligned} \ge\ & ||\underline{V}_3(\infty) + c_1\underline{V}_1(\infty) + c_2\underline{V}_2(\infty) + c_4\underline{V}_4(\infty)|| \\ & - \Big\{|c_{1min}(\xi)|\,||\underline{V}_1(\xi) - V_1(\infty)|| \\ & + |c_{2min}(\xi)|\,||\underline{V}_2(\xi) - \underline{V}_2(\infty)|| \\ & + |c_{4min}(\xi)|\,||\underline{V}_4(\xi) - \underline{V}_4(\infty)||\Big\} \quad . \end{aligned} \tag{76}$$

The term in braces on the right side of (76) tends to zero as $\xi \to \infty$. The first term on the right side of (76) is a positive constant. Hence, $||\underline{V}_3(\xi)+c_{1min}(\xi)\underline{V}_1(\xi)+\cdots||$ is uniformly bounded away from zero. Q.E.D.

LEMMA 3: If $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$ for $j = 1,2,3,$ and 4 and the roots of the dispersion relation (13) are equal, that is, $\lambda_1 = \lambda_2 = \frac{1}{2}$, then

$$\frac{||\underline{y}_j(\xi)||}{f_{j2}(\xi)} \le M_j < \infty \tag{77a}$$

for $j = 1,2,$ and 4 and $1 \le |\xi| < \infty$ and

$$\frac{||\underline{y}_3(\xi)||}{f_{32}(\xi)} \leq M\sqrt{|\xi|} \tag{77b}$$

for $1 \leq |\xi| < \infty$.

PROOF:

$$\underline{y}_1(\xi) \sim (\underline{b}_{0,1} + \cdots)\xi^{-\frac{1}{2}}$$

$$\underline{y}_2(\xi) \sim \left\{\left(\underline{b}_{0,2} + \frac{1}{2}\,\ell n\ \xi\ \underline{b}_{0,1}\right) + \cdots\right\}\xi^{-\frac{1}{2}}$$

$$\underline{y}_3(\xi) \sim \left(\underline{c}_{0,3} + \underline{c}_{1,3}/\sqrt{\xi} + \cdots\right)\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$$

$$\underline{y}_4(\xi) \sim \left(-\underline{c}_{0,3} + \underline{c}_{1,3}/\sqrt{\xi} + \cdots\right)\xi^{\frac{1}{4}}e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}} .$$

The analysis for conclusion (77a) is the same as the proof for (60a) if

$$\underline{V}_1(\xi) = \xi^{\frac{1}{2}}\underline{y}_1(\xi) ,$$

$$\underline{V}_2(\xi) = \frac{2}{\ell n\ \xi}\,\xi^{\frac{1}{2}}\underline{y}_2(\xi) ,$$

and $\underline{V}_3(\xi)$ and $\underline{V}_4(\xi)$ are defined in the proof of lemma 2. The analysis in lemma 2 will then establish conclusion (77a) for lemma 3.

In order to establish conclusion (77b) of lemma 3, consider

$$\underline{V}_1(\xi) = \xi^{\frac{1}{2}}\underline{y}_1(\xi)$$

$$\underline{V}_2(\xi) = \xi^{\frac{1}{2}}\left(\underline{y}_2(\xi) - \frac{1}{2}\ln \xi\underline{y}_1(\xi)\right) .$$

The vectors $\underline{V}_3(\xi)$ and $\underline{V}_4(\xi)$ remain the same, that is, $\underline{V}_3(\xi)$ and $\underline{V}_4(\xi)$ are defined in the proof of lemma 2. No other modifications need be made in the analysis of lemma 2 to complete the proof of lemma 3. Q.E.D.

LEMMA 4: If $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$ for $j = 1,2,3,$ and 4 and λ_1 and λ_2 are the roots of the dispersion relation (13), then

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_2(\xi)||} \leq M_{12} \frac{||\underline{y}_1(\xi_0)||}{||\underline{y}_2(\xi_0)||} \left(\frac{|\xi_0|}{|\xi|}\right)^{|Re\lambda_1 - Re\lambda_2|+c} , \quad (78a)$$

$$\frac{||\underline{y}_2(\xi)||}{||\underline{y}_1(\xi)||} \leq M_{21} \frac{||\underline{y}_2(\xi_0)||}{||\underline{y}_1(\xi_0)||} \left(\frac{|\xi_0|}{|\xi|}\right)^{|Re\lambda_1 - Re\lambda_2|+c} , \quad (78b)$$

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_4(\xi)||} \leq M_{14} \frac{||\underline{y}_1(\xi_0)||}{||\underline{y}_4(\xi_0)||} , \quad (78c)$$

$$\frac{||\underline{y}_2(\xi)||}{||\underline{y}_4(\xi)||} \leq M_{24} \frac{||\underline{y}_2(\xi_0)||}{||\underline{y}_4(\xi_0)||} , \qquad (78d)$$

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_1(\xi)||} \leq M_{31} \frac{||\underline{y}_3(\xi_0)||}{||\underline{y}_1(\xi_0)||} , \qquad (78e)$$

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_2(\xi)||} \leq M_{32} \frac{||\underline{y}_3(\xi_0)||}{||\underline{y}_2(\xi_0)||} , \qquad (78f)$$

and

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_4(\xi)||} \leq M_{34} \frac{||\underline{y}_3(\xi_0)||}{||\underline{y}_4(\xi_0)||} , \qquad (78g)$$

where the relations (78) are satisfied uniformly for $1 \leq |\xi| < \infty$ and the constants M_{ij} can be determined independently of $|\xi_0|$. If $\lambda_1 \neq \lambda_2$, then the constant c in relations (78a) and (78b) can be set equal to zero. If $\lambda_1 = \lambda_2 = \frac{1}{2}$, then the constant c can be chosen to be an arbitrarily small, positive number.

PROOF: If $|\xi_0|$ were bounded above by some finite constant, then lemma 4 would be trivial. Thus, it is necessary to consider only $|\xi_0|$ large.

For an arbitrary $\varepsilon > 0$ prescribed, there corresponds a finite constant $M(\xi)$ such that for $|\xi| \geq M(\varepsilon)$,

$$1 - \varepsilon \leq \frac{||\underline{y}_i(\xi)||}{||\underline{b}_{0,i}\xi^{-\lambda_i}||} \leq 1 + \varepsilon , \quad (i = 1 \text{ or } 2); \qquad (79)$$

for $\lambda_1 \neq \lambda_2$

$$1 - \varepsilon \leq \frac{||\underline{y}_1(\xi)||}{||b_{0,1}\xi^{-\frac{1}{2}}||} \leq 1 + \varepsilon \quad , \quad (\lambda_1 = \lambda_2); \tag{80}$$

$$1 - \varepsilon \leq \frac{||\underline{y}_2(\xi)||}{||\left(b_{0,2} + \frac{1}{2}\ \ell n\ \xi b_{0,1}\right)\xi^{-\frac{1}{2}}||} \leq 1 + \varepsilon \quad , \quad (\lambda_1 = \lambda_2); \tag{81}$$

$$1 - \varepsilon \leq \frac{||\underline{y}_3(\xi)||}{||\underline{c}_{0,3}\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}||} \leq 1 + \varepsilon \ ; \tag{82}$$

and $$1 - \varepsilon \leq \frac{||\underline{y}_4(\xi)||}{||-c_{0,3}\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}||} \leq 1 + \varepsilon \ . \tag{83}$$

Relations (79) through (83) are simply the basic asymptotic properties of the formal truncated asymptotic expansions. Thus, for $|\xi| \geq M(\varepsilon)$ it is necessary to consider only the asymptotic estimates since

$$\frac{||\hat{\underline{y}}_{i,0}(\xi)||}{||\hat{\underline{y}}_{j,0}(\xi)||}\ \frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{||\underline{y}_i(\xi)||}{||\underline{y}_j(\xi)||} \leq \frac{1 + \varepsilon}{1 - \varepsilon}\ \frac{||\hat{\underline{y}}_{i,0}(\xi)||}{||\underline{y}_{j,0}(\xi)||} \ , \tag{84}$$

for $|\xi| \geq M(\varepsilon)$.

Consider

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_2(\xi)||} \leq M_1 |\xi|^{Re\lambda_2 - Re\lambda_1} , \tag{85}$$

where $M_1 < \infty$ and $\lambda_1 \neq \lambda_2$. Relation (85) follows from relations (79) and (84). Since the solutions of (25) are capable of only a bounded growth on a bounded interval, it is clear that M_1 can be chosen sufficiently large so that relation (85) is valid for $1 \leq |\xi| < \infty$. In addition, relations (79) and (84) imply

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_2(\xi)||} \geq M_2 |\xi|^{Re\lambda_2 - Re\lambda_1}$$

for $0 < M_2 < \infty$. Thus,

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_2(\xi)||} \leq \frac{||\underline{y}_1(\xi_0)||}{||\underline{y}_2(\xi_0)||} \left(\frac{M_1}{M_2}\right) \left(\frac{|\xi_0|}{|\xi|}\right)^{|Re\lambda_2 - Re\lambda_1|}$$

Similarly, if $\lambda_1 = \lambda_2 = \frac{1}{2}$, then

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_2(\xi)||} \leq M \frac{||\underline{y}_1(\xi_0)||}{||\underline{y}_2(\xi_0)||} \left(\frac{|\xi_0|}{|\xi|}\right)^c$$

since $$|\ell n \, \xi| \leq \tilde{M} |\xi|^c$$

for $1 \leq |\xi| < \infty$ and arbitrarily small $c > 0$. Hence, conclusion (78a) follows. Similarly, conclusion (78b) follows.

Now consider $||\underline{y}_1(\xi)||/||\underline{y}_4(\xi)||$.

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_4(\xi)||} \leq M_3|\xi|^{-\left(\mathrm{Re}\lambda_1+\frac{1}{4}\right)} e^{\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}} \tag{86}$$

and

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_4(\xi)||} \geq M_4|\xi|^{-\left(\mathrm{Re}\lambda_1+\frac{1}{4}\right)} e^{\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}} . \tag{87}$$

Relations (86) and (87) are a consequence of (79), (83), and (84). The constants M_3 and M_4 satisfy $0 < M_3 < M_4 < \infty$; for the proper choice of M_3 and M_4 relations (86) and (87) are satisfied for $1 \leq |\xi| < \infty$. Thus,

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_4(\xi)||} \leq \left(\frac{M_3}{M_4}\right)\left(\frac{|\xi_0|}{|\xi|}\right)^{\mathrm{Re}\lambda_1+\frac{1}{4}} e^{\mathrm{Re}\frac{2\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})}$$

$$\times \frac{||\underline{y}_1(\xi_0)||}{||\underline{y}_4(\xi_0)||} . \tag{88}$$

However, for arbitrary λ_1 the function $\left(\frac{|\xi_0|}{|\xi|}\right)^{\mathrm{Re}\lambda_1+\frac{1}{4}} e^{\mathrm{Re}\frac{2\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})}$ is uniformly bounded for $1 \leq |\xi| \leq |\xi_0| < \infty$, since the factor $\left(\frac{|\xi_0|}{|\xi|}\right)^{\mathrm{Re}\lambda_1+\frac{1}{4}}$ is capable of only algebraic growth as $|\xi|$ is decreased from $|\xi_0|$ and $e^{\mathrm{Re}\frac{2\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})}$ decreases exponentially fast as

$|\xi|$ is decreased from $|\xi_0|$. Hence, relation (88) implies

$$\frac{||\underline{y}_1(\xi)||}{||\underline{y}_4(\xi)||} \leq M \frac{||\underline{y}_1(\xi_0)||}{||\underline{y}_4(\xi_0)||}$$

for $1 \leq |\xi| \leq |\xi_0|$, or conclusion (78c) is established. Similarly, conclusion (78d) can be established. For the case $\lambda_1 = \lambda_2 = \frac{1}{2}$ the analysis of (78d) must be modified. The function $\left(\frac{|\xi_0|}{|\xi|}\right)^{3/4+c} e^{\mathrm{Re}\frac{2\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})}$, where c is an arbitrarily small positive constant, must be considered. Since this function is uniformly bounded for $1 \leq |\xi| \leq |\xi_0| < \infty$ conclusion (78d) is established in all cases.

In order to establish conclusion (78e) consider $||y_3(\xi)||/||\underline{y}_1(\xi)||$.

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_1(\xi)||} \leq M_5|\xi|^{\mathrm{Re}\lambda_1+\frac{1}{4}} e^{\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}}$$

and

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_1(\xi)||} \geq M_6|\xi|^{\mathrm{Re}\lambda_1+\frac{1}{4}} e^{\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}},$$

where $0 < M_6 < M_5 < \infty$ and $1 \leq |\xi| < \infty$. Hence,

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_1(\xi)||} \le \left(\frac{M_5}{M_6}\right)\left(\frac{|\xi|}{|\xi_0|}\right)^{\frac{1}{4}+\mathrm{Re}\lambda_1} e^{\mathrm{Re}\frac{2\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})}$$

$$\times \frac{||\underline{y}_3(\xi_0)||}{||\underline{y}_1(\xi_0)||} ,$$

but $\left(\frac{|\xi|}{|\xi_0|}\right)^{\mathrm{Re}\lambda_1+\frac{1}{4}} e^{\mathrm{Re}\frac{2\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})}$ is uniformly bounded for $1 \le |\xi| \le |\xi_0| < \infty$. Thus,

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_1(\xi)||} \le M \frac{||\underline{y}_3(\xi_0)||}{||\underline{y}_1(\xi_0)||}$$

for $1 \le |\xi| \le |\xi_0| < \infty$, and conclusion (78e) is established. Similarly, conclusion (78f) can be established.

Consider $||\underline{y}_3(\xi)||/||\underline{y}_4(\xi)||$.

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_4(\xi)||} \le M_7 e^{\mathrm{Re}\,4\sigma\sqrt{\frac{\xi}{\gamma}}}$$

and

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_4(\xi)||} \ge M_8 e^{\mathrm{Re}\,4\sigma\sqrt{\frac{\xi}{\gamma}}} ,$$

where $0 < M_8 < M_7 < \infty$ and $1 \le |\xi| < \infty$. Hence,

$$\frac{||\underline{y}_3(\xi)||}{||\underline{y}_4(\xi)||} \le \frac{M_7}{M_8} e^{\mathrm{Re}\frac{4\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})} \frac{||\underline{y}_3(\xi_0)||}{||\underline{y}_4(\xi_0)||}$$

but $e^{\mathrm{Re}\frac{4\sigma}{\sqrt{\gamma}}(\sqrt{\xi}-\sqrt{\xi_0})} \le 1$ for $1 \le |\xi| \le |\xi_0| < \infty$. Q.E.D.

Theorem 7 is a consequence of lemmas 2, 3, and 4. Actually, estimate (59c) can be improved but it is unimportant in what follows.

Now consider the error $\underline{\varepsilon}_0$ in (45). Due to the asymptotic nature of $\hat{\underline{y}}_{j,L}(\xi)$, $R_{je}(\xi_0;\xi_0) \to 0$ for sufficiently large L as $|\xi_0| \to \infty$; that is, if $\underline{\varepsilon}_0$ is the vector error associated with the initial vector $\hat{\underline{y}}_{j,L}(\xi_0)$.

LEMMA 5: If $\hat{\underline{y}}_{j,L}(\xi)$ is the formal truncated expansion computed by method I in Appendix B.1 and $\underline{y}_{je}(\xi_0)$ in relation (45) is equal to $\hat{\underline{y}}_{j,L}(\xi_0)$, then $\underline{\varepsilon}_0$ in (45) satisfies

$$\frac{||\underline{\varepsilon}_0||}{||\underline{y}_j(\xi_0)||} = 0\left(\frac{1}{|\xi_0|^{\frac{L+1}{2}}}\right) ,$$

where $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$; that is,

$$\underline{y}_j(\xi) \sim \hat{\underline{y}}_{j,\infty}(\xi) \quad .$$

In addition, the ε_k in relation (58) satisfy

$$\frac{|\varepsilon_1|\ ||\underline{y}_1(\xi_0)||}{||\underline{y}_j(\xi_0)||} = 0\left(|\xi_0|^{c-\frac{L+1}{2}}\right) , \tag{89a}$$

$$\frac{|\varepsilon_2| \, ||\underline{y}_2(\xi_0)||}{||\underline{y}_j(\xi_0)||} = O\left(|\xi_0|^{c-\frac{L+1}{2}}\right) , \qquad (89b)$$

$$\frac{|\varepsilon_3| \, ||\underline{y}_3(\xi_0)||}{||\underline{y}_j(\xi_0)||} = O\left(|\xi_0|^{-\frac{L}{2}}\right) \qquad (89c)$$

and

$$\frac{|\varepsilon_4| \, ||\underline{y}_4(\xi_0)||}{||\underline{y}_j(\xi_0)||} = O\left(|\xi_0|^{-\frac{L}{2}}\right) \qquad (89d)$$

For $j = 1,2,3,$ and 4 . The constant c in relations (89a) and (89b) can be set equal to zero if the roots of the dispersion relation are distinct and set equal to an arbitrarily small positive constant if $\lambda_1 = \lambda_2 = \frac{1}{2}$.

PROOF:

$$\frac{||\underline{\varepsilon}_0||}{||\underline{y}_j(\xi_0)||} = O\left(\frac{1}{|\xi_0|^{\frac{L+1}{2}}}\right) \qquad (90)$$

as $\xi_0 \to \infty$. Relation (90) is a consequence of the asymptotic nature of the formal truncated expansion $\hat{\underline{y}}_{j,L}(\xi_0)$. It will be convenient for the analysis to introduce the following vectors:

$$\underline{V}_j(\xi) = \xi^{\lambda_j} \underline{y}_j(\xi) \qquad (j = 1 \text{ or } 2, \lambda_1 \neq \lambda_2) ,$$

$$\underline{V}_2(\xi) = \frac{1}{\ln \xi} \xi^{\frac{1}{2}} \underline{y}_2(\xi) \qquad \left(\lambda_1 = \lambda_2 = \frac{1}{2}\right) ,$$

$$\underline{V}_3(\xi) = \xi^{-\frac{1}{4}} e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}} \underline{y}_3(\xi) \quad ,$$

and
$$\underline{V}_4(\xi) = \xi^{-\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \underline{y}_4(\xi) \quad .$$

There exist constants c_1, c_2, c_3, and c_4 such that

$$\frac{\varepsilon_0}{||\underline{y}_j(\xi_0)||} = c_1\underline{V}_1(\xi_0) + c_2\underline{V}_2(\xi_0) + c_3\underline{V}_3(\xi_0) + c_4\underline{V}_4(\xi_0)$$

and
$$\frac{\varepsilon_k \underline{y}_k(\xi_0)}{||\underline{y}_j(\xi_0)||} = c_k\underline{V}_k(\xi_0) \quad .$$

Since the vectors $\underline{V}_j(\xi)$ are uniformly bounded for $j = 1,2,3$, and 4 , the lemma will be established if bounds can be found for $|c_j|$.

Consider the vectors $\tilde{\underline{V}}_j(\xi)$ defined by

$$\tilde{\underline{V}}_1(\xi) = \underline{V}_1(\xi) \quad ,$$

$$\tilde{\underline{V}}_2(\xi) = \left\{ \begin{array}{ll} \underline{V}_2(\xi) & \text{if } \lambda_1 \neq \lambda_2 \\ \xi^{\frac{1}{2}}\left(\underline{y}_2(\xi) - \frac{1}{2}\, \ell n\, \xi \underline{y}_1(\xi)\right) & \text{if } \lambda_1 = \lambda_2 = \frac{1}{2} \end{array} \right\} ,$$

$$\tilde{\underline{V}}_3(\xi) = \sqrt{\xi} \times \left(\frac{\underline{V}_3(\xi) + \underline{V}_4(\xi)}{2}\right) \quad ,$$

and $$\tilde{\underline{V}}_4(\xi) = \frac{\underline{V}_3(\xi) - \underline{V}_4(\xi)}{2} ,$$

then the vectors $\tilde{\underline{V}}_j(\xi)$ are uniformly bounded since

$$\left.\begin{array}{l} \underline{y}_1(\xi) \sim \xi^{-\frac{1}{2}}\left\{\underline{b}_{0,1} + \cdots\right\} \\ \underline{y}_2(\xi) \sim \xi^{-\frac{1}{2}}\left\{\left(\underline{b}_{0,2} + \frac{1}{2} \ln \xi \underline{b}_{0,1}\right) + \cdots\right\} \end{array}\right\} \text{if } \lambda_1 = \lambda_2 = \frac{1}{2} ,$$

$$\underline{y}_j(\xi) \sim \xi^{-\lambda_j}\left\{\underline{b}_{0,j} + \cdots\right\} \quad \text{if } \lambda_1 \neq \lambda_2,\ j = 1 \text{ or } 2 ,$$

$$\underline{y}_3(\xi) \sim \xi^{\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}\left\{\underline{c}_{0,3} + \underline{c}_{1,3} \frac{1}{\sqrt{\xi}} + \cdots\right\} ,$$

and $$\underline{y}_4(\xi) \sim \xi^{\frac{1}{4}} e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}\left\{-\underline{c}_{0,3} + \underline{c}_{1,3} \frac{1}{\sqrt{\xi}} + \cdots\right\} .$$

Thus, the vectors $\tilde{\underline{V}}_k(\xi)$ are uniformly bounded and constants d_k exist such that

$$\sum_{k=1}^{4} c_k \underline{V}_k(\xi_0) = \sum_{k=1}^{4} d_k \tilde{\underline{V}}_k(\xi_0) ,$$

where

$c_1 = d_1$, $c_2 = d_2$, if $\lambda_1 \neq \lambda_2$;

$$c_2 = \ln \xi \, d_2, \quad c_1 = d_1 - \frac{d_2 \ln \xi}{2}, \quad \text{if} \quad \lambda_1 = \lambda_2 = \frac{1}{2};$$

$$c_3 = d_4 + \sqrt{\xi}\, d_3 \quad ;$$

and $$c_4 = -d_4 + \sqrt{\xi}\, d_3 \quad .$$

The lemma will be established if it can be shown that

$$\frac{|d_k|}{(||\underline{\varepsilon}_0||/||\underline{y}_{j,L}(\xi_0)||)} \leq M$$

or

$$\frac{|d_k|}{(||\underline{\varepsilon}_0||/||\underline{y}_j(\xi_0)||)} \leq M$$

for $1 \leq |\xi_0| < \infty$. This can be established if the inverse of $\tilde{V}(\xi)$, where

$$\tilde{V}(\xi) = \left[\underline{\tilde{V}}_1(\xi),\ \underline{\tilde{V}}_2(\xi),\ \underline{\tilde{V}}_3(\xi),\ \underline{\tilde{V}}_4(\xi)\right] \quad ,$$

is shown to be uniformly bounded. However, $\tilde{V}^{-1}(\xi)$ can be constructed from cofactors of $\tilde{V}(\xi)$ divided by det $\tilde{V}(\xi)$. Since $\tilde{V}(\xi)$ is uniformly bounded and continuous, it follows that the cofactors and the determinant of $\tilde{V}(\xi)$ are continuous and uniformly bounded. In addition, the determinant of $\tilde{V}(\xi)$ is strictly nonzero since the columns of $\tilde{V}(\xi)$ are linearly independent. For large ξ the columns of $\tilde{V}(\xi)$ approach the vectors $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{1,3}$, and $c_{0,3}$ and these vectors are linearly independent (see lemma B1 in Appendix B.1). Hence,

$$\det \tilde{V}(\xi) \rightarrow \det [\underline{b}_{0,1}, \underline{b}_{0,2}, \underline{c}_{1,3}, \underline{c}_{0,3}] > 0$$

as $\xi \rightarrow \infty$. Thus, det $\tilde{V}(\xi)$ is uniformly bounded away from zero. Q.E.D.

THEOREM 8: The asymptotic continuation problem can be solved.

PROOF: Lemmas 2, 3, and 5 and theorem 7 imply that

$$R_{1e}(\xi;\xi_0) = O\left(|\xi_0|^{|Re\lambda_1 - Re\lambda_2|+c} + |\xi_0|^{\frac{1}{2}}\right) \times \frac{1}{|\xi_0|^{\frac{L+1}{2}}} , \tag{91}$$

$$R_{2e}(\xi;\xi_0) = O\left(|\xi_0|^{|Re\lambda_1 - Re\lambda_2|+c} + |\xi_0|^{\frac{1}{2}}\right) \times \frac{1}{|\xi_0|^{\frac{L+1}{2}}} , \tag{92}$$

$$R_{3e}(\xi;\xi_0) = O\left(|\xi_0|^{\frac{1-L}{2}}\right) , \tag{93}$$

and

$$R_{4e}(\xi;\xi_0) = O\left(|\xi_0|^{-\frac{L}{2}}\right) , \tag{94}$$

for $1 \leq |\xi| \leq |\xi_0| < \infty$. However, L can be chosen arbitrarily[18] large in relations (91) through (94). Hence, $R_{je}(\xi;\xi_0)$ can be made arbitrarily small by choosing L and ξ_0 sufficiently large. Q.E.D.

Thus, in principle, the continuation problem can be solved by continuing the initial vectors $\hat{\underline{y}}_{j,L}(\xi_0)$ to $\xi = \frac{1}{i}$ for sufficiently large L and ξ_0. Therefore, it would appear that only a numerical integration is required to obtain approximate values for $\underline{y}_j(\xi)$ at $\xi = \frac{1}{i}$. Theoretically this is true, but in practice this procedure may require more precision than is feasible.

For the theoretical investigation of the error $E_{je}(\xi;\xi_0)$, it was possible to neglect *errors* which propagate in an unbounded manner as ξ_0 tends to infinity. For example, consider $E_{1e}(\xi;\xi_0)$; then, due to theorems 2 and 3, it seemed reasonable to neglect errors which initiate multiples of $\underline{y}_4(\xi)$, the transition layer solution. Thus, from relation (58), it is easily shown that $E_{1e}(\xi;\xi_0) \leq |\varepsilon_1| \; ||\underline{y}_1(\xi)|| + |\varepsilon_2| \; ||\underline{y}_2(\xi)|| + |\varepsilon_3| \cdot ||\underline{y}_3(\xi)||$. However, $||\underline{y}_1(\xi)||$ and $||\underline{y}_2(\xi)||$ grow at essentially the same rates as $|\xi|$ is decreased and

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[18]If the roots of the dispersion relation differ by an integer, then the formal asymptotic solutions are not completely developed in Appendix B.1 (see case 3 in Appendix B.1.)

$||\underline{y}_3(\xi)||$ decreases exponentially fast. Hence, if $|\varepsilon_k|\ ||\underline{y}_k(\xi_0)||$ is made sufficiently small it is expected that $E_{1e}(\xi;\xi_0)$ can be made small. The analysis which culminated in theorem 8 justifies this heuristic argument.

From a numerical or practical point of view the *error* $|\varepsilon_4|\ ||\underline{y}_4(\xi)||$ cannot be ignored since the solution $\underline{y}_4(\xi)$ grows exponentially fast as $|\xi|$ is decreased. Thus, the term $|\varepsilon_4|\ ||\underline{y}_4(\xi)||$ could greatly exceed $||\underline{y}_1(\xi)||$ if $|\xi_0|$ is large and $|\xi| << |\xi_0|$. From a theoretical point of view it is unimportant how large $|\varepsilon_4|\ ||\underline{y}_4(\xi)||$ becomes, but from a numerical standpoint it is extremely important. Assuming that the numerical calculations are capable of maintaining a small relative accuracy (recall that relative accuracy is roughly a measure of the number of significant figures maintained in the calculations), then it is necessary to append some process which controls the multiple $\varepsilon_4\underline{y}_4(\xi)$ present in the approximate value of $\underline{y}_1(\xi)$. It is of the utmost importance to ensure that $|\varepsilon_4|\ ||\underline{y}_4(\xi)||$ cannot greatly exceed $||\underline{y}_1(\xi)||$ if only finitely many significant figures are maintained in the calculations. Otherwise, all of the significant figures may merely reflect the useless vector $\varepsilon_4\underline{y}_4(\xi)$.

Similarly, in order to compute approximate values of $\underline{y}_2(\xi)$ it is necessary to control the growth of $\varepsilon_4\underline{y}_4(\xi)$, that is, limit the multiple of the transition layer solution present in the approximate solution $\underline{y}_{2e}(\xi)$. In order to compute approximate values of $\underline{y}_3(\xi)$ it is necessary to limit the multiples of $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, and $\underline{y}_4(\xi)$ present in $\underline{y}_{3e}(\xi)$. This is difficult, since each of these solutions dominates $\underline{y}_3(\xi)$ for $|\xi|$ decreasing and intermediate rounding and truncation errors initiate multiples of $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, and $\underline{y}_4(\xi)$. The canonical form which was introduced in Section 4.1 (see theorems 4 and 5) provides the necessary error control.

If

$$||\underline{y}_{je}(\xi) - \underline{y}_j(\xi)|| \leq 10^{-N}||\underline{y}_j(\xi)|| \tag{96}$$

where $\underline{y}_{je}(\xi)$ is defined in (45), then

$$A_{je}(\xi;\xi_0) \leq 10^{-N} . \tag{97}$$

Relation (97) would require at least N significant figures to be carried in the calculations. Even if (97) can be maintained, no useful bound can be placed on $R_{je}(\xi;\xi_0)$ since

$$R_{je}(\xi;\xi_0) \leq 10^{-N} \frac{||\underline{y}_j(\xi)||}{f_{j2}(\xi)} \tag{98}$$

where $f_{j2}(\xi)$ is defined in lemma 1. Only for the case $j = 4$, the transition layer solution, can (97) be used to establish that

$$R_{4e}(\xi;\xi_0) \leq 10^{-N} \tag{99}$$

since $\{\underline{y}_4(\xi)\}$ consists of a single unique member $\underline{y}_4(\xi)$. For $j = 1,2,$ or 3 in relation (98), no useful bound results since the families $\{\underline{y}_j(\xi)\}$ contain arbitrarily large multiples of $\underline{y}_4(\xi)$. Hence, $||\underline{y}_j(\xi)||/f_{j2}(\xi)$ is in general unbounded, that is, not uniformly bounded for the family. Since it is desired to make $R_{je}(\xi;\xi_0)$ small for the numerical calculations, it is necessary to append some condition which ensures that $||\underline{y}_j(\xi)||/f_{j2}(\xi)$ is bounded uniformly on $1 \leq |\xi| < \infty$.

THEOREM 9: If the roots of the dispersion relation are distinct, then

$$\frac{||\underline{y}_j(\xi;\xi)||}{f_{j2}(\xi)} \leq M \tag{100}$$

for $j = 1,2,$ and 3 and $1 \leq |\xi| < \infty$ where $\underline{y}_j(\xi;\xi)$ are the canonical vectors (see theorems 4 and 5) and the functions $f_{j2}(\xi)$ are defined in lemma 1.

PROOF: Let $Y(\xi)$ be a fundamental solution of the differential equation (25) of the form

$$Y(\xi) = [\underline{y}_1(\xi), \underline{y}_2(\xi), \underline{y}_3(\xi), \underline{y}_4(\xi)]$$

where $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$. If $\underline{v}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$ and

$$V(\xi) = [\underline{v}_1(\xi), \underline{v}_2(\xi), \underline{v}_3(\xi), \underline{v}_4(\xi)] ,$$

then
$$v_4(\xi) \equiv \underline{y}_4(\xi) ,$$

$$\underline{y}_1(\xi) - \underline{v}_1(\xi) = c_1\underline{y}_4(\xi) ,$$

$$\underline{y}_2(\xi) - \underline{v}_2(\xi) = c_2\underline{y}_4(\xi) ,$$

and
$$\underline{y}_3(\xi) - \underline{v}_3(\xi) = a\underline{y}_1(\xi) + b\underline{y}_2(\xi) + c\underline{y}_4(\xi) .$$

Thus,
$$\det Y(\xi) \equiv \det V(\xi)$$

if $Y_c(\xi)$ is defined by

$$Y_c(\xi) = [\underline{y}_1(\xi;\xi), \underline{y}_T(\xi;\xi), \underline{y}_3(\xi;\xi), \underline{y}_4(\xi)]$$

and

$$Y_{min}(\xi) = [\underline{y}_{1min}(\xi), \underline{y}_{2min}(\xi), \underline{y}_{3min}(\xi), \underline{y}_{4min}(\xi)]$$

where $\underline{y}_j(\xi;\xi)$ is the canonical form of $\underline{y}_j(\xi)$ for $j = 1$ or 3 and $\underline{y}_T(\xi;\xi)$ is the temporary canonical form defined in relation (41). The vector $\underline{y}_T(\xi;\xi)$ differs from $\underline{y}_2(\xi)$ only in that multiples of $\underline{y}_4(\xi)$ and $\underline{y}_1(\xi)$ are added to $\underline{y}_2(\xi)$ to generate $\underline{y}_T(\xi;\xi)$. The vectors $\underline{y}_{jmin}(\xi)$ satisfy

$$||\underline{y}_{jmin}(\xi)|| = f_{j2}(\xi)$$

and for each fixed ξ ,

$$\underline{y}_{jmin}(\xi) = \underline{y}_j(\xi)$$

for some $\underline{y}_j(\xi)\varepsilon\{\underline{y}_j(\xi)\}$. However, $\underline{y}_{jmin}(\xi)$ is not included in $\{\underline{y}_j(\xi)\}$ since

$$\underline{y}_{jmin}(\xi) \not\equiv \underline{y}_j(\xi)$$

for one particular member $\underline{y}_j(\xi)\dot{\varepsilon}\{\underline{y}_j(\xi)\}$. Only for $j = 4$ does

$$y_{4min}(\xi) \equiv \underline{y}_4(\xi) .$$

Lemma 1 assures the existence of $\underline{y}_{jmin}(\xi)$. In addition,

$$\det Y_c(\xi) \equiv \det Y_{min}(\xi) \equiv \det Y(\xi) .$$

RECALL: $\| \ \|$ is the maximum norm.

$$|\det Y_{min}(\xi)| \leq 4!\,\|\underline{y}_{1min}(\xi)\|\,\|\underline{y}_{2min}(\xi)\|\,\|\underline{y}_{3min}(\xi)\| \times \|\underline{y}_{4min}(\xi)\|$$

$$|\det Y_c(\xi)| \equiv \|\underline{y}_1(\xi;\xi)\|\,\|\underline{y}_T(\xi;\xi)\|\,\|\underline{y}_3(\xi;\xi)\| \times \|\underline{y}_4(\xi)\|$$

since det $Y_c(\xi)$ is the product of only the maximum components of $\underline{y}_1(\xi;\xi)$, $\underline{y}_T(\xi;\xi)$, $\underline{y}_3(\xi;\xi)$, and $\underline{y}_4(\xi)$. Thus,

$$\frac{||\underline{y}_1(\xi;\xi)||}{||\underline{y}_{1min}(\xi)||} \times \frac{||\underline{y}_T(\xi;\xi)||}{||\underline{y}_{2min}(\xi)||} \times \frac{||\underline{y}_3(\xi;\xi)||}{||\underline{y}_{3min}(\xi)||} \times \frac{||\underline{y}_4(\xi)||}{||\underline{y}_{4min}(\xi)||} \leq 4! \qquad (101)$$

The first, third, and fourth factors in relation (101) are all greater than or equal to one. Thus, relation (101) will provide a bound for each of these terms if $||\underline{y}_T(\xi;\xi)||/||\underline{y}_{2min}(\xi)||$ is bounded away from zero uniformly for $1 \leq |\xi| < \infty$.

Consider the function $f_T(\xi)$ defined by

$$f_T(\xi) = \min_{\substack{\text{over all} \\ c_1(\xi) \text{ and } c_2(\xi)}} ||\underline{y}_2(\xi)+c_1(\xi)\underline{y}_1(\xi)+c_4(\xi)\underline{y}_4(\xi)|| .$$

It can be shown (analysis similar to theorem 6) that $f_T(\xi)$ exists and is continuous in ξ for $1 \leq |\xi| < \infty$. In addition,

$$\xi^{\lambda_1}\underline{y}_1(\xi) \rightarrow \underline{b}_{0,1} ,$$

$$\xi^{\lambda_2}\underline{y}_2(\xi) \rightarrow \underline{b}_{0,2} ,$$

and
$$\xi^{-\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}\underline{y}_4(\xi) \rightarrow -\underline{c}_{0,3}$$

as $\xi \to \infty$ (see Appendix B.2) and $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, and $\underline{c}_{0,3}$ are linearly independent. By analysis similar to lemma 2 it follows that

$$||\underline{y}_2(\xi)|| \leq Mf_T(\xi) \quad ,$$

but
$$||\underline{y}_2(\xi)|| \geq ||\underline{y}_{2min}(\xi)|| \quad .$$

Hence,
$$||\underline{y}_{2min}(\xi)|| \leq Mf_T(\xi)$$

or
$$\frac{||\underline{y}_T(\xi;\xi)||}{||\underline{y}_{2min}(\xi)||} \geq \frac{1}{M} \quad . \tag{102}$$

Thus, from relations (101) and (102)

$$\frac{||\underline{y}_j(\xi;\xi)||}{||\underline{y}_{jmin}(\xi)||} \leq 4! \times M$$

for $j = 1$ or 3 and

$$\frac{||\underline{y}_T(\xi;\xi)||}{||\underline{y}_{2min}(\xi)||} \leq 4! \quad .$$

Clearly, the subscripts $j = 1$ and $j = 2$ can be interchanged when the roots, λ_1 and λ_2, of the dispersion relation are distinct. This amounts merely to a renumbering of the inviscid solutions. Therefore, if

$\frac{||\underline{y}_1(\xi;\xi)||}{||\underline{y}_{1min}(\xi)||}$ is uniformly bounded for arbitrary λ_1 as long as $\lambda_1 \neq \lambda_2$, then $\frac{||\underline{y}_2(\xi;\xi)||}{||\underline{y}_{2min}(\xi)||}$ is uniformly bounded for arbitrary $\lambda_2 \neq \lambda_1$. Q.E.D.

If $\lambda_1 = \lambda_2 = \frac{1}{2}$, then

$$\frac{||\underline{y}_T(\xi;\xi)||}{||\underline{y}_{2min}(\xi)||} \geq \frac{M}{|\ell n\ \xi| + 1} \tag{103}$$

for $1 \leq |\xi| < \infty$. Thus,

$$\frac{||\underline{y}_j(\xi;\xi)||}{||\underline{y}_{jmin}(\xi)||} \leq \frac{4!}{M}(|\ell n\ \xi| + 1) \quad , \tag{104a}$$

for $j = 1$ or 3 and

$$\frac{||\underline{y}_2(\xi;\xi)||}{||\underline{y}_{2min}(\xi)||} \leq c(|\ell n\ \xi| + 1) \quad . \tag{104b}$$

Since $|\ell n\ \xi|$ grows very slowly there is very little additional difficulty for the case $\lambda_1 = \lambda_2 = \frac{1}{2}$.

Thus, in all cases the canonical form limits the ratio $||\underline{y}_j(\xi;\xi)||/||\underline{y}_{jmin}(\xi)||$. In so doing, if $\underline{y}_{je}(\xi;\xi)$ has a small relative accuracy, then it also has a small relative error.

The solutions of the differential equation (25) grow by only a bounded factor on any bounded interval; that is, if $\underline{y}(\xi)$ is a solution of (25) then

$$||\underline{y}(\xi_2)|| \quad \leq \quad ||\underline{y}(\xi_1)||e^{M|\xi_1-\xi_2|} \tag{105}$$

where $M < \infty$ can be determined for $1 \leq |\xi| < \infty$. Therefore, it is not necessary to continuously reduce $\underline{y}_{je}(\xi)$ to canonical form in order to control the *error* growth. It is only necessary to reduce $\underline{y}_{je}(\xi)$ to canonical form several times over the interval of numerical integration. The length of the interval over which a numerical integration can safely be performed without reduction to canonical form obviously depends on the parameters σ, k, and γ and the precision which is maintained in the calculations. To some extent experience is required, but a good first estimate of the length of an interval can be obtained by considering the different asymptotic rates of growth (see Section 4.1, step b in the numerical algorithm).

The accuracy of the calculations, which were performed for the viscous problem formulated in Section 2, is discussed in Section 5 and Appendix B.2.

5. COMPUTATIONS AND CONCLUSIONS

5.1 PRELIMINARY REMARKS AND ORGANIZATION OF THE COMPUTATIONS

The viscous problem is solved for small $\varepsilon > 0$ when equations (32) and (33) are solved. The most difficult numerical problem encountered is the determination of vectors at $\xi = \frac{1}{i}$ which specify the different asymptotic solutions. There are several cases which must be considered separately. The individual cases can be classified according to the character of the roots of the dispersion relation.

CASE 1: The roots of the dispersion relation, λ_1 and λ_2, are real and distinct. In addition, $\frac{\sigma^2}{\gamma} - k^2 \neq 0$.

Upon solving equation (32), constants e_1, e_2, e_3, and e_4 are determined such that

$$e_1\underline{DC}_1(\xi) + e_2\underline{DC}_2(\xi) = \underline{INV}_2(\xi) - e_3\underline{INV}_1(\xi) - e_4\underline{TLSOL}(\xi) \quad . \tag{106}$$

If $\lambda_1 > \frac{1}{2} > \lambda_2$, then

$$e_1\underline{DC}_1(\xi) + e_2\underline{DC}_2(\xi) = \left(\underline{b}_{0,2} - e_3\underline{b}_{0,1}\xi^{\lambda_2-\lambda_1} + 0\left(\frac{1}{\xi}\right)\right)\xi^{-\lambda_2} \quad . \tag{107}$$

However, $\xi^{\lambda_2 - \lambda_1}$ becomes negligible as $\varepsilon \to 0$, for $0 < z \leq B < \infty$. For $\varepsilon \approx 10^{-11}$ the vector $e_3 \underline{b}_{0,1} \xi^{\lambda_2 - \lambda_1}$ can be neglected below the transition region except for $\lambda_2 - \lambda_1 \approx 0$. The case $\sigma^2/\gamma - k^2 = 0$ is a special situation which is discussed in case 4.

Thus, above the boundary layer and below the transition layer, the solution of the viscous problem can be accurately approximated by a multiple of the inviscid solution with finite kinetic energy in an infinite column of fluid.

If $0 < A \leq z \leq B < \infty$, then (107) implies that the solution with finite kinetic energy is approached uniformly on this interval. The only calculations which must be performed for real λ_1 and λ_2 are for $\lambda_1 \approx \lambda_2$ (case 3) and $\sigma^2/\gamma - k^2 \approx 0$ (case 4).

CASE 2: The roots of the dispersion relation (13) are complex. For this case the results of Yanowitch [2,3] and Lindzen [4] imply that for small β, the dimensionless, vertical wave number defined in (20), the reflection coefficient is large and for β large the reflection coefficient is small. Thus, it is expected that reflection is significant near the boundaries of the shaded regions in figure 1.

||/

Aside from a scaling constant, the solution above the boundary layer and below the transition layer is approximately

$$\begin{bmatrix} U(z) \\ W(z) \end{bmatrix} \approx \begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix} + \kappa_R \begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix} , \qquad (108)$$

where $\begin{bmatrix} U_i(z) \\ W_i(z) \end{bmatrix}$ is a solution of the inviscid differential equation (12) for $i = 1$ or 2 and normalized so that $U_i(o) = 1$. In addition, the inviscid solutions are numbered such that $\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix}$ has upward energy flux and $\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix}$ has downward energy flux. The constant κ_R is, of course, the reflection coefficient.

If $\varepsilon > 0$ is specified, then κ_R is determined from the constant e_3 in (32). The scalar e_3 might be considered the *asymptotic* reflection coefficient. The constant e_3 is invariant (see theorem B2 in Appendix B.2) as $\varepsilon \to 0$. However, κ_R does change as $\varepsilon \to 0$. If $\delta = \ell n \left(\frac{1}{\varepsilon}\right)$, then

$|\kappa_R|$ is asymptotically invariant as $\varepsilon \to 0$ for fixed σ and k since $|e_3|$ is invariant. (109a)

Arg $\kappa_R-2\beta\delta$ is asymptotically invariant as $\varepsilon \to 0$, for fixed σ and k and the acoustic wave (109b)

$$\left(\frac{\sigma^2}{\gamma} - k^2 > 0\right) .$$

Arg $\kappa_R+2\beta\delta$ is asymptotically invariant as $\varepsilon \to 0$ for the gravity wave (109c)

$$\left(\frac{\sigma^2}{\gamma} - k^2 < 0\right) .$$

Due to relations (109), it is more useful to compute arg $\kappa_R \pm 2\beta\delta$ than arg κ_R.

It is reasonable to expect arg κ_R to change as $\varepsilon \to 0$ since the reflection takes place in the transition region. .he transition region is in the vicinity of $z = \ell n \left(\frac{1}{\varepsilon}\right)$ and this point recedes to infinity as $\varepsilon \to 0$. Thus, the reflection coefficient can be expected to undergo a phase change as $\varepsilon \to 0$.

In Appendix B (see B.2, relations (B74) through (B84)) it is shown that a fixed error tolerance (10^{-7}) on the asymptotic solutions resulted in an approximate value of ξ_I where

$$|\xi_I| \approx \left(\frac{11}{\sigma}\right)^2 \qquad (110)$$

and for $|\xi| \geq |\xi_I|$ the truncated asymptotic expansions have a minimum relative error less than or equal to 10^{-7}. For a fixed value of $k > 0$, σ is bounded away from zero in both the acoustic and gravity wave regions (see figure 1, Section 1.2) as $\beta \to 0$. If the initial vectors for the canonical numerical integration from ξ_I to $\xi = \frac{1}{i}$ are computed by method II (see B.1), then it is obvious that the asymptotic initial vectors with inviscid asymptotic behavior tend uniformly towards each other as $\beta \to 0$. If the vectors $\underline{DC}_1(\xi)$, $\underline{DC}_2(\xi)$, $\underline{INV}_1(\xi)$, and $\underline{TLSOL}(\xi)$ remain linearly independent as $\beta \to 0$, then

$$|\kappa_R(k,\beta)| \to 1 \tag{111}$$

as $\beta \to 0$ since $|e_3|$ in equation (32) tends to one as $\beta \to 0$. In addition, if $\varepsilon > 0$ is held fixed, then

$$\kappa_R(k,\beta) \to -1 \tag{112}$$

as $\beta \to 0$ since $e_3 \to 1$. Relations (111) and (112) are an immediate consequence of (31) and the above assumption on the linear independence of $\underline{DC}_1(\xi)$, $\underline{DC}_2(\xi)$, $\underline{INV}_1(\xi)$ and $\underline{TLSOL}(\xi)$.

CASE 3: The roots of the dispersion relation are equal, that is, $\lambda_1 = \lambda_2 = \frac{1}{2}$ or $\alpha = \frac{1}{4}$. For this case the solution of the viscous problem in the inviscid region is approximately

$$\begin{bmatrix} U(z) \\ W(z) \end{bmatrix} = A\left\{\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix} + B\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix}\right\} , \qquad (113)$$

where $\begin{bmatrix} U_i(z) \\ W_i(z) \end{bmatrix}$ is an inviscid solution and, for $i = 1$ or 2, is defined in (18). Since the asymptotic connection relations (32) are invariant as $\varepsilon \to 0$ it can be shown that $B \to \infty$ as $\varepsilon \to 0$. However,

$$|B| = 0\left(\ell n \left(\frac{1}{\varepsilon}\right)\right) \qquad (114)$$

as $\varepsilon \to 0$ and hence $\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix}$ cannot be neglected in the entire inviscid region. It should be noted that $A \to 0$ as $\varepsilon \to 0$ since the kinematic boundary condition ($W(0) = 1$) requires $A \times B$ to remain finite as $\varepsilon \to 0$.

CASE 4: The roots of the dispersion relation are $\lambda_1 = \frac{1}{\gamma}$ and $\lambda_2 = \frac{\gamma - 1}{\gamma}$ or $\frac{\sigma^2}{\gamma} - k^2 = 0$. In the inviscid region the solution of the viscous problem is approximately given by

$$\begin{bmatrix} U(z) \\ W(z) \end{bmatrix} \approx C\left\{\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix} + D\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix}\right\} , \qquad (115)$$

where $\begin{bmatrix} U_i(z) \\ W_i(z) \end{bmatrix}$ is an inviscid solution and, for $i = 1$ or 2, is defined in (16). The asymptotic connection relations (32) determine the constant D and since the asymptotic connection relations are invariant it follows that

$$|D| = O\left(\varepsilon^{\lambda_1 - \lambda_2}\right) \tag{116}$$

as $\varepsilon \to 0$. Hence $D \to 0$ as $\varepsilon \to 0$ since $\lambda_1 > \frac{1}{2} > \lambda_2$. If $\gamma = 1.4$, then

$$\lambda_1 \approx .714 \tag{117a}$$

and

$$\lambda_2 \approx .286 \quad . \tag{117b}$$

The kinematic boundary condition $(W(o) = 1)$ requires

$$W(z) = CW_2(z) + CDW_1(z) \approx 1 \tag{118}$$

near $z = 0$ as $\varepsilon \to 0$, but $W_2(z)$ is identically zero. Thus,

$$C = O\left(\left(\frac{1}{\varepsilon}\right)^{\lambda_1 - \lambda_2}\right) \tag{119}$$

as $\varepsilon \to 0$, that is, $C \to \infty$ as $\varepsilon \to 0$. A resonant situation develops as $\varepsilon \to 0$. For a finite value of ε

like 10^{-11} there should be a noticeable resonant peak for the scalar C in the vicinity of $\frac{\sigma^2}{\gamma} - k^2 = 0$.

In order to describe the resonant peak, it is useful to calculate the modulus of C in relation (115) for $\sigma^2/\gamma - k^2 \approx 0$, in addition to the obvious calculations required for $\sigma^2/\gamma - k^2 = 0$. For this purpose it is best to normalize the inviscid solutions so that solutions (16) are approached as $\sigma^2/\gamma - k^2 \to 0$. This is easily accomplished; consider

$$\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix} = \begin{bmatrix} \dfrac{-\left(\lambda_1^2 - \lambda_1 + \dfrac{\sigma^2}{\gamma}\right)}{k\left(\lambda_1 + \dfrac{1}{\gamma} - 1\right)} \\ 1 \end{bmatrix} e^{\lambda_1 z} \tag{120a}$$

and

$$\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\lambda_2 - \dfrac{1}{\gamma}\right)} \end{bmatrix} e^{\lambda_2 z} , \tag{120b}$$

where λ_1 and λ_2 are assumed real and

$$\lambda_2 < \frac{1}{2} < \lambda_1 \tag{121}$$

for $\gamma = 1.4$.

Not only is it important to organize the computations but it is also necessary to consider the actual computational devices which perform the calculations. The calculation of the DC solutions was accomplished on a UNIVAC 1108 in double-precision (about 18 significant figures). Three hundred terms were generated from the recursion relation (A10) in Appendix A and then summed at $\xi = \frac{1}{i}$. Due to certain compiler difficulties it was not possible to perform all the computations on the 1108. The calculation of asymptotic initial vectors and the canonical numerical integration was carried out on an IBM 7094 in single-precision (about eight significant figures). The error check (step 5 of the algorithm in Section 4.1) indicated that approximately four significant figures are maintained in the calculations.

5.2 NUMERICAL RESULTS

This section contains the numerical results for the viscous problem formulated in Section 2. The calculations were performed for $\varepsilon = 10^{-11}$, which is comparable to a value in the earth's atmosphere. The other dimensionless parameters, k and σ , when equal to unity, correspond to a horizontal wavelength of about 45 km and a frequency of 2.5 radians per minute, respectively.

CASE 1: The roots of the dispersion relation are real and distinct. In Section 5.1 it was shown that the only calculations required are for $\lambda_1 \approx \lambda_2$ (case 3) and $\sigma^2/\gamma - k^2 \approx 0$ (case 4).

CASE 2: The roots of the dispersion relation are complex or the inviscid solutions are wavelike in z. The reflection coefficient, κ_R in (108), is computed for various values of σ and k. Due to the remarks made in Section 1 about some recent research of Yanowitch, it will be worthwhile to compare $|\kappa_R|$ and $e^{-\pi\beta}$ or equivalently $\ell n\, |\kappa_R|$ and $-\pi\beta$. To avoid confusion κ_{RA} will denote the acoustic reflection coefficient and κ_{RG} will denote the gravity reflection coefficient. In order to construct figures 2 through 7 (pp. 119-124) the values of κ_{RG} and κ_{RA} were computed for $\beta = .01, .1, .2, .4, .8$, and 1.6. For $k = .5$, κ_{RA} was computed for the additional β values of .5, .6, .7, .9 and 1.0.

If the reflection coefficients are considered as functions of the horizontal and vertical wave numbers, then figures 2 through 7 imply

$$|\kappa_{RA}(k,\beta)| \leq |\kappa_{RG}(k,\beta)| \leq e^{-\pi\beta} \qquad (122)$$ [19]

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[19] The calculations which were performed for $k = .005$ lead to the relation $|\kappa_{RG}(.005,\beta)| > e^{-\pi\beta}$. However, $|\kappa_{RG}(.005,\beta)| - e^{-\pi\beta}$ was very small and about equal to the estimate of the error obtained from step 5 of the numerical algorithm.

In order to test relation (122) and determine any sensitivity to changes in γ , some computations were performed for $\gamma = 4$. Surprisingly, the acoustic reflection coefficient was in much better agreement with $e^{-\pi\beta}$ than the gravity reflection coefficient for $\gamma = 4$. Thus, it appears that (122) is a quantitative rather than a qualitative relation; nevertheless (122) is a useful *summary* of the calculations.

In order to completely specify κ_{RG} and κ_{RA} it is also necessary to determine the argument of these complex quantities. Due to the remarks in Section 5.1 it is best to compute arg $\kappa_{RG}(k,\beta)+2\beta\delta$ and arg $\kappa_{RA}(k,\beta)-2\beta\delta$ to obtain a useful profile of the argument of the reflection coefficient as $\varepsilon \to 0$ (tables I through VII).

TABLE I. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR $k = .005$

β	arg $\kappa_{RG}+2\beta\delta$	arg $\kappa_{RA}-2\beta\delta$
0.01	-3.12	-2.99
.1	-2.94	-1.68
.2	-2.67	- .515
.4	-1.71	.983
.8	1.71	2.26
1.6	- .567	2.44

TABLE II. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR $k = .05$

β	$\arg \kappa_{RG}+2\beta\delta$	$\arg \kappa_{RA}-2\beta\delta$
0.01	3.12	-3.00
.1	2.97	-1.75
.2	2.86	- .641
.4	2.99	.788
.8	-1.66	2.03
1.6	-1.34	2.20

TABLE III. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR $k = .25$

β	$\arg \kappa_{RG}+2\beta\delta$	$\arg \kappa_{RA}-2\beta\delta$
0.01	3.11	-3.02
.1	2.88	-2.00
.2	2.67	-1.09
.4	2.53	.099
.8	-3.03	1.14
1.6	1.17	1.20

TABLE IV. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR $k = .5$

β	$\arg \kappa_{RG} + 2\beta\delta$	$\arg \kappa_{RA} - 2\beta\delta$
.01	3.10	-3.11
.1	2.76	-2.79
.2	2.54	-2.41
.4	2.52	- .308
.5		.165
.6		.345
.7		.438
.8	-2.99	.487
.9		.507
1.0		.504
1.6	.547	.127

TABLE V. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR $k = .75$

β	$\arg \kappa_{RG} + 2\beta\delta$	$\arg \kappa_{RA} - 2\beta\delta$
0.01	3.10	3.12
.1	2.73	2.89
.2	2.54	2.64
.4	2.68	2.10
.8	-2.61	.804
1.6	.822	- .664

TABLE VI. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR k = 1.0

β	arg $\kappa_{RG}+2\beta\delta$	arg $\kappa_{RA}-2\beta\delta$
0.01	3.11	3.10
.1	2.83	2.75
.2	2.74	2.38
.4	3.06	1.74
.8	-1.97	.665
1.6	1.55	-1.24

TABLE VII. — ARGUMENT OF THE REFLECTION COEFFICIENTS FOR k = 1.5

β	arg $\kappa_{RG}+2\beta\delta$	arg $\kappa_{RA}-2\beta\delta$
0.01	3.13	3.08
.1	3.07	2.56
.2	-3.07	2.00
.4	-2.32	1.03
.8	- .441	- .409
1.6	-2.67	-2.59

The figures 2 through 7 and tables I through VII can be used to construct a multiple of the solution to the viscous problem above the boundary layer and below the transition layer. Thus, it is desirable to obtain some estimate of the lower boundary of the transition layer and the upper boundary of the boundary layer.

In Section 3.2, relation (30) implies that the boundary layer solution decreases by a factor $\frac{1}{e}$ as z is increased by $\sqrt{\frac{2\gamma\varepsilon}{\sigma}}$. For $\varepsilon = 10^{-11}$ a linear combination of inviscid solutions agrees with the solution to the viscous problem to four significant figures if $z_{b.\ell.} \leq z \leq z_{t.\ell.}$ and

$$z_{b.\ell.} = \frac{6 \times 10^{-5}}{\sqrt{\sigma}} \quad . \tag{123}$$

The upper boundary of the boundary layer varies from several meters for the gravity wave ($k = .005$, $\beta = 1.6$) to less than one-half meter ($k = 1.5$, $\beta = 1.6$) for the acoustic wave.

In order to obtain a more detailed description of the solution to the viscous problem, some computations were performed which illustrate how the transition region joins the inviscid and viscous regions. It is useful for this purpose to consider plots of $|U(\xi)| \times \sqrt{|\xi|}$ (ordinate) versus $\ell n \sqrt{|\xi|}$ (abscissa) and $|W(\xi)| \times \sqrt{|\xi|}$ versus $\ell n \sqrt{|\xi|}$.

The ordinates $|U(\xi)| \times \sqrt{|\xi|}$ and $|W(\xi)|\sqrt{|\xi|}$ are proportional to the square roots of the kinetic energies associated with the horizontal and vertical velocity components, respectively. The abscissa, $\ln \sqrt{|\xi|}$, is proportional to $z/2$ and hence a one-unit change in $\ln \sqrt{|\xi|}$ corresponds to two scale heights (see figures 8 through 14). It should be noted that convenient multiples of the viscous solutions are plotted in figures 8 through 14.

In figures 8 through 14 the inviscid region is easily identified. It appears that the inviscid region begins at about $z = 20$ (140 km) and extends downward from this height, that is, $z_{t.\ell.}$ is approximately 20 scale heights. The oscillation about the dotted line is caused by alternate constructive and destructive interference of the inviscid solutions with upward and downward energy propagation. The dotted line, in figures 8 through 14, corresponds to a multiple of the inviscid solution which has upward energy propagation and hence satisfies the radiation condition. The amplitude of the oscillation about this dotted line is an indication of how poor or good an approximation the radiation condition is. For $\beta = 1.6$ the radiation condition appears to be quite good (in the inviscid region); if $\beta = .4$ it is unsatisfactory. Relation (122) implies that the radiation condition introduces an error of about 4 percent for $\beta = 1$, and for larger values of β a smaller error is expected. Thus, if the vertical wavelength is less than 45 km ($\beta > 1$) the radiation condition is substantially correct.

CASE 3: The roots of the dispersion relation are equal or $\alpha = \frac{1}{4}$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. The constants A and B in relation (113) are listed in tables VIII and IX.

TABLE VIII. — THE CONSTANTS A AND B, DEFINED IN RELATION (113), FOR $\sigma^2/\gamma - k^2 < 0$ AND $\gamma = 1.4$.

k	σ	A	B	|B|	arg B
0.005	0.00535	-11.5-.935 i	-25.4+1.57 i	25.5	3.08
.05	.0534	-1.07-.078 i	-27.3+1.57 i	27.3	3.08
.25	.260	-.170-.013 i	-27.7+1.69 i	27.8	3.08
.5	.447	-.044-.0036 i	-28.5+1.87 i	28.5	3.08
.75	.505	-.018-.0013 i	-28.7+1.66 i	28.7	3.08
1.0	.520	-.012-.00083 i	-28.2+1.58 i	28.2	3.09
1.5	.529	-.0073-.00054 i	-27.0+1.57 i	27.0	3.08

TABLE IX. – THE CONSTANTS A AND B, DEFINED IN RELATION (113), for $\sigma^2/\gamma - k^2 > 0$ AND $\gamma = 1.4$.

k	σ	A	B	|B|	arg B
0.005	0.592	0.00015-.0000084 i	-34.0-1.57 i	34.0	-3.10
.05	.592	.00154-.000088 i	-33.6-1.59 i	33.6	-3.09
.25	.608	.0099-.00084 i	-32.3-2.27 i	32.4	-3.07
.5	.707	.043-.0083 i	-28.1-4.31 i	28.4	-2.99
.75	.939	.120-.015 i	-25.1-2.46 i	25.2	-3.04
1.0	1.22	.199-.019 i	-24.4-1.80 i	24.4	-3.07
1.5	1.79	.357-.032 i	-23.4-1.59 i	23.4	-3.07

CASE 4: The roots of the dispersion relation are $\lambda_1 = \frac{1}{\gamma}$ and $\lambda_2 = \frac{\gamma - 1}{\gamma}$ or $\sigma^2/\gamma - k^2 = 0$. In addition, computations were performed for

$$\sigma_{\pm} = (1 \pm .01)\sqrt{\gamma}\, k \quad . \qquad (124)$$

The results of the computations are summarized in figure 15.

There is a noticeable resonant peak for $\sigma^2/\gamma - k^2 = 0$. Varying σ by 1 percent results in a decrease of $|C|$ by approximately a factor of 10^3 (refer to relation (115) and figure 15).

The boundary layer solution is not capable of reducing U(z) in (115) to zero in $0 \leq z \leq z_{b.\ell.}$ with negligible effect on W(z). The value of $|C|$ computed for $\varepsilon = 10^{-11}$ depends on the peculiar properties of the boundary layer solution. Other effects in the earth's atmosphere more important than viscosity are being neglected near $z = 0$. Therefore, the results for this case are of dubious value.

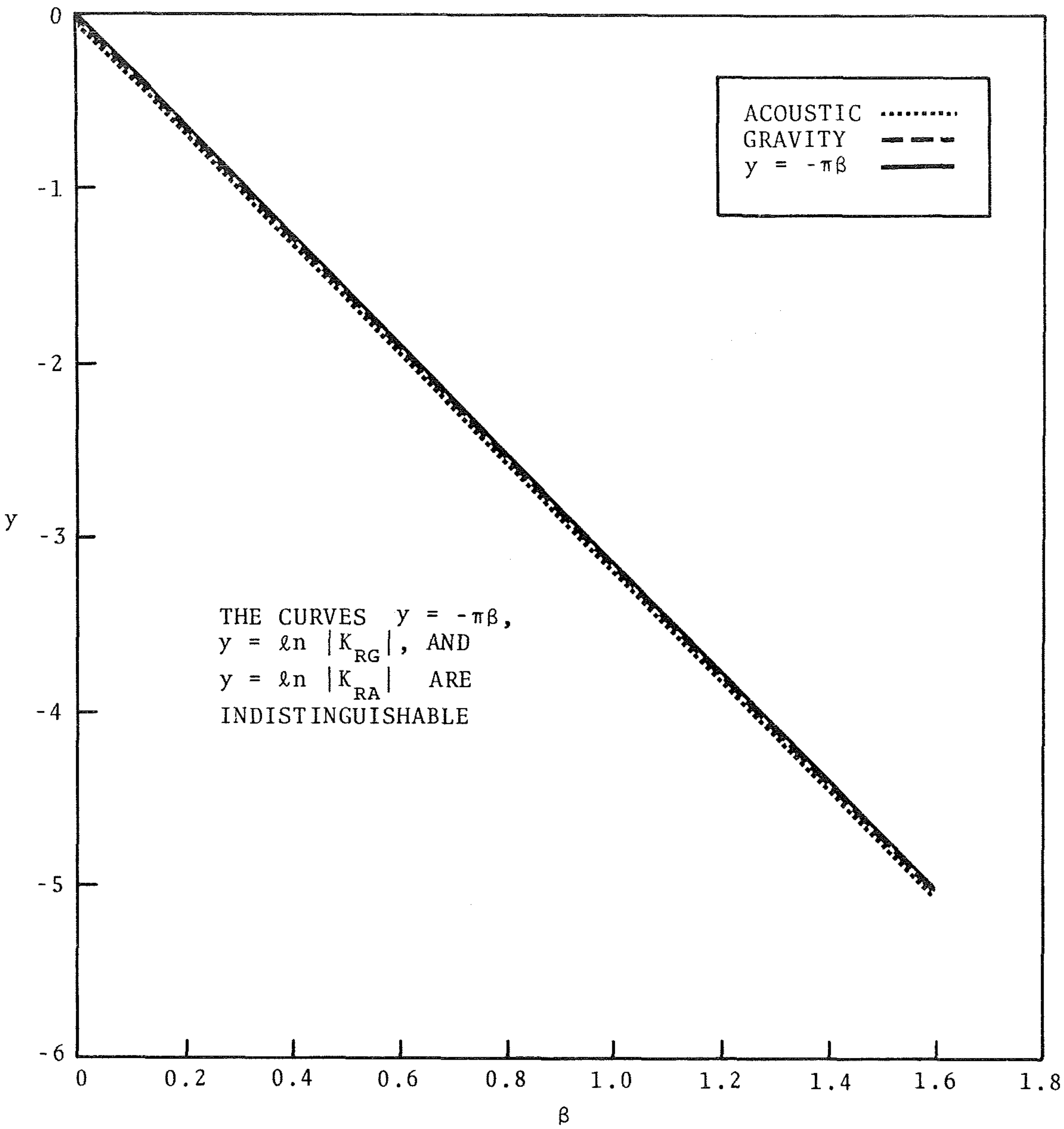

Figure 2. — Logarithm of the modulus of the reflection coefficients for $k = .005, .05$.

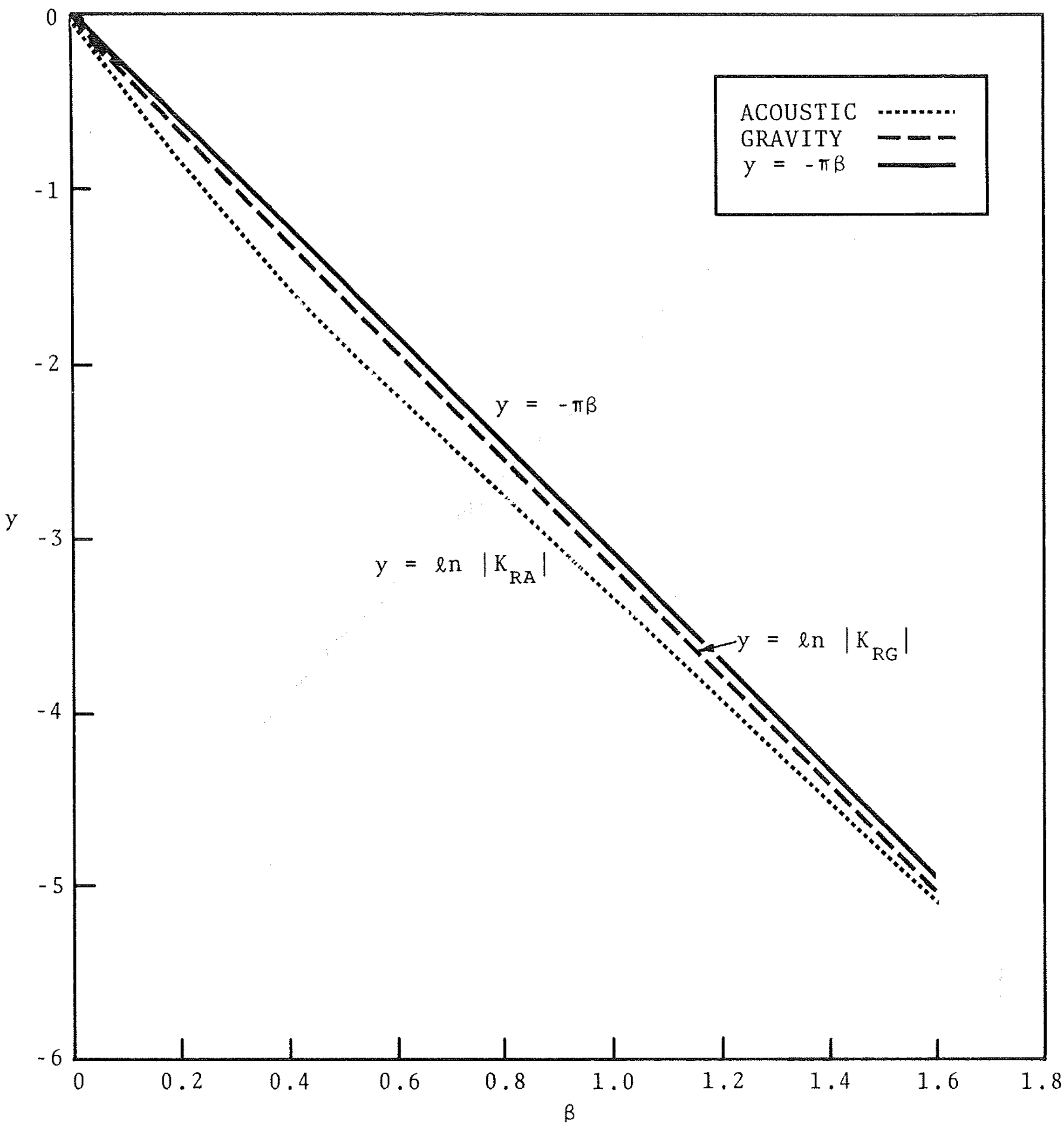

Figure 3. — Logarithm of the modulus of the reflection coefficients for $k = .25$.

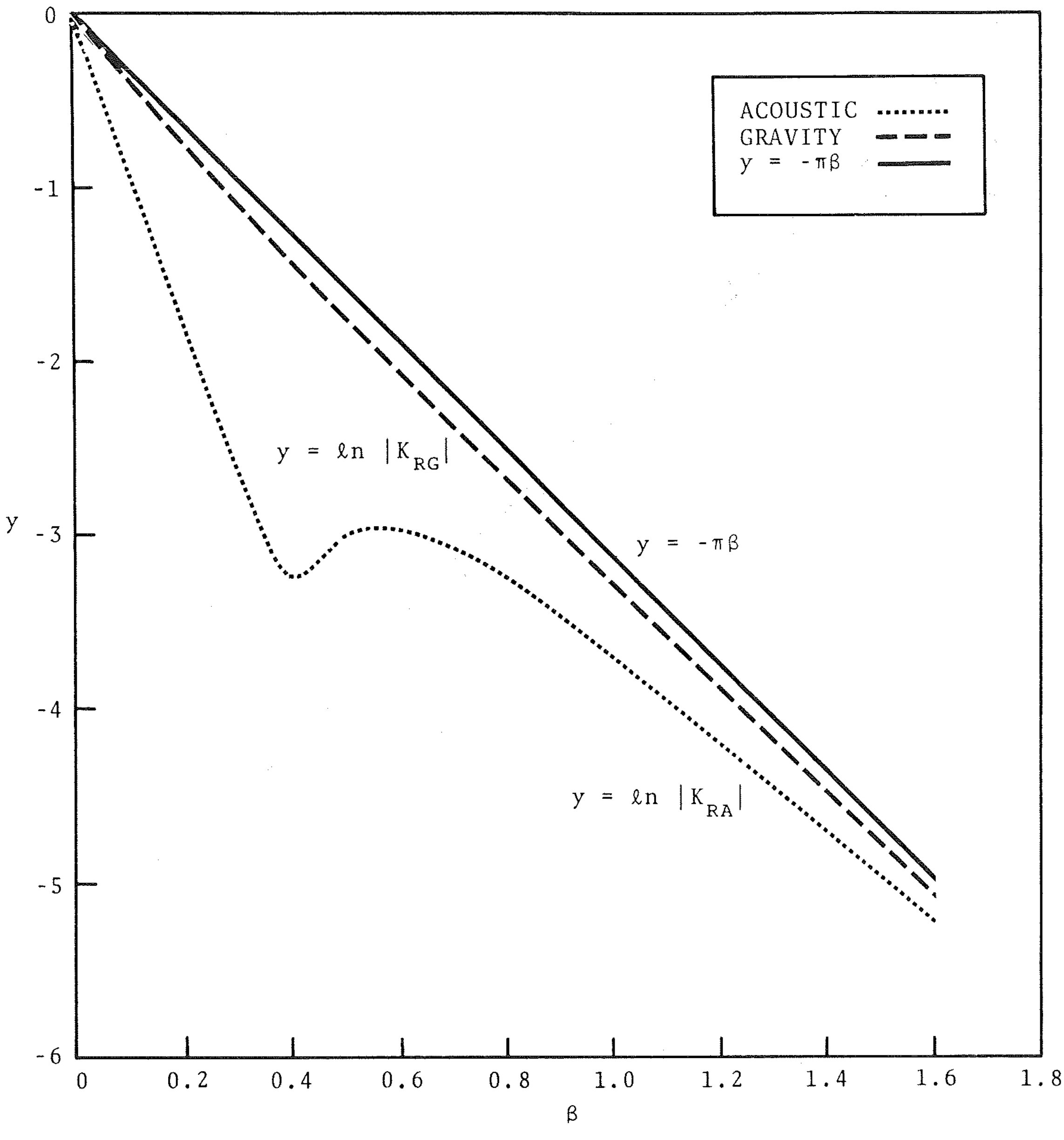

Figure 4. — Logarithm of the modulus of the reflection coefficients for $k = 0.5$.

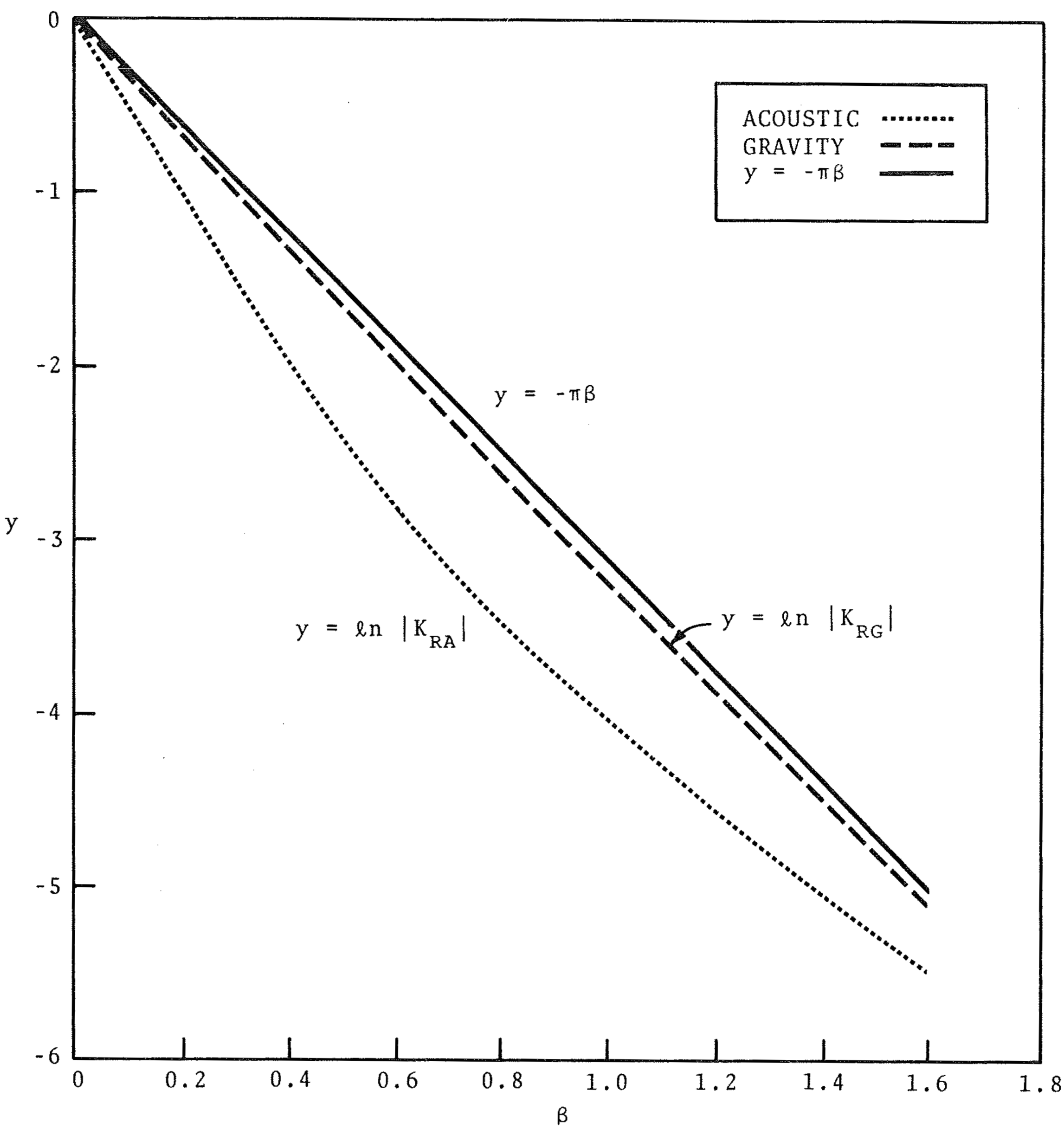

Figure 5. — Logarithm of the modulus of the reflection coefficients for $k = .75$.

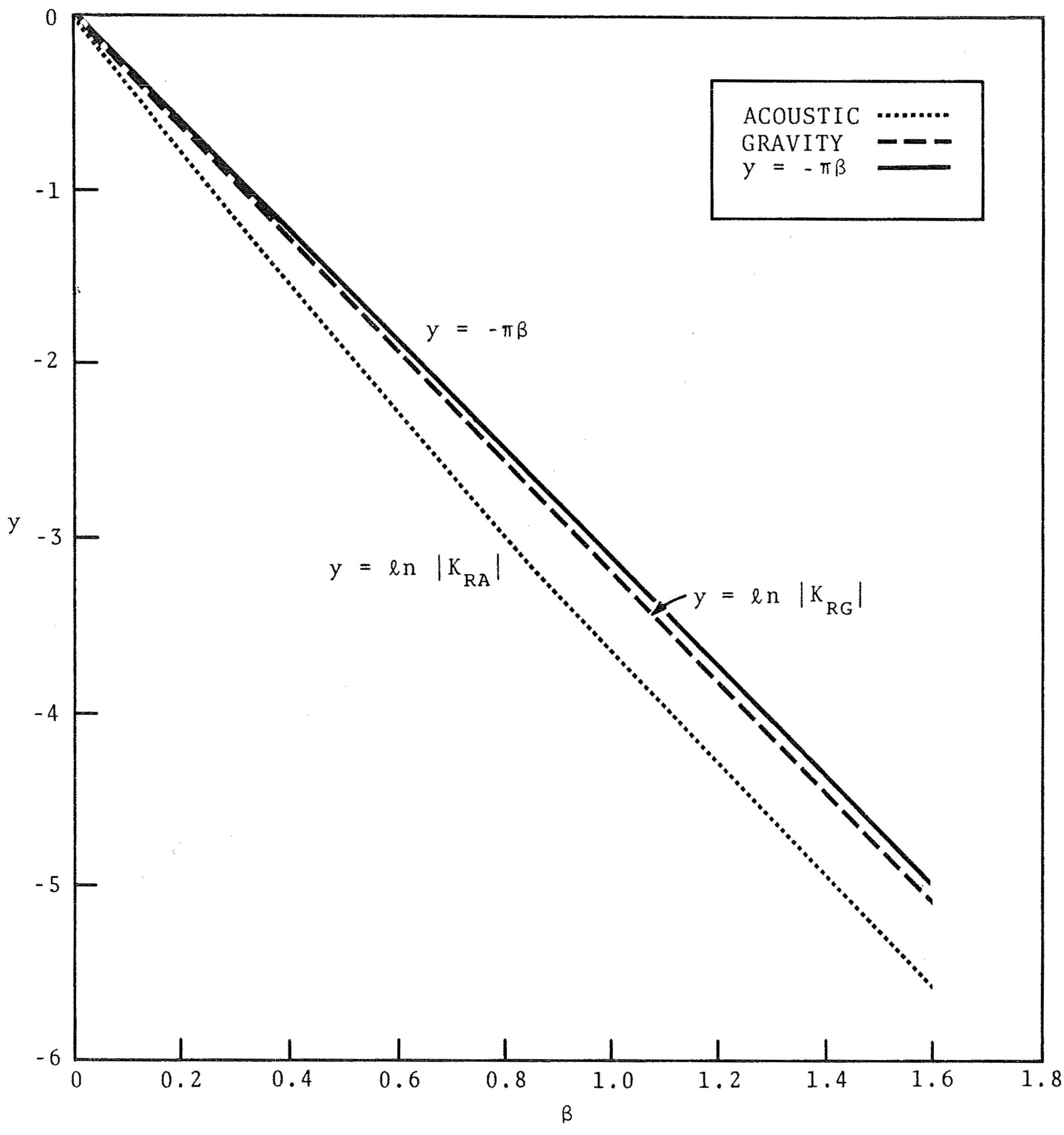

Figure 6. — Logarithm of the modulus of the reflection coefficients for $k = 1.0$.

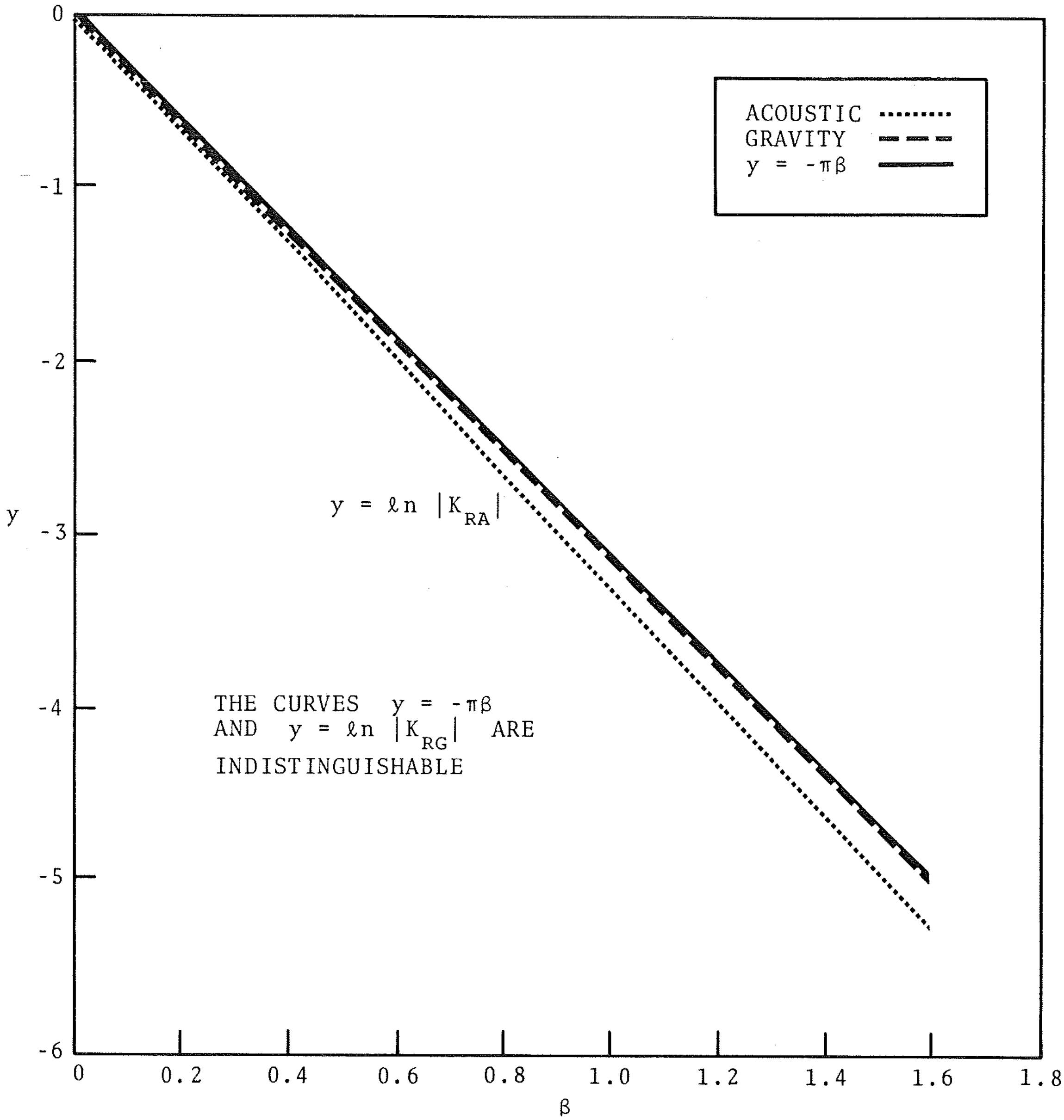

Figure 7. — Logarithm of the modulus of the reflection coefficients for $k = 1.5$.

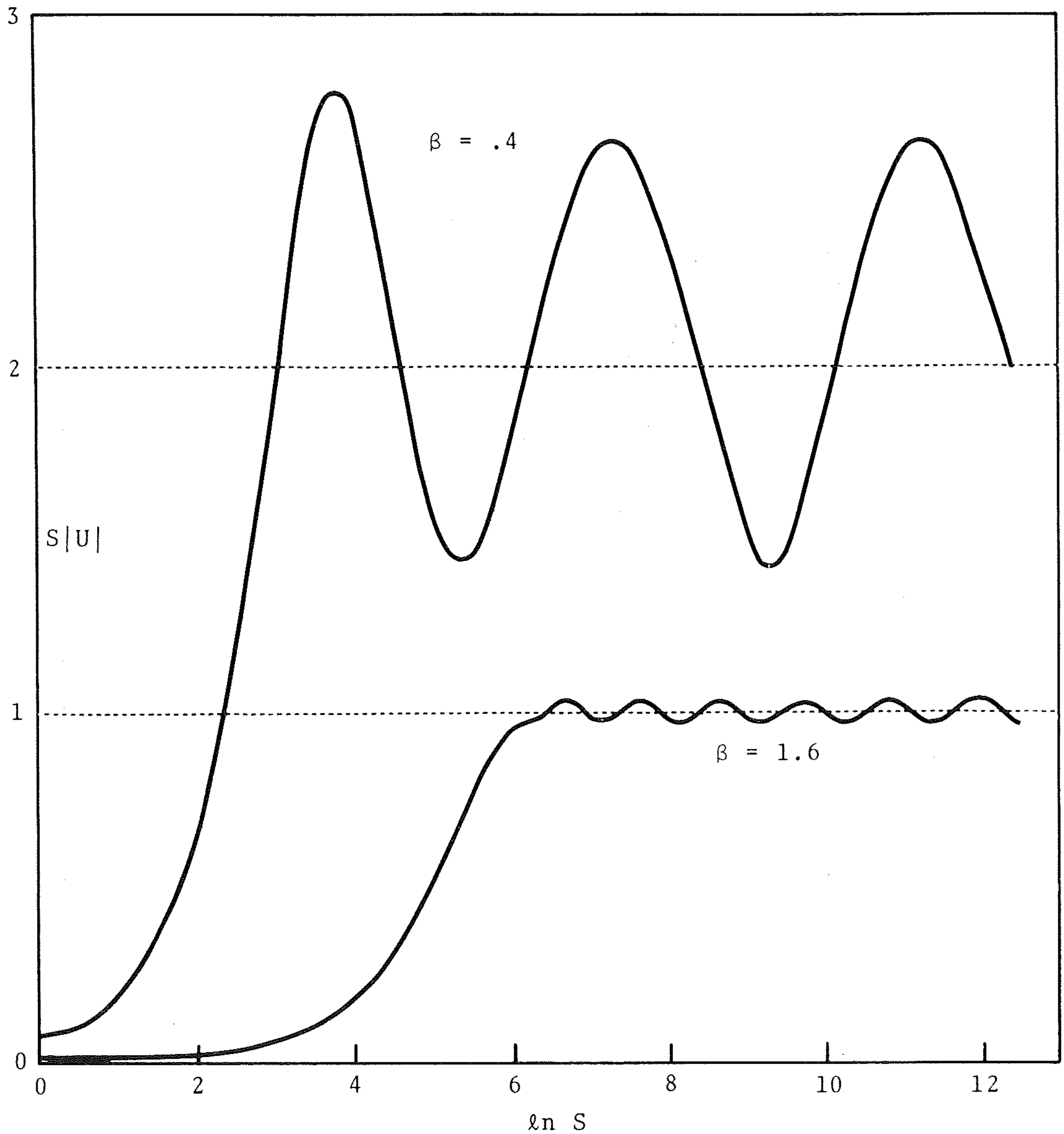

Figure 8. — Viscous gravity waves for k = .05 and β = .4, 1.6 . $S = \sqrt{|\xi|}$.

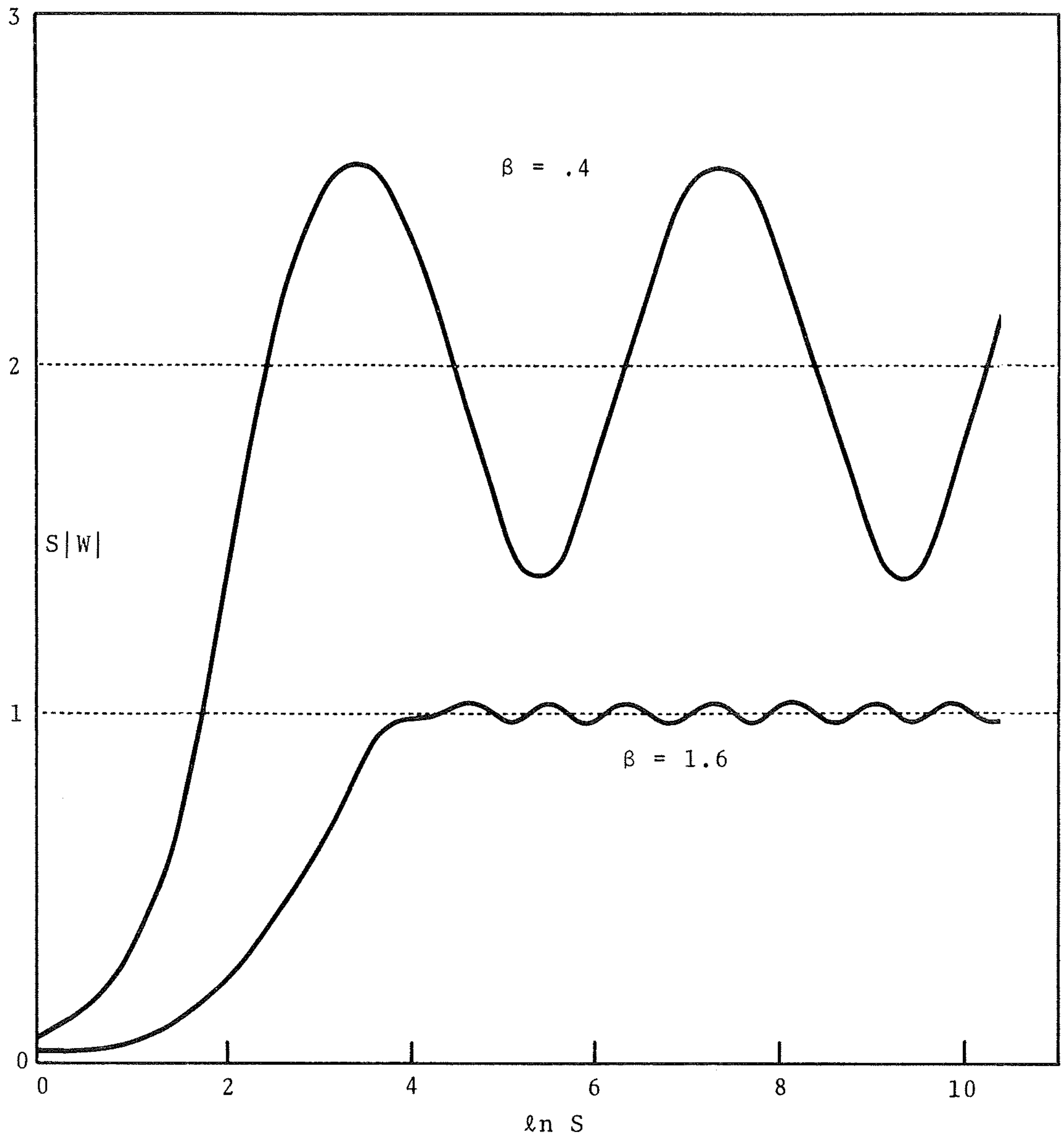

Figure 9. — Viscous gravity waves for $k = .5$ and $\beta = .4, 1.6$. $S = \sqrt{|\xi|}$.

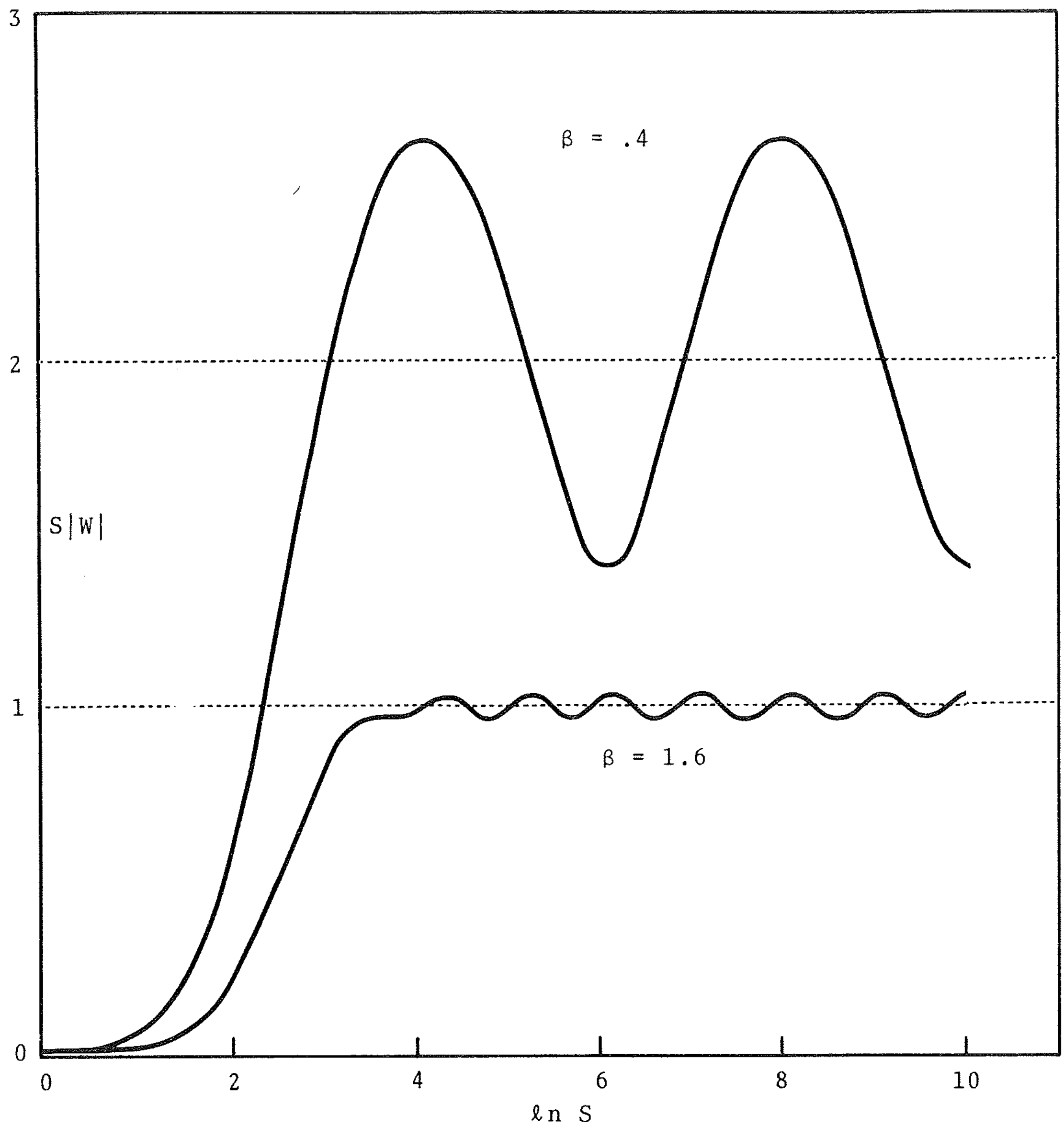

Figure 10. — Viscous gravity waves for $k = 1.5$ and $\beta = .4, 1.6$. $S = \sqrt{|\xi|}$.

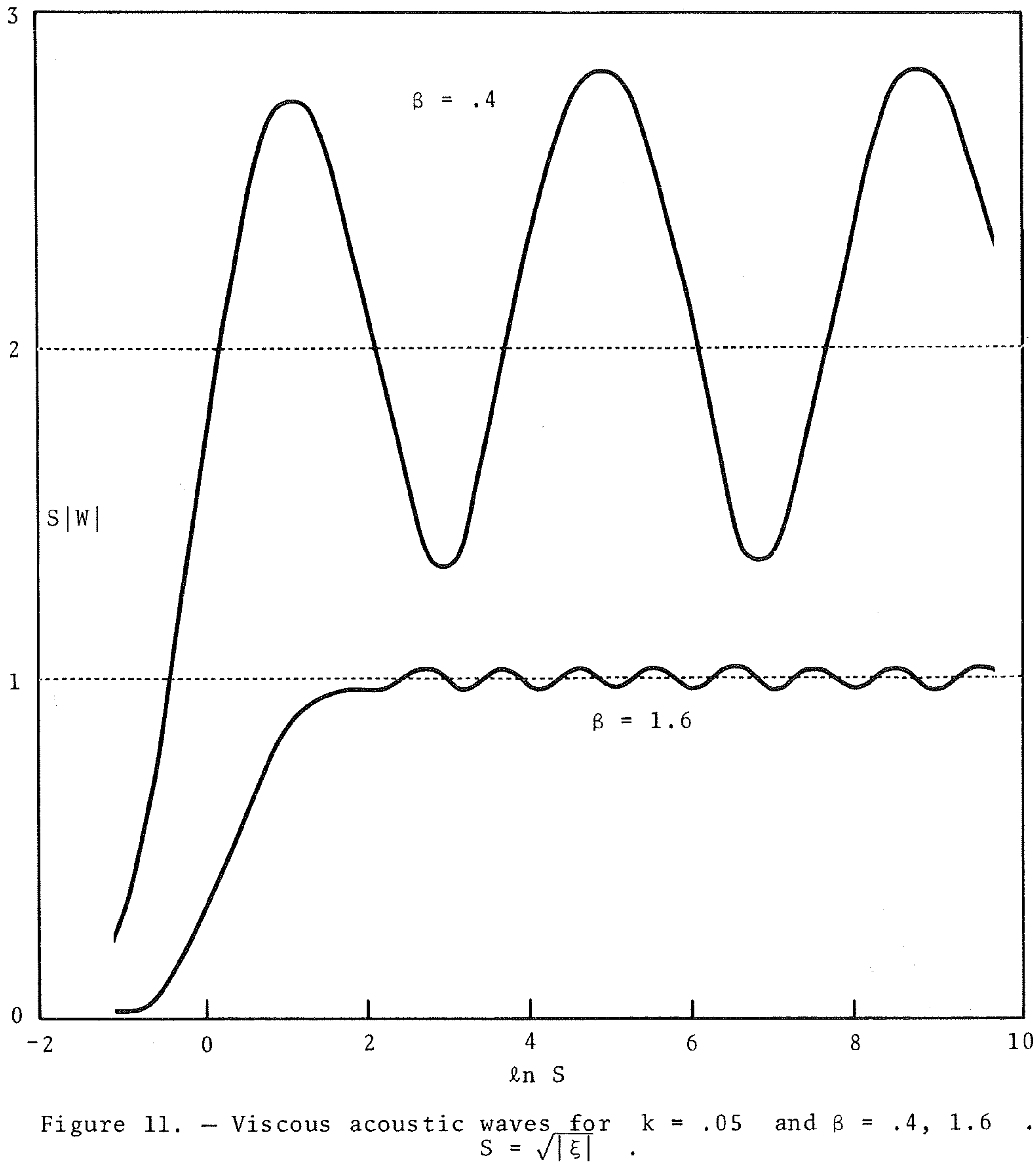

Figure 11. — Viscous acoustic waves for k = .05 and β = .4, 1.6 . $S = \sqrt{|\xi|}$.

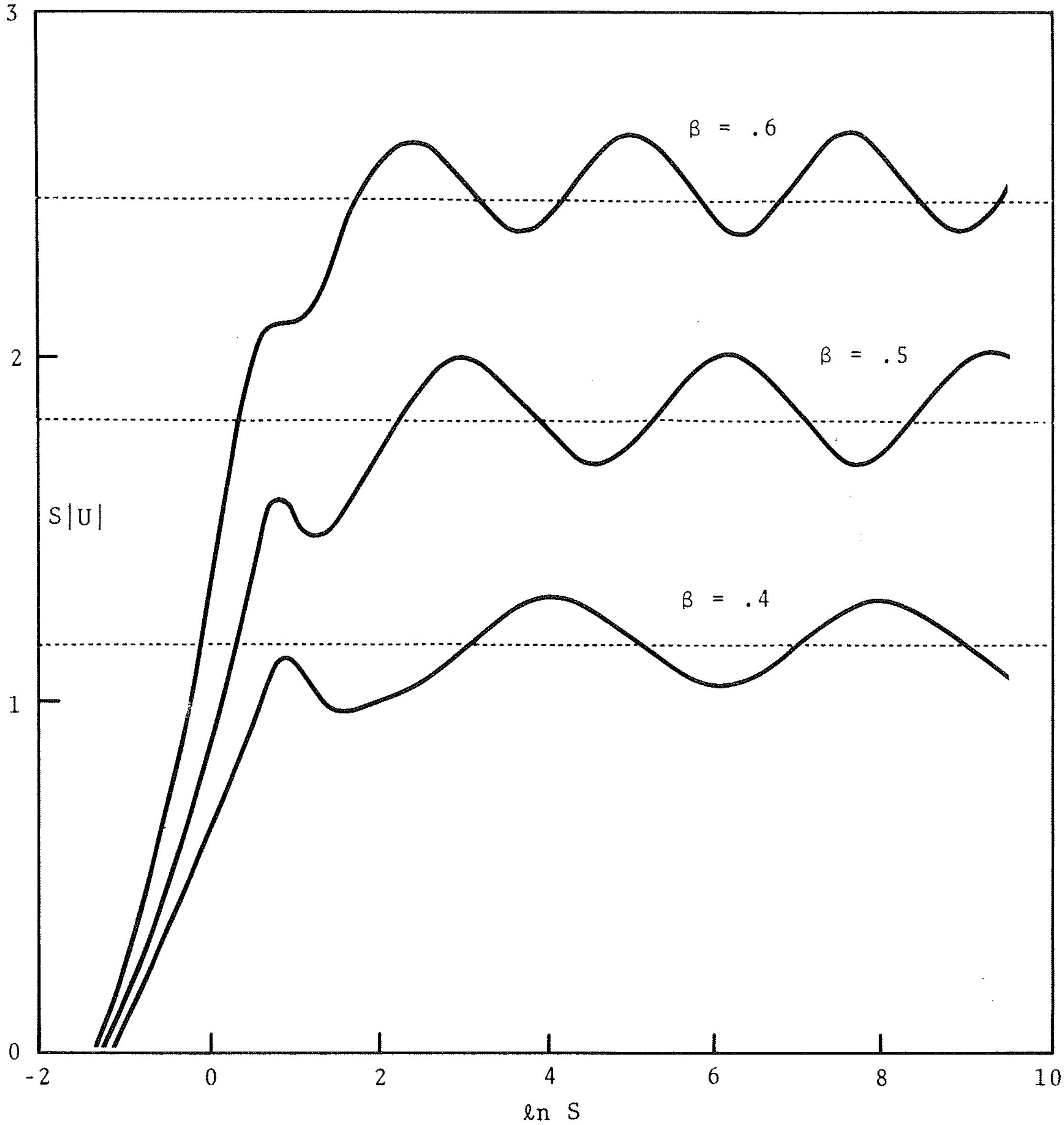

Figure 12. — Viscous acoustic waves for $k = .5$ and $\beta = .4, .5, .6$. $S = \sqrt{|\xi|}$.

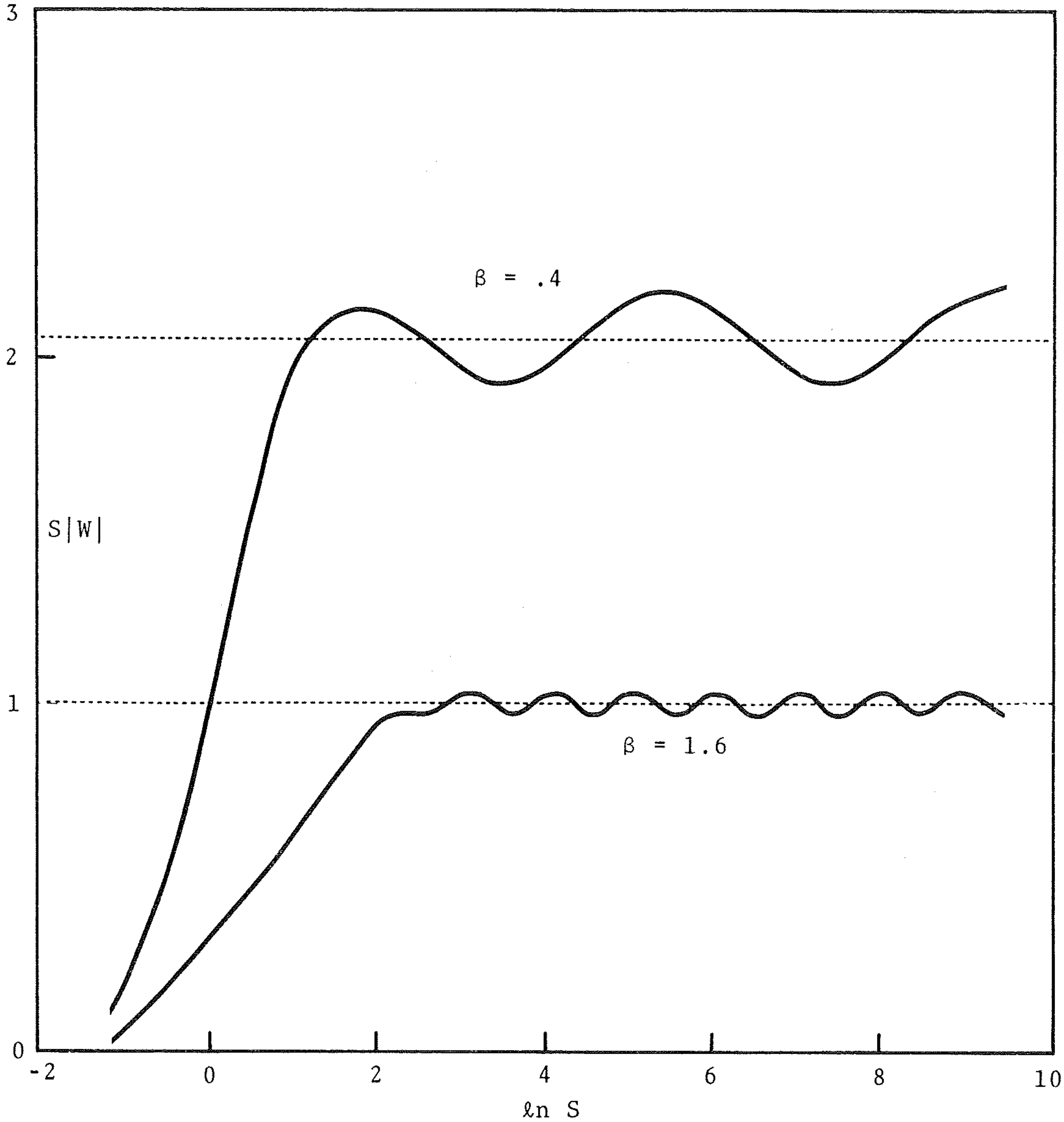

Figure 13. — Viscous acoustic waves for $k = .5$ and $\beta = .4, 1.6$. $S = \sqrt{|\xi|}$.

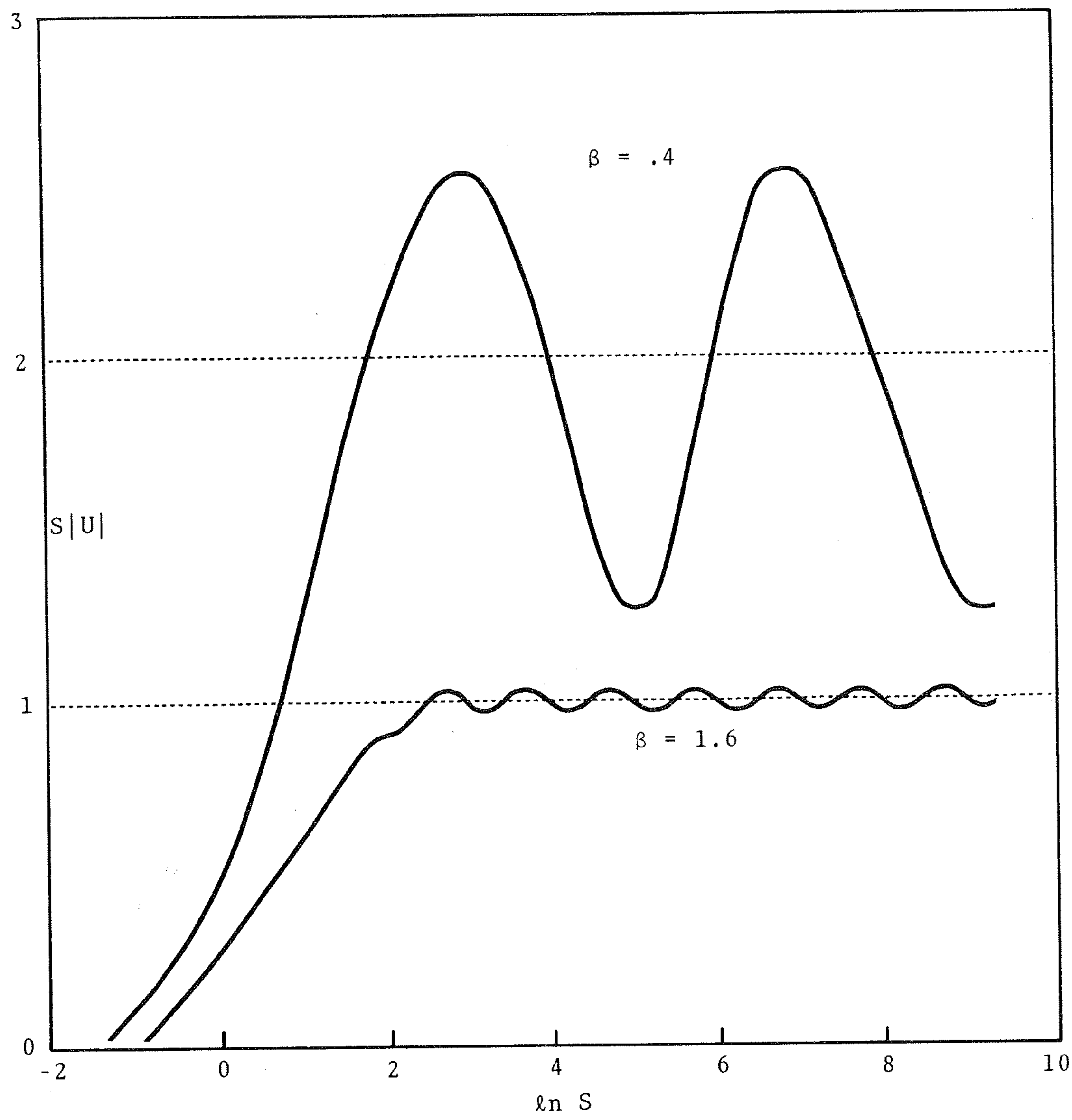

Figure 14. — Viscous acoustic waves for $k = 1.5$ and $\beta = .4, 1.6$. $S = \sqrt{|\xi|}$.

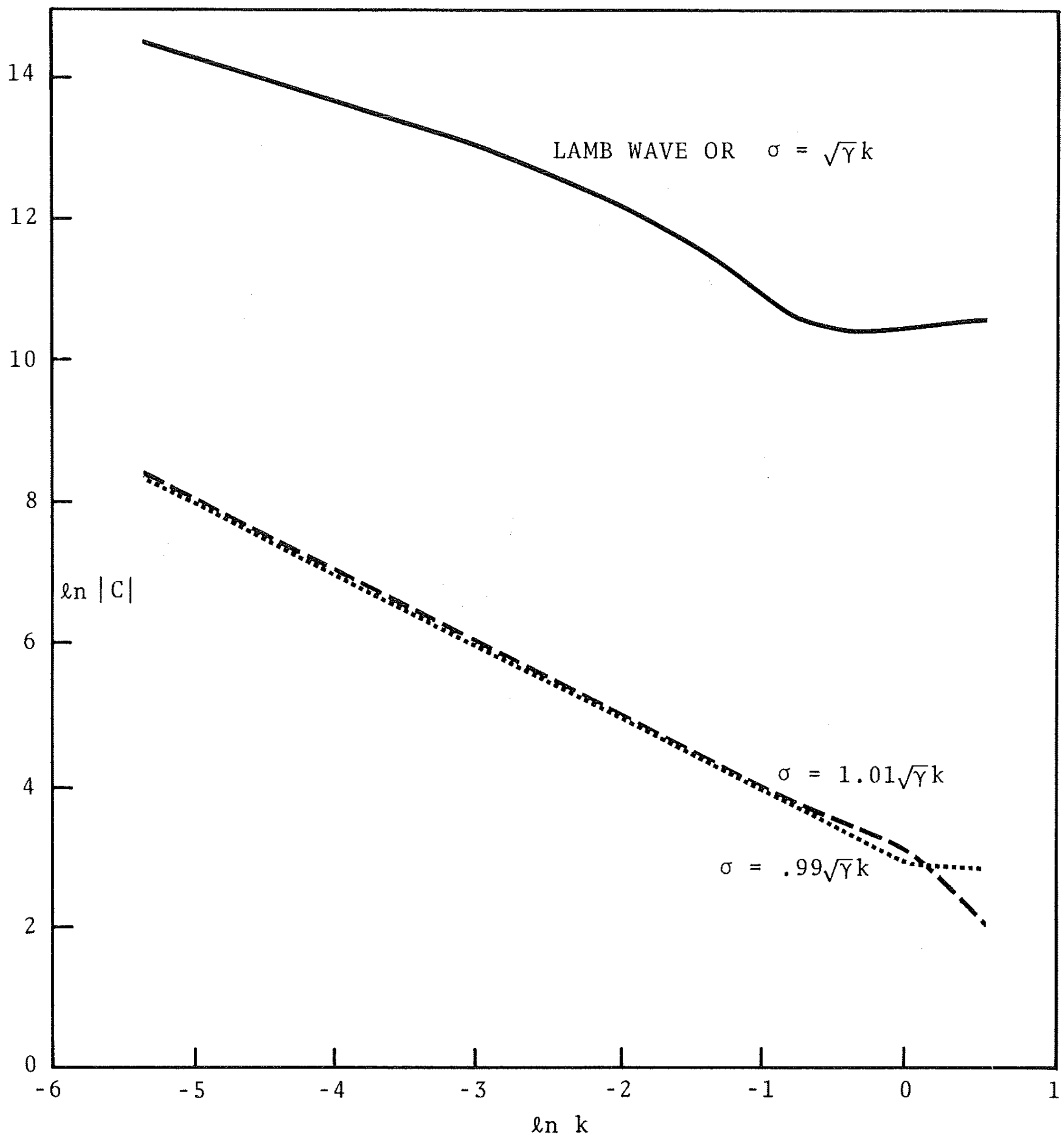

Figure 15. — The resonant Lamb wave and 1 percent variation in the frequency for $\varepsilon = 10^{-11}$ and $\gamma = 1.4$. The scalar C is defined in relation (115).

5.3 CONCLUSIONS

For the case of real distinct roots of the dispersion relation (13) and $\sigma^2/\gamma - k^2 \neq 0$, the solution of the viscous problem approaches a multiple of the inviscid solution with finite kinetic energy in a column of fluid of finite cross section as $\varepsilon \to 0$ or $\mu \to 0$. Moreover, the convergence is uniform on any interval $0 < A \leq z \leq B < \infty$ as $\mu \to 0$ (see Section 5.1, case 1). This conclusion is reasonable and in complete agreement with the results obtained by Yanowitch [2,3]. If $\sigma^2/\gamma - k^2 = 0$ a resonant situation develops and the limiting case, as $\varepsilon \to 0$, results in a viscous solution with a horizontal velocity amplitude which is which is $O\left(\left(\frac{1}{\varepsilon}\right)^{\frac{2-\gamma}{\gamma}}\right)$ near $z = 0$. A nonzero value of viscosity therefore limits the resonant peak (see case 4 in Sections 5.1 and 5.2).

By far the most interesting case concerns complex roots of the dispersion relation, that is, inviscid solutions which are wavelike in z. Due to the results of Yanowitch [2,3] and Lindzen [4] it seems reasonable to expect that

$$|\kappa_R| \approx e^{-\pi\beta} \quad . \tag{125}$$

However, figures 2 through 7 indicate that (125) is not satisfied for all horizontal wavelengths. Thus, the modulus of the reflection coefficient depends on the model selected; for example, the inclusion of compressibility appears to be quite significant.

A careful examination of figures 2 through 7 indicates that $f(k,\beta)$, where

$$f(k,\beta) \quad = \quad \left|\pi\beta + \ell n \left|\kappa_R(k,\beta)\right|\right| \quad , \qquad (126)$$

has a maximum near $k = \beta$. If more general problems are considered, it is probably true that the mathematical model will play a major role in determining $|\kappa_R|$, at least for small values of β .

Several qualitative statements can be made for the case of solutions which are wavelike in z .

a. Reflection is negligible for β large, that is, the radiation condition is substantially correct (refer to discussion of case 2 in Sections 5.1 and 5.2).

b. Reflection is important for β small and the reflection coefficient tends to the limiting value -1 as $\beta \to 0$ for fixed $\varepsilon > 0$ and $k > 0$ (refer to discussion in Section 5.1 and relation (111)).

c. The modulus of the gravity reflection coefficient more nearly equals $e^{-\pi\beta}$ for all values of k than does the corresponding acoustic reflection coefficient for $\gamma = 1.4$ (see figures 2 through 7).

The problem formulated in Section 2 is limited to small oscillations of the lower boundary $z = 0$. An estimate of the validity of the linearization can be obtained by determining the maximum value of $\left|\left|\begin{bmatrix} U(z) \\ W(z) \end{bmatrix}\right|\right|$, where $\begin{bmatrix} U(z) \\ W(z) \end{bmatrix}$ is a solution of the viscous problem. For the cases which were computed, the maximum of $\left|\left|\begin{bmatrix} U(z) \\ W(z) \end{bmatrix}\right|\right|$ satisfied

$$\left|\left|\begin{bmatrix} U(z) \\ W(z) \end{bmatrix}\right|\right| \leq 4 \times 10^5 \left|\left|\begin{bmatrix} U(z_{b.\ell.}) \\ W(z_{b.\ell.}) \end{bmatrix}\right|\right| , \qquad (127)[20]$$

where the boundary layer is given by $0 \leq z \leq z_{b.\ell.}$ and ε is assumed to have the value 10^{-11}.

The kinematic boundary condition, $W(o) = 1$, is equivalent to requiring an amplitude of $\frac{H}{\sigma}$ of the

[20]Estimate (127) is correct for $k = .05,\ 0.5,\ 1.5$, and $\beta = .4$ and $\beta = 1.6$ for both the acoustic and gravity waves. For real distinct roots of the dispersion relation, a considerably smaller bound should result since the inviscid solution with finite kinetic energy in an infinite column of fluid grows less rapidly than $e^{z/2}$ as z increases.

oscillation of the lower boundary. The amplitude of the oscillation scales the solution of the viscous problem. Hence, an oscillation of amplitude 10^{-6} H or 0.7 cm would be more reasonable, that is, consistent with the linearization. Of course the resonant case $\sigma^2/\gamma - k^2 = 0$ would require a much smaller amplitude in order for the linearization to be valid since $|U(z)|$ is very large.

APPENDIX A

THE REGULAR SINGULARITY $\xi = 0$

In this appendix the regular singularity $\xi = 0$ is investigated. Two linearly independent solutions of the viscous differential equation (25) are developed about $\xi = 0$, which satisfy the DC (11). In addition, it is shown that there exist two other solutions of equation (25) which violate the DC.

A.1 THE DC SOLUTIONS

Assume that a fundamental matrix solution of equation (25) can be expressed in the form

$$\Phi(\xi) = \left(\sum_{m=0}^{\infty} S_m \xi^m\right) \times \xi^J \quad , \tag{A1}$$

where

S_m are constant square matrices for $m = 0,1,2,\cdots$ (A2a)

J is the Jordan canonical form of R (defined in relations (24)) (A2b)

[21] In all subsequent contexts the functions $\ell n\xi$ will be defined to be the principal branch.

S_0 is a constant nonsingular matrix such (A2c)
that $S_0 J = RS_0$

$\xi^J = e^{\ell n \xi \times J}$ and $\ell n\xi$ is the principal branch (A2d)
of $\ell n\xi$

In order to construct S_0 the eigenvalues and eigenvectors of R will be determined. It is easily shown that the eigenvalues of R are +k and -k and each eigenvalue is of multiplicity two. Only one-parameter solutions are obtained upon solving

$$(R - kI)\underline{e}_1 = 0 \tag{A3a}$$

and

$$(R + kI)\underline{e}_3 = 0 \quad , \tag{A3b}$$

where

$$\underline{e}_1 = a \times \begin{bmatrix} 1 \\ k \\ 1 \\ k \end{bmatrix} , \quad \text{let} \quad a = 1 \tag{A4a}$$

and

$$\underline{e}_3 = b \times \begin{bmatrix} 1 \\ -k \\ -1 \\ k \end{bmatrix} , \quad \text{let} \quad b = 1 \ . \tag{A4b}$$

Thus, the Jordan canonical form of R is

$$J = \begin{bmatrix} k & 1 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & -k & 1 \\ 0 & 0 & 0 & -k \end{bmatrix} . \tag{A5}$$

In addition to the vectors $\underline{e}_1$ and $\underline{e}_3$ the generalized eigenvectors $\underline{e}_2$ and $\underline{e}_4$ (solutions of (A6)) must also be determined.

$$(R - kI)\underline{e}_2 = \underline{e}_1 \qquad \text{(A6a)}$$

and

$$(R + kI)\underline{e}_4 = \underline{e}_3 \quad . \qquad \text{(A6b)}$$

If the vectors $\underline{e}_2$ and $\underline{e}_4$ are normalized by setting the first components equal to one, then

$$\underline{e}_2 = \begin{bmatrix} 1 \\ k + 1 \\ \frac{k - 7}{k} \\ k - 6 \end{bmatrix} \qquad \text{(A7a)}$$

and

$$\underline{e}_4 = \begin{bmatrix} 1 \\ 1 - k \\ -\left(\frac{k + 7}{k}\right) \\ k + 6 \end{bmatrix} \quad . \qquad \text{(A7b)}$$

The vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, and $\underline{e}_4$ define the matrix S_0, that is,

$$S_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ k & k+1 & -k & 1-k \\ 1 & \frac{k-7}{k} & -1 & -\left(\frac{k+7}{k}\right) \\ k & k-6 & k & k+6 \end{bmatrix} \tag{A8}$$

If (A1) is substituted into (25) and S_0 is defined by (A8), then the recursion relation

$$S_{m+1}[(m+1)I + J] + KS_m[mI + J] = RS_{m+1} + DS_m \tag{A9}$$

is obtained. If $\underline{s}_m^{(j)}$ denotes the *jth* column of S_m, then (A9) can be written as

$$(m + 1 + \ell_j)\underline{s}_{m+1}^{(j)} + \delta_j\underline{s}_{m+1}^{(j-1)} + K\left[(m + \ell_j)\underline{s}_m^{(j)} + \delta_j\underline{s}_m^{(j-1)}\right]$$

$$= R\underline{s}_{m+1}^{(j)} + D\underline{s}_m^{(j)} \quad , \tag{A10}$$

where $\delta_1 = \delta_3 = 0 \quad , \quad \delta_2 = \delta_4 = 1$ (A11a)

and $\ell_1 = \ell_2 = k \quad , \quad \ell_3 = \ell_4 = -k \quad .$ (A11b)

The recursion relation (A10) can be placed in the form

$$\underline{s}_{m+1}^{(j)} = C_m^{(j)}\underline{s}_m^{(j)} + \delta_j D_m^{(j)}\left(\underline{s}_{m+1}^{(j-1)} + K\underline{s}_m^{(j-1)}\right) \quad , \tag{A12}$$

where $$D_m^{(j)} = -[(\ell_j + m + 1)I - R]^{-1} \tag{A13a}$$

and $$C_m^{(j)} = [(\ell_j + m + 1)I - R]^{-1}[D - (m + \ell_j)K] \quad . \tag{A13b}$$

Due to the DC (11) the vector solutions corresponding to $j = 3$ and 4 will be discarded. Therefore, it is only necessary to develop the recursion relation (A12) in detail for $j = 1$ and 2 .

For $j = 1$ or 2 , $\ell_j = k$. Thus, consider

$$[gI - R]^{-1} = \frac{1}{(g^2 - k^2)^2} \begin{bmatrix} \frac{g(3g^2 - 2k^2)}{3} & \frac{4g^2 - 3k^2}{4} & \frac{-k^3}{4} & \frac{-kg}{3} \\ \frac{k^2(4g^2 - 3k^2)}{3} & \frac{g(4g^2 - 3k^2)}{4} & \frac{-k^3 g}{4} & \frac{-kg^2}{3} \\ \frac{k^3}{3} & \frac{kg}{4} & \frac{g(4g^2 - 5k^2)}{4} & \frac{3g^2 - 4k^2}{3} \\ \frac{k^3 g}{3} & \frac{kg^2}{4} & \frac{k^2(3g^2 - 4k^2)}{4} & \frac{g(3g^2 - 4k^2)}{3} \end{bmatrix} \tag{A14a}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{4} \end{bmatrix} \tag{A14b}$$

and

$$[D - (m + k)K] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\sigma^2}{\gamma} - k^2 & 0 & \frac{k}{\gamma} & k \\ 0 & 0 & 0 & 0 \\ \frac{-3(\gamma - 1)k}{4\gamma} & -\frac{3k}{4} & \frac{3\sigma^2}{4\gamma} & \frac{3g}{4} \end{bmatrix} \quad \text{(A14c)}$$

where $$g = k + m + 1 \quad \text{(A14d)}$$

It is clear that the recursion relation (A12) is well defined[22] as long as the matrix $[(\ell_j+m+1)I-R]$ is nonsingular for $m = 0,1,2,\cdots$. Recall that $[-kI-R]$ and $[kI-R]$ are singular matrices since k and $-k$ are the eigenvalues of R . Moreover, k is the only nonnegative eigenvalue of R and, hence, $[(k+m+1)I-R]$ can never be singular for $m = 0,1,2,\cdots$. Thus, the recursion relation (A12) is always well defined for $j = 1$ or 2 .

Quite obviously, if k is an integral multiple of one-half, then the recursion relation (A12) breaks down for $j = 3$ or 4 since $[(-k+m+1)I-R]$ is singular

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[22]The recursion relation (A12) is well defined if the vectors $\underline{s}_o^{(j)}$, for $j = 1,2,3$ and 4 uniquely specify the vectors $\underline{s}_m^{(j)}$ for $j = 1,2,3$ and 4 and all positive integral subscripts m .

when $m = 2k - 1$. This difficulty can be overcome by introducing a shearing-type transformation [8, Chapter 4, Section 4] of the differential equation (25). This tactic is unnecessary since the solutions corresponding to $j = 3$ or 4 are to be discarded because they violate the DC. The approach adopted for $j = 3$ or 4 is one that is used over and over again in asymptotics and is discussed in Section A.2.

Since the recursion relation (A12) is well defined for $j = 1$ or 2 the first two columns of $\Phi(\xi)$, defined in (A1), are solutions of equation (25). Denote the first two columns of $\Phi(\xi)$ by $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$. Thus,

$$\underline{DC}_1(\xi) = \sum_{m=0}^{\infty} \underline{s}_m^{(1)} \xi^{m+k} \qquad \text{(A15a)}$$

and $$\underline{DC}_2(\xi) = \sum_{m=0}^{\infty} \underline{s}_m^{(2)} \xi^{m+k} + (\ln \xi)\underline{DC}_1(\xi) . \qquad \text{(A15b)}$$

As $\xi \to 0$, $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ exhibit the scalar growths ξ^k and $(\ln \xi)\xi^k$ or e^{-kz} and ze^{-kz} as $z \to \infty$. Hence, $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ satisfy the DC (11). The expansions (A15) converge for $0 < |\xi| < 4/3$ since the nearest nonzero singularity occurs at $\xi = 4/3$. The solutions $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ can be analytically continued to the whole ray $\left(\arg \xi = -\frac{\pi}{2}\right)$ since the differential equation (25) has no singularities for $0 < |\xi| < \infty$ and $\arg \xi = -\frac{\pi}{2}$.

It should be noted that in addition to the decay ξ^k and $(\ell n\ \xi)\xi^k$ the solutions $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ are not wavelike in z for sufficiently large z or equivalently for sufficiently small ξ ; that is, the variation in the argument of the components of $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ is bounded in the vicinity of $\xi = 0$. This is a consequence of

$$\arg(\xi^k) = \text{constant} \tag{A16a}$$

and

$$\arg(\ell n\ \xi) \to \text{constant} \tag{A16b}$$

as $\xi \to 0$ for $\arg \xi = -\frac{\pi}{2}$ and

$$\arg\left(\sum_{m=0}^{\infty} \underline{s}_m^{(j)} \xi^m\right) \to \text{constant} \tag{A17}$$

as $\xi \to 0$; that is, the components of $\sum_{m=0}^{\infty} \underline{s}_m^{(j)} \xi^m$ tend to constants as $\xi \to 0$. Thus, in the viscous region the wave motion is damped and not wavelike in z .

A.2 THE SOLUTIONS WHICH VIOLATE THE DC

In this section the vector solutions of (25) which violate the DC are investigated. If k is not an integral multiple of one-half, then the recursion relation

(A12) is well defined for $j = 3$ and 4 . The third and fourth columns of $\Phi(\xi)$ defined in (A1), exhibit the scalar growths ξ^{-k} and $(\ell n\ \xi)\xi^{-k}$ as $\xi \to 0$ or e^{kz} and ze^{kz} as $z \to \infty$. Hence the third and fourth columns of $\Phi(\xi)$ violate the DC (11). If k is an integral multiple of one-half, then the third and fourth columns of $\Phi(\xi)$ cannot be constructed from (A1) since the recursion relation (A12) is not well defined. However, it can still be shown that there exist two solutions, of the differential equation (25), which exhibit the scalar growths ξ^{-k} and $(\ell n\ \xi)\xi^{-k}$ as $\xi \to 0$. Thus, for all values of k there exist precisely two linearly independent solutions which satisfy the DC (11) and two other solutions which violate the DC.

Consider

$$\hat{\Phi}(\xi) = S_0 \xi^J \quad , \tag{A18}$$

where

$$S_0 \xi^J = \left[\underline{e}_1 \xi^k, (\underline{e}_2 + (\ell n\ \xi)\underline{e}_1)\xi^k, \underline{e}_3 \xi^{-k}, (\underline{e}_4 + (\ell n\ \xi)\underline{e}_3)\xi^{-k}\right] \tag{A19}$$

and $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ and $\underline{e}_4$ are the column vectors which define S_0 (see (A8)). It is easily verified that

$$\frac{d}{d\xi} \hat{\Phi}(\xi) = \left(\frac{R}{\xi}\right) \hat{\Phi}(\xi) \quad . \tag{A20}$$

Suppose $\Phi(\xi)$ is a fundamental solution of equation (25), then

$$\frac{d}{d\xi}\,\Phi(\xi) = \left(\frac{R}{\xi} + O(1)\right)\Phi(\xi) \quad . \tag{A21}$$

The symbol O(1) is used in (A21) to denote that the remaining description of the coefficient matrix is bounded as $\xi \to 0$.

Suppose the *jth* column of $\hat{\Phi}(\xi)$ is denoted by $\hat{\underline{\phi}}_j(\xi)$ and the *jth* column of $\Phi(\xi)$ is denoted by $\underline{\phi}_j(\xi)$. If the first two columns of $\Phi(\xi)$ are chosen to be the convergent expansions (A15), then it is easily verified that

$$||\underline{\phi}_j(\xi) - \hat{\underline{\phi}}_j(\xi)|| = o(1) \times ||\hat{\underline{\phi}}_j(\xi)|| \quad \text{as} \quad \xi \to 0 \quad , \tag{A22}$$ [23]

for $j = 1$ or 2 . If it can also be shown that there exist solutions $\underline{\phi}_3(\xi)$ and $\underline{\phi}_4(\xi)$ of equation (25) such that relation (A22) is valid for $j = 3$ and 4 , then it immediately follows that $\underline{\phi}_3(\xi)$ and $\underline{\phi}_4(\xi)$ violate the DC (11). Thus, $\Phi(\xi)$ would consist of two column vectors which satisfy the DC and two solutions of (25) which violate the DC. The remainder of this section is devoted

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[23]The maximum norm is used throughout this paper.

to establishing the existence of solutions $\underline{\phi}_3(\xi)$ and $\underline{\phi}_4(\xi)$ which satisfy (A22).

The method of attack is to treat the differential equation (A21) as though the term $O(1)\times\Phi(\xi)$, on the right-hand side of (A21), is an inhomogenous term of the differential equation. The homogenous part of the right side of (A21) is $\left(\frac{R}{\xi}\right)\Phi(\xi)$, that is, the homogenous differential equation is (A20). If the term $O(1)\times\Phi(\xi)$ were a known function, then a closed-form solution of the differential equation (A21) could be obtained by the method of variation of parameters. Since $O(1)\times\Phi(\xi)$ is not known, an indefinite integral equation is obtained.

Attempt to determine a vector solution $\underline{\phi}(\xi)$, of equation (A21), of the form

$$\underline{\phi}(\xi) = \hat{\Phi}(\xi)\underline{c}(\xi) \quad . \tag{A23}$$

If equation (A23) is substituted into the differential equation (A21), then

$$\underline{\phi}(\xi) = \int^{\xi} \hat{\Phi}(\xi)\hat{\Phi}^{-1}(S)O(1)\underline{\phi}(S)\, dS \tag{A24}$$

is obtained, where $O(1)$ is defined in (A21).

If

$$\hat{\Phi}_1(S) + \hat{\Phi}_2(S) = \hat{\Phi}^{-1}(S) \tag{A25}$$

and $\underline{\phi}_j(\xi)$ is a continuous solution of

$$\underline{\phi}_j(\xi) = \hat{\underline{\phi}}_j(\xi) + \int_0^\xi \hat{\Phi}(\xi)\hat{\Phi}_1(S)O(1)\underline{\phi}_j(S)\, dS$$

$$+ \int_a^\xi \hat{\Phi}(\xi)\hat{\Phi}_2(S)O(1)\underline{\phi}_j(S)\, dS \quad , \qquad (A26)$$

then $\underline{\phi}_j(\xi)$ is a differentiable solution of equation (A21). Let

$$\hat{\Phi}_1(S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S^k & -(\ell n\ S)S^k \\ 0 & 0 & 0 & S^k \end{bmatrix} S_0^{-1} \qquad (A27a)$$

and $$\hat{\Phi}_2(S) = \begin{bmatrix} S^{-k} & -(\ell n\ S)S^{-k} & 0 & 0 \\ 0 & S^{-k} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} S_0^{-1} \quad . \qquad (A27b)$$

Now consider the Picard iteration

$$\underline{\phi}_j^{(0)}(\xi) \equiv \underline{0} \qquad (A28a)$$

and

$$\underline{\phi}_j^{(n+1)}(\xi) = \hat{\underline{\phi}}_j(\xi) + \int_0^\xi \hat{\Phi}(\xi)\hat{\Phi}_1(S)O(1)\underline{\phi}_j^{(n)}(S)\, dS \qquad (A28b)$$

$$+ \int_a^\xi \hat{\Phi}(\xi)\hat{\Phi}_2(S)O(1)\underline{\phi}_j^{(n)}(S)\, dS \quad , \qquad (A28b)$$

where $0 < |\xi| < |a|$ and ξ, a, and S lie on the ray $\arg S = -\frac{\pi}{2}$ and $j = 3$ or 4 .

LEMMA A1: The Picard iteration, defined in (A28), converges for $|a| > 0$ sufficiently small. The convergence is uniform for each compact subset of the line segment $0 < |\xi| \leq |a|$ and $\arg \xi = -\frac{\pi}{2}$. In addition, if $|a| > 0$ is sufficiently small, then the iterants satisfy the inequality

$$||\underline{\phi}_j^{(n+1)}(\xi) - \underline{\phi}_j^{(n)}(\xi)|| \leq \left(\frac{1}{2}\right)^n |\xi|^{-k} m_j(\xi) \quad , \qquad \text{(A29a)}$$

where

$$m_j(\xi) = \begin{cases} ||\underline{e}_3|| & \text{if } j = 3 \\ ||\underline{e}_4|| + |\ell n\ \xi|\,||\underline{e}_4|| & \text{if } j = 4 \end{cases} . \qquad \text{(A29b)}$$

PROOF: (induction)

$$\underline{\phi}_j^{(1)}(\xi) - \underline{\phi}_j^{(o)}(\xi) \equiv \hat{\underline{\phi}}_j(\xi)$$

Thus, for $n = 0$ the relation (A29) is satisfied. The lemma will now be established for all values of n by induction. Assume relation (A29) holds for some $n - 1 \geq -1$ and consider

$$||\underline{\phi}_j^{(n+1)}(\xi) - \underline{\phi}_j^{(n)}(\xi)|| \leq \int_0^\xi M_1\left(\left|\ln \frac{\xi}{S}\right| + 1\right)\left|\frac{S}{\xi}\right|^k ||\underline{\phi}_j^{(n)}(S) - \underline{\phi}_j^{(n-1)}(S)|| \times |dS| + \int_a^\xi M_2\left(\left|\ln \frac{\xi}{S}\right| + 1\right)\left|\frac{\xi}{S}\right|^k ||\underline{\phi}_j^{(n)}(S) - \underline{\phi}_j^{(n-1)}(S)|| \times |dS| \qquad (A30)$$

where M_1 and M_2 are chosen so that

$$||\hat{\Phi}(S)\hat{\Phi}_1(S)O(1)|| \leq M_1\left(\left|\ln \frac{\xi}{S}\right| + 1\right)\left|\frac{S}{\xi}\right|^k ,$$

for $0 < |S| \leq |\xi| \leq |a|$ for some $|a| > 0$.

Similarly

$$||\hat{\Phi}(\xi)\hat{\Phi}_2(S)O(1)|| \leq M_2\left(\left|\ln \frac{\xi}{S}\right| + 1\right)\left|\frac{\xi}{S}\right|^k$$

for $0 < |\xi| \leq |S| \leq |a|$. The inductive hypothesis can be applied to the integrand in (A30). Thus

$$||\underline{\phi}_j^{(n+1)}(\xi) - \underline{\phi}_j^{(n)}(\xi)|| \leq \int_0^\xi M_1\left(\left|\ln \frac{\xi}{S}\right| + 1\right)\left|\frac{S}{\xi}\right|^k \left(\frac{1}{2}\right)^{n-1} |S|^{-k} \times m_j(\xi)|dS| + \int_a^\xi M_2\left(\left|\ln \frac{\xi}{S}\right| + 1\right) \times \left|\frac{\xi}{S}\right|^k \left(\frac{1}{2}\right)^{n-1} |S|^{-k}_{m_j}(\xi)|dS| .$$

(A31)

The first integral $\int_0^\xi$ on the right side of (A31) has a singular integrand (logarithmic growth near $S = 0$). However, a logarithmic singularity is very weak and any power of $\ell n\, S$ is integrable on a finite portion of the ray $\left(\arg S = -\frac{\pi}{2}\right)$ passing through $S = 0$. Thus, the integral from zero to ξ in (A31) satisfies

$$\int_0^\xi = o(1) \times |\xi|^{-k} \times \left(\frac{1}{2}\right)^{n-1} \tag{A32}$$

as $|a| \to 0$.

The integral from a to ξ in (A31) can be bounded by

$$\int_a^\xi \leq \left(\frac{1}{2}\right)^{n-1} M_2 |\xi|^{-k} \int_a^\xi \left(\left|\ell n\, \frac{\xi}{S}\right| + 1\right) \left|\frac{\xi}{S}\right|^k m_j(S)\,|dS| \quad . \tag{A33}$$

However, $\left|\frac{\xi}{S}\right|^k$ tends to zero faster than any power of $\left|\ell n\, \frac{\xi}{S}\right|$ tends to infinity as $\left|\frac{\xi}{S}\right|$ tends to zero. Thus, the integrand on the right side of (A33) is uniformly bounded for $0 < |\xi| \leq |S| \leq |a|$. Therefore,

$$\int_a^\xi = o(1) \times |\xi|^{-k} \times \left(\frac{1}{2}\right)^{n-1} \tag{A34}$$

as $|a| \to 0$.

If $|a| > 0$ is chosen sufficiently small, then relations (A32) and (A34) imply

$$\left|\int_0^\xi + \int_a^\xi\right| \le \left(\frac{1}{2}\right)^n \times |\xi|^{-k} \times m_j(\xi) \quad .$$

NOTE: The value of the constant a can be chosen independent of the superscript n since a must only be chosen sufficiently small to satisfy

$$\frac{1}{2} > M_2\int_a^\xi \left(\left|\ln \frac{\xi}{S}\right| + 1\right)\left|\frac{\xi}{S}\right|^k |dS| + M_1\int_0^\xi \left(\left|\ln \frac{\xi}{S}\right| + 1\right)$$

$$\times\; m_j(\xi)\,|dS|$$

for $0 < |\xi| \le |a|$ and clearly this can be done since the constants M_1 and M_2 and functions $\left|\ln \frac{\xi}{S}\right|$, $\left|\frac{\xi}{S}\right|^k$, and $m_j(\xi)$ do not depend on the superscript n.

Thus, relation (A29) is established by induction, but (A29) implies

$$||\underline{\phi}_j^{(M+1+K)}(\xi) - \underline{\phi}_j^{(M)}(\xi)|| \le \sum_{i=M+1}^{M+K+1} \left(\frac{1}{2}\right)^{i-1} \times |\xi|^{-k} m_j(\xi)$$

since

$$||\underline{\phi}_j^{(M+1+K)}(\xi) - \underline{\phi}_j^{(M)}(\xi)|| \leq ||\underline{\phi}_j^{(M+1+K)}(\xi) - \underline{\phi}_j^{(M+K)}(\xi)|| + ||\underline{\phi}_j^{(M+K)}(\xi) - \underline{\phi}_j^{(M+K-1)}(\xi)|| + \cdots + ||\underline{\phi}_j^{(M+1)}(\xi) - \underline{\phi}_j^{(M)}(\xi)|| \quad .$$

Hence, the Picard iteration converges uniformly in ξ on each compact subset of $0 < |\xi| \leq |a|$ and $\arg \xi = -\frac{\pi}{2}$. Q.E.D.

THEOREM A1: The Picard iterants $\underline{\phi}_3^{(n)}(\xi)$ and $\underline{\phi}_4^{(n)}(\xi)$, defined in relations (A28), converge uniformly to the solutions $\underline{\phi}_3(\xi)$ and $\underline{\phi}_4(\xi)$ of the differential equation (25), respectively, and

$$||\underline{\phi}_j(\xi) - \hat{\underline{\phi}}_j(\xi)|| = o(1) \times ||\hat{\underline{\phi}}_j(\xi)|| \tag{A35}$$

as $\xi \to 0$ and $\arg \xi = -\frac{\pi}{2}$.

PROOF: Since the Picard iterants $\underline{\phi}_j^{(n)}(\xi)$ converge uniformly for each compact subset of $0 < |\xi| \leq |a|$ and $\arg \xi = -\frac{\pi}{2}$, thus the limit function $\underline{\phi}_j(\xi)$, where

$$\underline{\phi}_j(\xi) = \lim_{n\to\infty} \underline{\phi}_j^{(n)}(\xi)$$

is a solution of the differential equation (25) for $0 < |\xi| \leq |a|$. This follows because $\underline{\phi}_j(\xi)$ is a continuous solution of the integral equation (A26) and, hence, is a differentiable solution of the differential equation (25). In addition, the estimate (A29) implies

$$||\underline{\phi}_j(\xi)|| \leq 2|\xi|^{-k} m_j(\xi) \quad ,$$

where $m_j(\xi)$ is defined in lemma A1. Thus,

$$||\underline{\phi}_j(\xi) - \hat{\underline{\phi}}_j(\xi)|| \leq \int_0^\xi 2M_1\left(\left|\ell n \frac{\xi}{S}\right| + 1\right)\left|\frac{S}{\xi}\right|^k |S|^{-k} m_j(S)|dS| + \int_a^\xi 2M_2\left(\left|\ell n \frac{\xi}{S}\right| + 1\right)\left|\frac{\xi}{S}\right|^k |S|^{-k} m_j(S) \cdot |dS| \quad . \qquad \text{(A36)}$$

The integrals on the right side of (A36) must be estimated separately as $\xi \to 0$.

$$\int_0^\xi \leq |\xi|^{-k} \int_0^\xi 2M_1\left(\left|\ell n \frac{\xi}{S}\right| + 1\right) m_j(S)|dS| = o(1) \times |\xi|^{-k}$$

since any power of $\left|\ell n \frac{\xi}{S}\right|$ is integrable near $\frac{\xi}{S} = 0$. The integration from a to ξ on the right side of (A36) is not handled so easily. For $|\xi| > 0$ sufficiently small the following inequality is satisfied:

$$0 < |\xi| < \sqrt[3]{|\xi|} < |a| \quad .$$

Consider $\int_a^\xi$ on the right side of (A36); then

$$\int_a^\xi = \int_a^{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}} + \int_{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}}^\xi ,$$

and

$$\int_a^{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}} \le 2M_2|\xi|^{-2k/3}m_j(\xi)\int_a^{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}}\left(\left|\ell n\,\frac{\xi}{S}\right| + 1\right)$$

$$\times\ |\xi|^k|dS| \quad . \tag{A37}$$

The integrand on the right side of (A37) is bounded as $|\xi| \to 0$ since $|\xi|^k\left(\left|\ell n\,\frac{\xi}{S}\right| + 1\right)$ is bounded. The factor in front of the integral on the right side of (A37) is obviously $o(1)\times|\xi|^{-k}m_j(\xi)$ as $|\xi| \to 0$.

All that remains to be shown is that

$$\int_{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}}^\xi = o(1) \times |\xi|^{-k}m_j(\xi) \quad \text{as} \quad |\xi| \to 0 .$$

$$\int_{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}}^\xi \le 2M_2|\xi|^{-k}m_j(\xi)\int_{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}}^\xi \left(\left|\ell n\,\frac{\xi}{S}\right| + 1\right)\left|\frac{\xi}{S}\right|^{2k}|dS| \quad , \tag{A38}$$

but the integrand on the right side of (A38) is uniformly bounded for $0 < \left|\frac{\xi}{S}\right| \leq 1$. The integrand is integrated over an interval which shrinks to zero as $\xi \to 0$.

Thus,

$$\int_{\sqrt[3]{|\xi|}\frac{\xi}{|\xi|}}^{\xi} = o(1) \times |\xi|^{-k} m_j(\xi) \quad \text{as} \quad |\xi| \to 0 \; .$$

Q.E.D.

APPENDIX B

THE IRREGULAR SINGULARITY $\xi = \infty$

In this appendix the formal asymptotic expansions about $\xi = \infty$ are developed and investigated. Two methods are considered. One method of obtaining asymptotic solutions, referred to as method I, consists of transforming the differential equation (25) until a *guess* can be made concerning the form of a fundamental set of asymptotic expansions. The other method, called method II, avoids the many transformations required in method I. However, method II only determines the asymptotic expansions with algebraic growth in ξ, that is, only the asymptotic expansions with lead terms which correspond to the inviscid solutions are determined. Multiples of the boundary layer and transition layer solutions can be determined by numerical integration. Thus, for large values of ξ a fundamental set of approximate solutions of equation (25) can be obtained by method II.

The proof of the existence of actual solutions of the differential equation (25) which are asymptotic to the formal expansions about the irregular singularity, $\xi = \infty$, follows a rather standard format [8, Chapter 5]. For this reason no attempt will be made to establish the asymptotic nature of the formal expansions, but this property will be used freely.

B.1 FORMAL ASYMPTOTIC EXPANSIONS

In this section the formal expansions of the differential equation (25) about the irregular singularity $\xi = \infty$ are developed for four distinct cases:

CASE 1: The roots of the dispersion relation (13) are distinct. In addition, $2(\lambda_1 - \lambda_2) \neq$ integer and $\sigma^2/\gamma - k^2 \neq 0$.

CASE 2: The roots of the dispersion relation are equal or $\lambda_1 = \lambda_2 = \frac{1}{2}$.

CASE 3: The roots of the dispersion relation equal an integral multiple of one-half.

CASE 4[24]: $\sigma^2/\gamma - k^2 = 0$.

CASE 1: $2(\lambda_1 - \lambda_2) \neq$ integer and $\sigma^2/\gamma - k^2 \neq 0$.

Consider the differential equation (25) which can be written in the form

$$\frac{d\underline{y}(\xi)}{d\xi} = \left(\sum_{n=0}^{\infty} A_n \xi^{-n}\right)\underline{y}(\xi) \quad , \tag{B1}$$

where $\sum_{n=0}^{\infty} A_n \xi^{-n}$ converges for $|\xi| > \frac{4}{3}$ and

[24]Method I is used for cases 1, 2, and 3; method II is used for case 4.

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\sigma^2}{\gamma} - k^2 & 0 & \frac{k}{\gamma} & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \qquad \text{(B2a)}$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{4k^2}{3} & 0 & 0 & -\frac{k}{3} \\ 0 & 0 & 0 & 1 \\ \frac{\gamma - 1}{\gamma} k & k & -\frac{\sigma^2}{\gamma} & -1 \end{bmatrix} , \qquad \text{(B2b)}$$

and for $n \geq 2$

$$A_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \left(\frac{4}{3}\right)^{n-1} \frac{\gamma - 1}{\gamma} k & \left(\frac{4}{3}\right)^{n-2} k & -\left(\frac{4}{3}\right)^{n-2} \left(\frac{4\sigma^2}{3\gamma} + k^2\right) & -\left(\frac{4}{3}\right)^{n-1} \end{bmatrix} .$$

(B2c)

Since $A_0^2 = 0$, equation (B1) falls in the class of differential equations with a nilpotent lead coefficient matrix. The formal asymptotic solutions are not trivially determined for this class of differential equations. One approach is to transform (B1) so that the transformed equation has a form for which an immediate guess can be made concerning the asymptotic solutions [10]. This approach (method I) will now be developed.

The first transformation to be considered is a similarity transformation. Let

$$\underset{\sim}{y}(\xi) = E\tilde{\underset{\sim}{y}}(\xi) \qquad \text{(B3a)}$$

where

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{\gamma}{k}\left(k^2 - \frac{\sigma^2}{\gamma}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{k}\left(k^2 - \frac{\sigma^2}{\gamma}\right) & 0 & \frac{1}{k} \end{bmatrix} \qquad \text{(B3b)}$$

and

$$E^{-1} = \begin{bmatrix} 0 & 0 & \frac{k}{\gamma k^2 - \sigma^2} & 0 \\ 1 & 0 & \frac{k}{\sigma^2 - \gamma k^2} & 0 \\ k^2 - \frac{\sigma^2}{\gamma} & 1 & -\frac{k}{\gamma} & -k \\ \frac{\sigma^2}{\gamma} - k^2 & 0 & \frac{k}{\gamma} & k \end{bmatrix}. \qquad \text{(B3c)}$$

The vector $\tilde{\underset{\sim}{y}}(\xi)$ then satisfies the differential equation

$$\frac{d\tilde{\underset{\sim}{y}}(\xi)}{d\xi} = \left(\sum_{n=0}^{\infty} \tilde{A}_n \xi^{-n}\right) \tilde{\underset{\sim}{y}}(\xi) \quad , \qquad \text{(B4a)}$$

where

$$\tilde{A}_n = E^{-1} A_n E \quad . \tag{B4b}$$

Thus,

$$\tilde{A}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \tag{B5a}$$

$$\tilde{A}_1 = \begin{bmatrix} 0 & \frac{1}{\gamma} & 0 & \frac{1}{\gamma k^2 - \sigma^2} \\ 0 & -\frac{1}{\gamma} & 1 & 1 + \frac{1}{\sigma^2 - \gamma k^2} \\ \frac{4k^2}{3} - \sigma^2\alpha & k^2 + \frac{\sigma^2}{\gamma}\left(\frac{1}{\gamma} - \frac{2}{3}\right) & -\frac{\sigma^2}{\gamma} & \frac{2}{3} - \frac{\sigma^2 + 1}{\gamma} \\ \sigma^2\alpha & \frac{\sigma^2(\gamma - 1)}{\gamma^2} & \frac{\sigma^2}{\gamma} & \frac{\sigma^2 + 1}{\gamma} - 1 \end{bmatrix}$$

(B5b)

and for $n \geq 2$

$$\tilde{A}_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\left(\frac{4}{3}\right)^{n-2}\left\{\frac{4}{3}\frac{\gamma-1}{\gamma}k^2 + (\sigma^2 - \gamma k^2)\left(\frac{4\sigma^2}{3\gamma} + k^2\right)\right. & -\left(\frac{4}{3}\right)^{n-1}\left(\frac{\sigma^2-k^2}{\gamma}\right) & -\left(\frac{4}{3}\right)^{n-2}k^2 & \left(\frac{4}{3}\right)^{n-2}\left(\frac{4}{3} - k^2\right) \\ \left(\frac{4}{3}\right)^{n-2}\left\{\frac{4}{3}\frac{\gamma-1}{\gamma}k^2 + (\sigma^2 - \gamma k^2)\left(\frac{4\sigma^2}{3\gamma} + k^2\right)\right. & \left(\frac{4}{3}\right)^{n-1}\left(\frac{\sigma^2-k^2}{\gamma}\right) & \left(\frac{4}{3}\right)^{n-2}k^2 & \left(\frac{4}{3}\right)^{n-2}\left(k^2 - \frac{4}{3}\right) \end{bmatrix} \tag{B5c}$$

and the scalar α is defined in (13).

Now consider the shearing transformation

$$\tilde{\underset{\sim}{y}}(\xi) = \text{diagonal}\left(1,\ 1,\ 1,\ \frac{1}{\tau}\right)\tilde{\tilde{\underset{\sim}{y}}}(\tau) \quad , \tag{B6a}$$

where

$$\tau = \sqrt{|\xi|}\ e^{i\frac{\arg\ \xi}{2}} \tag{B6b}$$

and

$$\text{diagonal}\left(1,\ 1,\ 1,\ \frac{1}{\tau}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau} \end{bmatrix} \quad , \tag{B6c}$$

The vector $\tilde{\tilde{\underset{\sim}{y}}}(\tau)$ satisfies the differential equation

$$\frac{d\tilde{\tilde{\underset{\sim}{y}}}(\tau)}{d\tau} = \left(\sum_{n=0}^{\infty}\tilde{\tilde{A}}_n\tau^{-n}\right)\tilde{\tilde{\underset{\sim}{y}}}(\tau) \quad , \tag{B7a}$$

where

$$\sum_{n=0}^{\infty} \tilde{\tilde{A}}_n \tau^{-n} = 2\tau \text{ diagonal}(1,1,1,\tau)\left(\sum_{n=0}^{\infty} \tilde{A}_n \tau^{-2n}\right) \cdot \text{diagonal}\left(1,1,1,\frac{1}{\tau}\right) + \text{diagonal}\left(0,0,0,\frac{1}{\tau}\right) . \tag{B7b}$$

Thus,

$$\tilde{\tilde{A}}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2\sigma^2\alpha & \frac{2\sigma^2(\gamma - 1)}{\gamma^2} & \frac{2\sigma^2}{\gamma} & 0 \end{bmatrix} \tag{B8a}$$

$$\tilde{\tilde{A}}_1 = \begin{bmatrix} 0 & \frac{2}{\gamma} & 0 & 0 \\ 0 & -\frac{2}{\gamma} & 2 & 0 \\ \frac{8k^2}{3} & 2\left[k^2 + \frac{\sigma^2}{\gamma}\left(\frac{1}{\gamma} - \frac{2}{3}\right)\right] & -\frac{2\sigma^2}{\gamma} & 0 \\ 0 & 0 & 0 & \frac{2\sigma^2 + 2}{\gamma} - 1 \end{bmatrix} \tag{B8b}$$

$$\tilde{\tilde{A}}_2 = \begin{bmatrix} 0 & 0 & 0 & \dfrac{2}{\gamma k^2 - \sigma^2} \\ 0 & 0 & 0 & 2 + \dfrac{2}{\sigma^2 - \gamma k^2} \\ 0 & 0 & 0 & \dfrac{4}{3} - \dfrac{2\sigma^2 + 2}{\gamma} \\ 2\left[\dfrac{4}{3}\,\dfrac{\gamma-1}{\gamma}\,k^2 + \gamma\left(\dfrac{\sigma^2}{\gamma} - k^2\right)\left(\dfrac{4\sigma^2}{3\gamma} + k^2\right)\right] & \dfrac{8}{3}\left(\dfrac{\sigma^2 - k^2}{\gamma}\right) & 2k^2 & 0 \end{bmatrix}$$

(B8c)

and for $n \geq 2$

$$\tilde{\tilde{A}}_{2n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\left(\frac{4}{3}\right)^{n-2}\left[\frac{4}{3}\,\frac{\gamma-1}{\gamma}\,k^2 + (\sigma^2 - \gamma k^2)\left(\frac{4\sigma^2}{3\gamma} + k^2\right)\right] & -2\left(\frac{4}{3}\right)^{n-1}\left(\frac{\sigma^2-k^2}{\gamma}\right) & -2\left(\frac{4}{3}\right)^{n-2}k^2 & 0 \\ 0 & 0 & 0 & 2\left(\frac{4}{3}\right)^{n-2}\left(k^2 - \frac{4}{3}\right) \end{bmatrix}$$

(B8d)

and

$$\tilde{\tilde{A}}_{2n} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\left(\frac{4}{3}\right)^{n-2}\left(\frac{4}{3} - k^2\right) \\ 2\left(\frac{4}{3}\right)^{n-1}\left[\frac{4}{3}\,\frac{\gamma-1}{\gamma}\,k^2 + (\sigma^2 - \gamma k^2)\left(\frac{4\sigma^2}{3\gamma} + k^2\right)\right] & 2\left(\frac{4}{3}\right)^{n}\left(\frac{\sigma^2-k^2}{\gamma}\right) & 2\left(\frac{4}{3}\right)^{n-1}k^2 & 0 \end{bmatrix}.$$

(B8e)

Now consider another similarity transformation.

Let

$$\tilde{\tilde{y}}(\tau) = F\tilde{\tilde{\tilde{y}}}(\tau) \quad , \qquad \text{(B9a)}$$

where

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -\frac{\gamma^2}{\gamma - 1}\alpha & 0 & 0 & 0 \\ 0 & -\gamma\alpha & \frac{\sqrt{\gamma}}{\sigma} & -\frac{\sqrt{\gamma}}{\sigma} \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \text{(B9b)}$$

and

$$F^{-1} = \begin{bmatrix} 0 & -\frac{\gamma - 1}{\gamma^2\alpha} & 0 & 0 \\ 1 & \frac{\gamma - 1}{\gamma^2\alpha} & 0 & 0 \\ \frac{\sqrt{\gamma}\,\sigma\alpha}{2} & \frac{\sqrt{\gamma}\,\sigma(\gamma - 1)}{2\gamma^2} & \frac{\sigma}{2\sqrt{\gamma}} & \frac{1}{2} \\ -\frac{\sqrt{\gamma}\,\sigma\alpha}{2} & -\frac{\sqrt{\gamma}\,\sigma(\gamma - 1)}{2\gamma^2} & -\frac{\sigma}{2\sqrt{\gamma}} & \frac{1}{2} \end{bmatrix} \qquad \text{(B9c)}$$

then $\tilde{\tilde{\tilde{y}}}(\tau)$ satisfies the differential equation

$$\frac{d\tilde{\tilde{\underline{y}}}(\tau)}{d\tau} = \left(\sum_{n=0}^{\infty} \tilde{\tilde{\tilde{A}}}_n \tau^{-n}\right) \tilde{\tilde{\underline{y}}}(\tau) \quad , \tag{B10a}$$

where

$$\tilde{\tilde{\tilde{A}}}_n = F^{-1} \tilde{\tilde{A}}_n F \quad . \tag{B10b}$$

Thus,

$$\tilde{\tilde{\tilde{A}}}_0 = \text{diagonal}\left(0, 0, \frac{2\sigma}{\sqrt{\gamma}}, -\frac{2\sigma}{\sqrt{\gamma}}\right) \tag{B11a}$$

$$\tilde{\tilde{\tilde{A}}}_1 = \begin{bmatrix} -\frac{2}{\gamma} & \frac{2(\gamma-1)}{\gamma} & \frac{-2\sqrt{\gamma}(\gamma-1)}{\gamma^2\sigma\alpha} & \frac{2\sqrt{\gamma}(\gamma-1)}{\gamma^2\sigma\alpha} \\ \frac{2}{\gamma} - \frac{2\gamma\alpha}{(\gamma-1)} & -\frac{2(\gamma-1)}{\gamma} & \frac{2\sqrt{\gamma}(\gamma-1)}{\gamma^2\sigma\alpha} & \frac{-2\sqrt{\gamma}(\gamma-1)}{\gamma^2\sigma\alpha} \\ -\frac{\gamma^{3/2}\sigma\alpha^2}{\gamma-1} + \frac{\sigma}{\sqrt{\gamma}}\left\{\alpha + \frac{4k^2}{3} - \left[1 + \frac{\left(k^2 + \frac{\sigma^2}{\gamma}\left(\frac{1}{\gamma} - \frac{2}{3}\right)\right)\gamma^2}{\sigma^2(\gamma-1)}\right]\sigma^2\alpha\right\} & \frac{\sigma}{\sqrt{\gamma}}\left(\frac{4k^2}{3} - (\gamma-1)\alpha\right) & \frac{1}{2} & \frac{2\sigma^2+2}{\gamma} - \frac{3}{2} \\ -(3,1)\text{ ELEMENT} & -(3,2)\text{ ELEMENT} & \frac{2\sigma^2+2}{\gamma} - \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

(B11b)

$$\tilde{\tilde{A}}_2 = \begin{bmatrix} 0 & 0 & -2H\,\frac{\gamma-1}{\gamma^2\alpha} & +(1,3)\ \text{ELEMENT} \\ 0 & 0 & \frac{2}{\gamma k^2-\sigma^2} + 2H\,\frac{\gamma-1}{\gamma^2\alpha} & +(2,3)\ \text{ELEMENT} \\ G + \frac{4}{3}\,\frac{\gamma}{\gamma-1}\,(k^2-\sigma^2)\alpha & G - \gamma k^2\alpha & \frac{\sqrt{\gamma}\,k^2}{\sigma} + \frac{\sqrt{\gamma}\,\sigma\alpha}{\gamma k^2-\sigma^2} + H\,\frac{\sqrt{\gamma}\,\sigma(\gamma-1)}{\gamma^2} + \frac{\sigma}{\sqrt{\gamma}}\left(\frac{2}{3} - \frac{\sigma^2+1}{\gamma}\right) & -(4,3)\ \text{ELEMENT} \\ +(3,1)\ \text{ELEMENT} & +(3,2)\ \text{ELEMENT} & \frac{\sqrt{\gamma}\,k^2}{\sigma} - \frac{\sqrt{\gamma}\,\sigma\alpha}{\gamma k^2-\sigma^2} - H\,\frac{\sqrt{\gamma}\,\sigma(\gamma-1)}{\gamma^2} - \frac{\sigma}{\sqrt{\gamma}}\left(\frac{2}{3} - \frac{\sigma^2+1}{\gamma}\right) & -(3,3)\ \text{ELEMENT} \end{bmatrix}$$

(B11c)

and for $n \geq 2$

$$\tilde{\tilde{A}}_{2n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\sigma}{\sqrt{\gamma}}\left[-\left(\frac{4}{3}\right)^{n-2} G + \left(\frac{4}{3}\right)^{n-1}\frac{(\sigma^2-k^2)\gamma\alpha}{(\gamma-1)}\right] & \frac{\sigma}{\sqrt{\gamma}}\left(\frac{4}{3}\right)^{n-2}[k^2\gamma\alpha - G] & -\left(\frac{4}{3}\right)^{n-1} & \left(\frac{4}{3}\right)^{n-2}\left(2k^2 - \frac{4}{3}\right) \\ -(3,1)\ \text{ELEMENT} & -(3,2)\ \text{ELEMENT} & \left(\frac{4}{3}\right)^{n-2}\left(2k^2 - \frac{4}{3}\right) & -\left(\frac{4}{3}\right)^{n-1} \end{bmatrix}$$

(B11d)

$$\tilde{\tilde{A}}_{2n} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \left(\frac{4}{3}\right)^{n-1} G - \left(\frac{4}{3}\right)^{n} \frac{(\sigma^2 - k^2)\gamma\alpha}{\gamma - 1} & \left(\frac{4}{3}\right)^{n-1} [G - k^2\gamma\alpha] & \left(\frac{4}{3}\right)^{n-2} \left\{ \left(\frac{4}{3} - k^2\right) \frac{\sigma}{\sqrt{\gamma}} + \frac{4}{3} \frac{k^2\sqrt{\gamma}}{\sigma} \right\} & -(4,3) \text{ ELEMENT} \\ +(3,1) \text{ ELEMENT} & +(3,2) \text{ ELEMENT} & \left(\frac{4}{3}\right)^{n-2} \left\{ \left(k^2 - \frac{4}{3}\right) \frac{\sigma}{\sqrt{\gamma}} + \frac{4}{3} \frac{k^2\sqrt{\gamma}}{\sigma} \right\} & -(3,3) \text{ ELEMENT} \end{bmatrix} \tag{B11e}$$

where

$$G = \frac{4}{3} \frac{\gamma - 1}{\gamma} k^2 + (\sigma^2 - \gamma k^2)\left(\frac{4\sigma^2}{3\gamma} + k^2\right) \tag{B11f}$$

$$H = 1 + \frac{1}{\sigma^2 - \gamma k^2} \ . \tag{B11g}$$

Sibuya-type transformations are now considered. The purpose of these transformations is to reduce the off-diagonal elements of the coefficient matrix to zero. This procedure will be modified. Only the off-diagonal elements of the matrix associated with $\frac{1}{\tau}$ will be reduced to zero. This is a simpler task and ensures the convergence of the transformed coefficient matrix [10, Chapter IV].

Let

$$\underset{\sim}{\tilde{\tilde{y}}}(\tau) = P(\tau)\underline{X}(\tau) \quad , \tag{B12a}$$

where

$$P(\tau) = P_0 + \frac{P_1}{\tau} , \tag{B12b}$$

$$P_0 = I \qquad \text{(identity matrix)}, \tag{B12c}$$

and

$$P_1 = \begin{bmatrix} 0 & 0 & -\left(\frac{\gamma-1}{\gamma\sigma^2\alpha}\right) & -\left(\frac{\gamma-1}{\gamma\sigma^2\alpha}\right) \\ 0 & 0 & \left(\frac{\gamma-1}{\gamma\sigma^2\alpha}\right) & \left(\frac{\gamma-1}{\gamma\sigma^2\alpha}\right) \\ \frac{\gamma^2\alpha^2}{2(\gamma-1)} - \frac{1}{2}\left\{\alpha + \frac{4k^2}{3} - \left[1 + \frac{\left(k^2 + \frac{\sigma^2}{\gamma}\left(\frac{1}{\gamma} - \frac{2}{3}\right)\right)\gamma^2}{\sigma^2(\gamma-1)}\right]\sigma^2\alpha\right\} & \frac{(\gamma-1)\alpha}{2} - \frac{2k^2}{3} & 0 & 0 \\ +(3,1)\ \text{ELEMENT} & \frac{(\gamma-1)\alpha}{2} - \frac{2k^2}{3} & 0 & 0 \end{bmatrix},$$

(B12d)

then $\underline{X}(\tau)$ satisfies the differential equation

$$\frac{d\underline{X}(\tau)}{d\tau} = \left(\sum_{n=0}^{\infty} C_n \tau^{-n}\right)\underline{X}(\tau) , \tag{B13a}$$

where

$$C_0 = \text{diagonal}\left(0,\ 0,\ \frac{2\sigma}{\sqrt{\gamma}},\ -\frac{2\sigma}{\sqrt{\gamma}}\right) , \tag{B13b}$$

$$C_1 = \begin{bmatrix} -\frac{2}{\gamma} & \frac{2(\gamma-1)}{\gamma} & 0 & 0 \\ \frac{2}{\gamma} - \frac{2\gamma\alpha}{\gamma-1} & \frac{-2(\gamma-1)}{\gamma} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{2\sigma^2+2}{\gamma} - \frac{3}{2} \\ 0 & 0 & \frac{2\sigma^2+2}{\gamma} - \frac{3}{2} & \frac{1}{2} \end{bmatrix} , \tag{B13c}$$

$$C_2 = \tilde{\tilde{A}}_2 + \tilde{\tilde{A}}_1 P_1 - P_1 C_1 + P_1 , \tag{B13d}$$

and for $n \geq 3$

$$C_n = \tilde{\tilde{A}}_n + \tilde{\tilde{A}}_{n-1} P_1 - P_1 C_{n-1} . \tag{B13e}$$

One additional transformation, a combination of a similarity and a Sibuya transformation, reduces C_1 to diagonal form and leaves C_0 unchanged. Consider

$$\underline{X}(\tau) = \tilde{P}(\tau)\underline{Z}(\tau) , \tag{B14a}$$

where

$$\tilde{P}(\tau) = \tilde{P}_0 + \frac{\tilde{P}_1}{\tau} , \tag{B14b}$$

$$\tilde{P}_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \dfrac{1 - \lambda_1\gamma}{\gamma - 1} & \dfrac{1 - \lambda_2\gamma}{\gamma - 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{(B14c)}$$

$$\tilde{P}_0^{-1} = \begin{bmatrix} \dfrac{1 - \lambda_2\gamma}{\gamma(\lambda_1 - \lambda_2)} & \dfrac{\gamma - 1}{\gamma(\lambda_2 - \lambda_1)} & 0 & 0 \\ \dfrac{\lambda_1\gamma - 1}{\gamma(\lambda_1 - \gamma_2)} & \dfrac{\gamma - 1}{\gamma(\lambda_1 - \lambda_2)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{(B14d)}$$

and

$$\tilde{P}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\sqrt{\gamma}}{4\sigma}\left(\dfrac{3}{2} - \dfrac{2\sigma^2 + 2}{\gamma}\right) \\ 0 & 0 & \dfrac{\sqrt{\gamma}}{4\sigma}\left(\dfrac{2\sigma^2 + 2}{\gamma} - \dfrac{3}{2}\right) & 0 \end{bmatrix}. \qquad \text{(B14e)}$$

RECALL: λ_1 and λ_2 are the roots of the dispersion relation (13).

Thus,

$$\frac{d\underline{Z}(\tau)}{d\tau} = \left(\sum_{n=0}^{\infty} B_n \tau^{-n}\right) \underline{Z}(\tau) \quad , \tag{B15}$$

where

$$B_0 = \text{diagonal}\left(0,\ 0,\ \frac{2\sigma}{\sqrt{\gamma}},\ -\frac{2\sigma}{\sqrt{\gamma}}\right) \quad , \tag{B16a}$$

$$B_1 = \text{diagonal}\left(-2\lambda_1,\ -2\lambda_2,\ \frac{1}{2},\ \frac{1}{2}\right) \quad , \tag{B16b}$$

$$B_2 = \tilde{P}_0^{-1}\{C_2\tilde{P}_0 + C_1\tilde{P}_1 - \tilde{P}_1 B_1 + \tilde{P}_1\} \quad , \tag{B16c}$$

and for $n \geq 3$

$$B_n = \tilde{P}_0^{-1}\{C_n\tilde{P}_0 + C_{n-1}\tilde{P}_1 - \tilde{P}_1 B_{n-1}\} \quad . \tag{B16d}$$

For sufficiently large $|\tau|$ the expansion $\sum_{n=0}^{\infty} B_n \tau^{-n}$ converges [10, pp. 54,55]. It is not important for the present analysis to determine how large $|\tau|$ must be to ensure convergence. It should be noted that the transformations E, F, diagonal $\left(1, 1, 1, \frac{1}{\tau}\right)$, $P(\tau)$, and $\tilde{P}(\tau)$ are nonsingular for $\lambda_1 \neq \lambda_2$, $\frac{\sigma^2}{\gamma} - k^2 \neq 0$, and $|\tau|$ sufficiently large.

The differential equation (B15) can be used to develop formal expansions about $\xi = \infty$. The matrices B_0 and B_1 suggest attempting solutions of the form

$$\hat{\underline{Z}}_{1,L}(\tau) = \sum_{n=0}^{L} \underline{p}_n \tau^{-(n+2\lambda_1)} \tag{B17a}$$

$$\hat{\underline{Z}}_{2,L}(\tau) = \sum_{n=0}^{L} \underline{q}_n \tau^{-(n+2\lambda_2)} \tag{B17b}$$

$$\hat{\underline{Z}}_{3,L}(\tau) = \sum_{n=0}^{L} \underline{r}_n \tau^{\frac{1}{2}-n} e^{\frac{2\sigma\tau}{\sqrt{\gamma}}} , \tag{B17c}$$

and

$$\hat{\underline{Z}}_{4,L}(\tau) = \sum_{n=0}^{L} \underline{s}_n \tau^{\frac{1}{2}-n} e^{-\frac{2\sigma\tau}{\sqrt{\gamma}}} , \tag{B17d}$$

where $L = \infty$. For a finite value of L the formal truncated solutions are obtained. NOTE: The circumflex ($\hat{\ }$) is introduced to denote that $\hat{\underline{Z}}_{i,\infty}(\tau)$ is only a formal expansion, that is, the full infinite expansions may diverge. A vector $\underline{Z}(\tau)$ without the circumflex will denote an actual solution of (B15).

If formal solutions (B17) are substituted into the differential equation (B15), then the following relations are obtained:

$$\underline{p}_0 = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \tag{B18a}$$

$$\underline{q}_0 = C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \qquad \text{(B18b)}$$

$$\underline{r}_0 = C_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \qquad \text{(B18c)}$$

and

$$\underline{s}_0 = C_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \qquad \text{(B18d)}$$

If the constants C_i in (B18) are all set equal to one, then the lead asymptotic vectors, $\underline{p}_0$, $\underline{q}_0$, $\underline{r}_0$, and $\underline{s}_0$, recursively determine the remaining vectors, $\underline{p}_n$, $\underline{q}_n$, $\underline{r}_n$, and $\underline{s}_n$, for $n = 1,2,\ldots$.

If

$$\underline{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{p}_n = \begin{bmatrix} {}_1p_n \\ {}_2p_n \\ {}_3p_n \\ {}_4p_n \end{bmatrix} , \qquad \text{(B19a)}$$

$$\underline{q}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{q}_n = \begin{bmatrix} {}_1q_n \\ {}_2q_n \\ {}_3q_n \\ {}_4q_n \end{bmatrix}, \qquad \text{(B19b)}$$

$$\underline{r}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{r}_n = \begin{bmatrix} {}_1r_n \\ {}_2r_n \\ {}_3r_n \\ {}_4r_n \end{bmatrix}, \qquad \text{(B19c)}$$

and $$\underline{s}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{s}_n = \begin{bmatrix} {}_1s_n \\ {}_2s_n \\ {}_3s_n \\ {}_4s_n \end{bmatrix}, \qquad \text{(B19d)}$$

then

$${}_1p_n = -\frac{{}_1\pi_{n-1}}{n}, \qquad \text{(B20a)}$$

$${}_2p_n = \frac{{}_2\pi_{n-1}}{2\lambda_2 - (2\lambda_1 + n)}, \qquad \text{(B20b)}$$

$$ {}_3p_{n+1} = \frac{-\sqrt{\gamma}}{2\sigma}\left\{{}_3\pi_{n-1} + \left(2\lambda_1 + n + \frac{1}{2}\right){}_3p_n\right\} , \quad \text{(B20c)} $$

$$ {}_4p_{n+1} = \frac{\sqrt{\gamma}}{\sigma}\left\{{}_4\pi_{n-1} + \left(2\lambda_1 + n + \frac{1}{2}\right){}_4p_n\right\} \quad \text{(B20d)} $$

where

$$ \underline{\pi}_{n-1} = \begin{bmatrix} {}_1\pi_{n-1} \\ {}_2\pi_{n-1} \\ {}_3\pi_{n-1} \\ {}_4\pi_{n-1} \end{bmatrix} = B_2\underline{p}_{n-1} + \cdots + B_{n+1}\underline{p}_0 , \quad \text{(B20e)} $$

$$ {}_1q_n = \frac{{}_1\rho_{n-1}}{2\lambda_1 - (2\lambda_2 + n)} , \quad \text{(B21a)} $$

$$ {}_2q_n = -\frac{{}_2\rho_{n-1}}{n} , \quad \text{(B21b)} $$

$$ {}_3q_n = \frac{-\sqrt{\gamma}}{2\sigma}\left\{{}_3\rho_{n-1} + \left(2\dot{\lambda}_2 + n + \frac{1}{2}\right){}_3q_n\right\} , \quad \text{(B21c)} $$

$$ {}_4q_n = \frac{\sqrt{\gamma}}{2\sigma}\left\{{}_4\rho_{n-1} + \left(2\lambda_2 + n + \frac{1}{2}\right){}_4q_n\right\} , \quad \text{(B21d)} $$

where

$$\underline{\rho}_{n-1} = \begin{bmatrix} {}_1\rho_{n-1} \\ {}_2\rho_{n-1} \\ {}_3\rho_{n-1} \\ {}_4\rho_{n-1} \end{bmatrix} = B_2\underline{q}_{n-1} + \cdots + B_{n+1}\underline{q}_0 \ , \tag{B21e}$$

$${}_1r_{n+1} = \frac{\sqrt{\gamma}}{2\sigma}\left\{ {}_1\nu_{n-1} + \left[n - \left(2\lambda_1 + \frac{1}{2}\right)\right]{}_1r_n \right\} \ , \tag{B22a}$$

$${}_2r_{n+1} = \frac{\sqrt{\gamma}}{2\sigma}\left\{ {}_2\nu_{n-1} + \left[n - \left(2\lambda_2 + \frac{1}{2}\right)\right]{}_2r_n \right\} \ , \tag{B22b}$$

$${}_3r_n = \frac{-{}_3\nu_{n-1}}{n} \ , \tag{B22c}$$

$${}_4r_{n+1} = \frac{\sqrt{\gamma}}{4\sigma}\left\{ {}_4\nu_{n-1} + n \times {}_4r_n \right\} \ , \tag{B22d}$$

where

$$\underline{\nu}_{n-1} = \begin{bmatrix} {}_1\nu_{n-1} \\ {}_2\nu_{n-1} \\ {}_3\nu_{n-1} \\ {}_4\nu_{n-1} \end{bmatrix} = B_2\underline{r}_{n-1} + \cdots + B_{n+1}\underline{r}_0 \ , \tag{B22e}$$

$${}_1s_{n+1} = \frac{-\sqrt{\gamma}}{2\sigma}\left\{ {}_1\psi_{n-1} + \left[n - \left(2\lambda_1 + \frac{1}{2}\right)\right]{}_1s_n \right\} \ , \tag{B23a}$$

$$ {}_2s_{n+1} = -\frac{\sqrt{\gamma}}{2\sigma}\left\{ {}_2\psi_{n-1} + \left[n - \left(2\lambda_2 + \frac{1}{2}\right)\right] {}_2s_n \right\} \quad , \qquad \text{(B23b)} $$

$$ {}_3s_{n+1} = -\frac{\sqrt{\gamma}}{4\sigma}\left\{ {}_3\psi_{n-1} + n \times {}_2s_n \right\} \quad , \qquad \text{(B23c)} $$

$$ {}_4s_n = \frac{-{}_4\psi_{n-1}}{n} \quad , \qquad \text{(B23d)} $$

where

$$ \underline{\psi}_{n-1} = \begin{bmatrix} {}_1\psi_{n-1} \\ {}_2\psi_{n-1} \\ {}_3\psi_{n-1} \\ {}_4\psi_{n-1} \end{bmatrix} = B_2\underline{s}_{n-1} + \cdots + B_{n+1}\underline{s}_0 \quad . \qquad \text{(B23e)} $$

The formal truncated solutions satisfy the differential equation (B15) approximately. More precisely,

$$ \frac{d\hat{\underline{Z}}_{i,L}(\tau)}{d\tau} = \left\{\left(\sum_{n=0}^{L} B_n \tau^{-n}\right) + O\left(\frac{1}{\tau^{L+1}}\right)\right\} \hat{\underline{Z}}_{i,2}(\tau) \quad . \qquad \text{(B24)} $$

The recursion relations (B20) through (B23) are well defined if $2(\lambda_1 - \lambda_2) \neq$ integer . In addition the transformations E, F, diagonal $\left(1, 1, 1, \frac{1}{\tau}\right)$, $P(\tau)$, and $\tilde{P}(\tau)$ require that $\frac{\sigma^2}{\gamma} - k^2 \neq 0$ and $|\tau|$ be sufficiently large in order for the differential equation (B15) to have a convergent expansion for its coefficient matrix.

CASE 2: The roots of the dispersion relation (13) are repeated. This is equivalent to requiring $\alpha = \frac{1}{4}$ in relation (13). The transformations E, diagonal $\left(1, 1, 1, \frac{1}{\tau}\right)$, F, and $P(\tau)$ developed for case 1 can be used for this case also. However, the transformation $\tilde{P}(\tau)$ must be modified since $\lambda_1 = \lambda_2 = \frac{1}{2}$.

Let

$$\tilde{P}(\tau) = \tilde{P}_0 + \frac{\tilde{P}_1}{\tau} , \tag{B25a}$$

where

$$\tilde{P}_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \frac{2-\gamma}{2(\gamma-1)} & \frac{1}{\gamma-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{B25b}$$

$$\tilde{P}_0^{-1} = \begin{bmatrix} \frac{2}{\gamma} & \frac{-2(\gamma-1)}{\gamma} & 0 & 0 \\ \frac{-(2-\gamma)}{\gamma} & \frac{2(\gamma-1)}{\gamma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{B25c}$$

and

$$\tilde{P}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\gamma}}{4\sigma}\left(\frac{3}{2} - \frac{2\sigma^2 + 2}{\gamma}\right) \\ 0 & 0 & \frac{\sqrt{\gamma}}{4\sigma}\left(\frac{2\sigma^2 + 2}{\gamma} - \frac{3}{2}\right) & 0 \end{bmatrix} . \tag{B25d}$$

Consider

$$\underline{X}(\tau) = \tilde{P}(\tau)\underline{Z}(\tau) \quad ; \tag{B26}$$

then

$$\frac{d\underline{Z}(\tau)}{d\tau} = \left(\sum_{n=0}^{\infty} B_n \tau^{-n}\right)\underline{Z}(\tau) \quad , \tag{B27}$$

where

$$B_0 = \text{diagonal}\left(0,\ 0,\ \frac{2\sigma}{\sqrt{\gamma}},\ -\frac{2\sigma}{\sqrt{\gamma}}\right) \quad , \tag{B28a}$$

$$B_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} , \tag{B28b}$$

$$B_2 = \tilde{P}_0^{-1}\{C_2\tilde{P}_0 + C_1\tilde{P}_1 - \tilde{P}_1B_1 + \tilde{P}_1\} \quad , \tag{B28c}$$

and for $n \geq 3$

$$B_n = \tilde{P}_0^{-1}\{C_n\tilde{P}_0 + C_{n-1}\tilde{P}_1 - \tilde{P}_1B_{n-1}\} \quad . \tag{B28d}$$

The rather simple form of B_0 and B_1 suggests attempting formal solutions

$$\hat{\underline{Z}}_{1,\infty}(\tau) = \sum_{n=0}^{\infty} \underline{p}_n \tau^{-(n+1)} \quad , \tag{B29a}$$

$$\hat{\underline{Z}}_{2,\infty}(\tau) = \sum_{n=0}^{\infty} \underline{q}_n \tau^{-(n+1)} + (\ell n\ \tau)\hat{\underline{Z}}_{1,\infty}(\tau) \quad , \tag{B29b}$$

$$\hat{\underline{Z}}_{3,\infty}(\tau) = \sum_{n=0}^{\infty} \underline{r}_n \tau^{\frac{1}{2}-n} e^{\frac{2\sigma\tau}{\sqrt{\gamma}}} \quad , \tag{B29c}$$

and

$$\hat{\underline{Z}}_{4,\infty}(\tau) = \sum_{n=0}^{\infty} \underline{s}_n \tau^{\frac{1}{2}-n} e^{-\frac{2\sigma\tau}{\sqrt{\gamma}}} \quad . \tag{B29d}$$

If $\alpha = \frac{1}{4}$ and expansions (B29) are formally substituted into (B27), then the following relations are obtained:

$$\underline{p}_0 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \tag{B30a}$$

$$\underline{q}_0 = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2' \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \qquad \text{(B30b)}$$

$$\underline{r}_0 = c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \qquad \text{(B30c)}$$

and

$$\underline{s}_0 = c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \qquad \text{(B30d)}$$

If the scalars c_i in (B30) are set equal to one and $c_2' = 0$, then the following recursion relations are obtained:

$$_2p_n = -\frac{_2\pi_{n-1}}{n} , \qquad \text{(B31a)}$$

$$_1p_n = -\left\{\frac{_1\pi_{n-1} + {_2p_n}}{n}\right\} , \qquad \text{(B31b)}$$

$$_3p_{n+1} = -\frac{\sqrt{\gamma}}{2\sigma} \left\{ {_3\pi_{n-1}} + \left(n + \frac{3}{2}\right) {_3p_n} \right\} , \qquad \text{(B31c)}$$

$$_4p_{n+1} = \frac{\sqrt{\gamma}}{2\sigma} \left\{ {_4\pi_{n-1}} + \left(n + \frac{3}{2}\right) {_4p_n} \right\} , \qquad \text{(B31d)}$$

where

$$\underline{\pi}_{n-1} = \begin{bmatrix} {}_1\pi_{n-1} \\ {}_2\pi_{n-1} \\ {}_3\pi_{n-1} \\ {}_4\pi_{n-1} \end{bmatrix} = B_2\underline{p}_{n-1} + \cdots + B_{n+1}\underline{p}_0 \quad , \qquad \text{(B31e)}$$

$${}_2q_n = -\left\{\frac{{}_2\rho_{n-1} - {}_2p_n}{n}\right\} \quad , \qquad \text{(B32a)}$$

$${}_1q_n = -\left\{\frac{{}_1\rho_{n-1} + {}_2q_n - {}_1p_n}{n}\right\} \quad , \qquad \text{(B32b)}$$

$${}_3q_{n+1} = -\frac{\sqrt{\gamma}}{2\sigma}\left\{\left(n + \frac{3}{2}\right){}_3q_n - {}_3p_n + {}_3\rho_{n-1}\right\} \quad , \qquad \text{(B32c)}$$

$${}_4q_{n+1} = \frac{\sqrt{\gamma}}{2\sigma}\left\{\left(n + \frac{3}{2}\right){}_4q_n - {}_4p_n + {}_4\rho_{n-1}\right\} \quad , \qquad \text{(B32d)}$$

where

$$\underline{\rho}_{n-1} = \begin{bmatrix} {}_1\rho_{n-1} \\ {}_2\rho_{n-1} \\ {}_3\rho_{n-1} \\ {}_4\rho_{n-1} \end{bmatrix} = B_2\underline{q}_{n-1} + \cdots + B_{n+1}\underline{q}_0 \quad , \qquad \text{(B32e)}$$

$${}_1r_{n+1} = \frac{\sqrt{\gamma}}{2\sigma}\left\{{}_1\nu_{n-1} + \left(n - \frac{3}{2}\right){}_1r_n + {}_2r_n\right\} , \qquad \text{(B33a)}$$

$${}_2r_{n+1} = \frac{\sqrt{\gamma}}{2\sigma}\left\{{}_2\nu_{n-1} + \left(n - \frac{3}{2}\right){}_2r_n\right\} , \qquad \text{(B33b)}$$

$${}_3r_n = -\frac{{}_3\nu_{n-1}}{n} , \qquad \text{(B33c)}$$

$${}_4r_{n+1} = \frac{\sqrt{\gamma}}{4\sigma}\left\{{}_4\nu_{n-1} + n \times {}_4r_n\right\} , \qquad \text{(B33d)}$$

where

$$\underline{\nu}_{n-1} = \begin{bmatrix} {}_1\nu_{n-1} \\ {}_2\nu_{n-1} \\ {}_3\nu_{n-1} \\ {}_4\nu_{n-1} \end{bmatrix} = B_2\underline{r}_{n-1} + \cdots + B_{n+1}\underline{r}_0 , \qquad \text{(B33e)}$$

$${}_1s_{n+1} = -\frac{\sqrt{\gamma}}{2\sigma}\left\{{}_1\psi_{n-1} + \left(n - \frac{3}{2}\right){}_1s_n + {}_2s_n\right\} , \qquad \text{(B34a)}$$

$${}_2s_{n+1} = -\frac{\sqrt{\gamma}}{2\sigma}\left\{{}_2\psi_{n-1} + \left(n - \frac{3}{2}\right){}_2s_n\right\} , \qquad \text{(B34b)}$$

$${}_3s_{n+1} = -\frac{\sqrt{\gamma}}{4\sigma}\left\{{}_3\psi_{n-1} + n \times {}_3s_n\right\} , \qquad \text{(B34c)}$$

$${}_4s_n = \frac{{}_4\psi_{n-1}}{n} , \qquad \text{(B34d)}$$

and

$$\underline{\psi}_{n-1} = \begin{bmatrix} {}_1\psi_{n-1} \\ {}_2\psi_{n-1} \\ {}_3\psi_{n-1} \\ {}_4\psi_{n-1} \end{bmatrix} = B_2\underline{s}_{n-1} + \cdots + B_{n+1}\underline{s}_0 \ . \quad \text{(B34e)}$$

CASE 3: $2(\lambda_1 - \lambda_2)$ = integer

If λ_1 and λ_2 differ by an odd multiple of one-half, then method II can be used. If λ_1 and λ_2 differ by an integer, then both method I and method II are difficult procedures to implement.

If method I is attempted, then additional shearing and similarity transformations must be developed. This approach is considered in [8, Chapter 4].

The main difficulty is that the recursion relations (B20) and (B21) are not both well defined if λ_1 and λ_2 differ by an integral multiple of one-half. In addition to the transformations E, diagonal $\left(1, 1, 1, \frac{1}{\tau}\right)$, F, $P(\tau)$, and $\tilde{P}(\tau)$, it is necessary to introduce a product of shearing, Sibuya, and similarity transformations.

Consider

$$H(\tau) = \prod_{i=1}^{N} \left\{ \mathrm{Sim}_i \times \mathrm{Sib}_i(\tau) \times \text{diagonal}\left(\frac{1}{\tau}, 1, 1, 1\right) \right\} \quad \text{(B35)}$$

and $$\underline{Z}(\tau) = H(\tau)\underline{\tilde{Z}}(\tau) \quad . \tag{B36}$$

The shearing transformation, diagonal $\left(\frac{1}{\tau}, 1, 1, 1\right)$, does not affect B_0 in (B15). However B_1 is replaced by a matrix with eigenvalues $-2\lambda_1+1$, $-2\lambda_2$, $\frac{1}{2}$, $\frac{1}{2}$. The matrix $\mathrm{Sib}_i(\tau)$ is constructed so that the new B_1 matrix is block diagonal with precisely two 2×2 blocks on the diagonal. The matrix Sim_1 reduces the (1,1) block to Jordan form. This process is repeated N times, where

$$N = 2(\lambda_1 - \lambda_2) \quad . \tag{B37}$$

The net result is that

$$\frac{d\underline{\tilde{Z}}(\tau)}{d\tau} = \left(\tilde{B}_0 + \frac{\tilde{B}_1}{\tau} + \cdots\right)\underline{\tilde{Z}}(\tau) \quad , \tag{B38a}$$

$$\tilde{B}_0 = B_0 \quad , \tag{B38b}$$

and

$\tilde{B}_1$ has eigenvalues $-2\lambda_1+N$, $-2\lambda_2$, $\frac{1}{2}$, and $\frac{1}{2}$ and $\tilde{B}_1$ is in Jordan canonical form. (B38c)

This is essentially the same problem as case 2 since

$$-2\lambda_2 = -2\lambda_1 + N \quad . \tag{B39}$$

This procedure of developing several shearing, Sibuya, and similarity transformations is necessary only if full infinite formal expansions are desired. However, for the purposes of this research it is necessary only to be assured that the lead terms in the asymptotic expansions, with algebraic growth, correspond to inviscid solutions and that there exist actual solutions of (25) which agree with the lead asymptotic terms to arbitrarily many significant figures as $\tau \to \infty$ along $\arg \tau = -\frac{\pi}{4}$. If the transformation $H(\tau)$ defined in (B35) is omitted, then the only loss is that full infinite formal expansions are not obtained because the recursion relations (B20) and (B21) are not both well defined. Since no computations for this case are necessary (see comments in Section 5.1), it is not worthwhile to pursue this case any further.

The development of formal expansions about the irregular singularity $\xi = \infty$ appears to be quite complex. A direct approach (method II) which avoids the complicated transformations of the differential equation (25) will now be developed. The success of this approach is based on the observation that it is necessary only to determine multiples of the boundary layer and transition layer solutions, in equations (32) and (33), in order to solve the viscous problem.

Consider the differential equation (25) which can be written in the form

$$\frac{d\underline{y}(\xi)}{d\xi} = A(\xi)\underline{y}(\xi) \quad , \tag{B40a}$$

where

$$A(\xi) = \frac{1}{\xi}\begin{bmatrix} 0 & 1 & 0 & 0 \\ \left(\frac{\sigma^2}{\gamma} - k^2\right)\xi + \frac{4k^2}{3} & 0 & \frac{k}{\gamma}\xi & k\left(\xi - 1/3\right) \\ 0 & 0 & 0 & 1 \\ \frac{\gamma - 1}{\gamma}\left(\frac{k\xi}{\xi - 4/3}\right) & k\left(\frac{\xi - 1/3}{\xi - 4/3}\right) & -\left(\frac{\frac{\sigma^2}{\gamma}\xi + k^2}{\xi - 4/3}\right) & -\left(\frac{\xi}{\xi - 4/3}\right) \end{bmatrix} . \tag{B40b}$$

The characteristic equation of $A(\xi)$ for large ξ has the following approximate roots ℓ_i :

$$\ell_1 \approx -\frac{\lambda_1}{\xi} \quad , \tag{B41a}$$

$$\ell_2 \approx -\frac{\lambda_2}{\xi} \quad , \tag{B41b}$$

$$\ell_3 \approx \frac{\sigma}{\sqrt{\gamma\xi}} \quad , \tag{B41c}$$

and

$$\ell_4 \approx -\frac{\sigma}{\sqrt{\gamma\xi}} \quad , \tag{B41d}$$

where λ_1 and λ_2 are the roots of the dispersion relation (13). In addition, the eigenvectors corresponding to the roots ℓ_i tend toward constants vectors as $\xi \to \infty$. The eigenvectors corresponding to ℓ_1 and ℓ_2 tend towards linearly independent vectors if the roots of the dispersion relation (13) are distinct. The eigenvectors corresponding to ℓ_3 and ℓ_4 tend towards linearly dependent vectors. Relations (B41) suggest attempting solutions[25] of the form

$$\hat{\hat{\underline{y}}}_{i,L}(\xi) = \sum_{n=0}^{L} \underline{a}_{n,i}\xi^{-(n+\lambda_i)} , \tag{B42}$$

for $i = 1$ or 2 and $L = \infty$. If formal solutions with exponential growth $\left(e^{\pm 2\sigma\sqrt{\frac{\xi}{\gamma}}}\right)$ are substituted into (B40), then the recursion relations are not well defined. Thus, method II does not yield formal solutions with exponential growth. However, there is no difficulty in determining formal solutions of the form (B42).

It will be convenient to substitute directly into (22) rather than (B40). Consider

$$\begin{bmatrix} U_i \\ W_i \end{bmatrix} = \sum_{n=0}^{\infty} \vec{\ell}_{n,i}\xi^{-(n+\lambda_i)} , \tag{B43a}$$

where

$$\vec{\ell}_{n,i} = \begin{bmatrix} {}_1\ell_{n,i} \\ {}_2\ell_{n,i} \end{bmatrix} \tag{B43b}$$

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[25]The double circumflex ($\hat{\hat{}}$) in relation (B42) denotes that $\hat{\hat{\underline{y}}}_{i,L}(\xi)$ is a formal solution of (25) which is obtained by direct substitution into the original differential equation (method II).

and

$$\underline{a}_{n,i} \;=\; \begin{bmatrix} {}_1\ell_{n,i} \\ -(n + \lambda_i)\,{}_1\ell_{n,i} \\ {}_2\ell_{n,i} \\ -(n + \lambda_i)\,{}_2\ell_{n,i} \end{bmatrix} . \qquad \text{(B43c)}$$

Substituting (B43) into (22) leads to

$$0 \;=\; \det \begin{bmatrix} k^2 - \dfrac{\sigma^2}{\gamma} & k\left(\lambda_i - \dfrac{1}{\gamma}\right) \\ k\left(\lambda_i - \dfrac{\gamma - 1}{\gamma}\right) & \lambda_i^2 - \lambda_i + \dfrac{\sigma^2}{\gamma} \end{bmatrix} \qquad \text{(B44a)}$$

or

$$0 \;=\; -\frac{\sigma^2}{\gamma}\left(\lambda_i^2 - \lambda_i + \frac{\sigma^2}{\gamma} - k^2 + \frac{\gamma - 1}{\gamma}\,\frac{k^2}{\sigma^2}\right) \qquad \text{(B44b)}$$

and

$$\vec{\ell}_{0,i} \qquad \begin{bmatrix} 1 \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\lambda_i - \dfrac{1}{\gamma}\right)} \end{bmatrix} \qquad \text{(B45a)}$$

$$\begin{bmatrix} k^2 - \frac{\sigma^2}{\gamma} & k\left(\lambda_i + n + 1 - \frac{1}{\gamma}\right) \\ k\left(\lambda_i + n + 1 - \frac{\gamma - 1}{\gamma}\right) & (\lambda_i + n + 1)(\lambda_i + n) + \frac{\sigma^2}{\gamma} \end{bmatrix} \vec{\ell}_{n+1,i}$$

$$= \begin{bmatrix} (\lambda_i + n)^2 - \frac{4k^2}{3} & -k\left(\frac{\lambda_i + n}{3}\right) \\ -k\left(\frac{\lambda_i + n}{3}\right) & k^2 - \frac{4(\lambda_i + n)^2}{3} \end{bmatrix} \vec{\ell}_{n,i} \quad .$$

(B45b)

The matrix coefficient of $\vec{\ell}_{n+1,i}$ is nonsingular in (B45b) for all n if and only if λ_1 and λ_2 do not differ by an integer. Thus, it is possible to determine formal solutions of (22) via the recursion relation (B45) for $\lambda_1 - \lambda_2 \neq$ integer .

CASE 4: $\frac{\sigma^2}{\gamma} - k^2 = 0$

Different normalizations of the lead vectors, $\vec{\ell}_{0,i}$ (i = 1 or 2), are required.

$$\vec{\ell}_{0,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad \lambda_2 = \frac{\gamma - 1}{\gamma} \qquad \text{(B46a)}$$

and $$\vec{\ell}_{0,1} = \begin{bmatrix} \dfrac{\frac{1}{\gamma}\left(\frac{1}{\gamma} - 1\right) + k^2}{k\left(\frac{2 - \gamma}{\gamma}\right)} \\ 1 \end{bmatrix} , \quad \lambda_1 = \frac{1}{\gamma} , \qquad \text{(B46b)}$$

and the remaining vectors ($\vec{\ell}_{n,i}$, i = 1 or 2) can be determined from the recursion relation (B45b).

Thus, there is little difficulty in determining formal solutions of the differential equation (22) about the irregular singularity $\xi = \infty$. However a fundamental set of formal solutions has not been obtained. Approximate multiples of the boundary layer and transition layer solutions can be found instead of determining two additional formal expansions.

Suppose, due to (B41), it is assumed that there exist two solutions, $\underline{y}_3(\xi)$ and $\underline{y}_4(\xi)$, of the differential equation (25) which satisfy

$$M_1|\xi|^{-c_1} e^{\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}} \leq ||\underline{y}_3(\xi)|| \leq M_2|\xi|^{c_2} e^{\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}} , \tag{B47a}$$

and

$$M_3|\xi|^{-c_3} e^{-\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}} \leq ||\underline{y}_4(\xi)|| \leq M_4|\xi|^{c_4} e^{-\mathrm{Re}\,2\sigma\sqrt{\frac{\xi}{\gamma}}} \tag{B47b}$$

for some positive constants M_1, M_2, M_3, M_4, c_1, c_2, c_3, and c_4.

If such solutions exist, then there should be very little difficulty in determining multiples of these solutions numerically. The solution $\underline{y}_3(\xi)$ grows more rapidly than $\underline{y}_4(\xi)$ for increasing $|\xi|$ and $\arg \xi = -\frac{\pi}{2}$. In addition, there are solutions $\underline{y}_1(\xi)$ and $\underline{y}_2(\xi)$ which are asymptotic to $\hat{\hat{\underline{y}}}_{1,\infty}(\xi)$ and $\hat{\hat{\underline{y}}}_{2,\infty}(\xi)$ and hence grow in norm essentially like $|\xi|^{-\mathrm{Re}\lambda_1}$ and $|\xi|^{-\mathrm{Re}\lambda_2}$ respectively. Thus, $\underline{y}_3(\xi)$ grows more rapidly than $\underline{y}_1(\xi)$ and $\underline{y}_2(\xi)$ as $|\xi|$ increases along the ray $\arg \xi = -\frac{\pi}{2}$.

Consider a numerical integration in the direction of increasing $|\xi|$. For an arbitrary initial vector it is expected that the initial vector will contain nonzero multiples of all four solutions, $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, $\underline{y}_3(\xi)$, and $\underline{y}_4(\xi)$. Since $\underline{y}_3(\xi)$ dominates in growth as $|\xi|$ is increased, a numerical integration of (25) over a sufficiently large $|\xi|$ interval in the direction of increasing $|\xi|$ should result in nearly a multiple of $\underline{y}_3(\xi)$. Similarly, if a numerical integration of (25) is performed in the direction of decreasing $|\xi|$, then a multiple of $\underline{y}_4(\xi)$ can be obtained since $\underline{y}_4(\xi)$ has dominant growth for decreasing $|\xi|$ and $\arg \xi = -\frac{\pi}{2}$.

The procedure (method II) leads to very good agreement with the asymptotic solutions obtained by method I, that is, multiples of $\underline{y}_3(\xi)$ and $\underline{y}_4(\xi)$ are obtained. The agreement of these two approaches provides a good check on all the transformations performed in method I.

B.2 PROPERTIES OF THE ASYMPTOTIC SOLUTIONS

In this section the formal asymptotic solutions developed in Section B.1 are investigated. All of the various properties of the asymptotic expansions mentioned in Sections 3, 4, and 5 are developed.

Associated with the formal asymptotic solutions in Section B.1 there are four distinct rates of growth,[26]

$\tau^{-2\lambda_1}$, $\tau^{-2\lambda_2}$, $\tau^{\frac{1}{2}}e^{\frac{2\sigma\tau}{\sqrt{\gamma}}}$, and $\tau^{\frac{1}{2}}e^{-\frac{2\sigma\tau}{\sqrt{\gamma}}}$ or equivalently $\xi^{-\lambda_1}$, $\xi^{-\lambda_2}$, $\xi^{\frac{1}{4}}e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$, and $\xi^{\frac{1}{4}}e^{-2\sigma\sqrt{\frac{\xi}{\gamma}}}$. Recall that $\xi = \frac{e^{-z}}{i\varepsilon\sigma}$ and $\tau = \sqrt{\xi}$; thus

$$\tau^{-2\lambda_1} = (i\varepsilon\sigma)^{\lambda_1}e^{\lambda_1 z} \quad , \tag{B48a}$$

$$\tau^{-2\lambda_2} = (i\varepsilon\sigma)^{\lambda_2}e^{\lambda_2 z} \quad , \tag{B48b}$$

$$\tau^{\frac{1}{2}}e^{2\sigma\frac{\tau}{\sqrt{\gamma}}} = \left(\frac{e^{-z}}{i\varepsilon\sigma}\right)^{\frac{1}{4}} \exp\left\{\frac{2\sigma}{\sqrt{\gamma}}\left(\frac{e^{-z/2}}{\sqrt{i\varepsilon\sigma}}\right)\right\} \quad , \tag{B48c}$$

and
$$\tau^{\frac{1}{2}}e^{-2\sigma\frac{\tau}{\sqrt{\gamma}}} = \left(\frac{e^{-z}}{i\varepsilon\sigma}\right)^{\frac{1}{4}} \exp\left\{-\frac{2\sigma}{\sqrt{\gamma}}\left(\frac{e^{-z/2}}{\sqrt{i\varepsilon\sigma}}\right)\right\} \quad . \tag{B48d}$$

---

[26] It is tacitly assumed that $\lambda_1 \neq \lambda_2$ and $\sigma^2/\gamma - k^2 \neq 0$. The several special cases will also be investigated.

In Section 3.2 the scalar growths $\tau^{\frac{1}{2}} e^{\pm\frac{2\sigma\tau}{\sqrt{\gamma}}}$ were investigated and it was shown that the solutions with these asymptotic growths are important only in the boundary layer and transition layer. The scalar growths $(i\varepsilon\sigma)^{\lambda_1} e^{\lambda_1 z}$ and $(i\varepsilon\sigma)^{\lambda_2} e^{\lambda_2 z}$ are obviously the same as the scalar growths associated with the inviscid solutions. The factors $(i\varepsilon\sigma)^{\lambda_1}$ and $(i\varepsilon\sigma)^{\lambda_2}$ are merely scaling constants for a prescribed $\varepsilon > 0$. In addition to the scalar growths it is also important to investigate the lead vectors in the formal asymptotic expansions.

Let

$$T_0 \;=\; E \text{ diagonal}(1, 1, 1, 0) F P_0 \tilde{P}_0 \qquad \text{(B49a)}$$

$$T_1 \;=\; E\{\text{diagonal}(1, 1, 1, 0)[F P_0 \tilde{P}_1 + F P_1 \tilde{P}_0] + \text{diagonal}(0, 0, 0, 1) F P_0 \tilde{P}_0\} \qquad \text{(B49b)}$$

and

$$T(\tau) \;=\; E \text{ diagonal}\left(1, 1, 1, \frac{1}{\tau}\right) F P(\tau) \tilde{P}(\tau) \quad , \qquad \text{(B49c)}$$

where the matrices E, diagonal $\left(1, 1, 1, \frac{1}{\tau}\right)$, F, $P(\tau)$, $\tilde{P}(\tau)$, P_0, P_1, $\tilde{P}_0$, $\tilde{P}_1$ are defined in Section B.1.

Consider

$$\hat{\underline{y}}_{i,L}(\xi) = T(\tau)\hat{\underline{Z}}_{i,L}(\tau) \qquad (i=1,2,3,\text{ or } 4) \tag{B50a}$$

and

$$\underline{y}_i(\xi) = T(\tau)\underline{Z}_i(\tau) \quad , \tag{B50b}$$

where $\xi = \tau^2$ and $\hat{\underline{Z}}_{i,L}(\tau)$ is defined in relation (B17). Thus, $\hat{\underline{y}}_{i,\infty}(\xi)$ is a formal solution of (25) since $\hat{\underline{Z}}_{i,\infty}(\tau)$ is a formal solution of (B15). If $\underline{Z}_i(\tau)$ is an actual solution of the differential equation (B15), then $\underline{y}_i(\xi)$ is an actual solution of the differential equation (25).

The formal truncated solution $\hat{\underline{y}}_{i,L}(\xi)$ is of the form

$$\hat{\underline{y}}_{i,L}(\xi) = \left\{\sum_{n=0}^{L}\underline{b}_{n,i}\xi^{-n/2} + O\left(\xi^{-\left(\frac{L+1}{2}\right)}\right)\right\}\xi^{-\lambda_i} \quad ,$$

$$\text{for } i=1 \text{ or } 2 \tag{B51a}$$

and

$$\hat{\underline{y}}_{j,L}(\xi) = \left\{\sum_{n=0}^{L}\underline{c}_{n,j}\xi^{-n/2} + O\left(\xi^{-\left(\frac{L+1}{2}\right)}\right)\right\}\xi^{\frac{1}{4}}e^{d_j 2\sigma\sqrt{\frac{\xi}{\gamma}}} \quad , \tag{B51b}$$

for $j = 3$ or 4 and $d_3 = +1$ and $d_4 = -1$. The vectors $\underline{b}_{n,i}$ and $\underline{c}_{n,j}$ are determined by the transformation $T(\tau)$ and the formal truncated expansions

$\underline{\hat{Z}}_{i,L}(\tau)$ (B17). In particular, the vectors $\underline{b}_{0,i}$ and $\underline{c}_{0,j}$ are given by

$$\underline{b}_{0,1} = T_0\underline{p}_0 = \begin{bmatrix} \frac{\gamma}{\gamma - 1}(1 - \lambda_1 - \gamma\alpha) \\ -\gamma\alpha\left(\frac{1 - \lambda_1\gamma}{\gamma - 1}\right) \\ \frac{\gamma^2}{k(\gamma - 1)}(1 - \lambda_1)\left(k^2 - \frac{\sigma^2}{\gamma}\right) \\ -\frac{\gamma^2}{k(\gamma - 1)}\alpha\left(k^2 - \frac{\sigma^2}{\gamma}\right) \end{bmatrix}, \tag{B52a}$$

$$\underline{b}_{0,2} = T_0\underline{q}_0 = \begin{bmatrix} \frac{\gamma}{\gamma - 1}(1 - \lambda_2 - \gamma\alpha) \\ -\gamma\alpha\left(\frac{1 - \lambda_2\gamma}{\gamma - 1}\right) \\ \frac{\gamma^2}{k(\gamma - 1)}(1 - \lambda_2)\left(k^2 - \frac{\sigma^2}{\gamma}\right) \\ -\frac{\gamma^2}{k(\gamma - 1)}\alpha\left(k^2 - \frac{\sigma^2}{\gamma}\right) \end{bmatrix}, \tag{B52b}$$

$$\underline{c}_{0,3} = T_0\underline{r}_0 = \frac{\sqrt{\gamma}}{\sigma}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{B52c}$$

and

$$\underline{c}_{0,4} = T_0\underline{s}_0 = -\frac{\sqrt{\gamma}}{\sigma}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \tag{B52d}$$

Notice that the lead vectors $\underline{c}_{0,3}$ and $\underline{c}_{0,4}$ are multiples of each other. This partially explains the difficulty in determining the formal asymptotic solutions with exponential growth.

The formal solutions (B42) can be determined from relations (B43) and (B45). The lead asymptotic vector $\underline{a}_{0,i}$ is given by

$$\underline{a}_{0,i} = \begin{bmatrix} 1 \\ -\lambda_i \\ \dfrac{\frac{\sigma^2}{\gamma} - k^2}{k\left(\lambda_i - \frac{1}{\gamma}\right)} \\ -\lambda_i\left\{\dfrac{\sigma^2/\gamma - k^2}{k\left(\lambda_i - \frac{1}{\gamma}\right)}\right\} \end{bmatrix} \tag{B53}$$

for $i = 1$ or 2 . Thus, the formal solutions (B42) have the lead asymptotic terms $\underline{a}_{0,1}\xi^{-\lambda_1}$ and $\underline{a}_{0,2}\xi^{-\lambda_2}$. It has already been established that the scalar growths $\xi^{-\lambda_1}$

and $\xi^{-\lambda_2}$ correspond to the inviscid growths $e^{\lambda_1 z}$ and $e^{\lambda_2 z}$, respectively. Comparison of relations (B53) and (15) establishes that the lead asymptotic vectors $\underline{a}_{0,1}$ and $\underline{a}_{0,2}$ correspond to the inviscid vectors. Note that the first and third components of $\underline{a}_{0,i}\xi^{-\lambda_i}$ should be compared with $U_i(z)$ and $W_i(z)$ in (15), respectively, due to relations (24). The second and fourth components of $\underline{a}_{0,i}\xi^{-\lambda_i}$ should be compared with $-\frac{dU_i(z)}{dz}$ and $-\frac{dW_i(z)}{dz}$. These comparisons are easily made since the normalizations of the lead asymptotic vectors $\underline{a}_{0,1}$ and $\underline{a}_{0,2}$ consisted of specifying the first components of each of these vectors to be one. The formal solutions (B17) are normalized by specifying the constants c_i in relations (B18) to be identically one for $i = 1, 2, 3,$ and 4. The lead asymptotic vectors $\underline{b}_{0,i}$ given in (B52) have a rather complicated form. However, it can be verified that

$$\underline{b}_{0,i} = \frac{\gamma}{\gamma - 1}(1 - \lambda_i - \gamma\alpha)\underline{a}_{0,i} \tag{B54}$$

for $i = 1$ or 2.

Thus, the formal solutions (B17) for $i = 1$ or 2 are asymptotic developments of actual solutions of (25) and the lead terms in these formal solutions correspond to multiples of the inviscid solutions (15).

For the case $\sigma^2 = \gamma k^2$, the normalization (B46) leads to a development of $\underline{a}_{0,i}$ which corresponds to the inviscid solutions (i6).

One additional case must be considered, namely the case $\lambda_1 = \lambda_2 = \frac{1}{2}$. The formal truncated solutions (B51) are of the form

$$\hat{\underline{y}}_{1,L}(\xi) = \left\{ \sum_{n=0}^{L} \underline{b}_{n,1} \xi^{-n/2} + O\left(\xi^{-\frac{L+1}{2}}\right) \right\} \xi^{-\frac{1}{2}} , \tag{B55a}$$

$$\hat{\underline{y}}_{2,L}(\xi) = \left\{ \sum_{n=0}^{L} \underline{b}_{n,2} \xi^{-n/2} + O\left(\xi^{-\frac{L+1}{2}}\right) \right\} \xi^{-\frac{1}{2}} + \frac{1}{2} \ln \xi \hat{\underline{y}}_{1,L}(\xi) , \tag{B55b}$$

and

$$\hat{\underline{y}}_{j,L}(\xi) = \left\{ \sum_{n=0}^{L} \underline{c}_{n,j} \xi^{-n/2} + O\left(\xi^{-\frac{L+1}{2}}\right) \right\} \xi^{\frac{1}{4}} e^{d_j 2\sigma \sqrt{\frac{\xi}{\gamma}}} , \tag{B55c}$$

where $j = 3$ or 4, $d_3 = +1$, and $d_4 = -1$.

Once again it is of interest to investigate the lead asymptotic terms which do not exhibit exponential growth or decay. Consider $\underline{b}_{0,1} \xi^{-\frac{1}{2}}$ and $\left\{ \underline{b}_{0,2} + \frac{1}{2} \ln \xi \underline{b}_{0,1} \right\} \xi^{-\frac{1}{2}}$.

$$\underline{b}_{0,1}\xi^{-\frac{1}{2}} = (i\varepsilon\sigma)^{\frac{1}{2}}\,\frac{\gamma}{4}\left(\frac{2-\gamma}{\gamma-1}\right)\begin{bmatrix} 1 \\ -\frac{1}{2} \\ \dfrac{\frac{\sigma^2}{\gamma} - k^2}{k\left(\frac{1}{2} - \frac{1}{\gamma}\right)} \\ -\frac{1}{2}\left\{\dfrac{\frac{\sigma^2}{\gamma} - k^2}{k\left(\frac{1}{2} - \frac{1}{\gamma}\right)}\right\} \end{bmatrix} e^{z/2} \quad . \tag{B56}$$

If the first and third components are compared with the inviscid solution $\begin{bmatrix} U_1(z) \\ W_1(z) \end{bmatrix}$ in equation (18), then it is easily seen that $\underline{b}_{0,1}\xi^{-\frac{1}{2}}$ is a multiple of this inviscid solution. Now consider

$$\begin{aligned}\left\{\underline{b}_{0,2} + \frac{1}{2}(\ell n\ \xi)\underline{b}_{0,1}\right\}\xi^{-\frac{1}{2}} &= (i\varepsilon\sigma)^{\frac{1}{2}} e^{z/2}\left\{\underline{b}_{0,2} - \frac{z}{2}\,\underline{b}_{0,1}\right\} \\ &+ (i\varepsilon\sigma)^{\frac{1}{2}} e^{z/2}\left(\frac{1}{2}\ \ell n\left(\frac{1}{\varepsilon\sigma}\right) - i\,\frac{\pi}{4}\right)\underline{b}_{0,1}\end{aligned} \tag{B57}$$

where

$$\underline{b}_{0,2} = \begin{bmatrix} \frac{\gamma}{\gamma - 1}\left(\frac{4 - \gamma}{4}\right) \\ -\frac{\gamma}{4(\gamma - 1)} \\ \frac{\gamma}{\gamma - 1}\left(\frac{\gamma}{k}\right)\left(k^2 - \frac{\sigma^2}{\gamma}\right) \\ -\frac{\gamma}{4(\gamma - 1)}\left(\frac{\gamma}{k}\right)\left(k^2 - \frac{\sigma^2}{\gamma}\right) \end{bmatrix} . \tag{B58}$$

The first and third components of $\underline{b}_{0,2}$ do not compare with the expression for $\begin{bmatrix} U_2(z) \\ W_2(z) \end{bmatrix}$ in relation (18). However, the two-dimensional vector which is constructed from the first and third components of $(i\varepsilon\sigma)^{\frac{1}{2}} e^{z/2}\{\underline{b}_{0,2} - \frac{z}{2}\,\underline{b}_{0,1}\}$ is a solution of the differential equation (12). Hence $\left\{\underline{b}_{0,2} + \frac{1}{2}\,(\ell n\ \xi)\underline{b}_{0,1}\right\}\xi^{-\frac{1}{2}}$ compares with a multiple of an inviscid solution.

Thus, in all cases the lead terms in the formal expansions with algebraic growth correspond to multiples of the solutions to the inviscid differential equation (12), that is, the lead terms are merely inviscid solutions.

In order to distinguish the asymptotic expansions for the boundary and transition layer solutions, it is necessary to determine additional terms in the asymptotic expansions. The relations (B22), (B23), (B49), and (B51) yield

$$T_0\underline{r}_1 + T_1\underline{r}_0 = \underline{c}_{1,3} = \begin{bmatrix} \frac{\gamma}{\sigma^2} \\ \frac{\sqrt{\gamma}}{\sigma}\left({}_3r_1 - \tilde{p}_1^{43}\right) - \frac{\gamma}{\sigma^2} + 1 \\ 0 \\ \frac{1}{k}\left(1 + \frac{\gamma}{\sigma^2}\left(k^2 - \frac{\sigma^2}{\gamma}\right)\right) \end{bmatrix} \tag{B59a}$$

and

$$T_0\underline{s}_1 + T_1\underline{s}_0 = \underline{c}_{1,4} = \underline{c}_{1,3} \; , \tag{B59b}$$

where $\tilde{p}_1^{43}$ is the (4,3) element of $\tilde{P}_1$ defined in relation (B14). The same result (B59) is obtained for the case $\lambda_1 = \lambda_2 = \frac{1}{2}$.

LEMMA B1: The vectors $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, and $\underline{c}_{1,3}$ defined in relations (51) are linearly independent if $\lambda_1 - \lambda_2 \neq$ integer and $\sigma^2/\gamma - k^2 \neq 0$.

PROOF:

det $[\underline{b}_{0,1}, \underline{b}_{0,2}, \underline{c}_{0,3}, \underline{c}_{1,3}]$

$$= \frac{\sqrt{\gamma}}{\sigma}\left(\frac{\gamma}{\gamma - 1}\right)^2 (\lambda_1 - \lambda_2)\left(k^2 - \frac{\sigma^2}{\gamma}\right)\frac{\gamma^2}{k^2}\alpha \ . \qquad \text{(B60)}$$

If $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, and $\underline{c}_{1,3}$ are linearly dependent, then (B60) implies

$$k^2 - \frac{\sigma^2}{\gamma} = 0 \qquad \text{(B61a)}$$

or
$$\lambda_1 = \lambda_2 \qquad \text{(B61b)}$$

or
$$\alpha = 0 \Rightarrow \lambda_1 = 1 \ , \quad \lambda_2 = 0 \ . \qquad \text{(B61c)}$$

The hypothesis of lemma B1 excludes relations (B61). Hence, $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, and $\underline{c}_{1,3}$ are linearly independent. Q.E.D.

For computational purposes it is important to avoid near-linear dependence or ill-conditioning of the vectors $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, and $\underline{c}_{1,3}$. Although lemma B1 implies that the vectors $\underline{b}_{0,1}$, $\underline{b}_{0,2}$, $\underline{c}_{0,3}$, and $\underline{c}_{1,3}$ are linearly independent for all k , it was found that for $k \geq 5$, near-linear dependence destroyed the calculations. The error check, step 5 in the algorithm developed in section 4.1, implied that no significant figures were obtained for these calculations ($k \geq 5$). To avoid this difficulty the range of k values was restricted ($k \leq 1.5$). Similarly, for $\lambda_1 = \lambda_2 = \frac{1}{2}$ and the Lamb wave ($\sigma^2/\gamma - k^2 = 0$), good accuracy was obtained for $k \leq 1.5$, that is, the problem of ill-conditioning was avoided.

Relations (B52) and (B59) imply

$$\underline{\text{BLSOL}}(\xi) \approx \left(\underline{c}_{0,3} + \frac{\underline{c}_{1,3}}{\sqrt{\xi}}\right)\xi^{\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \tag{B62}$$

and

$$\underline{\text{TLSOL}}(\xi) \approx \left(-\underline{c}_{0,3} + \frac{\underline{c}_{1,3}}{\sqrt{\xi}}\right)\xi^{\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \tag{B63}$$

for large ξ and $\arg \xi = -\frac{\pi}{2}$. Notice that the form of the vectors $\underline{c}_{0,3}$ and $\underline{c}_{1,3}$ implies that the modulus of the third component of the boundary layer solution is much smaller than the modulus of the first component. In fact, as $\xi \to \infty$ along the ray $\arg \xi = -\frac{\pi}{2}$.

$$\frac{\text{third component of } \underline{\text{BLSOL}}(\xi)}{\text{first component of } \underline{\text{BLSOL}}(\xi)} = O\left(\frac{1}{\sqrt{\xi}}\right) . \tag{B64}$$

For $\varepsilon = 10^{-11}$ the ratio (B64) was approximately 10^{-6} at $z = 0$. Thus, in most cases the boundary layer solution which arises in equation (33) affects only the horizontal component of the velocity.[27] The boundary layer solution reduces U(z) to zero in the boundary layer and has a small effect on W(z). If the boundary layer is ignored, then the constant d_2 in equation (33) can be accurately determined so that the kinematic boundary condition $W(0) = 1$ is satisfied, that is, it is only necessary to satisfy

$$\underline{\chi}_{VP}(\xi_1) = d_2\left(\underline{INV}_2(\xi_1) - e_3\underline{INV}_1(\xi_1)\right) . \tag{B65}$$

Equation (B65) will determine d_2 to about five significant figures for $\varepsilon = 10^{-11}$. Of course only the kinematic boundary condition $W(0) = 1$ will be satisfied by (B65).

Thus, the boundary layer solution has negligible effect on W(z) for all z , including $z = 0$. In addition, it was shown in Section 3.2 that the boundary layer solution has negligible effect on U(z) outside a thin boundary layer near $z = 0$.

In Section B.1 two distinct methods were developed for computing formal solutions which have lead terms corresponding to multiples of the inviscid solutions. For computational purposes it is necessary to consider additional terms in the formal expansions. Quite naturally

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[27]The only exception is the Lamb wave.

the question arises as to whether or not the remaining terms in the two distinct developments (method I and method II) are related in any way.

So far it has been shown that

$$\underline{b}_{0,i} = c_i \times \underline{a}_{0,i} \tag{B66}$$

where the vectors $\underline{b}_{n,i}$ and $\underline{a}_{n,i}$ are defined in (B51) and (B42), respectively.

THEOREM B1: If relation (B66) is satisfied and $2(\lambda_1 - \lambda_2) \neq$ integer and $\sigma^2/\gamma - k^2 \neq 0$, then

$$\underline{b}_{2n,i} = c_i \times \underline{a}_{n,i} \tag{B67a}$$

and

$$\underline{b}_{2n+1,i} = \underline{0} \ , \tag{B67b}$$

for $i = 1$ or 2 and $n = 0,1,2,\cdots$.

NOTE: If theorem B1 can be established, then there is very little difference in using $\hat{\hat{\underline{y}}}_{i,L}(\xi)$ defined in (B42) or $\hat{\underline{y}}_{i,2L}(\xi)$ defined in (B51) for computational purposes.

PROOF: Let $\underline{y}_i(\xi)$ be a solution of the differential equation (25) such that $\underline{y}_i(\xi)$ is asymptotic to $\hat{\underline{y}}_{i,\infty}(\xi)$ as $\xi \to \infty$ along the ray $\arg \xi = -\frac{\pi}{2}$, that is,

$$||\underline{y}_i(\xi) - \hat{\underline{y}}_{i,L}(\xi)|| = O\left(\xi^{-\frac{L+1}{2}}\right)|\xi|^{-\mathrm{Re}\lambda_i} \ , \tag{B68}$$

for $i = 1$ or 2 and all $L = 0,1,2,\cdots$. For a proof of the existence of an actual solution $\underline{y}_i(\xi)$ of the differential equation (25) which is asymptotic to $\hat{\underline{y}}_{i,\infty}(\xi)$, see [8, Chapter 5]. Similarly, let $\underline{Y}_i(\xi)$ be a solution of the differential equation (25) such that $\underline{Y}_i(\xi)$ is asymptotic to $\hat{\hat{\underline{y}}}_{i,\infty}(\xi)$, that is,

$$||\underline{Y}_i(\xi) - \hat{\underline{y}}_{i,k}(\xi)|| = O\left(\xi^{-(k+1)}\right)|\xi|^{-\mathrm{Re}\lambda_i} , \qquad \text{(B69)}$$

for $i = 1$ or 2 and all $k = 0,1,2,3,\cdots$.

Consider $\underline{w}_i(\xi)$ defined by

$$\underline{w}_i(\xi) = \underline{Y}_i(\xi) - c_i \underline{y}_i(\xi) . \qquad \text{(B70)}$$

Since (25) is a linear homogenous differential equation, $\underline{w}_i(\xi)$ is a solution of (25). The hypothesis and relations (B68), (B69), and (B70) imply

$$||\underline{w}_i(\xi)|| = O\left(\xi^{-\frac{1}{2}}\right)|\xi|^{-\mathrm{Re}\lambda_i} .$$

If

$$\underline{b}_{1,1} \neq \underline{0} , \qquad \text{(B71)}$$

then

$$\underline{w}_1(\xi) \sim \left(\underline{b}_{1,1}\frac{1}{\sqrt{\xi}} + (\underline{b}_{2,1} - c_1 \times \underline{a}_{1,1})\frac{1}{\xi} + \cdots\right)\xi^{-\lambda_1} . \qquad \text{(B72)}$$

If $2(\lambda_1 - \lambda_2) \neq$ integer, then (B71) and (B72) imply $\underline{w}_1(\xi)$, $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, $\underline{y}_3(\xi)$, and $\underline{y}_4(\xi)$ are linearly independent $\big(\underline{y}_3(\xi)$ and $\underline{y}_4(\xi)$ are defined in relation (B50)$\big)$, since each of these solutions of the differential equation (25) has a different asymptotic growth. However, this is impossible since a system of four first-order, linear differential equations has only four linearly independent solutions. Relation (B72) is a consequence of relation (B71); thus (B71) must be invalid or $\underline{b}_{1,1} = \underline{0}$.

All the coefficients in the asymptotic development (B72) must be zero; otherwise $\underline{w}_1(\xi)$, $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, $\underline{y}_3(\xi)$, and $\underline{y}_4(\xi)$ are linearly independent. Similarly,

$$\underline{w}_2(\xi) \sim \left(\underline{b}_{1,2}\frac{1}{\sqrt{\xi}} + (\underline{b}_{2,2} - c_2 \times \underline{a}_{1,2})\frac{1}{\xi} + \cdots\right)\xi^{-\lambda_2} . \tag{B73}$$

If any of the coefficients in the asymptotic development (B73) are nonzero, then $\underline{w}_2(\xi)$, $\underline{y}_1(\xi)$, $\underline{y}_2(\xi)$, $\underline{y}_3(\xi)$, and $\underline{y}_4(\xi)$ are linearly independent. Since this is impossible it follows that

$$\underline{b}_{1,1} = (\underline{b}_{2,1} - c_1 \times \underline{a}_{1,1}) = \cdots = \underline{0}$$

and $\underline{b}_{1,2} = (\underline{b}_{2,2} - c_2 \times \underline{a}_{1,2}) = \cdots = \underline{0}$ Q.E.D.

Thus, for computational purposes there is little difference in using either the formal expansions (B17) and transformations (B49) to obtain approximate values of the solutions of (25) with inviscid growth or the expansions (B42).

Since the asymptotic solutions are used for computational purposes, it is necessary to obtain some error estimates. Consider

$$||\underline{Z}_i(\tau) - \hat{\underline{Z}}_{i,L}(\tau)|| = O(\tau^{-L-1})||\hat{\underline{Z}}_{i,o}(\tau)|| \quad , \qquad \text{(B74)}$$

as $\tau \to \infty$,, where $\hat{\underline{Z}}_{i,L}(\tau)$ is defined in (B17) and $\underline{Z}_i(\tau)$ is a solution of (B15). If $\varepsilon > 0$ is prescribed, then there corresponds a constant $M(\varepsilon)$ such that

$$\frac{||\underline{p}_{L+1}|| - \varepsilon}{|\tau|^{L+1}} \left|\tau^{-2\lambda_1}\right| \leq ||Z_1(\tau) - \hat{\underline{Z}}_{i,L}(\tau)||$$

$$\leq \frac{||\underline{p}_{L+1}|| + \varepsilon}{|\tau|^{L+1}} \left|\tau^{-2\lambda_1}\right| \quad , \qquad \text{(B75a)}$$

$$\frac{||\underline{q}_{L+1}|| - \varepsilon}{|\tau|^{L+1}} \left|\tau^{-2\lambda_2}\right| \leq ||\underline{Z}_2(\tau) - \hat{Z}_{2,L}(\tau)||$$

$$\leq \frac{||\underline{q}_{L+1}|| + \varepsilon}{|\tau|^{L+1}} \left|\tau^{-2\lambda_2}\right| \quad , \qquad \text{(B75b)}$$

$$\frac{||\underline{r}_{L+1}|| - \varepsilon}{|\tau|^{L+1}} |\tau|^{\frac{1}{2}} e^{\text{Re}\frac{2\sigma\tau}{\sqrt{\gamma}}} \leq ||Z_3(\tau) - \hat{\underline{Z}}_{3,L}(\tau)||$$

$$\leq \frac{||\underline{r}_{L+1}|| + \varepsilon}{|\tau|^{L+1}} |\tau|^{\frac{1}{2}} e^{\text{Re}\frac{2\sigma\tau}{\sqrt{\gamma}}} \tag{B75c}$$

and

$$\frac{||\underline{s}_{L+1}|| - \varepsilon}{|\tau|^{L+1}} |\tau|^{\frac{1}{2}} e^{-\text{Re}\frac{2\sigma\tau}{\sqrt{\gamma}}} \leq ||\underline{Z}_4(\tau) - \hat{\underline{Z}}_{4,L}(\tau)||$$

$$\leq \frac{||\underline{s}_{L+1}|| + \varepsilon}{|\tau|^{L+1}} |\tau|^{\frac{1}{2}} e^{-\text{Re}\frac{2\sigma\tau}{\sqrt{\gamma}}} , \tag{B75d}$$

for $|\tau| \geq M(\varepsilon)$ and $\arg \tau = -\frac{\pi}{4}$. It is difficult to determine $M(\varepsilon)$. However, an indication, not a bound, of the error can be found by setting $\varepsilon = 0$ in (B75). For a specific value of τ it would be reasonable to truncate the expansions (B17) at a value of k such that $r(\tau,k)$, defined by

$$r(\tau,k) = \max \left\{ \frac{||\underline{p}_k||}{|\tau|^k} , \frac{||\underline{q}_k||}{|\tau|^k} , \frac{||\underline{r}_k||}{|\tau|^k} , \frac{||\underline{s}_k||}{|\tau|^k} \right\} , \tag{B76}$$

is a minimum; that is, the expansions (B17) should be truncated at $L = N$, where

$$r(\tau,N) \leq r(\tau,k) \quad \text{for all} \quad k = 0,1,2,\cdots \tag{B77}$$

and τ is held fixed. Of course N will vary with the value of τ prescribed.

Notice that $r(\tau,N)$ provides an estimate of the error. For a specific value of τ there will correspond a certain degree of accuracy. However, it is reasonable to require a certain accuracy from the expansions (B17) rather than accept any accuracy that is possible at a prescribed value of τ. For example, if it is desired to satisfy

$$r(\tau,N) \leq 10^{-7} \quad , \tag{B78}$$ [28]

then τ and N must be determined. Due to many calculations it was empirically established that (B78) is satisfied if

$$|\tau| \approx \left(\frac{11}{\sigma}\right) \tag{B79a}$$

and

$$N \approx 20 \quad . \tag{B79b}$$

It is possible to verify (B79) by examining the recursion relation (B45) or

$$\vec{\ell}_{n+1,i} = \begin{bmatrix} \dfrac{\gamma n^2}{\sigma^2} + 0(n) & 0(n) \\ 0(n) & 0(n) \end{bmatrix} \vec{\ell}_{n,i} \quad . \tag{B80}$$

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[28] Most of the calculations summarized in Section 5 were performed on an IBM 7094 in single precision (about eight significant figures) and relation (B78) was satisfied.

A crude estimate can be obtained for N and $|\tau|$ by examining $\frac{\gamma^n (n!)^2}{\sigma^{2n}} |\xi|^{-n}$. Recall that $N = 20$ for expansions (B17) is equivalent to summing 10 terms of the formal expansion (B42). It is easily verified that

$$\frac{(1.4)^{10}(10!)^2}{\sigma^{20}} \times |\xi|^{-10} < 10^{-6} \tag{B81}$$

if

$$|\xi| = \left(\frac{11}{\sigma}\right)^2 . \tag{B82}$$

More generally, if it is required that $r(\tau,N)$ satisfy

$$r(\tau,N) \leq 10^{-p} , \tag{B83}$$

then

$$|\tau| \approx \frac{A}{\sigma} \quad \text{(A is some positive constant)} \tag{B84a}$$

and N varies only slightly. (B84b)

Since the function $r(\tau,L)$ does not vary significantly for L near the value N defined in (B77), it is not critical to truncate the expansions (B17) at precisely the minimum term. Hence, it is possible to fix the value of N with very little change in the determination of the initial vectors for the canonical numerical integration procedure.

In Section 3.2, equation (32) was developed. Suppose the same notation is used for the solutions of the differential equation (25), namely $\underline{DC}_1(\xi)$, $\underline{DC}_2(\xi)$, $\underline{INV}_1(\xi)$, $\underline{INV}_2(\xi)$, and $\underline{TLSOL}(\xi)$, and suppose the scalars e_1, e_2, e_3, and e_4 are defined in (32). It is of interest to investigate the solutions of (32) as $\varepsilon \to 0$. Thus, it is natural to investigate the dependence of e_1, e_2, e_3, and e_4 on the parameter ε for fixed values of σ and k.

THEOREM B2: If the solutions of the differential equation (25), which are asymptotic to multiples of the inviscid solutions, are required to satisfy

$$\underline{INV}_1(\xi) \sim e^{\frac{\pi}{2}\mathrm{Im}(\lambda_1)} T(\tau)\hat{\underline{Z}}_{1,\infty}(\tau) \qquad \text{(B85a)}$$

and

$$\underline{INV}_2(\xi) \sim e^{\frac{\pi}{2}\mathrm{Im}(\lambda_2)} T(\tau)\hat{\underline{Z}}_{2,\infty}(\tau) \quad , \qquad \text{(B85b)}$$

where λ_1 and λ_2 are the roots of the dispersion relation (13), $T(\tau)$ is defined in (B49), and $\hat{\underline{Z}}_{i,\infty}(\lambda)$ is defined in (B17), then the scalars e_1, e_2, and e_3 are invariant as $\varepsilon \to 0$ if $\underline{DC}_1(\xi)$, $\underline{DC}_2(\xi)$, $\underline{INV}_1(\xi)$, and $\underline{TLSOL}(\xi)$ are linearly independent.

PROOF: The vectors $\underline{DC}_1(\xi)$ and $\underline{DC}_2(\xi)$ are uniquely defined. The vectors which satisfy (B85a) form a one-parameter family of solutions such that any two members of the family differ by a multiple of the transition layer solution. Similarly, the vectors which satisfy (B85b) form a one-parameter family of solutions. Thus, consider

$$e_1\underline{DC}_1(\xi) + e_2\underline{DC}_2(\xi) + e_3\underline{INV}_1(\xi) + e_4\underline{TLSOL}(\xi) = \underline{INV}_2(\xi)$$

and

$$f_1\underline{DC}_1(\xi) + f_2\underline{DC}_2(\xi) + f_3\{\underline{INV}_1(\xi) + A\ \underline{TLSOL}(\xi)\} + f_4\underline{TLSOL}(\xi)$$
$$= \underline{INV}_2(\xi) + B\ \underline{TLSOL}(\xi) \quad . \qquad \text{(B86)}$$

The theorem is established if for arbitrary scalars A and B, $f_i = e_i$ for $i = 1,2,$ and 3. However, equation (B86) can be rewritten as

$$f_1\underline{DC}_1(\xi) + f_2\underline{DC}_2(\xi) + f_3\underline{INV}_1(\xi)$$
$$+ \{f_4 + f_3A - B\}\underline{TLSOL}(\xi) = \underline{INV}_2(\xi) \quad .$$

If $\underline{DC}_1(\xi)$, $\underline{DC}_2(\xi)$, $\underline{INV}_1(\xi)$, and $\underline{TLSOL}(\xi)$ are linearly independent, then $f_1 = e_1$, $f_2 = e_2$, $f_3 = e_3$, and $f_4 + f_3A - B = e_4$. Q.E.D.

In general, if conditions (B85) are satisfied, then the scalar e_4 can vary. In addition, if $\underline{INV}_1(\xi)$ and $\underline{INV}_2(\xi)$ are reduced to canonical form at $\xi = \frac{1}{i}$, then the scalar e_4 is also uniquely determined.

THEOREM B3:

If

$\underline{DC}_1(\xi)$ or $\underline{DC}_2(\xi)$ is asymptotic to a nonzero multiple of the boundary layer solution (B87a)

and

whenever $c_1\underline{DC}_1(\xi)+c_2\underline{DC}_2(\xi)$ is not asymptotic to a nonzero multiple of the boundary layer solution, then

$$c_1\underline{DC}_1(\xi)+c_2\underline{DC}_2(\xi) \sim f_1\hat{\underline{Y}}_{1,\infty}(\xi)+f_2\hat{\underline{Y}}_{2,\infty}(\xi) \ , \tag{B87b}$$

where f_1 and f_2 are nonzero if $|c_1| + |c_2| > 0$,

then the viscous problem has one and only one solution corresponding to every sufficiently small $\varepsilon > 0$ for the following cases

a. $\lambda_1 > \frac{1}{2} > \lambda_2$, $\frac{\sigma^2}{\gamma} - k^2 \neq 0$

b. $\lambda_1 = \lambda_2 = \frac{1}{2}$

c.[29] $\lambda_2 < \lambda_1$, $\frac{\sigma^2}{\gamma} - k^2 = 0$, $\gamma = 1.4$,

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[29]For case c, hypotheses (B87a) and (B87b) are modified slightly.

where λ_1 and λ_2 are the roots of the dispersion relation (13) and $\hat{\underline{y}}_{i,\infty}(\xi)$ is defined in (B50).

PROOF: Due to hypothesis (B87a), either $\underline{DC}_1(\xi)$ or $\underline{DC}_2(\xi)$ is asymptotic to a multiple of the boundary layer solution and, hence, simply by interchanging subscripts it is always possible to assume $\underline{DC}_1(\xi)$ is asymptotic to a multiple of the boundary layer solution. Thus, without loss in generality assume $\underline{DC}_1(\xi)$ is asymptotic to a multiple of the boundary layer solution, that is,

$$\underline{DC}_1(\xi) \sim a\hat{\underline{y}}_{3,\infty}(\xi) \quad , \tag{B88}$$

where $a \neq 0$ and $\hat{\underline{y}}_{3,\infty}(\xi)$ is defined in (B50).

Consider

$$\underline{\widetilde{DC}}_2(\xi) = \underline{DC}_2(\xi) + b\underline{DC}_1(\xi) \quad .$$

For a unique value of b it is possible to eliminate the boundary layer solution from $\underline{\widetilde{DC}}_2(\xi)$. If $\underline{DC}_2(\xi)$ is not asymptotic to a nonzero multiple of the boundary layer solution, then $b = 0$ and if $\underline{DC}_2(\xi)$ is asymptotic to, say, $\tilde{a}\times\hat{\underline{y}}_{3,\infty}(\xi)$, then $b = -\frac{\tilde{a}}{a}$. Thus, for the proper choice of b, $\underline{\widetilde{DC}}_2(\xi)$ is not asymptotic to a nonzero multiple of the boundary layer solution and hypothesis (B87b) implies

$$\underline{\widetilde{DC}}_2(\xi) \sim c\hat{\underline{y}}_{1,\infty}(\xi) + d\hat{\underline{y}}_{2,\infty}(\xi) \quad , \tag{B89}$$

where c and d are nonzero.

Now let $\vec{dc}_1(\xi)$ and $\vec{\tilde{dc}}_2(\xi)$ be constructed from the first and third components of $\underline{DC}_1(\xi)$ and $\underline{\tilde{DC}}_2(\xi)$, respectively. Clearly (see theorem 1 in Section 3.2), if it can be shown that $\vec{dc}_1(\xi)$ and $\vec{\tilde{dc}}_2(\xi)$ are linearly independent for sufficiently large ξ, then the theorem is established.

CASE 1: $\lambda_1 > \frac{1}{2} > \lambda_2$, $\frac{\sigma^2}{\gamma} - k^2 \neq 0$

Relations (B52), (B59), (B88), and (B89) imply

$$\vec{dc}_1(\xi) = e\begin{bmatrix} 1 + 0\left(\frac{1}{\sqrt{\xi}}\right) \\ 0\left(\frac{1}{\sqrt{\xi}}\right) \end{bmatrix} \xi^{-\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \tag{B90}$$

and

$$\vec{\tilde{dc}}_2(\xi) = f\begin{bmatrix} 1 + 0\left(\frac{1}{\xi}\right) \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\lambda_1 - \frac{1}{\gamma}\right)} + 0\left(\frac{1}{\xi}\right) \end{bmatrix} \xi^{-\lambda_1} + g\begin{bmatrix} 1 + 0\left(\frac{1}{\xi}\right) \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\lambda_2 - \frac{1}{\gamma}\right)} + 0\left(\frac{1}{\xi}\right) \end{bmatrix} \xi^{-\lambda_2}$$

Hence,

$$\det\,[\vec{dc}_1(\xi),\ \vec{\tilde{dc}}_2(\xi)] = eg\left(\frac{\sigma^2/\gamma - k^2}{k\left(\lambda_2 - \frac{1}{\gamma}\right)}\right)\xi^{-\left(\frac{1}{4}+\lambda_2\right)} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$$

$$+ \ o(1)\,\xi^{-\left(\frac{1}{4}+\lambda_2\right)} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \tag{B91}$$

and by assumption $\lambda_2 < \frac{1}{2}$, $\frac{\sigma^2}{\gamma} - k^2 \neq 0$, $e \neq 0$ and $g \neq 0$. Thus, relation (B91) implies that for sufficiently large ξ , $\overrightarrow{dc}_1(\xi)$ and $\overrightarrow{\tilde{dc}}_2(\xi)$ are linearly independent.

CASE 2: $\lambda_1 = \lambda_2 = \frac{1}{2}$

Relations (B55), (B56), and (B89) imply

$$\overrightarrow{\tilde{dc}}_2(\xi) = h\begin{bmatrix} 1 + o(1) \\ \dfrac{\sigma^2/\gamma - k^2}{k\left(\frac{1}{2} - \frac{1}{\gamma}\right)} + o(1) \end{bmatrix} (\ell n\ \xi)\xi^{-\frac{1}{2}} .$$

$$\det[\overrightarrow{dc}_1(\xi), \overrightarrow{\tilde{dc}}_2(\xi)] = eh \frac{\sigma^2/\gamma - k^2}{k\left(\frac{1}{2} - \frac{1}{\gamma}\right)} (\ell n\ \xi)\xi^{-\frac{3}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} + o(1)\xi^{-\frac{3}{4}}(\ell n\ \xi)e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} . \qquad \text{(B92)}$$

Hence, relation (B92) implies that $\overrightarrow{dc}_1(\xi)$ and $\overrightarrow{\tilde{dc}}_2(\xi)$ are linearly independent for sufficiently large ξ .

CASE 3: $\sigma^2/\gamma - k^2 = 0$, $\gamma = 1.4$

For this case the formal expansions were obtained by method II. However, by investigating the eigenvectors of the matrix $A(\xi)$ defined in relation (B40), it is expected that relation (B90) is still satisfied, that is, $\underline{DC}_1(\xi)$ satisfies (B90). Thus, hypothesis (B87a) is slightly modified and relation (B90) is assumed to hold. In addition, hypothesis (B87b) must be modified; that is, assume

$$c_1\underline{DC}_1(\xi) + c_2\underline{DC}_2(\xi) \sim f_1\hat{\hat{\underline{y}}}_{1,\infty}(\xi) + f_2\hat{\hat{\underline{y}}}_{2,\infty}(\xi) \quad , \qquad \text{(B93)}$$

if $c_1\underline{DC}_1(\xi)+c_2\underline{DC}_2(\xi)$ is not asymptotic to a nonzero multiple of the boundary layer solution, where $\hat{\hat{\underline{y}}}_{i,\infty}(\xi)$ is defined in relations (B42), (B43), (B45), and (B46). Thus,

$$\overrightarrow{\widetilde{dc}}_2(\xi) = \begin{bmatrix} \left(\ell + O\left(\frac{1}{\xi^{\lambda_1-\lambda_2}}\right)\right)\xi^{-\lambda_2} \\ \left(m + O\left(\frac{1}{\xi^{1+\lambda_2-\lambda_1}}\right)\right)\xi^{-\lambda_1} \end{bmatrix}$$

and

$$\det\,[\overrightarrow{dc}_1(\xi),\ \overrightarrow{\widetilde{dc}}_2(\xi)] = \left\{em\xi^{-\lambda_1} + O\left(\xi^{-\frac{1}{2}+\lambda_2}\right)\right\} \times \xi^{-\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \quad .$$

However, $e \neq 0$ (relation (B90) assumed), $m \neq 0$, and $\lambda_1 < \frac{1}{2} + \lambda_2$ for $\gamma > \frac{4}{3}$ (recall $\gamma = 1.4$); thus, $\overrightarrow{dc}_1(\xi)$ and $\overrightarrow{\widetilde{dc}}_2(\xi)$ are linearly independent for sufficiently large ξ . Q.E.D.

THEOREM B4:

If

λ_1 and λ_2 are complex roots of the dispersion relation (13), (B94a)

$\underline{DC}_1(\xi)$ or $\underline{DC}_2(\xi)$ is asymptotic to a nonzero multiple of the boundary layer solution, (B94b)

$\hat{\underline{y}}_{1,0}(\xi)$ equals a multiple of the inviscid solution with upward energy flux and $\hat{\underline{y}}_{2,0}(\xi)$ equals a multiple of the inviscid solution with downward energy flux, (B94c)

and

whenever $c_1\underline{DC}_1(\xi)+c_2\underline{DC}_2(\xi)$ is not asymptotic to a nonzero multiple of the boundary layer solution, then

$$c_1\underline{DC}_1(\xi) + c_2\underline{DC}_2(\xi) \sim a\left(\hat{\underline{y}}_{1,\infty}(\xi)e^{\frac{\pi}{2}\mathrm{Im}(\lambda_1)} + \kappa_R\hat{\underline{y}}_{2,\infty}(\xi)e^{\frac{\pi}{2}\mathrm{Im}(\lambda_2)}\right), \quad \text{(B94d)}$$

where $a \neq 0$ if $|c_1| + |c_2| > 0$, $|\kappa_R| < 1$ and $\hat{\underline{y}}_{i,\infty}(\xi)$ is defined in (B50), then for sufficiently small $\varepsilon > 0$ the viscous problem has one and only one solution.

PROOF: Without loss in generality, assume $\underline{DC}_1(\xi)$ is asymptotic to a nonzero multiple of the boundary layer solution; that is,

$$\underline{DC}_1(\xi) \sim c\hat{\underline{y}}_{3,\infty}(\xi) \quad , \tag{B95}$$

where $\hat{\underline{y}}_{3,\infty}(\xi)$ is defined in (B50) and $c \neq 0$.

For a unique value of b , the vector $\underline{\widetilde{DC}}_2(\xi)$ defined by

$$\underline{\widetilde{DC}}_2(\xi) = \underline{DC}_2(\xi) + b\underline{DC}_1(\xi) \tag{B96}$$

satisfies

$$\underline{\widetilde{DC}}_2(\xi) \sim a\left(\hat{\underline{y}}_{1,\infty}(\xi)e^{\frac{\pi}{2}\mathrm{Im}(\lambda_1)} + \kappa_R\hat{\underline{y}}_{2,\infty}(\xi)e^{\frac{\pi}{2}\mathrm{Im}(\lambda_2)}\right) \quad ,$$

where $a \neq 0$ and $|\kappa_R| < 1$ (due to hypothesis (B94d)). Let $\vec{dc}_1(\xi)$ and $\vec{\widetilde{dc}}_2(\xi)$ consist of the first and third components of $\underline{DC}_1(\xi)$ and $\underline{\widetilde{DC}}_2(\xi)$, respectively. Relations (B52), (B59), (B95), and (B96) imply

$$\vec{dc}_1(\xi) = c\begin{bmatrix} \frac{\gamma}{\sigma^2} + O\left(\frac{1}{\sqrt{\xi}}\right) \\ O\left(\frac{1}{\sqrt{\xi}}\right) \end{bmatrix} \xi^{-\frac{1}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}}$$

and

$$\vec{dc}_2(\xi) \;=\; a\,\frac{\gamma}{\gamma - 1}\left\{\begin{bmatrix} 1 - \lambda_1 - \gamma\alpha + O\left(\frac{1}{\xi}\right) \\ \\ \frac{\gamma}{k}(1 - \lambda_1)\left(k^2 - \frac{\sigma^2}{\gamma}\right) + O\left(\frac{1}{\xi}\right)\end{bmatrix}\xi^{-\lambda_1}e^{\frac{\pi}{2}\mathrm{Im}(\lambda_1)}\right.$$

$$\left. + \kappa_R\begin{bmatrix} 1 - \lambda_2 - \gamma\alpha + O\left(\frac{1}{\xi}\right) \\ \\ \frac{\gamma}{k}(1 - \lambda_2)\left(k^2 - \frac{\sigma^2}{\gamma}\right) + O\left(\frac{1}{\xi}\right)\end{bmatrix}\xi^{-\lambda_2}e^{\frac{\pi}{2}\mathrm{Im}(\lambda_2)}\right\},$$

where $c \neq 0$, $a \neq 0$ and $|\kappa_R| < 1$.

NOTE:

$$|1 - \lambda_1| \;=\; |1 - \lambda_2| \;\neq\; 0 \qquad \text{(B97)}$$

and

$$\left|\xi^{-\lambda_1}e^{\frac{\pi}{2}\mathrm{Im}(\lambda_1)}\right| \;=\; \left|\xi^{-\lambda_2}e^{\frac{\pi}{2}\mathrm{Im}(\lambda_2)}\right| \;=\; |\xi|^{-\frac{1}{2}} \quad . \qquad \text{(B98)}$$

Thus,

$$|\det [\vec{dc}_1(\xi), \vec{\tilde{dc}}_2(\xi)]| \geq |c||a| \frac{\gamma^3}{(\gamma - 1)\sigma^2 k} \left|k^2 - \frac{\sigma^2}{\gamma}\right|$$

$$\times \left||1 - \lambda_1| - |\kappa_R||1 - \lambda_2|\right|$$

$$\times |\xi|^{-\frac{3}{4}} e^{2\sigma \mathrm{Re}\sqrt{\frac{\xi}{\gamma}}} + o(1)|\xi|^{-\frac{3}{4}} e^{2\sigma\sqrt{\frac{\xi}{\gamma}}} \quad \text{(B99)}$$

For $|\kappa_R| < 1$, relations (B97), (B98), and (B99) imply that $\vec{dc}_1(\xi)$ and $\vec{\tilde{dc}}_2(\xi)$ are linearly independent for sufficiently large ξ. Hence, the viscous problem has a unique solution for sufficiently small $\varepsilon > 0$. Q.E.D.

It has been tacitly assumed that $k > 0$. However, the limiting case $k = 0$ is of value. If σ is bounded away from zero as $k \to 0$, then relation (B53) implies that the inviscid solutions tend towards solutions with only vertical motion. Thus, as $k \to 0$ the wave motion approaches the more restrictive case of vertical oscillations of an isothermal atmosphere. The limiting case $k = 0$ has already been solved by Yanowitch [3].

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