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THE USE OF STATISTICS IN GUIDANCE ANALYSIS

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ABSTRACT

A statistical analysis of space guidance systems, based on the properties of multivariate Gaussian distributions and linear perturbation theory, is discussed. The results of this analysis can be applied to any mission regardless of the nature or complexity of the trajectory or the guidance system, as long as the linear perturbation theory is valid. The theory needed to statistically describe the injection errors, trajectory dispersions at the target due to the injection errors, and the calculation of the average midcourse maneuver of a guided spacecraft are reviewed.

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SUMMARY

A statistical analysis of space guidance systems, based on the properties of multivariate Gaussian distributions and linear perturbation theory, is discussed. The results of this analysis can be applied to any mission regardless of the nature or complexity of the trajectory or the guidance system, as long as the linear perturbation theory is valid. The theory needed to statistically describe the injection errors, trajectory dispersions at the target due to the injection errors, and the calculation of the average midcourse maneuver of a guided spacecraft are reviewed.

SECTION I. INTRODUCTION

The application of statistics to guidance analysis has received considerable attention during the past few years. Noton [1] has pointed out that the treatment of injection errors as independent random variables may sometimes be a poor approximation, and cross-correlations between injection errors must be taken into account. This paper reviews the theory (taking into account the cross-correlations between injection errors) needed to statistically describe the dispersions at the target and to determine the magnitude of the average midcourse maneuver for a typical interplanetary flight. To accomplish this, a statistical knowledge of the system errors, e.g., gyro drift, accelerometer errors, engine shut-down errors, etc., is used to statistically describe the injection errors. This representation of the injection errors, along with a state transition matrix, is used to describe the dispersion at the target planet. Such information is directly related to the probability of mission success.

The analysis used is credited to Noton [1] and is based on the properties of multivariate Gaussian distributions and linear perturbation theory. The results of this analysis can be applied to any mission regardless of the nature or complexity of the trajectory or the guidance system, as long as linear perturbation theory is valid.

SECTION II. JUSTIFICATION FOR USING LINEAR COMBINATIONS OF SOURCE ERRORS TO APPROXIMATE TRAJECTORY ERRORS

Injection error analysis is often based on the method of linearized perturbations. The approach taken is to represent the injection errors as linear functions of the source errors. A statistical description of the source errors will then yield a statistical representation of the injection errors.

It is assumed that the injection coordinates $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ are functions of the source errors $e_i (1 \leq i \leq n)$ and assume nominal values when $e_i = 0 (1 \leq i \leq n)$. That is,

$$\begin{aligned} x &= x(e_1, e_2, \ldots, e_n) \\ y &= y(e_1, e_2, \ldots, e_n) \\ &\vdots \\ \dot{z} &= \dot{z}(e_1, e_2, \ldots, e_n) \end{aligned} \qquad (1)$$

and

$$\begin{aligned} x &= x(0, 0, \ldots, 0) \\ y &= y(0, 0, \ldots, 0) \\ &\vdots \\ \dot{z} &= \dot{z}(0, 0, \ldots, 0) \end{aligned}$$

are nominal values of the injection coordinates.

Consider a Taylor series expansion of the first of equations (1) about the point (0, 0, ..., 0). Then,

$$x(e_1, e_2, \ldots, e_n) = x(0, 0, \ldots, 0) + \left.\frac{\partial x}{\partial e_1}\right|_0 e_1 + \ldots + \left.\frac{\partial x}{\partial e_n}\right|_0 e_n + R,$$

where

$$\left.\frac{\partial x}{\partial e_i}\right|_0$$

denotes the partial derivative of x with respect to e_i evaluated at (0, 0, ..., 0), and R is the remainder term in the Taylor series expansion.

For sufficiently small source errors, denoted by δe_i, one can write since R approaches zero,

$$\delta x = x(e_1, e_2, \ldots, e_n) - x(0, 0, \ldots, 0)$$

$$= a_{11}\delta e_1 + a_{12}\delta e_2 + \ldots + a_{1n}\,\delta e_n,$$

where

$$a_{1j} = \left.\frac{\partial x_1}{\partial e_j}\right|_0 .$$

Thus, for sufficiently small $e_i (1 \leq j \leq n)$, the injection error, δx, is accurately approximated by a linear combination of the e_i. Similar results can be shown for y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$.

Then

$$\delta x = a_{11}\delta e_1 + a_{12}\delta e_2 + \ldots + a_{1n}\,\delta e_n$$

$$\delta y = a_{21}\delta e_1 + a_{22}\delta e_2 + \ldots + a_{2n}\,\delta e_n$$

$$\vdots$$

$$\delta \dot{z} = a_{61}\delta e_1 + a_{62}\delta e_2 + \ldots + a_{6n}\,\delta e_n$$

or in matrix notation

$$\delta X = A\delta e, \qquad (2)$$

where

$$\delta X = \begin{bmatrix} \delta x \\ \delta y \\ \vdots \\ \delta z \end{bmatrix}, \quad A = [a_{ij}], \quad \delta e = \begin{bmatrix} \delta e_1 \\ \delta e_2 \\ \vdots \\ \delta e_n \end{bmatrix}.$$

The terms in the A matrix would be computed on a digital computer in a comprehensive trajectory program. For n error sources the calculation of the A matrix would require n + 1 machine runs. On the first of these runs, all source errors are zero and, of course, x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ are nominal. On the second and subsequent runs, each independent error source is perturbed, one at a time, and perturbed values of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ are found. Each such run yields a column of A. For instance,

$$a_{1j} = \frac{x(0, 0, \ldots, \delta e_j, 0, \ldots 0) - x(0, 0, \ldots, 0)}{\delta e_j}$$

$$\vdots$$

$$a_{6j} = \frac{\dot{z}(0, 0, \ldots, \delta e_j, 0, \ldots 0) - \dot{z}(0, 0, \ldots, 0)}{\delta e_j}.$$

Thus, under the assumption that linear perturbation theory is valid, trajectory errors can be accurately approximated by linear combinations of source errors.

SECTION III. DERIVATION OF THE PROBABILITY DENSITY FUNCTION INJECTION ERRORS

The injection errors are described statistically by their joint probability density function, $f(\delta X)$ (see Appendix A). The derivation of $f(\delta X)$ is based on the properties of multivariate normal distributions (Appendix B) and the results of the linearization procedure of Section II.

The hardware errors $e_i (1 \leq i \leq n)$ are assumed to be independent random variables, each normally distributed with mean $\mu_i = 0$, and variance, σ_i^2. The zero mean assumption is valid, since if the random hardware errors have non-zero means, they could be corrected before launch. In practice, the statistics of the individual errors can be satisfactorily approximated by normal distributions. Furthermore, since the number of system errors is large and the errors are often independent of one another, the central limit theorem asserts that the sum of these errors approaches a normal distribution in the limit regardless of the distribution of the individual system errors.

Using the fact that linear combinations of independent normally distributed random variables are normally distributed (see Appendix C), and the results of Section II, it follows that the joint distribution of the injection errors is six-dimensional normal. That is,

$$f(\delta X) = \frac{1}{(2\pi)^3\sqrt{M}} \; e^{-1/2(\delta X-B)^T \, M^{-1}(\delta X-B)}$$

where

$$B = \begin{bmatrix} E(\delta x) \\ E(\delta y) \\ \vdots \\ E(\delta \dot{z}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and M, the covariance matrix of injection errors, is given by

$$M = E[(\delta X)(\delta X)^T] = A\Lambda A^T, \text{ (Appendix I)},$$

where

$$\Lambda = E[(\delta e)(\delta e)^T] = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ 0 & 0 & & \sigma_n^2 \end{bmatrix}.$$

Λ may be determined from pre-flight laboratory tests. Once A is found, the probability density function of injection errors can be readily obtained.

SECTION IV. DISPERSIONS AT THE TARGET

In the absence of post injection guidance, the dispersion at the target is a function of the errors at injection. The statistical description of injection errors, therefore, can be employed to obtain a convenient pre-flight description of trajectory dispersion at the target.

The dispersion at the target may be measured in terms of miss coordinates. The miss coordinates, M_1 and M_2, are measured in a plane perpendicular to the approach velocity vector.

From perturbations on the standard trajectory a matrix, U, may be formed such that

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = U\delta X = UA\delta e,$$

where A and δe are obt..ned from Section II.

Since linear combinations of independent normally distributed random variables are normally distributed, it follows that the joint distribution of M_1 and M_2 is bivariate normal. That is,

$$f(M_1, M_2) = \frac{1}{2\pi(|C|)^{1/2}} e^{-1/2\left[(M_1-\mu_{M_1}, M_2-\mu_{M_2})C^{-1}\begin{pmatrix} M_1-\mu_{M_1} \\ M_2-\mu_{M_2} \end{pmatrix}\right]} \tag{3}$$

where $\mu_{M_1} = \mu_{M_2} = 0$, and C, the covariance matrix of the miss components is given by

$$C = E[(U\delta X)(U\delta X)^T] = UMU^T,$$

where M is the covariance matrix of injection errors.

In order to determine a convenient description of trajectory dispersion, let $f(M_1, M_2)$ = CONSTANT. Then the exponent in (3) is also a constant. That is, rewriting the exponent of (3) gives

$$\frac{1}{1-\rho^2}\left[\left(\frac{M_1}{\sigma_{M_1}}\right)^2 - 2\rho\left(\frac{M_1}{\sigma_{M_1}}\right)\left(\frac{M_2}{\sigma_{M_2}}\right) + \left(\frac{M_2}{\sigma_{M_2}}\right)^2\right] = k^2, \qquad (4)$$

where k^2 is a constant, ρ is the coefficient of correlation between M_1 and M_2, and σ_{M_1}, σ_{M_2} are the standard deviations of M_1 and M_2, respectively.

Equation (4) is the equation of an ellipse (Appendix IV) having semimajor and semiminor axes $k\lambda_1$ and $k\lambda_2$, where

$$\lambda_1^2 = \frac{1}{2}\left(\sigma_{M_1}^2 + \sigma_{M_2}^2\right) + \left[\left(\frac{\sigma_{M_1}^2 - \sigma_{M_2}^2}{2}\right)^2 + \left(\rho\sigma_{M_1}\sigma_{M_2}\right)^2\right]^{1/2} \qquad (5)$$

$$\lambda_2^2 = \frac{1}{2}\left(\sigma_{M_1}^2 + \sigma_{M_2}^2\right) - \left[\left(\frac{\sigma_{M_1}^2 - \sigma_{M_2}^2}{2}\right)^2 + \left(\rho\sigma_{M_1}\sigma_{M_2}\right)^2\right]^{1/2}. \qquad (6)$$

Each of these ellipses has its major axis inclined at an angle θ to the M_1 axis, where

$$\theta = \frac{1}{2}\tan^{-1}\frac{2\rho\,\sigma_{M_1}\sigma_{M_2}}{\sigma_{M_1}^2 - \sigma_{M_2}^2} \qquad \text{for} \quad \sigma_{M_1}^2 \neq \sigma_{M_2}^2$$

$$\theta = \pi/4 \qquad \text{for} \quad \sigma_{M_1}^2 = \sigma_{M_2}^2. \qquad (7)$$

The probability of the miss being within such an ellipse is

$$P = 1 - e^{-1/2\,k^2}$$

(when k = 1, 2, 3, P = 0.40, 0.86, 0.99) (Appendix IV).

SECTION V. DETERMINATION OF THE MAGNITUDE OF THE AVERAGE MIDCOURSE MANEUVER

For a given mission, the analysis of the preceding section may indicate a high probability of excessively large trajectory errors at the target. It may then become necessary to allot propellant for making midcourse corrections. Let P be the probability that enough fuel is carried on the spacecraft to perform the required corrective maneuver. Since the magnitude of the midcourse maneuver depends on the covariance matrix of injection errors, preflight knowledge of the probable magnitude of the midcourse maneuver can be obtained. This makes it possible to allot a sufficient amount of propellant to insure a probability, P, of accomplishing the midcourse maneuver without carrying an excessive amount of fuel.

For a midcourse point on a given trajectory, it can be shown that, to a first order of approximation, the three components of the maneuver are given by

$$\bar{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = K\delta X = KA\delta e,$$

where K is a (3 x 6) matrix consisting of elements computed on a standard trajectory. Since $E(V_x) = E(V_y) = E(V_z) = 0$, we have

$$E[\bar{V}\ \bar{V}^T] = KE[\delta X\ \delta X^T]\ K^T$$

$$= KMK^T.$$

KMK^T is the covariance matrix of the three-dimensional normal distribution of the components V_x, V_y, and V_z. Although V_x, V_y, and V_z satisfy a joint normal distribution, the distribution of $(V_x^2 + V_y^2 + V_z)^{1/2} = V$ is not normal.

Now, there is an orthogonal transformation which diagonalizes the covariance matrix KMK^T so that the elements of the diagonal are the eigenvalues of KMK^T [5]. In this new coordinate system let V_{x_1}, V_{y_1}, and V_{z_1} be the random variables denoting the components of the midcourse maneuver. The eigenvalues of KMK^T are $\sigma_{x_1}^2$, $\sigma_{y_1}^2$, and $\sigma_{z_1}^2$, the variances of V_{x_1}, V_{y_1}, and V_{z_1} respectively. The following procedure (obtained from ref. 8) may be used to approximate the ΔV capability required to insure a probability, P, of accomplishing the midcourse maneuver.

Let k_1 denote the largest eigenvalue of KMK^T. If $\sqrt{k_1}$ is at least an order of magnitude larger than both $\sqrt{k_2}$ and $\sqrt{k_3}$, a one-dimensional normal distribution of

$$V_1 = \begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix}$$

is assumed. We have

$$P = \int_{-n\sqrt{k_1}}^{n\sqrt{k_1}} \frac{1}{\sqrt{2\pi k_1}} e^{-\frac{1}{2}\frac{x^2}{k_1}} dx,$$

where n can be determined from Tables of Normal Distribution. Then, $V = n\sqrt{k_1}$.

If $\sqrt{k_1}$ and $\sqrt{k_2}$ are of the same order of magnitude and are at least an order of magnitude larger than $\sqrt{k_3}$, a two-dimensional normal distribution of V_1 is assumed. Thus, in polar coordinates,

$$P = \int_0^{n\sqrt{k_1}} \int_0^{2\pi} \frac{1}{2\pi k_1} e^{-\frac{1}{2}\frac{r^2}{k_1}} r d\theta dr$$

$$= 1 - e^{-n^2/2}$$

and

$$n = \sqrt{\ell n(1/1-P)^2} \quad \text{and} \quad V = n\sqrt{k_1}.$$

If $\sqrt{k_1}$, $\sqrt{k_2}$, and $\sqrt{k_3}$ are all of the same order of magnitude, a three-dimensional normal distribution of V_1 is assumed. Then, in spherical coordinates,

$$P = \int_0^{n\sqrt{k_1}} \int_0^{\pi} \int_0^{2\pi} \frac{1}{(2\pi k_1)^{3/2}} e^{-\frac{1}{2}\frac{r^2}{k_1}} r^2 \sin \emptyset \, d\theta \, d\emptyset \, dr$$

$$= \int_{-n\sqrt{k_1}}^{n\sqrt{k_1}} \frac{1}{\sqrt{2\pi k_1}} e^{-\frac{1}{2}\frac{r^2}{k_1}} dr - n\sqrt{2/\pi}\; e^{-n^2/2}$$

and

$$V = n\sqrt{k_1}.$$

Thus, a preflight statistical analysis can be used to obtain a good estimate of the magnitude of the midcourse maneuver.

APPENDIX A

REVIEW OF ELEMENTARY STATISTICAL THEORY

A random variable x is one whose value can be predicted only on a probabilistic basis. A function f(x), such that

$$P[x \text{ is in } (a,b)] = \int_a^b f(x)\,dx,$$

is called the probability density function of the random variable x. The distribution function, F(x), is then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

A physical interpretation is given f(x) and F(x) by considering a unit mass distributed along a straight line such that the fraction of mass concentrated to the left of X = x is F(x). Then

$$\frac{dF(x)}{dx} = f(x),$$

the density of the unit mass at point x.

The mean or expected value of x, E(x), or μ, is given by

$$\mu = E(x) = \int_{-\infty}^{\infty} xf(x)\,dx.$$

E(x), the coordinate of the center of gravity of the mass distribution, provides a measure of the location of the distribution.

The variance of x, V(x) is given by

$$V(x) = \int_{-\infty}^{\infty} [x - E(x)]^2 f(x)\,dx;$$

and again V(x) is the moment of inertia (or the second moment about the mean) of the associated mass distribution. V(x) is a measure of variation of the distribution.

In many cases, the result of a random process is not expressed by one observed quantity, but by a certain number of simultaneously observed quantities. For example, in a study of guidance errors, the random experiment involves the simultaneous observation of six random variables - the errors in three components of position and three components of velocity.

The one-dimensional concepts are now generalized, and we are concerned with the probability that the random variables $x_1, x_2, \ldots, x_n$ will simultaneously belong to the intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, respectively. That is,

$$P(a_1 \leq x \leq b_1, \ldots, a_n \leq x_n \leq b_n)$$

$$= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_1, \ldots, dx_n \ ,$$

where $f(x_1, x_2, \ldots, x_n)$ is the joint probability density function of $x_1, \ldots, x_n$. In general, the random variables will not be independent of one another. That is, $P[a_i \leq x_i \leq b_i]$ will depend upon the values assumed by all the other random variables in the joint distribution.

The mean or expected values, $E(x_i)$, of the x_i are defined by

$$m_i = E(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_i f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \ .$$

If the second order moments are computed about the mean values, then the covariance $\sigma_{x_i x_j}$ is obtained, that is

$$\sigma_{x_i x_j} = E[(x_i - m_i)(x_j - m_j)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - m_i)(x_j - m_j)$$

$$f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \ .$$

If $i = j$, this second order moment is called the variance $\sigma^2_{x_i}$. The standard deviation is simply the square root of the variance, and is used as a measure of the dispersion about the mean.

Correlation coefficients of the variables x_i and x_j are defined as follows:

$$\rho_{x_i x_j} = \sigma_{x_i x_j} / \sigma_{x_i} \sigma_{x_j} \ .$$

The correlation coefficients are bounded between -1 and 1 and are equal to 1 for i = j.

If x_i and x_j are independent, then $\rho_{x_i x_j} = 0$. If $\rho = \pm 1$, there is a complete linear dependence between the variables x_i and x_j. Correlation coefficients provide a measure of the degree of linear dependence between the respective variables.

Let

$$A^{[mxn]} = [a_{ij}].$$

The expected value of the matrix A, E[A], is defined by

$$E[A] = [E(a_{ij})].$$

Let

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then,

$$E(x) = \begin{bmatrix} E(x_1) \\ \vdots \\ E(x_n) \end{bmatrix}.$$

The covariance matrix, C, of the random vector x is defined by

$$C = E[(x - E(x)) (x - E(x))^T].$$

Then

$$C^{[nxn]} = E[x_i - E(x_i)) (x_j - E(x_j))]^{[nxn]},$$

or

$$C = \begin{bmatrix} \sigma^2_{x_1} & & & \text{symmetric} \\ \rho_{x_1x_2}\,\sigma_{x_1}\,\sigma_{x_2} & \sigma^2_{x_2} & & \\ \vdots & & & \\ \rho_{x_1x_n}\,\sigma_{x_1}\,\sigma_{x_n} & \rho_{x_2x_n}\,\sigma_{x_2}\,\sigma_{x_n} & \cdots & \sigma^2_{x_n} \end{bmatrix}.$$

It is also necessary to consider linear combinations of the random vector x and to determine the new covariance matrix. For simplicity, let $E(x) = 0$, then $C = E[xx^T]$. Suppose

$$y^{[m\text{x}1]} = H^{[m\text{x}n]}\, x^{[n\text{x}1]}.$$

The covariance matrix of y is

$$M^{[m\text{x}m]} = E\{yy^T\}$$

$$= E\{Hxx^TH^T\}\ .$$

Let $xx^T = D$; then

$$M^{[m\text{x}m]} = E\{HDH^T\}$$

$$= E\left\{\left[\sum_{j=1}^{n}\ \sum_{k=1}^{n} h_{ij}\ d_{jk}\ h_{ik}\right]\right\},$$

where

$$H = [h_{ij}], \quad D = [d_{ij}], \quad H^T = [h_{ji}].$$

Since

$$E\{a_{x_1} + b_{x_2}\} = aE(x_1) + bE(x_2),$$

$$M = \left[\sum_{j=1}^{n} \sum_{k=1}^{n} h_{ij}\ E\{d_{jk}\}\ h_{ik}\right]$$

$$= H\ E\{D\}\ H^T$$

$$= H\ E\{xx^T\}\ H^T$$

$$= H\ C\ H^T.$$

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APPENDIX B

NORMAL DISTRIBUTION

A fundamental role in probability theory is played by the function $\phi(x)$, defined as follows: For any real number x,

$$\phi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \tag{8}$$

A random variable x is said to be normally distributed if its probability density $\phi(x)$ is given by (1) for any real number x. That is the probability that $a \leq x \leq b$,

$$P[a \leq x \leq b] = \int_a^b \phi(x)\, dx.$$

The parameters μ and σ are the mean and standard deviation of x, respectively.

The probability that x will fall in the interval $\mu \pm k\sigma$ is given by

$$P = \int_{\mu-k\sigma}^{\mu+k\sigma} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Let

$$t = \frac{x-\mu}{k\sigma} .$$

Then

$$P = \int_{-k}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

When $k = 1, 2, 3$, it is found from tables of the normal density function that $P = .68$, $.95$, and $.99$, respectively.

We now consider the normal distribution of two random variables. The two-dimensional random variable (x, y) is said to have a bivariate normal distribution if its joint density function, $f(x, y)$, is given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} . \tag{9}$$

That is,

$$P[(x,y)\in A] = \int_A f(x,y)\, dy\, dx,$$

where $A \subset R^{(2)}$, $\mu_x, \mu_y, \sigma_x, \sigma_y$ are the mean and variance of x and y, respectively, and ρ is the coefficient of correlation between x and y. Notice that $f(x,y)$ may be written in the form

$$f(x,y) = \frac{1}{\sqrt{(2\pi)^2}(|M|)^{1/2}} e^{-\frac{1}{2}\left\{[x-\mu_x \;\; y-\mu_y]M^{-1}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\right\}} ,$$

where

$$M = \begin{bmatrix} \sigma_x^2 & \rho\,\sigma_x\sigma_y \\ \rho\,\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}.$$

M is the covariance or moment matrix of x and y. Again the probability that $\mu_x - \sigma_x \leq x \leq \mu_x + \sigma_x$ and $\mu_y - \sigma_y \leq y \leq \mu_y + \sigma_y$ is given by

$$P = \int_{\mu_x-\sigma_x}^{\mu_x+\sigma_x} \int_{\mu_y-\sigma_y}^{\mu_y+\sigma_y} f(x,y)\, dy\, dx.$$

It can be shown that

$$P[\mu_x - K\sigma_x \leq x \leq \mu_x + K\sigma_x, \mu_y - K\sigma_y \leq y \leq \mu_y + K\sigma_y] = 1 - e^{\frac{1}{2}K^2},$$

when K = 1, 2, 3, P = .40, .86, and .99, respectively (Reference 4).

The preceding discussion may be generalized to n-dimensions. Let

$$x = \{x_1, x_2, \ldots, x_n\}$$

be a random vector. We shall say that x is normally distributed in n dimensions if its probability density function is

$$f(x) = \frac{1}{(2\pi)^{n/2}\sqrt{|M_n|}} e^{-\frac{1}{2}(x-B)^T M_n^{-1}(x-B)}, \qquad (10)$$

where

$$B = E\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and M_n is the covariance matrix of $x_1, x_2, \ldots, x_n$.

Equation (10) is completely determined once M_n and B are known. Also,

$$P[x \in A] = \int_A f(x)\, dx_1 \ldots dx_n,$$

where $A \subset R^{(n)}$.

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APPENDIX C

LINEAR COMBINATIONS OF NORMAL RANDOM VARIABLES

This appendix shows that a linear combination of independent normally distributed random variables is normally distributed. The proof is based on the properties of characteristic functions.

Let $x_1, x_2, \ldots, x_n$ be independent normally distributed variables, the parameters of x_j being μ_j and σ_j. The characteristic function of x_j is given by

$$E(e^{itx_j}) = e^{\mu_j it - \frac{1}{2}\sigma_j^2 t^2}.$$

Let

$$x = x_1 + x_2 + \ldots + x_n.$$

The characteristic function of x is the product of the characteristic functions of all the x_j. Thus,

$$E(e^{itx}) = E(e^{itx_1} \cdot e^{itx_2} \ldots e^{itx_n})$$

$$= \prod_{p=1}^{n} E(e^{itx_j})$$

$$= \prod_{j=1}^{n} e^{\mu_j it - \frac{1}{2}\sigma_j^2 t^2}$$

$$= e^{\sum_{j=1}^{n} \mu_j it - \frac{1}{2}\sum_{j=1}^{n} \sigma_j^2 t^2}.$$

This is the characteristic function of a normal distribution with parameters

$$\mu = \sum_{j=1}^{n} \mu_j \quad \text{and} \quad \sigma = \sum_{j=1}^{n} \sigma_j^2.$$

Thus, the sum of independent normally distributed variables is itself normally distributed.

Since any linear function of a normal variable is itself normal, it follows, by the preceding paragraph, that a linear function

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n + b$$

of independent normal variables is itself normal with parameters

$$\mu = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n + b$$

and

$$\sigma^2 = a_1^2\,\sigma_1^2, + \ldots + a_n^2\,\sigma_n^2.$$

APPENDIX D

DISPERSION ELLIPSES

In this section it is shown that the probability that the miss coordinates (M_1, M_2) are contained in the ellipse

$$k^2 = \frac{1}{1 - \rho^2} \left[\frac{M_1^2}{\sigma_{M_1}^2} - \frac{2\rho M_1 M_2}{\sigma_{M_1}\sigma_{M_2}} + \frac{M_2^2}{\sigma_{M_2}^2}\right] \qquad (11)$$

is given by $1 - e^{-1/2\, k^2}$.

Equation (11) represents an ellipse in the (M_1, M_2) plane with center at the origin, or aim point, and axes that are not parallel to the coordinates axes as long as $\rho \neq 0$.

The coordinate system may be rotated so that the axes of the ellipse are parallel to the new coordinate axes. The linear transformation required is defined by

$$y_1 = M_1 \cos\theta + M_2 \sin\theta$$

$$y_2 = -M_1 \sin\theta + M_2 \cos\theta,$$

(Reference 4)

where

$$\tan 2\theta = \frac{2\rho\, \sigma_{M_1} \sigma_{M_2}}{\sigma_{M_1}^2 - \sigma_{M_2}^2} \qquad \text{for } \sigma_{M_1} \neq \sigma_{M_2}$$

and

$$\theta = \frac{\pi}{4} \qquad \text{for } \sigma_{M_1} = \sigma_{M_2}.$$

Carrying out this rotation gives

$$k^2 = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2},$$

where λ_1 and λ_2 are as defined in section III.

Now the quantity

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2}$$

is distributed as an χ^2 variable with two degrees of freedom. Thus, the probability, P, that (M_1, M_2) lies in the ellipse defined by (1) is given by

$$\frac{1}{2\Gamma(\mu)} \int_0^{k^2} (\chi^2/2)^{\mu-1} \; e^{-1/2\ \chi^2} \; d(\chi^2),$$

where $\mu = 1$. Thus,

$$P = \int_0^{k^2} \frac{1}{2} e^{-1/2\ \chi^2} \, d\chi^2$$

$$= 1 - e^{-1/2\ k^2}.$$

For k = 1, 2, 3, P = .40, .86, and .99, respectively.

APPENDIX E

EXAMPLE PROBLEM

In order to illustrate the use of the material presented in the body of this report, the following example problem is presented. The hardware error sources, and the matrices A, U, and K represent realistic but not necessarily exact values.

The 1σ values of six significant error sources (three each of constant gyro drift rate, and accelerometer constant bias error), and the matrices A, U, and K are given. From these the covariance matrix of injection errors, trajectory dispersion at the target in the absence of post injection guidance, and the probable magnitude of a midcourse maneuver for a typical interplanetary flight are determined.

The platform hardware error sources are

<u>Constant Gyro Drift Rate</u>

	σ	
GX	0.025	deg/hr
GY	0.025	deg/hr
GZ	0.333	deg/hr

<u>Accelerometer Bias</u>

	σ	
BX	0.1×10^{-6}	km/sec^2
BY	0.1×10^{-6}	km/sec^2
BZ	0.1×10^{-6}	km/sec^2

Then, in the notation of Section II,

$$\Lambda = \begin{bmatrix} .000625 & 0 & 0 & 0 & 0 & 0 \\ 0 & .000625 & 0 & 0 & 0 & 0 \\ 0 & 0 & .001089 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^{-14} \end{bmatrix}$$

The hardware errors are linearly related to the injection errors by the matrix A, where

$$A = \begin{bmatrix} -4.609 & -66.994 & 3.662 & -.48066 \times 10^{6} & -.27300 & -19.123 \\ -5.785 & -1.122 & -30.257 & .96666 \times 10^{4} & -.45600 & -.24966 \\ -4.104 & -17.717 & 0.365 & .35666 \times 10^{6} & -.11933 & -9.3346 \\ 2.306 & -16.240 & 28.133 & .11833 \times 10^{7} & .12666 & 8.46000 \\ 33.186 & -0.133 & -195.173 & .66666 \times 10^{4} & .57000 & -.40666 \\ -26.826 & 81.386 & 0.520 & -.42333 \times 10^{6} & -.30333 & -20.623 \end{bmatrix}.$$

A may be found by the method described in Section II.

The covariance matrix of injection errors is given by $M = A \Lambda A^T$. Thus,

$$M = \begin{bmatrix} 6.493 & & & \text{Symmetric} & & \\ -0.008 & 1.021 & & & & \\ 2.538 & 0.039 & 1.079 & & & \\ -0.838 & -0.945 & -0.600 & 1.759 & & \\ -0.792 & 6.309 & -0.124 & -5.964 & 42.176 & \\ 0.618 & 0.075 & 1.091 & -2.598 & -0.591 & 8.845 \end{bmatrix}.$$

The matrix U, relating injection errors to trajectory errors at the target, is taken to be

$$U = \begin{bmatrix} -90.744 & -50.985 & -1.314 & -1746.410 & -99.936 & -.733 \\ -17.058 & 28.050 & -92.698 & -37.376 & -575.802 & -1873.930 \end{bmatrix}.$$

The covariance matrix of the miss components M_1 and M_2 is given by

$$C = UMU^T$$

$$= \begin{bmatrix} 3,370,249 & -11,188,598 \\ -11,188,598 & 38,535,805 \end{bmatrix}.$$

Using (3), (4), (5) of Section IV gives

$$\lambda_1 = 6464.81 \text{ km}$$

$$\lambda_2 = 335.00 \text{ km}$$

$$\theta = 16.23 \text{ deg.}$$

Thus, the probability of the miss being within an ellipse, centered at the aim point, having semimajor axis λ_1, inclined at an angle θ to the M_1 axis, and semiminor axis λ_2 is 0.40.

The matrix K, relating injection errors to the components of the midcourse maneuver, is

$$K = 10^{-8} \begin{bmatrix} -.216 & .0216 & -.0096 & -12736.55 & 1196.78 & -537.22 \\ .0202 & .0560 & -.0030 & 1150.61 & 3317.27 & -179.25 \\ -.00916 & -.0029 & .1784 & -537.94 & -178.28 & 10587.67 \end{bmatrix} .$$

The covariance matrix of the components of the midcourse maneuver is

$$KMK^T = \begin{bmatrix} .495 \times 10^{-05} & & \text{Symmetric} \\ .383 \times 10^{-5} & .422 \times 10^{-5} & \\ .285 \times 10^{-} & -.836 \times 10^{-6} & .102 \times 10^{-4} \end{bmatrix} .$$

The eigenvalues of KMK^T are

$$k_1 = .11593 \times 10^{-4}$$

$$k_2 = .90764 \times 10^{-7}$$

$$k_3 = .77415 \times 10^{-5}.$$

Since $\sqrt{k_1}$, $\sqrt{k_3}$ are of the same order of magnitude and are at least an order of magnitude larger than $\sqrt{k_2}$, a two-dimensional normal distribution of V_1 is assumed. Then, to approximate the ΔV capability required to insure a probability of .99 of accomplishing the midcourse maneuver, let P = .99 in

$$n = \sqrt{\ell n \ (1/1-P)^2}$$

and

$$\Delta V = n \sqrt{k_1}.$$

The required ΔV capability is 10.33 m/sec.

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