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DERIVATION OF TWELVE-BY-TWELVE STIFFNESS MATRIX FOR SHEAR PANEL UNDERGOING PARABOLIC DEFORMATION

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RESEARCH AND DEVELOPMENT OPERATIONS

AERO-ASTRODYNAMICS INTERNAL NOTE NO. 3-67

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ABSTRACT

Equations are derived for finding the stiffness matrix for a twelve-node shear panel undergoing parabolic deformation. The deflections of a hypothetical cantilever beam composed of five twelve-node shear panels are then compared with those of six other equivalent cantilever beams. Two are classical strength of materials beams, one excluding shear effects, and the other including shear effects. The third is a theory of elasticity beam. The last three beams are composed of eight-node linear deformation shear panels, one composed of one row of five panels, the second composed of two rows of five panels per row, and the third composed of three rows of five panels per row. Tip deflection of the five twelve-node shear panels beam is 93 percent of the strength of materials beam excluding shear, whereas that of the best eight-node beam is 73 percent.

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DEFINITION OF SYMBOLS

Symbol	Definition
A, B, T	length, width, and thickness of a shear panel
C_e, C_f, C_g, C_h	interval between the S- and Y-axes
D_e, D_f, D_g, D_h	interval between the R- and X-axes
E	modulus of elasticity
e, f, g, h	denote the left, right, lower, and upper surfaces of a shear panel
F_{ij}	flexibility matrix element defined as the deflection at i caused by a unit force at j
K_e, K_f, K_g, K_h	latus rectum of a parabolic surface
P_i	force at i
R, S	Cartesian coordinate system with one rotational and two translational degrees of freedom and origin attached to and tangent to the vertex of a parabolic surface
R_e, R_f, R_g, R_h S_e, S_f, S_g, S_h	coordinates of the end points of a parabolic surface in the R,S system
R_ϕ, R_ρ, R_ψ, R_η S_ϕ, S_ρ, S_ψ, S_η	Coordinates of the point of intersection of a δ_9-, δ_{10}-, δ_{11}-, or δ_{12}-vector with a respective e-, f-, g-, or h -surface
S_{ij}	stiffness matrix element defined as the force at i caused by a unit deflection at j
u	strain energy per unit volume
U	total shear panel strain energy
X, Y	fixed Cartesian coordinate system

Symbol	Definition
δ_i	deflection at i
ϵ_x, ϵ_y	linear strain parallel to the X-, Y-axes
ϕ, ρ, ψ, η	angle of rotation of the R,S coordinate system relative to the X,Y system
γ	angular strain
μ	Poisson's ratio

INTRODUCTION

An eight-node shear panel typical of figure 1a has been in use for several years in computer matrix analysis of aircraft and missile

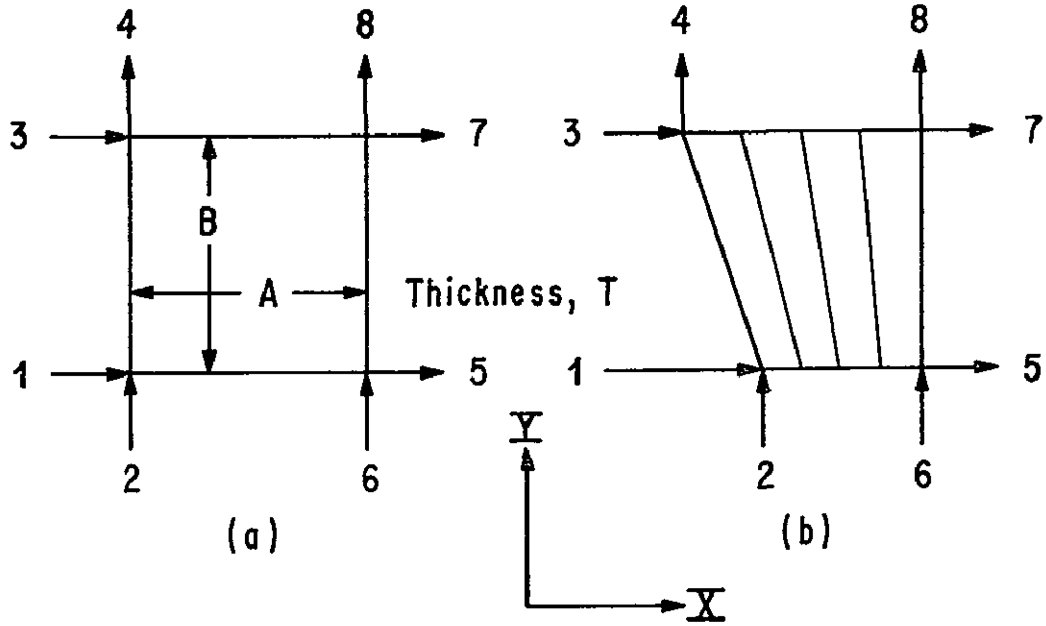

Figure 1. Eight-Node Shear Panel

structures. However, this particular model is needlessly unrealistic in that it simply assumes that surfaces deform linearly as shown in figure 1b. Actually, surface 1-3 deforms in some nonlinear shape f(x,y) which would be difficult to determine experimentally or derive theoretically without extensive recourse to the formal theory of elasticity. This paper proposes to elaborate on the model of figure 1 by adding four nodes and defining f(x,y) to be parabolic as in figure 2. The nodes 9, 10, 11, and 12 are assumed to be located initially at the mid-points of their respective surfaces, and the distribution of strain (figure 2) remains linear as in figure 1b.

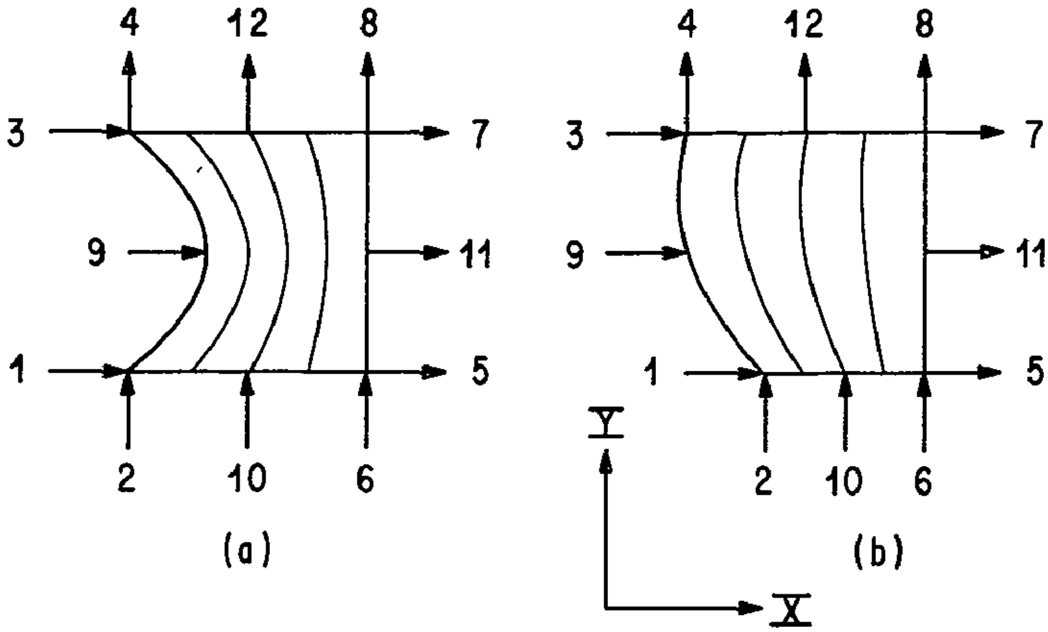

Figure 2. Twelve-Node Shear Panel

DERIVATIONS

With δ_9 nonzero, locate the origin of the X-Y coordinate system at the 1-2 node, and attach an R-S coordinate system to the vertex of the e-surface (figure 3). Thus, the equation of the e-surface is either

$$R = S^2/K_e \tag{1}$$

or

$$e = \frac{(y - D_e)^2}{K_e} + C_e, \tag{2a}$$

where $D_e = \frac{1}{2} B$, $C_e = \delta_9$, and $K_e = -B^2/4\delta_9$. A point on the e-surface is (e,y), and shear on the e-surface is

$$\gamma_e = \tan^{-1}(de/dy) = \tan^{-1} [2(y - D_e)/K_e]. \tag{3a}$$

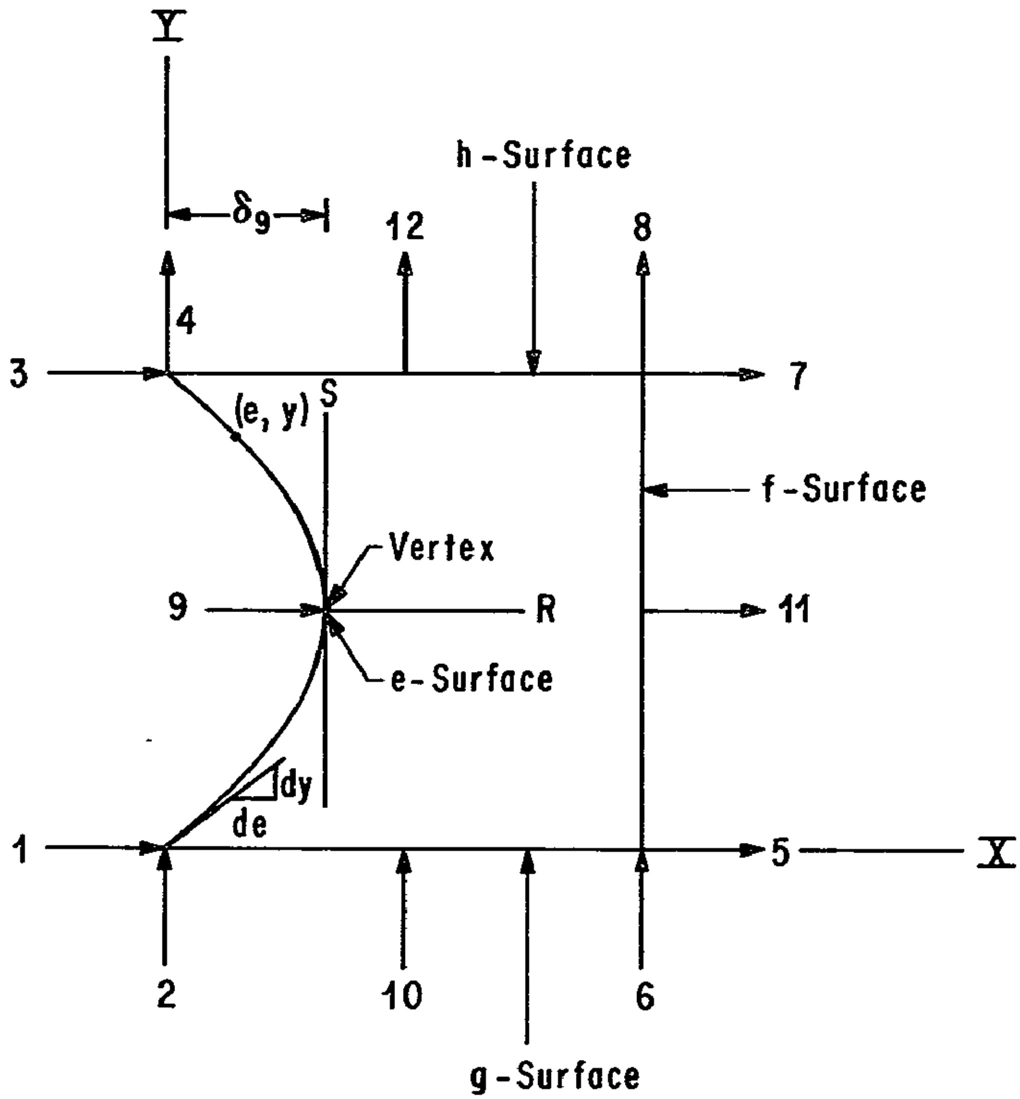

Figure 3. Location of Coordinate Systems and Surface Notation

Progressively, nonzero δ_1 and δ_3 introduce an angle of rotation ϕ such that

$$\phi = \tan^{-1}\left(\frac{\delta_1 - \delta_3}{B}\right) . \tag{4a}$$

Figure 4 shows that the δ_9-vector is no longer orthogonal to the e-surface but remains always parallel to and midway between the δ_1- and δ_3-vectors. Also, the vertex of the parabolic e-surface has rotated through the angle ϕ while maintaining the point $[1/2(\delta_1 + \delta_3), 1/2\ B]$ as the common point of intersection of (1) the 1-3 line, (2) the δ_9-vector, and (3) the R-axis.

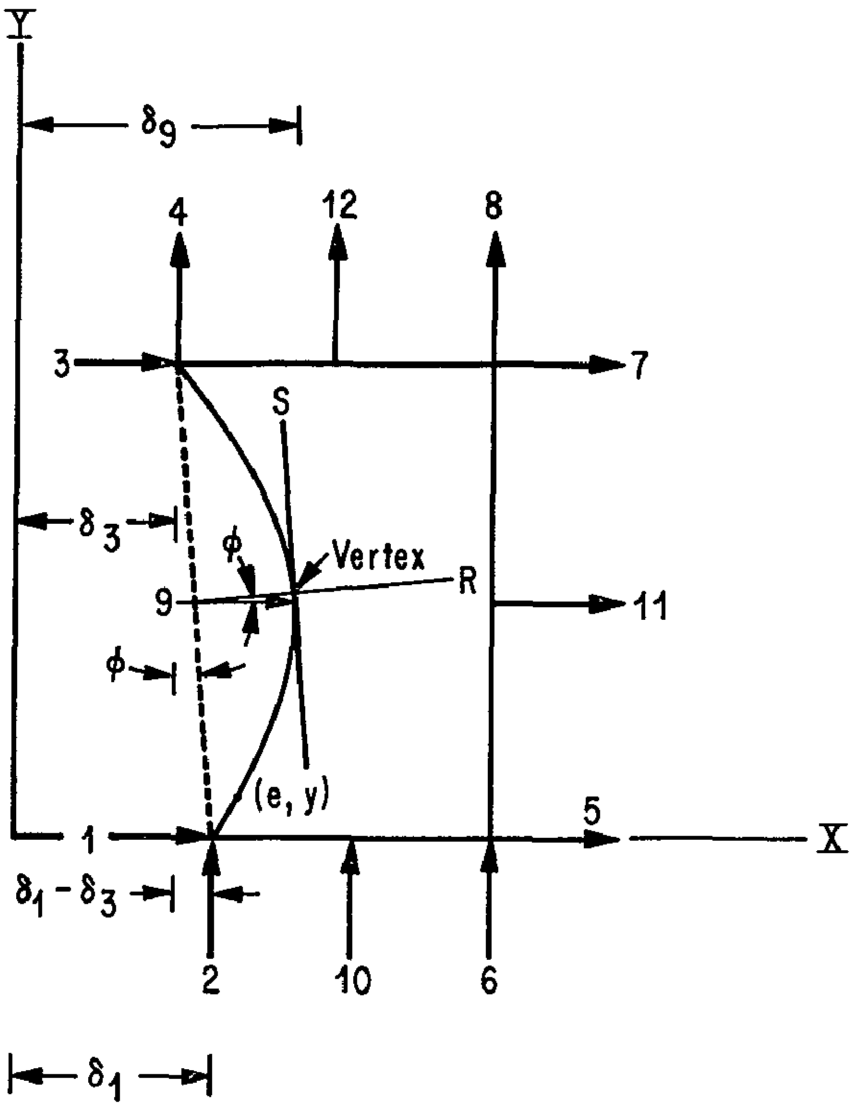

Figure 4. Rotation of R-S Coordinate System

Since all the five displacement vectors attached to a surface determine its total geometry, set δ_2 and δ_4 nonzero. Thus, by inspection of figure 5, it may be observed that

$$\phi = \tan^{-1}\left(\frac{\delta_1 - \delta_3}{B + \delta_4 - \delta_2}\right) \tag{4b}$$

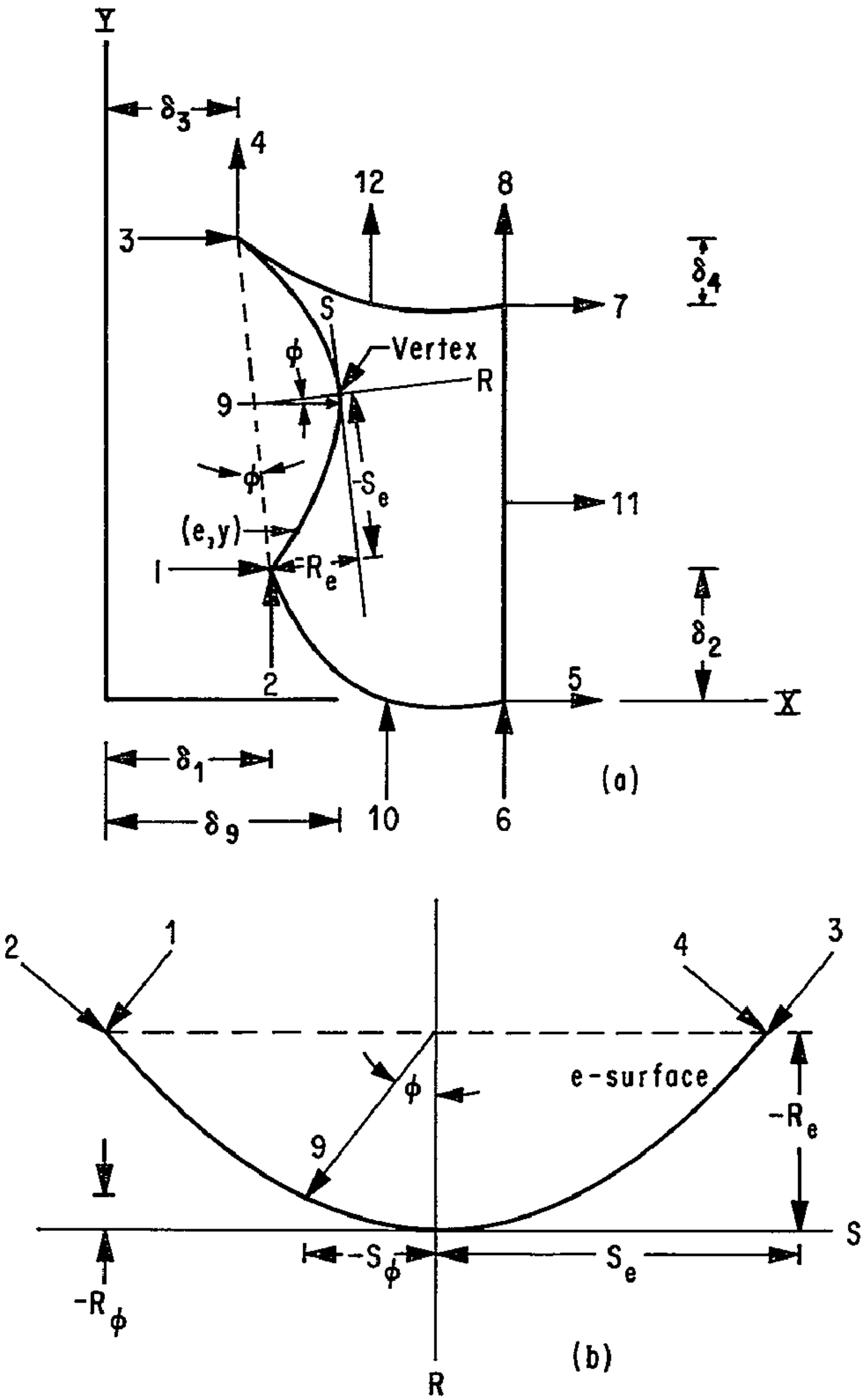

Figure 5. Total Geometry of the e-Surface

$$S_{\phi} = \frac{1}{2}\ (2\delta_9 - \delta_1 - \delta_3)\ \sin\ \phi \tag{5}$$

$$S_e = \left(\frac{B + \delta_4 - \delta_2}{2\ \cos\ \phi}\right) \tag{6}$$

$$R_e = -\ \frac{1}{2}(2\delta_9 - \delta_1 - \delta_3)\ \cos\ \phi + R_{\phi}. \tag{7}$$

From geometrical similarity, $R_{\phi}\ S_e^2 \equiv R_e\ S_{\phi}^2$, so that

$$R_{\phi} = -\ \frac{(2\delta_9 - \delta_1 - \delta_3)^3\ \sin^2\phi\ \cos^3\phi}{2[(B + \delta_4 - \delta_2)^2 - (2\delta_9 - \delta_1 - \delta_3)^2\ \sin^2\phi\ \cos^2\phi]} \tag{8}$$

and

$$K_e = -\ \frac{[(B + \delta_4 - \delta_2)^2 - (2\delta_9 - \delta_1 - \delta_3)^2\ \sin^2\phi\ \cos^2\phi]}{2(2\delta_9 - \delta_1 - \delta_3)\ \cos^3\phi}\ . \tag{9a}$$

The equation of the S-axis in the X-Y coordinate system is

$$C_e = \frac{1}{2}(\delta_1 + \delta_3) - R_e/\cos\ \phi - \left(y - \frac{1}{2}\ (B + \delta_2 + \delta_4)\right)\ \tan\ \phi. \tag{10a}$$

The equation of the R-axis in the X-Y coordinate system is

$$D_e = \frac{1}{2}(B + \delta_2 + \delta_4) - R_e\ \sin\ \phi + \left(x - \frac{1}{2}(\delta_1 + \delta_3)\right)\ \tan\ \phi. \tag{11a}$$

Similarly, for the f-surface of figure 6,

$$\rho = \tan^{-1}\left(\frac{\delta_5 - \delta_7}{B + \delta_8 - \delta_6}\right) \tag{12a}$$

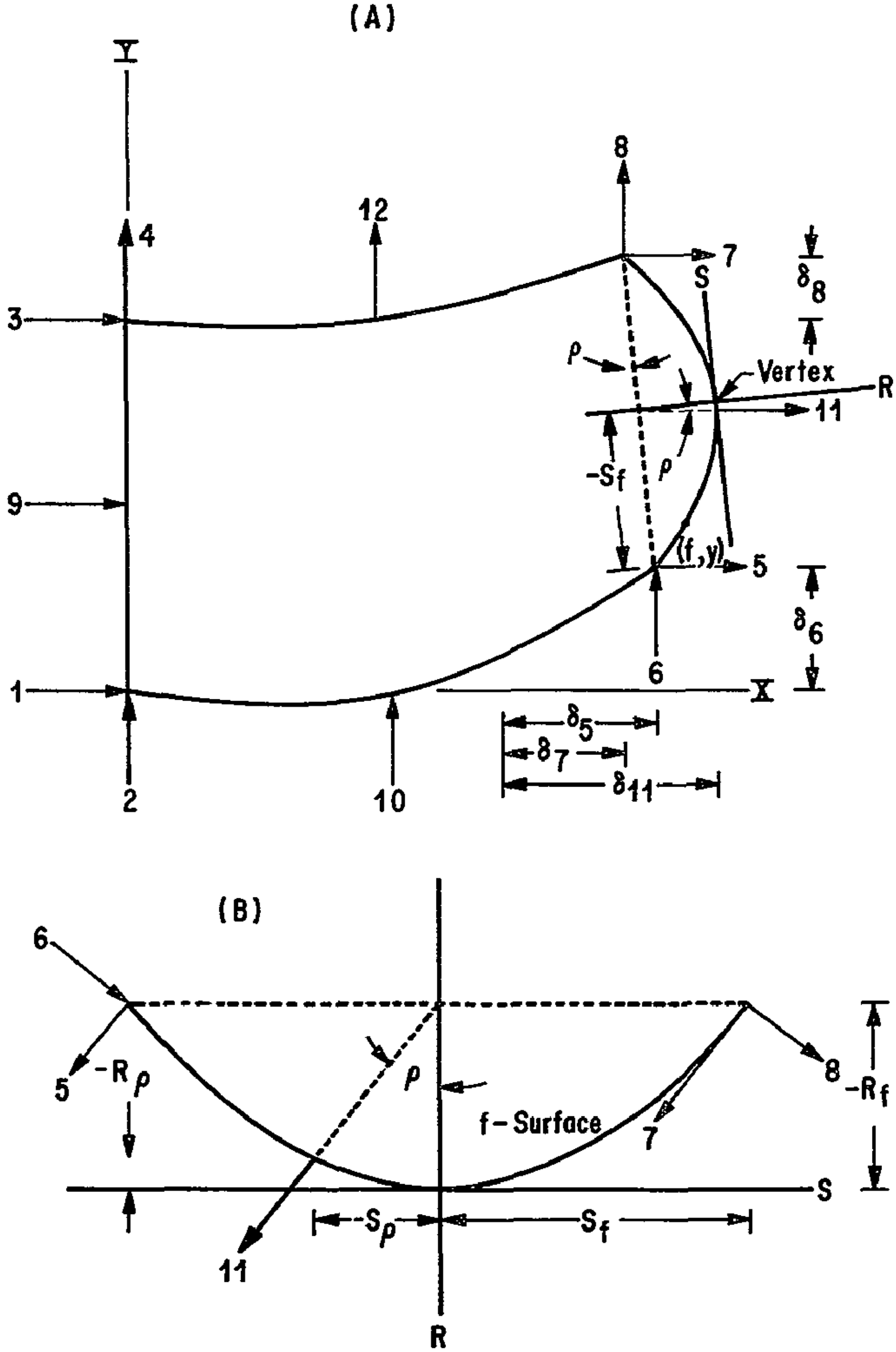

Figure 6. Total Geometry of the f-Surface

$$S_\rho = \frac{1}{2}\,(2\delta_{11} - \delta_5 - \delta_7)\,\sin\rho \tag{13}$$

$$S_f = \left(\frac{B + \delta_8 - \delta_6}{2\cos\rho}\right) \tag{14}$$

$$R_f = -\frac{1}{2}\,(2\delta_{11} - \delta_5 - \delta_7)\,\cos\rho + R_\rho. \tag{15}$$

From geometrical similarity, $R_\rho\, S_f^2 \equiv R_f\, S_\rho^2$, so that

$$R_\rho = -\frac{(2\delta_{11} - \delta_5 - \delta_7)^3 \sin^2\rho \cos^3\rho}{2[(B + \delta_8 - \delta_6)^2 - (2\delta_{11} - \delta_5 - \delta_7)^2 \sin^2\rho \cos^2\rho]} \tag{16}$$

and

$$K_f = -\frac{[(B + \delta_8 - \delta_6)^2 - (2\delta_{11} - \delta_5 - \delta_7)^2 \sin^2\rho \cos^2\rho]}{2(2\delta_{11} - \delta_5 - \delta_7)\cos^3\rho}. \tag{17a}$$

The equation of the S-axis in the X-Y coordinate system is

$$C_f = A + \frac{1}{2}(\delta_5 + \delta_7) - R_f/\cos\rho - \left(y - \frac{1}{2}(B + \delta_8 + \delta_6)\right)\tan\rho. \tag{18a}$$

The equation of the R-axis in the X-Y coordinate system is

$$D_f = \frac{1}{2}(B + \delta_8 + \delta_6) - R_f \sin\rho + \left(x - \frac{1}{2}(\delta_5 + \delta_7) - A\right)\tan\rho \tag{19a}$$

For the g-surface of figure 7,

$$\psi = \tan^{-1}\left(\frac{\delta_6 - \delta_2}{A + \delta_5 - \delta_1}\right) \tag{20a}$$

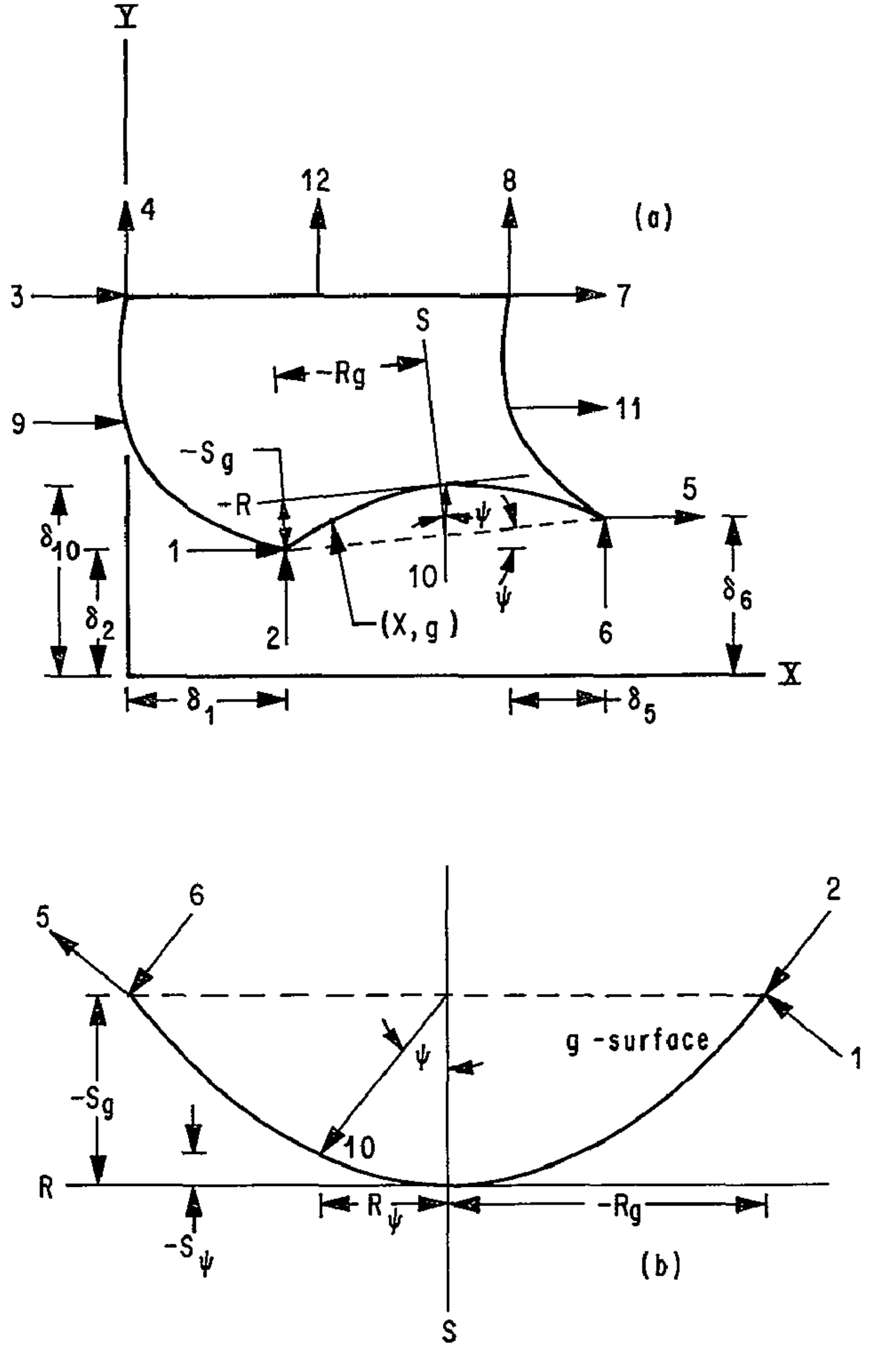

Figure 7. Total Geometry of the g -Surface

$$R_\psi = \frac{1}{2}\,(2\delta_{10} - \delta_2 - \delta_6)\,\sin\psi \tag{21}$$

$$R_g = \left(\frac{A + \delta_5 - \delta_1}{2\cos\psi}\right) \tag{22}$$

$$S_g = -\frac{1}{2}\,(2\delta_{10} - \delta_2 - \delta_6)\,\cos\psi + S_\psi. \tag{23}$$

From geometrical similarity, $S_\psi R_g^2 \equiv S_g\ R_\psi^2$, so that

$$S_\psi = -\frac{(2\delta_{10} - \delta_2 - \delta_6)^3 \sin^2\psi \cos^3\psi}{2[(A + \delta_5 - \delta_1)^2 - (2\delta_{10} - \delta_2 - \delta_6)^2 \sin^2\psi \cos^2\psi]} \tag{24}$$

and

$$K_g = -\frac{[(A + \delta_5 - \delta_1)^2 - (2\delta_{10} - \delta_2 - \delta_6)^2 \sin^2\psi \cos^2\psi]}{2(2\delta_{10} - \delta_2 - \delta_6)\cos^3\psi}. \tag{25a}$$

The equation of the S-axis in the X-Y coordinate system is

$$C_g = \frac{1}{2}(A + \delta_1 + \delta_5) - S_g\ \sin\psi - \left(y - \frac{1}{2}(\delta_2 + \delta_6)\right)\tan\psi. \tag{26a}$$

The equation of the R-axis in the X-Y coordinate system is

$$D_g = \frac{1}{2}(\delta_2 + \delta_6) - S_g\,/\cos\psi + \left(x - \frac{1}{2}(A + \delta_1 + \delta_5)\right)\ \tan\psi \tag{27a}$$

Finally, for the h-surface of figure 8,

$$\eta = \tan^{-1}\left(\frac{\delta_8 - \delta_4}{A + \delta_7 - \delta_3}\right) \tag{28a}$$

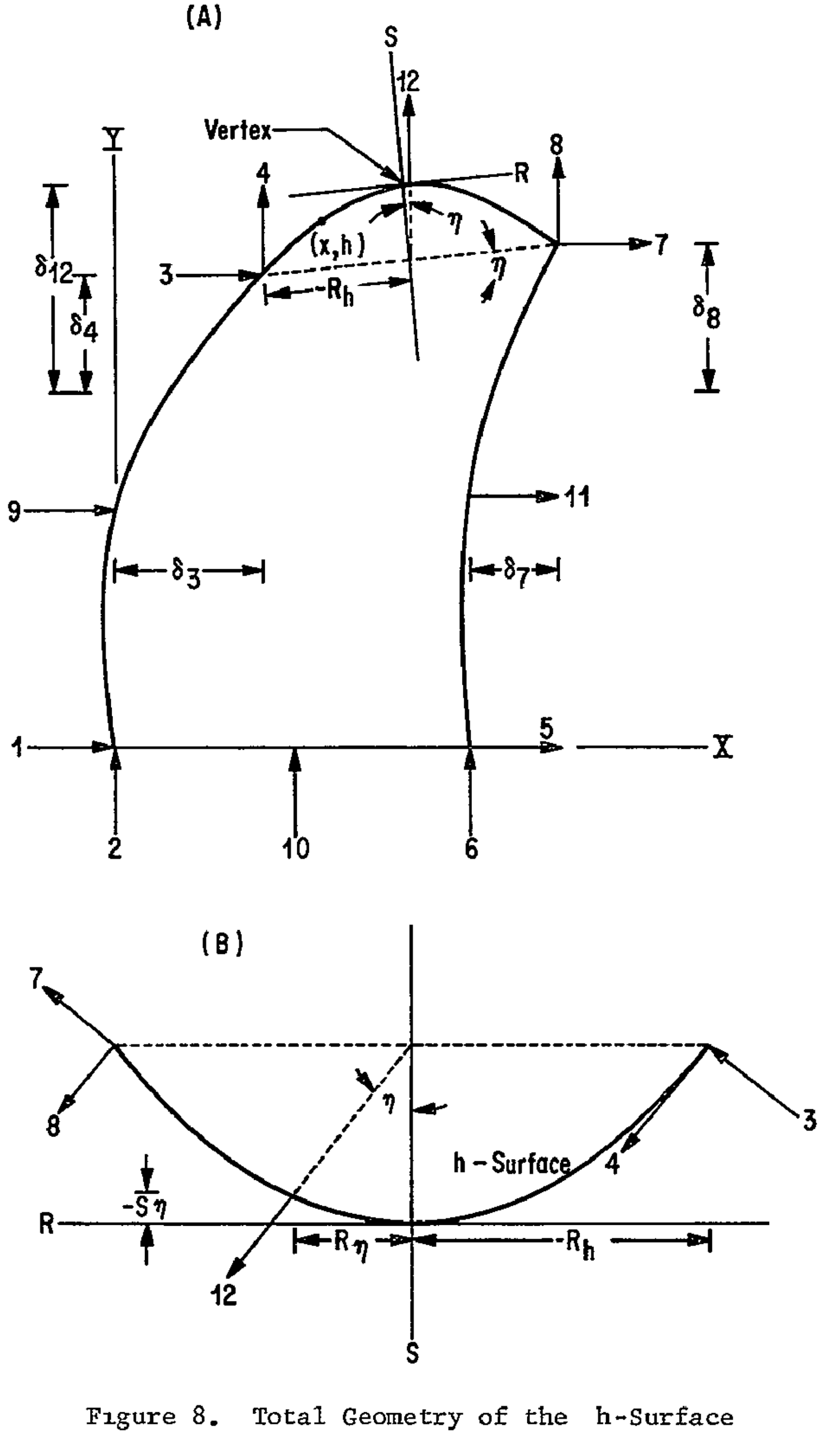

Figure 8. Total Geometry of the h-Surface

$$R_\eta = \frac{1}{2}(2\delta_{12} - \delta_4 - \delta_8) \sin \eta \qquad (29)$$

$$R_h = \left(\frac{A + \delta_7 - \delta_3}{2 \cos \eta}\right) \qquad (30)$$

$$S_h = -\frac{1}{2}(2\delta_{12} - \delta_4 - \delta_8) \cos \eta + S_\eta . \qquad (31)$$

From geometrical similarity, $S_\eta R_h^2 \equiv S_h R_\eta^2$, so that

$$S_\eta = -\frac{(2\delta_{12} - \delta_4 - \delta_8)^3 \sin^2\eta \cos^3\eta}{2[(A + \delta_7 - \delta_3)^2 - (2\delta_{12} - \delta_4 - \delta_8)^2 \sin^2\eta \cos^2\eta]} , \qquad (32)$$

and

$$K_h = -\frac{[(A + \delta_7 - \delta_3)^2 - (2\delta_{12} - \delta_4 - \delta_8)^2 \sin^2\eta \cos^2\eta]}{2(2\delta_{12} - \delta_4 - \delta_8) \cos^3\eta} . \qquad (33a)$$

The equation of the S-axis in the X-Y coordinate system is

$$C_h = \frac{1}{2}(A + \delta_3 + \delta_7) - S_h \sin \eta - \left(y - \frac{1}{2}(\delta_4 + \delta_8) - B\right) \tan \eta . \qquad (34a)$$

The equation of the R-axis in the X-Y coordinate system is

$$D_h = B + \frac{1}{2}(\delta_4 + \delta_8) - S_h/\cos \eta + \left(x - (\frac{1}{2} A + \delta_3 + \delta_7)\right) \tan \eta . \qquad (35a)$$

If it is reasonably assumed that deflections will generally be much less than A or B, $\delta_i \ll A$ or B, then

$$\sin\alpha \doteq \alpha \qquad \sin^2\alpha \doteq 0$$

$$\cos\alpha \doteq 1 \qquad \cos^2\alpha \doteq 1$$

$$\tan\alpha \doteq \alpha \qquad \cos^3\alpha \doteq 1$$

$$\delta_i^2 \doteq 0$$

where $\alpha = \emptyset$, ρ, ψ, or η, and

$$\emptyset = \frac{\delta_1 - \delta_3}{B + \delta_4 - \delta_2} \tag{4c}$$

$$\rho = \frac{\delta_5 - \delta_7}{B + \delta_8 - \delta_6} \tag{12b}$$

$$\psi = \frac{\delta_6 - \delta_2}{A + \delta_5 - \delta_1} \tag{20b}$$

$$\eta = \frac{\delta_8 - \delta_4}{A + \delta_7 - \delta_3} \tag{28b}$$

$$K_e = -\left[\frac{(B + \delta_4 - \delta_2)^2}{2(2\delta_9 - \delta_1 - \delta_3)}\right] \tag{9b}$$

$$K_f = -\left[\frac{(B + \delta_8 - \delta_6)^2}{2(2\delta_{11} - \delta_5 - \delta_7)}\right] \tag{17b}$$

$$K_g = -\left[\frac{(A + \delta_5 - \delta_1)^2}{2(2\delta_{10} - \delta_2 - \delta_6)}\right] \tag{25b}$$

$$K_h = -\left[\frac{(A + \delta_7 - \delta_3)^2}{2(2\delta_{12} - \delta_4 - \delta_8)}\right] \tag{33b}$$

$$C_e = \frac{1}{2}(2\delta_9 + \delta_1 - \delta_3) - y\phi \tag{10b}$$

$$C_f = A + \frac{1}{2}(2\delta_{11} + \delta_5 - \delta_7) - y\rho \tag{18b}$$

$$C_g = \frac{1}{2}(A + \delta_1 + \delta_5) \tag{26b}$$

$$C_h = \frac{1}{2}(A + \delta_3 + \delta_7) \tag{34b}$$

$$D_e = \frac{1}{2}(B + \delta_2 + \delta_4) \tag{11b}$$

$$D_f = \frac{1}{2}(B + \delta_6 + \delta_8) \tag{19b}$$

$$D_g = \frac{1}{2}(2\delta_{10} + \delta_2 - \delta_6) + x\psi \tag{27b}$$

$$D_h = B + \frac{1}{2}(2\delta_{12} + \delta_4 - \delta_8) + x\eta\ . \tag{35b}$$

The equations for each surface and its associated shear become

$$e = \frac{(y - D_e)^2}{K_e} + C_e \tag{2b}$$

$$f = \frac{(y - D_f)^2}{K_f} + C_f \tag{36}$$

$$g = \frac{(x - C_g)^2}{K_g} + D_g \tag{37}$$

$$h = \frac{(x - C_h)^2}{K_h} + D_h \tag{38}$$

$$\gamma_e = \frac{2(y - D_e)}{K_e} - \phi \tag{3b}$$

$$\gamma_f = \frac{2(y - D_f)}{K_f} - \rho \tag{39}$$

$$\gamma_g = \frac{2(x - C_g)}{K_g} + \psi \tag{40}$$

$$\gamma_h = \frac{2(x - C_h)}{K_h} + \eta . \tag{41}$$

Recalling that $\delta_1 \ll A$ or B and strain is assumed to be linearly distributed, shear panel strains (figure 9) are

$$\epsilon_x = \frac{1}{A}\left[\frac{(y - D_f)^2}{K_f} - \frac{(y - D_e)^2}{K_e} + C_f - C_e - A\right] \tag{42}$$

$$\epsilon_y = \frac{1}{B}\left[\frac{(x - C_h)^2}{K_h} - \frac{(x - C_g)^2}{K_g} + D_h - D_g - B\right] \tag{43}$$

$$\begin{aligned} \gamma = &\left[2\left(\frac{y - D_e}{K_e}\right) - \phi\right]\left(1 - \frac{x}{A}\right) + \left[2\left(\frac{y - D_f}{K_f}\right) - \rho\right]\left(\frac{x}{A}\right) \\ &+ \left[2\left(\frac{x - C_g}{K_g}\right) + \psi\right]\left(1 - \frac{y}{B}\right) + \left[2\left(\frac{x - C_h}{K_h}\right) + \eta\right]\left(\frac{y}{B}\right) . \end{aligned} \tag{44}$$

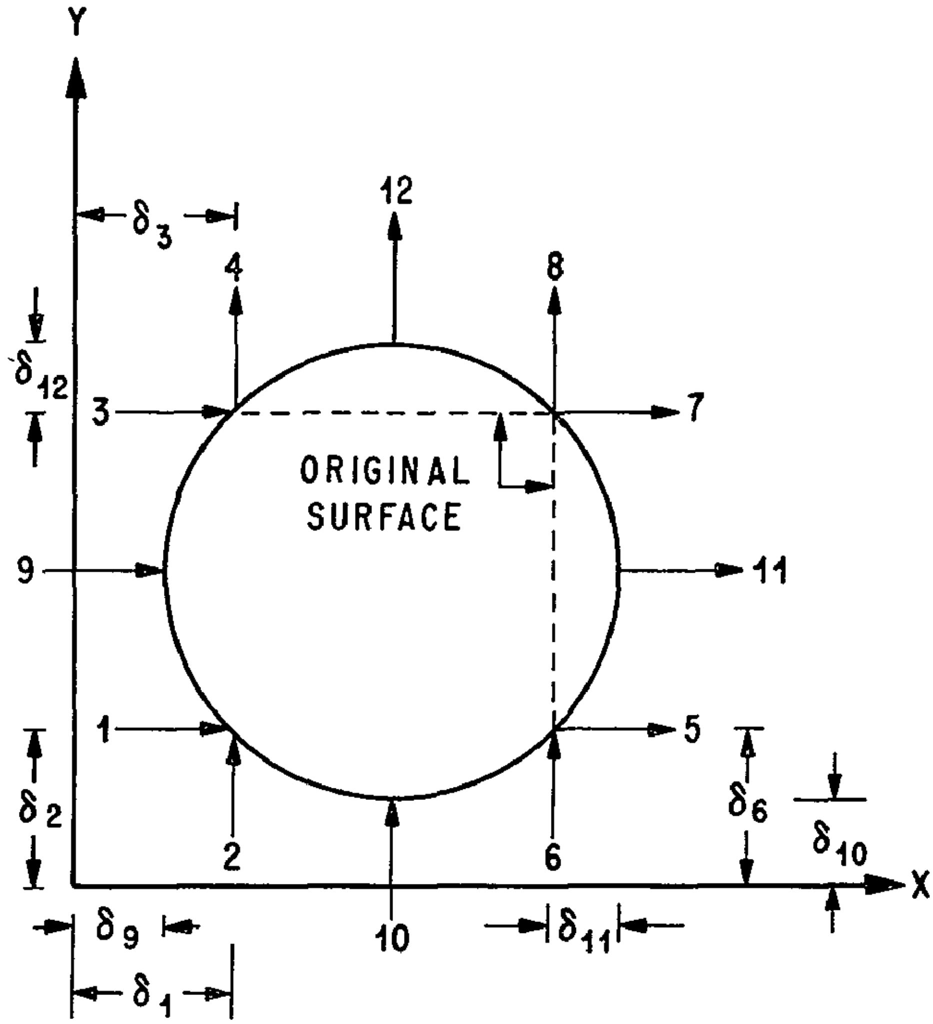

Figure 9. Completely Deformed Twelve-Node Shear Panel

The stiffness matrix elements may now be calculated by means of equation (45),

$$S_{ij} = \frac{ET}{1 - \mu^2} \int_0^A \int_0^B \left[\epsilon_x \frac{\partial \epsilon_x}{\partial \delta_i} + \mu \left(\epsilon_x \frac{\partial \epsilon_y}{\partial \delta_i} + \epsilon_y \frac{\partial \epsilon_x}{\partial \delta_i} \right) + \epsilon_y \frac{\partial \epsilon_y}{\partial \delta_i} \right.$$

$$\left. + \frac{1}{2} \gamma \frac{\partial \gamma}{\partial \delta_i} (1 - \mu) \right] dy\, dx. \tag{45}$$

To calculate an element, examples 1 and 2, first set all δ_K, $K \neq i$ or j, equal to zero in equations (42), (43), and (44); second, take the partials of equations (42), (43), and (44) with respect to δ_i, and third, set $\delta_i = 0$ and $\delta_j = 1$ in the expressions for ϵ_x, ϵ_y, γ, $\partial\epsilon_x/\partial\delta_i$, $\partial\epsilon_y/\partial\delta_i$, and $\partial\gamma/\partial\delta_i$. Substitute these expressions into equation (45) and carry out the indicated integration.

Example 1, $S_{9,1}$

$$\phi = \delta_1/B \qquad \rho = \psi = \eta = 0$$

$$K_e = \frac{-B^2}{2(2\delta_9 - \delta_1)} \qquad K_f = K_g = K_h = -\infty$$

$$C_e = \delta_9 + \delta_1\left(\frac{1}{2} - \frac{y}{B}\right) \qquad D_e = D_f = \frac{1}{2}B$$

$$C_f = A \qquad D_g = 0$$

$$C_g = C_h = \frac{1}{2}A \qquad D_h = B$$

$$\epsilon_x = \frac{1}{A}\left[\frac{2(2\delta_9 - \delta_1)(y - \frac{1}{2}B)^2}{B^2} - \delta_9 - \delta_1\left(\frac{1}{2} - \frac{y}{B}\right)\right]$$

$$\epsilon_x \Big/ \begin{matrix}\delta_1 = 1 \\ \delta_9 = 0\end{matrix} = \frac{1}{A}\left[-\frac{2(y - \frac{1}{2}B)^2}{B^2} - \frac{1}{2} + \frac{y}{B}\right]$$

$$\frac{\partial\epsilon_x}{\partial\delta_9} = \frac{1}{A}\left[\frac{4(y - \frac{1}{2}B)^2}{B^2} - 1\right]$$

$$\epsilon_y = \frac{\partial\epsilon_y}{\partial\delta_9} = 0$$

$$\gamma = \left[- \frac{4(2\delta_9 - \delta_1)(y - \frac{1}{2}B)}{B^2} - \frac{\delta_1}{B}\right]\left(1 - \frac{x}{A}\right)$$

$$\gamma \Big/ \begin{matrix}\delta_1 = 1 \\ \delta_9 = 0\end{matrix} = \left[\frac{4(y - \frac{1}{2}B)}{B^2} - \frac{1}{B}\right]\left(1 - \frac{x}{A}\right)$$

$$\frac{\partial\gamma}{\partial\delta_9} = \left[- \frac{8(y - \frac{1}{2}B)}{B^2}\right]\left(1 - \frac{x}{A}\right)$$

$$\epsilon_x \frac{\partial\epsilon_x}{\partial\delta_9} = \frac{1}{A^2}\left[- \frac{8(y - \frac{1}{2}B)^4}{B^4} + \frac{4y(y - \frac{1}{2}B)^2}{B^3} + \frac{1}{2} - \frac{y}{B}\right]$$

$$\epsilon_x \frac{\partial\epsilon_y}{\partial\delta_9} = \epsilon_y \frac{\partial\epsilon_x}{\partial\delta_9} = \epsilon_y \frac{\partial\epsilon_y}{\partial\delta_9} = 0$$

$$\gamma \frac{\partial\gamma}{\partial\delta_9} = \left[- \frac{32(y - \frac{1}{2}B)^2}{B^4} + \frac{8(y - \frac{1}{2}B)}{B^3}\right]\left(1 - \frac{x}{A}\right)^2$$

$$S_{9,1} = \frac{ET}{1 - \mu^2} \int_0^A\int_0^B \left[\epsilon_x \frac{\partial\epsilon_x}{\partial\delta_9} + \frac{1}{2}\gamma \frac{\partial\gamma}{\partial\delta_9}(1 - \mu)\right] dy\, dx$$

$$S_{9,1} = \frac{ET}{1 - \mu^2}\left[\frac{1}{15}\frac{B}{A} - \frac{1}{2}(8/9)\frac{A}{B}(1 - \mu)\right]$$

$$S_{9,1} = \frac{ET}{1 - \mu^2}\left[\frac{3\ B/A - 20\ A/B\ (1 - \mu)}{45}\right] \tag{46}$$

Example 2, $S_{10,9}$

$$\emptyset = \rho = \psi = \eta = 0$$

$$K_e = -\frac{B^2}{4\delta_9} \qquad K_g = -\frac{A^2}{4\delta_{10}} \qquad K_f = K_h = -\infty$$

$$C_e = \delta_9 \qquad D_e = D_f = \frac{1}{2} B$$

$$C_f = A \qquad D_g = \delta_{10}$$

$$C_g = C_h = \frac{1}{2} A \qquad D_h = B$$

$$\epsilon_x = \frac{1}{A}\left[\frac{4\delta_9 \; (y - \frac{1}{2} B)^2}{B^2} - \delta_9\right]$$

$$\epsilon_x \Big/ \begin{matrix} \delta_9 = 1 \\ \delta_{10} = 0 \end{matrix} = \frac{1}{A}\left[\frac{4(y - \frac{1}{2} B)^2}{B^2} - 1\right]$$

$$\frac{\partial \epsilon_x}{\partial \delta_{10}} = 0$$

$$\epsilon_y = \frac{1}{B}\left[\frac{4\delta_{10} \; (x - \frac{1}{2} A)^2}{A^2} - \delta_{10}\right]$$

$$\epsilon_y \Big/ \begin{matrix} \delta_9 = 1 \\ \delta_{10} = 0 \end{matrix} = 0$$

$$\frac{\partial \epsilon_y}{\partial \delta_{10}} = \frac{1}{B}\left[\frac{4(x - \frac{1}{2}A)^2}{A^2} - 1\right]$$

$$\gamma = \left[-\frac{8\delta_9\ (y - \frac{1}{2}B)}{B^2}\right]\left(1 - \frac{x}{A}\right) + \left[-\frac{8\delta_{10}\ (x - \frac{1}{2}A)}{A^2}\right]\left(1 - \frac{y}{B}\right)$$

$$\gamma \Big/ \begin{matrix}\delta_9 = 1 \\ \delta_{10} = 0\end{matrix} = \left[-\frac{8(y - \frac{1}{2}B)}{B^2}\right]\left(1 - \frac{x}{A}\right)$$

$$\frac{\partial \gamma}{\partial \delta_{10}} = \left[-\frac{8(x - \frac{1}{2}A)}{A^2}\right]\left(1 - \frac{y}{B}\right)$$

$$\epsilon_x \frac{\partial \epsilon_x}{\partial \delta_{10}} = \epsilon_y \frac{\partial \epsilon_x}{\partial \delta_{10}} = \epsilon_y \frac{\partial \epsilon_y}{\partial \delta_{10}} = 0$$

$$\epsilon_x \frac{\partial \epsilon_y}{\partial \delta_{10}} = \frac{1}{AB}\left[\frac{16(x - \frac{1}{2}A)^2(y - \frac{1}{2}B)^2}{A^2B^2} - \frac{4(x - \frac{1}{2}A)^2}{A^2} - \frac{4(y - \frac{1}{2}B)^2}{B^2} + 1\right]$$

$$\gamma \frac{\partial \gamma}{\partial \delta_{10}} = \frac{64(x - \frac{1}{2}A)(y - \frac{1}{2}B)}{A^2B^2}\left[1 - \frac{x}{A} - \frac{y}{B} + \frac{xy}{AB}\right]$$

$$S_{10,9} = \frac{ET}{1 - \mu^2}\int_0^A\int_0^B \left[\mu\epsilon_x \frac{\partial \epsilon_y}{\partial \delta_{10}} + \frac{1}{2}\gamma \frac{\partial \gamma}{\partial \delta_{10}}(1 - \mu)\right] dy\ dx$$

$$S_{10,9} = \frac{ET}{1 - \mu^2}\left[\frac{4}{9}\mu + \frac{1}{2}\left(\frac{4}{9}\right)(1 - \mu)\right]$$

$$S_{10,9} = \frac{ET}{1 - \mu^2}\left[\frac{2 + 2\mu}{9}\right]. \tag{47}$$

Thus, column 1, which is typical of loads caused by a corner deflection (figure 2b), is

$$S_{1,1} = \frac{ET}{1 - \mu^2}\left[\frac{12\ B/A + 35\ A/B\ (1 - \mu)}{90}\right] \tag{48}$$

$$S_{2,1} = \frac{ET}{1 - \mu^2}\left(\frac{25 - 23\mu}{72}\right) \tag{49}$$

$$S_{3,1} = \frac{-ET}{1 - \mu^2}\left[\frac{3\ B/A - 5\ A/B\ (1 - \mu)}{90}\right] \tag{50}$$

$$S_{4,1} = \frac{ET}{1 - \mu^2}\left(\frac{5 - 7\mu}{72}\right) \tag{51}$$

$$S_{5,1} = \frac{-ET}{1 - \mu^2}\left[\frac{24\ B/A - 35\ A/B\ (1 - \mu)}{180}\right] \tag{52}$$

$$S_{6,1} = \frac{-ET}{1 - \mu^2}\left(\frac{5 - 7\mu}{72}\right) \tag{53}$$

$$S_{7,1} = \frac{ET}{1 - \mu^2}\left[\frac{6\ B/A + 5\ A/B\ (1 - \mu)}{180}\right] \tag{54}$$

$$S_{8,1} = \frac{-ET}{1 - \mu^2}\left(\frac{1 + \mu}{72}\right) \tag{55}$$

$$S_{9,1} = \frac{ET}{1 - \mu^2}\left[\frac{3\ B/A - 20\ A/B\ (1 - \mu)}{45}\right] \tag{46}$$

$$S_{10,1} = \frac{-ET}{1 - \mu^2}\left(\frac{5 - 7\mu}{18}\right) \tag{57}$$

$$S_{11,1} = \frac{-ET}{1 - \mu^2} \left[\frac{3\ B/A + 10\ A/B\ (1 - \mu)}{45}\right] \tag{57}$$

$$S_{12,1} = \frac{-ET}{1 - \mu^2} \left(\frac{1 + \mu}{18}\right) \tag{58}$$

and column 9, which is typical of loads caused by a midpoint deflection (figure 2a), is

$$S_{1,9} = \frac{ET}{1 - \mu^2} \left[\frac{3\ B/A - 20\ A/B\ (1 - \mu)}{45}\right] \tag{59}$$

$$S_{2,9} = \frac{-ET}{1 - \mu^2} \left(\frac{5 - 7\mu}{18}\right) \tag{60}$$

$$S_{3,9} = \frac{ET}{1 - \mu^2} \left[\frac{3\ B/A - 20\ A/B\ (1 - \mu)}{45}\right] \tag{61}$$

$$S_{4,9} = \frac{ET}{1 - \mu^2} \left(\frac{5 - 7\mu}{18}\right) \tag{62}$$

$$S_{5,9} = \frac{-ET}{1 - \mu^2} \left[\frac{3\ B/A + 10\ A/B\ (1 - \mu)}{45}\right] \tag{63}$$

$$S_{6,9} = \frac{ET}{1 - \mu^2} \left(\frac{1 + \mu}{18}\right) \tag{64}$$

$$S_{7,9} = \frac{-ET}{1 - \mu^2} \left[\frac{3\ B/A + 10\ A/B\ (1 - \mu)}{45}\right] \tag{65}$$

$$S_{8,9} = \frac{-ET}{1 - \mu^2} \left(\frac{1 + \mu}{18} \right) \tag{66}$$

$$S_{9,9} = \frac{ET}{1 - \mu^2} \left[\frac{24\ B/A + 40\ A/B\ (1 - \mu)}{45} \right] \tag{67}$$

$$S_{10,9} = \frac{ET}{1 - \mu^2} \left(\frac{2 + 2\mu}{9} \right) \tag{47}$$

$$S_{11,9} = \frac{-ET}{1 - \mu^2} \left[\frac{24\ B/A - 20\ A/B\ (1 - \mu)}{45} \right] \tag{68}$$

$$S_{12,9} = \frac{-ET}{1 - \mu^2} \left(\frac{2 + 2\mu}{9} \right) \tag{69}$$

By consideration of the geometry of the shear panel, the remaining 120 stiffness matrix elements may be substituted in terms of those already calculated Figure 10 graphically illustrates these substitutions. Those with a bar over them essentially represent a rotation of 90 degrees of the shear panel. For example,

$$S_{2,2} = S_{1,1}/90^\circ = \frac{ET}{1 - \mu^2} \left[\frac{12\ A/B + 35\ B/A\ (1 - \mu)}{90} \right] . \tag{70}$$

Actually, since the stiffness matrix is positive definite, i.e.,

$$\sum_{i=1}^{12} \sum_{j=1}^{12} S_{ij}\ \delta_i\ \delta_j > 0 \qquad \text{for any arbitrary } \delta_i \text{ when not all } \delta_i = 0,$$

$$= 0 \qquad \text{for all } \delta_i = 0,$$

	j = 1	2	3	4	5	6	7	8	9	10	11	12
i = 1	1,1	2,1	3,1	-6,1	5,1	-4,1	7,1	8,1	1,9	-4,9	7,9	-6,9
2	2,1	$\overline{1,1}$	-4,1	$\overline{5,1}$	-6,1	$\overline{3,1}$	8,1	$\overline{7,1}$	2,9	$\overline{3,9}$	8,9	$\overline{5,9}$
3	3,1	6,1	1,1	-2,1	7,1	-8,1	5,1	4,1	3,9	-8,9	5,9	-2,9
4	4,1	$\overline{5,1}$	-2,1	$\overline{1,1}$	-8,1	$\overline{7,1}$	6,1	$\overline{3,1}$	4,9	$\overline{7,9}$	6,9	$\overline{1,9}$
5	5,1	4,1	7,1	-8,1	1,1	-2,1	3,1	6,1	5,9	-2,9	3,9	-8,9
6	6,1	$\overline{3,1}$	-8,1	$\overline{7,1}$	-2,1	$\overline{1,1}$	4,1	$\overline{5,1}$	6,9	$\overline{1,9}$	4,9	$\overline{7,9}$
7	7,1	8,1	5,1	-4,1	3,1	-6,1	1,1	2,1	7,9	-6,9	1,9	-4,9
8	8,1	$\overline{7,1}$	-6,1	$\overline{3,1}$	-4,1	$\overline{5,1}$	2,1	$\overline{1,1}$	8,9	$\overline{5,9}$	2,9	$\overline{3,9}$
9	9,1	10,1	9,1	-10,1	11,1	-12,1	11,1	12,1	9,9	-12,9	11,9	-10,9
10	10,1	$\overline{9,1}$	-12,1	$\overline{11,1}$	-10,1	$\overline{9,1}$	12,1	$\overline{11,1}$	10,9	$\overline{9,9}$	12,9	$\overline{11,9}$
11	11,1	12,1	11,1	-12,1	9,1	-10,1	9,1	10,1	11,9	-10,9	9,9	-12,9
12	12,1	$\overline{11,1}$	-10,1	$\overline{9,1}$	-12,1	$\overline{11,1}$	10,1	$\overline{9,1}$	12,9	$\overline{11,9}$	10,9	$\overline{9,9}$

Figure 10 Stiffness Matrix Substitutions for Twelve-Node Shear Panel

only column 1 and the last four elements of column 9 are necessary to arrive at a lower triangular matrix, $i \geqq j$, (although equations (59) to (66) serve as accuracy checks) with the upper triangular matrix given by Maxwell's reciprocity relationship, $S_{ij} = S_{ji}$ The matrix is also inherently singular since, to satisfy static equilibrium

$$\sum_{i=1}^{12} S_{ij} \delta_j \equiv 0$$

for any arbitrary j, $1 \leqq j \leqq 12$, $\delta_j \neq 0$

RESULTS AND CONCLUSIONS

Suppose, now, that the twelve-node stiffness matrix has been derived, and that seven cantilever beams are machined from aluminum with the dimensions and properties of table I

TABLE I

Beam	A (cm)	B (cm)	T (cm)	E (dy/cm^2)	I (cm^4)	μ
SOM w/o shear	10	2	1 1/2	6.895×10^{11}	1	.3
SOM with shear	10	2				
TOE	10	2				
5/8	2	2				
10/8	2	1				
15/8	2	2/3				
5/12	2	2				

The first two beams (figure 11c) are analyzed using common strength of materials equations, excluding shear effects (SOMEXS), and including shear (SOMINS). The third (also figure 11c) is analyzed from theory of elasticity (TOE). The next three are composed entirely of eight-node shear panels analyzed by matrix algebra (appendix C), one composed of the first five panels of figure 11a (this beam hereafter noted by "5/8"), another composed of the first ten panels (10/8), and the third composed of all fifteen eight-node panels (15/8). The last beam is composed of five twelve-node panels (5/12) of figure 11b. All beams are anchored to zero-deflection "foundations" at the left end ($\delta_1 = \delta_7 = \delta_{13} = \delta_{19} = \delta_{25} = \delta_{31} = \delta_{37} = \delta_{43} \equiv 0$ for the 5/8, 10/8, and 15/8 beams or $\delta_1 = \delta_7 = \delta_{13} = \delta_{19} = \delta_{30}$ for the 5/12 beam). All beams are end-loaded with 2×10^7 dynes of downward acting force (P_{30} and P_{29} for figure 11a and 11b, respectively). Results are presented in tables II and III and figures 12 to 16.

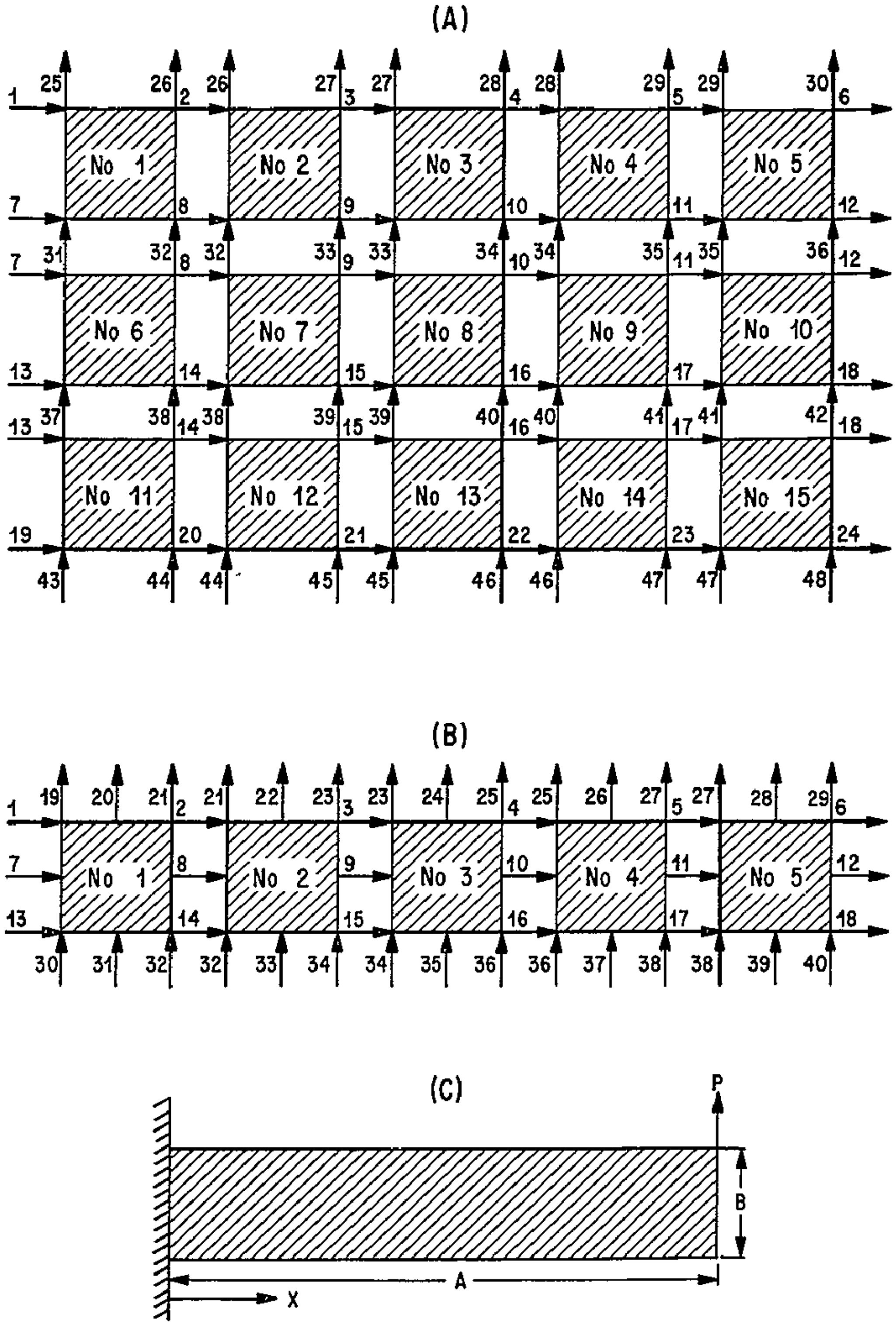

Figure 11. Cantilever Beam Notation

TABLE II

SOM And TOE Cantilever Beam Deflections

X (cm)	δ - SOM w/o Shear (mm)	δ - SOM with Shear (mm)	δ - TOE (mm)
0	.0000	.0000	.0000
1	-.0014	-.0016	-.0018
2	- 0054	-.0059	-.0062
3	-.0117	-.0125	- 0129
4	-.0201	-.0210	- 0216
5	-.0302	-.0314	-.0321
6	-.0418	-.0432	-.0440
7	-.0545	- 0562	-.0571
8	-.0681	-.0700	- 0711
9	-.0822	-.0844	- 0856
10	-.0967	-.0992	- 1005
Tip Rotation (Radians)	- 00145	-.00147	- 00149

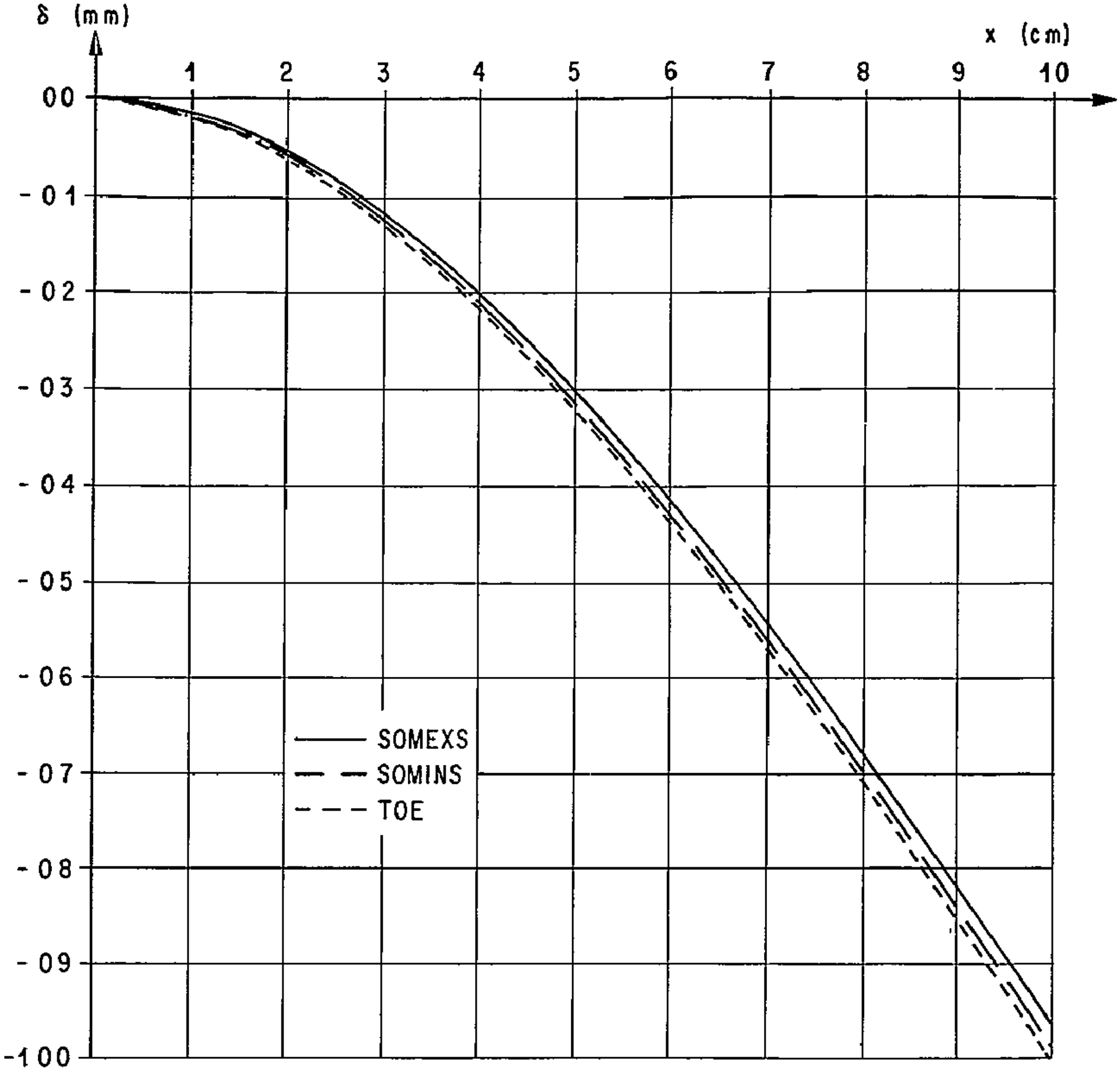

Figure 12. SOMEX, SOMINS, and TOE Beam Deflections

TABLE III

Shear Panel Cantilever Beams External Loads and Deflections

	5/8		10/8		15/8		5/12	
i	P_i (dynes)	δ_i (mm)	P_i (dynes)	δ_i (mm)	P_i (dynes)	δ_i (mm)	P_i (dynes)	δ_i (mm)
1	-1×10^8	.0000	-1×10^8	.0000	-8.58×10^7	.0000	-1×10^8	.0000
2	0	0035	0	.0036	0	.0037	0	.0048
3	0	0063	0	0065	0	.0066	0	.0084
4	0	.0082	0	0086	0	0087	0	.0111
5	0	.0094	0	0098	0	.0099	0	0127
6	0	0098	0	.0103	0	.0104	0	0133
7	1×10^8	.0000	3.15×10^3	.0000	$-4\ 27 \times 10^7$	.0000	$-2\ 30 \times 10^3$	0000
8	0	- 0035	0	.0000	0	.0012	0	-.0000
9	0	-.0063	0	- 0000	0	0022	0	.0000
10	0	-.0082	0	0000	0	0029	0	- 0000
11	0	-.0094	0	- 0000	0	0033	0	.0000
12	0	-.0097	0	0000	0	0034	0	-.0000
13	-	-	1×10^8	.0000	4.27×10^7	0000	1×10^8	0000
14	-	-	0	-.0036	0	- 0012	0	- 0048
15	-	-	0	-.0065	0	- 0022	0	- 0084
16	-	-	0	- 0086	0	- 0029	0	- 0111
17	-	-	0	-.0098	0	- 0033	0	- 0127
18	-	-	0	-.0102	0	- 0034	0	- 0131
19	-	-	-	-	8.58×10^7	0000	1×10^7	0000
20	-	-	-	-	0	- 0037	0	-.0014
21	-	-	-	-	0	- 0066	0	- 0053
22	-	-	-	-	0	- 0087	0	- 0112
23	-	-	-	-	0	- 0099	0	- 0190
24	-	-	-	-	0	-.0103	0	- 0283
25	1×10^7	.0000	2.43×10^7	.0000	2.86×10^7	.0000	0	- 0390
26	0	-.0040	0	- 0043	0	-.0044	0	- 0507
27	0	- 0143	0	-.0149	0	- 0150	0	- 0633
28	0	-.0293	0	-.0305	0	-.0308	0	-.0763
29	0	-.0473	0	-.0493	0	-.0497	-2×10^7	- 0899
30	-2×10^7	-.0672	-2×10^7	-.0700	-2×10^7	-.0707	1×10^7	0000
31	1×10^7	0000	-2.86×10^7	0000	-1.86×10^7	0000	0	- 0014
32	0	- 0040	0	-.0040	0	-.0041	0	- 0053
33	0	-.0143	0	- 0147	0	-.0149	0	-.0112
34	0	-.0293	0	-.0303	0	-.0306	0	- 0190
35	0	-.0474	0	-.0493	0	- 0497	0	- 0283
36	0	-.0669	0	-.0698	0	- 0705	0	-.0390
37	-	-	2.43×10^7	.0000	-1.86×10^7	0000	0	-.0507
38	-	-	0	-.0043	0	- 0041	0	-.0632
39	-	-	0	-.0149	0	-.0149	0	-.0763
40	-	-	0	-.0305	0	- 0306	0	-.0893
41	-	-	0	-.0493	0	- 0498	-	-
42	-	-	0	-.0697	0	- 0704	-	-
43	-	-	-	-	2.86×10^7	0000	-	-
44	-	-	-	-	0	- 0044	-	-
45	-	-	-	-	0	- 0150	-	-
46	-	-	-	-	0	-.0308	-	-
47	-	-	-	-	0	-.0498	-	-
48	-	-	-	-	0	-.0704	-	-

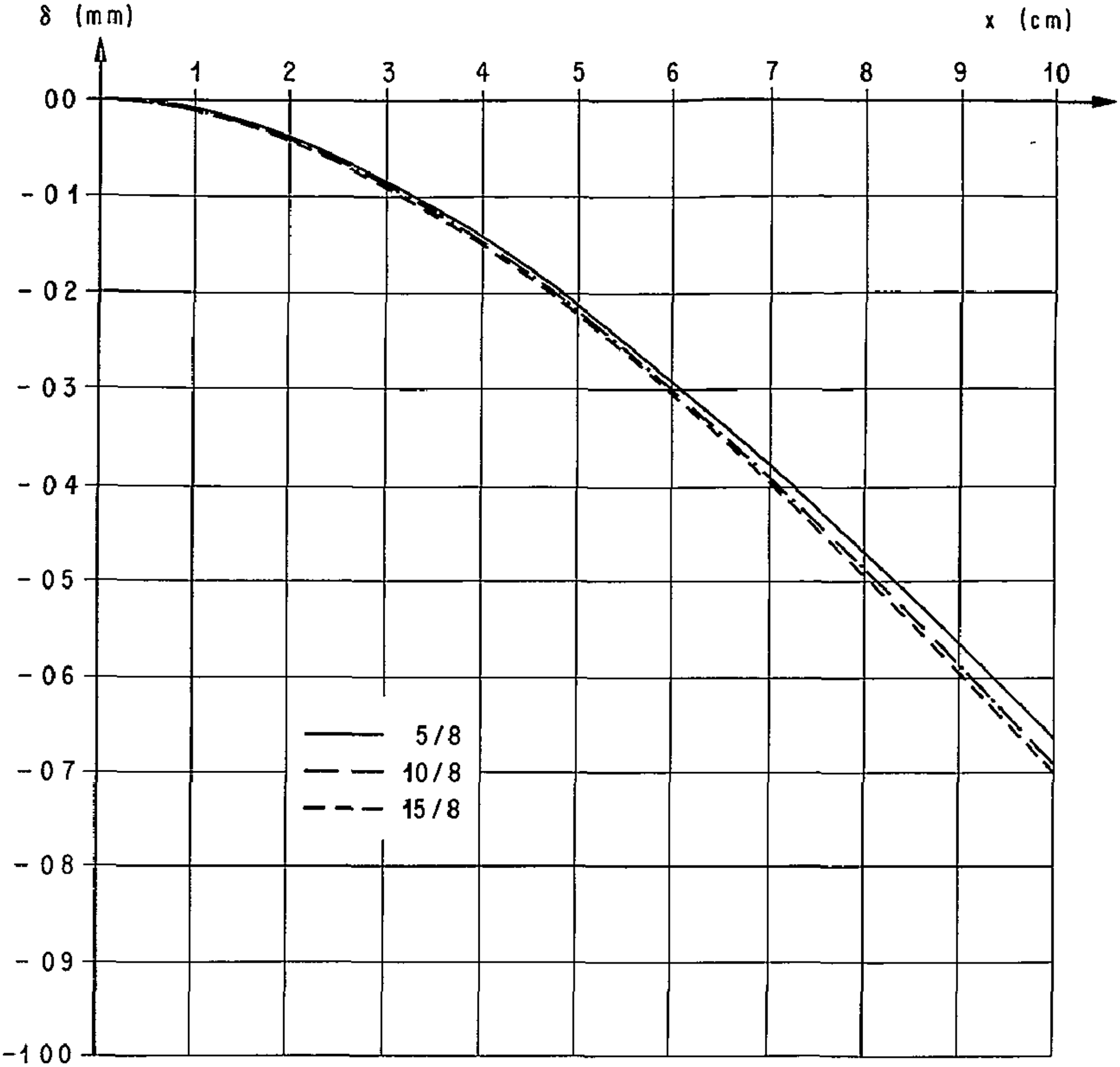

Figure 13. Eight-Node Shear Panel Beam Deflections

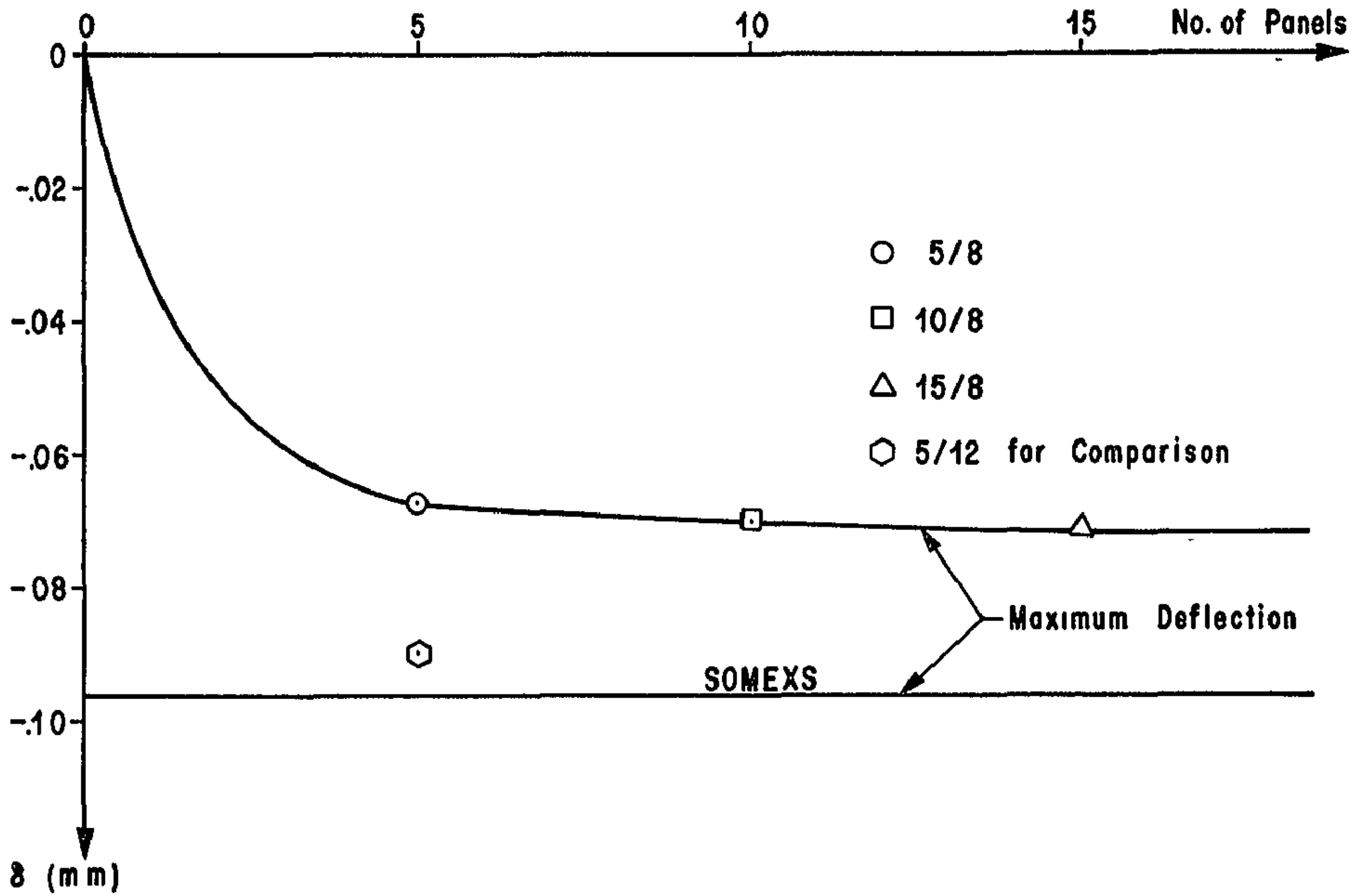

Figure 14. Improvement of Eight-Node Shear Panel Beams versus Number of Panels

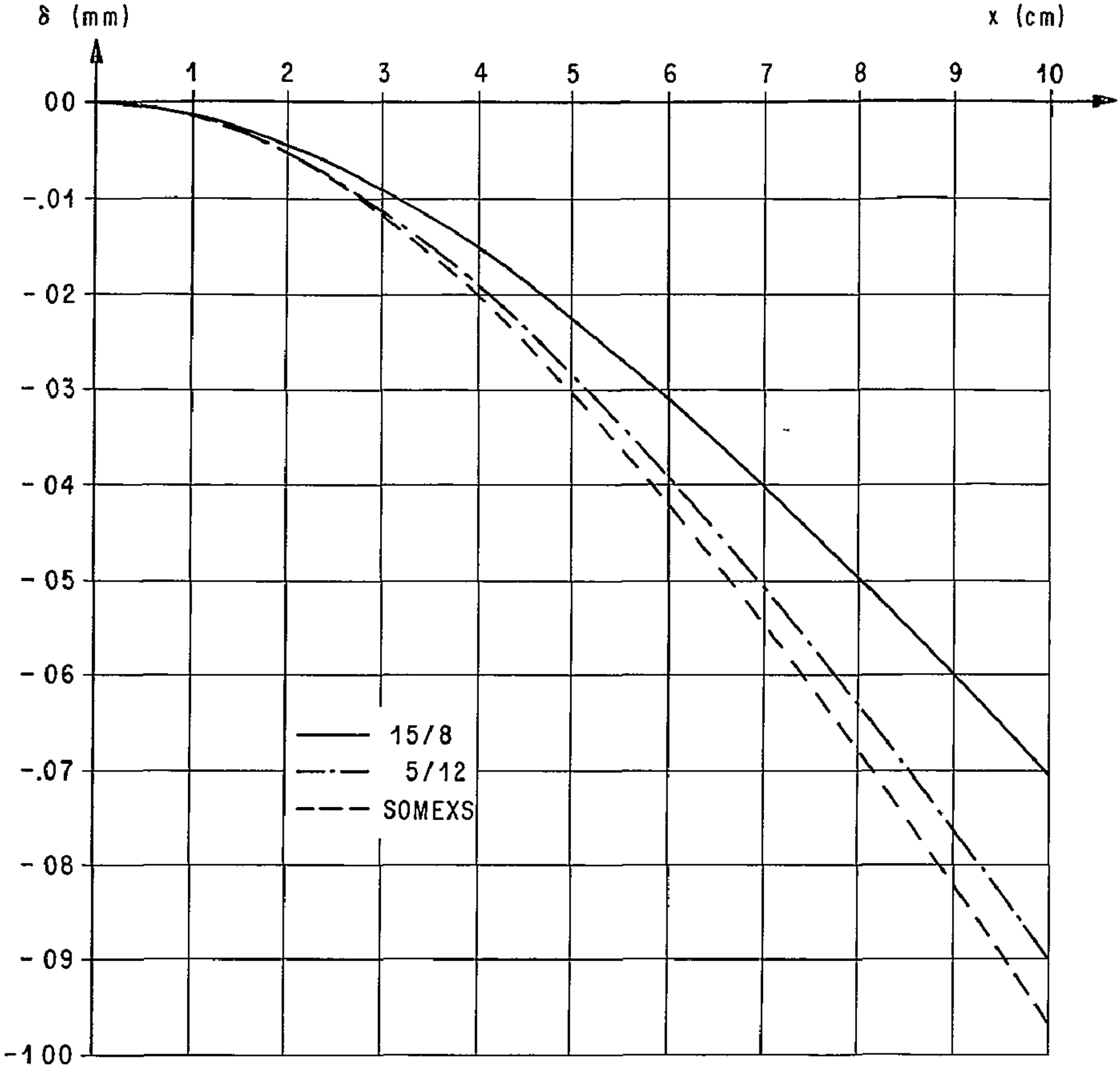

Figure 15. 15/8, 5/12, and SOMEXS Beam Deflections

Four conclusions are drawn: (1) Of primary interest - the twelve-node shear panel is superior to the eight-node panel as the 5/12 beam tip deflection is 93 percent of the tip deflection of the SOMEXS beam, whereas that of the best eight-node panel beam is 73 percent (figure 15); (2) all the shear panel beams (figures 13 and 15) are stiffer than the SOM and TOE beams (figure 12); (3) increasing the number of shear panels composing a structure, i.e., decreasing "grain size," generally increases the accuracy of the results (figure 13), but the additional accuracy gained by adding more rows rapidly approaches a tradeoff with the size of the stiffness matrix as illustrated in figure 14 and table III; and (4) plane sections initially orthogonal to the neutral axis tend essentially to remain plane after loading (figure 16).

Table IV lists the shear panel stiffness matrices in units of dynes/centimeter.

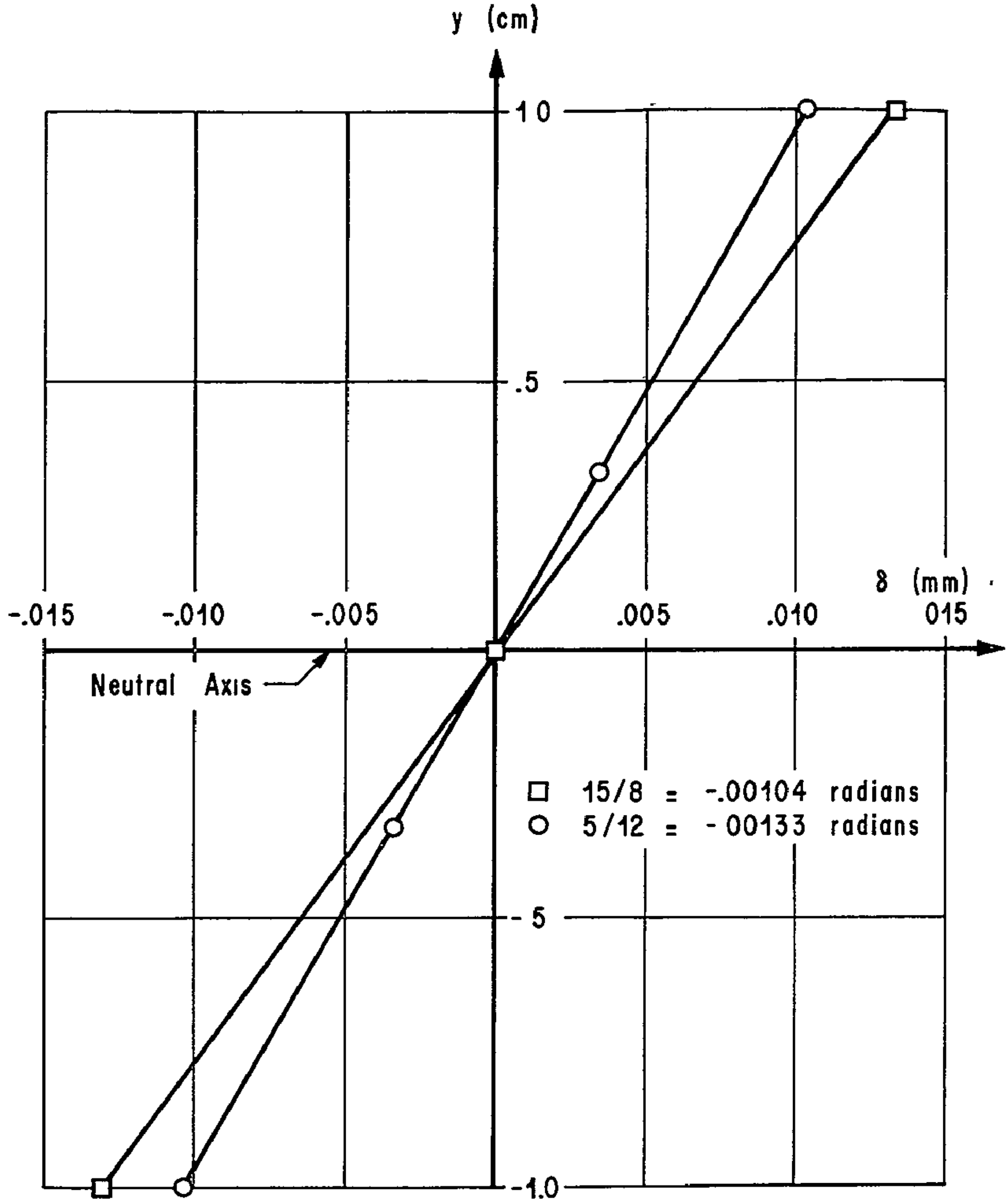

Figure 16. Tip Rotation of 15/8 and 5/12 Shear Panel Beams

TABLE IV SHEAR PANEL STIFFNESS MATRICES

	5/8	10/8	15/8	5/12
S[1, 1]=	.51144231E 12	.45461538E 12	.52407051E 12	.46092949E 12
S[1, 2]=	.18468750E 12	.18468750E 12	.18468750E 12	.28571314E 12
S[1, 3]=	.56826923E 11	-.17048077E 12	-.33464744E 12	.63141026E 10
S[1, 4]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[1, 5]=	-.31254808E 12	-.56826923E 11	.72612179E 11	.31570513E 10
S[1, 6]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[1, 7]=	-.25572115E 12	-.22730769E 12	-.26203526E 12	.59983974E 11
S[1, 8]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.20520833E 11
S[1, 9]=				-.27782051E 12
S[1,10]=				-.18310897E 12
S[1,11]=				-.25256410E 12
S[1,12]=				-.82083333E 11
S[2, 1]=	.18468750E 12	.18468750E 12	.18468750E 12	.28571314E 12
S[2, 2]=	.51144231E 12	.82399038E 12	.11807372E 13	.46092949E 12
S[2, 3]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[2, 4]=	-.31254808E 12	-.72454327E 12	-.11144391E 13	.31570513E 10
S[2, 5]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[2, 6]=	.56826923E 11	.31254808E 12	.52407051E 12	.63141026E 10
S[2, 7]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.20520833E 11
S[2, 8]=	-.25572115E 12	-.41199519E 12	-.59036659E 12	.59983974E 11
S[2, 9]=				-.18310897E 12
S[2,10]=				-.27782051E 12
S[2,11]=				-.82083333E 11
S[2,12]=				-.25256410E 12
S[3, 1]=	.56826923E 11	-.17048077E 12	-.33464744E 12	.63141026E 10
S[3, 2]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[3, 3]=	.51144231E 12	.45461538E 12	.52407051E 12	.46092949E 12
S[3, 4]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.28571314E 12
S[3, 5]=	-.25572115E 12	-.22730769E 12	-.26203526E 12	.59983974E 11
S[3, 6]=	.18468750E 12	.18468750E 12	.18468750E 12	.20520833E 11
S[3, 7]=	-.31254808E 12	-.56826923E 11	.72612179E 11	.31570513E 10
S[3, 8]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[3, 9]=				-.27782051E 12
S[3,10]=				.82083333E 11
S[3,11]=				-.25256410E 12
S[3,12]=				.18310897E 12
S[4, 1]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[4, 2]=	-.31254808E 12	-.72454327E 12	-.11144391E 13	.31570513E 10
S[4, 3]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.28571314E 12
S[4, 4]=	.51144231E 12	.82399038E 12	.11807372E 13	.46092949E 12
S[4, 5]=	.18468750E 12	.18468750E 12	.18468750E 12	.20520833E 11
S[4, 6]=	-.25572115E 12	-.41199519E 12	-.59036859E 12	.59983974E 11
S[4, 7]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[4, 8]=	.56826923E 11	.31254808E 12	.52407051E 12	.63141026E 10
S[4, 9]=				.18310897E 12
S[4,10]=				-.25256410E 12
S[4,11]=				.82083333E 11
S[4,12]=				-.27782051E 12

TABLE IV [CONTINUED]

	5/8	10/8	15/8	5/12
S[5, 1]=	-.31254808E 12	-.56826923E 11	.72612179E 11	.31570513E 10
S[5, 2]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[5, 3]=	-.25572115E 12	-.22730769E 12	-.26203526E 12	.59983974E 11
S[5, 4]=	.18468750E 12	.18468750E 12	.18468750E 12	.20520833E 11
S[5, 5]=	.51144231E 12	.45461538E 12	.52407051E 12	.46092949E 12
S[5, 6]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.28571314E 12
S[5, 7]=	.56826923E 11	-.17048077E 12	-.33464744E 12	.63141026E 10
S[5, 8]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[5, 9]=				-.25256410E 12
S[5,10]=				.18310897E 12
S[5,11]=				-.27782051E 12
S[5,12]=				.82083333E 11
S[6, 1]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[6, 2]=	.56826923E 11	.31254808E 12	.52407051E 12	.63141026E 10
S[6, 3]=	.18468750E 12	.18468750E 12	.18468750E 12	.20520833E 11
S[6, 4]=	-.25572115E 12	-.41199519E 12	-.59036859E 12	.59983974E 11
S[6, 5]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.28571314E 12
S[6, 6]=	.51144231E 12	.82399038E 12	.11807372E 13	.46092949E 12
S[6, 7]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[6, 8]=	-.31254808E 12	-.72454327E 12	-.11144391E 13	.31570513E 10
S[6, 9]=				.82083333E 11
S[6,10]=				-.27782051E 12
S[6,11]=				.18310897E 12
S[6,12]=				-.25256410E 12
S[7, 1]=	-.25572115E 12	-.22730769E 12	-.26203526E 12	.59983974E 11
S[7, 2]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.20520833E 11
S[7, 3]=	-.31254808E 12	-.56826923E 11	.72612179E 11	.31570513E 10
S[7, 4]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[7, 5]=	.56826923E 11	-.17048077E 12	-.33464744E 12	.63141026E 10
S[7, 6]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[7, 7]=	.51144231E 12	.45461538E 12	.52407051E 12	.46092949E 12
S[7, 8]=	.18468750E 12	.18468750E 12	.18468750E 12	.28571314E 12
S[7, 9]=				-.25256410E 12
S[7,10]=				-.82083333E 11
S[7,11]=				-.27782051E 12
S[7,12]=				-.18310897E 12
S[8, 1]=	-.18468750E 12	-.18468750E 12	-.18468750E 12	-.20520833E 11
S[8, 2]=	-.25572115E 12	-.41199519E 12	-.59036859E 12	.59983974E 11
S[8, 3]=	.14206731E 11	.14206731E 11	.14206731E 11	.45777244E 11
S[8, 4]=	.56826923E 11	.31254808E 12	.52407051E 12	.63141026E 10
S[8, 5]=	-.14206731E 11	-.14206731E 11	-.14206731E 11	-.45777244E 11
S[8, 6]=	-.31254808E 12	-.72454327E 12	-.11144391E 13	.31570513E 10
S[8, 7]=	.18468750E 12	.18468750E 12	.18468750E 12	.28571314E 12
S[8, 8]=	.51144231E 12	.82399038E 12	.11807372E 13	.46092949E 12
S[8, 9]=				-.82083333E 11
S[8,10]=				-.25256410E 12
S[8,11]=				-.18310897E 12
S[8,12]=				-.27782051E 12

TABLE IV [CONTINUED]

	5/8	10/8	15/8	5/12
S[9, 1]=				-.27782051E 12
S[9, 2]=				-.18310897E 12
S[9, 3]=				-.27782051E 12
S[9, 4]=				.18310897E 12
S[9, 5]=				-.25256410E 12
S[9, 6]=				.82083333E 11
S[9, 7]=				-.25256410E 12
S[9, 8]=				-.82083333E 11
S[9, 9]=				.13133333E 13
S[9,10]=				.25256410E 12
S[9,11]=				-.25256410E 12
S[9,12]=				-.25256410E 12
S[10, 1]=				-.18310897E 12
S[10, 2]=				-.27782051E 12
S[10, 3]=				.82083333E 11
S[10, 4]=				-.25256410E 12
S[10, 5]=				.18310897E 12
S[10, 6]=				-.27782051E 12
S[10, 7]=				-.82083333E 11
S[10, 8]=				-.25256410E 12
S[10, 9]=				.25256410E 12
S[10,10]=				.13133333E 13
S[10,11]=				-.25256410E 12
S[10,12]=				-.25256410E 12
S[11, 1]=				-.25256410E 12
S[11, 2]=				-.82083333E 11
S[11, 3]=				-.25256410E 12
S[11, 4]=				.82083333E 11
S[11, 5]=				-.27782051E 12
S[11, 6]=				.18310897E 12
S[11, 7]=				-.27782051E 12
S[11, 8]=				-.18310897E 12
S[11, 9]=				-.25256410E 12
S[11,10]=				-.25256410E 12
S[11,11]=				.13133333E 13
S[11,12]=				.25256410E 12
S[12, 1]=				-.82083333E 11
S[12, 2]=				-.25256410E 12
S[12, 3]=				.18310897E 12
S[12, 4]=				-.27782051E 12
S[12, 5]=				.82083333E 11
S[12, 6]=				-.25256410E 12
S[12, 7]=				-.18310897E 12
S[12, 8]=				-.27782051E 12
S[12, 9]=				-.25256410E 12
S[12,10]=				-.25256410E 12
S[12,11]=				.25256410E 12
S[12,12]=				.13133333E 13

STOP

APPENDIX A

Eight-Node Shear Panel Stiffness Matrix

By placing the origin of the X-Y coordinate system at the 1-2 node of the shear panel of figure 1a of the text, retaining the stated assumptions, and setting all $\delta_i \neq 0$, the strains may be easily formulated.

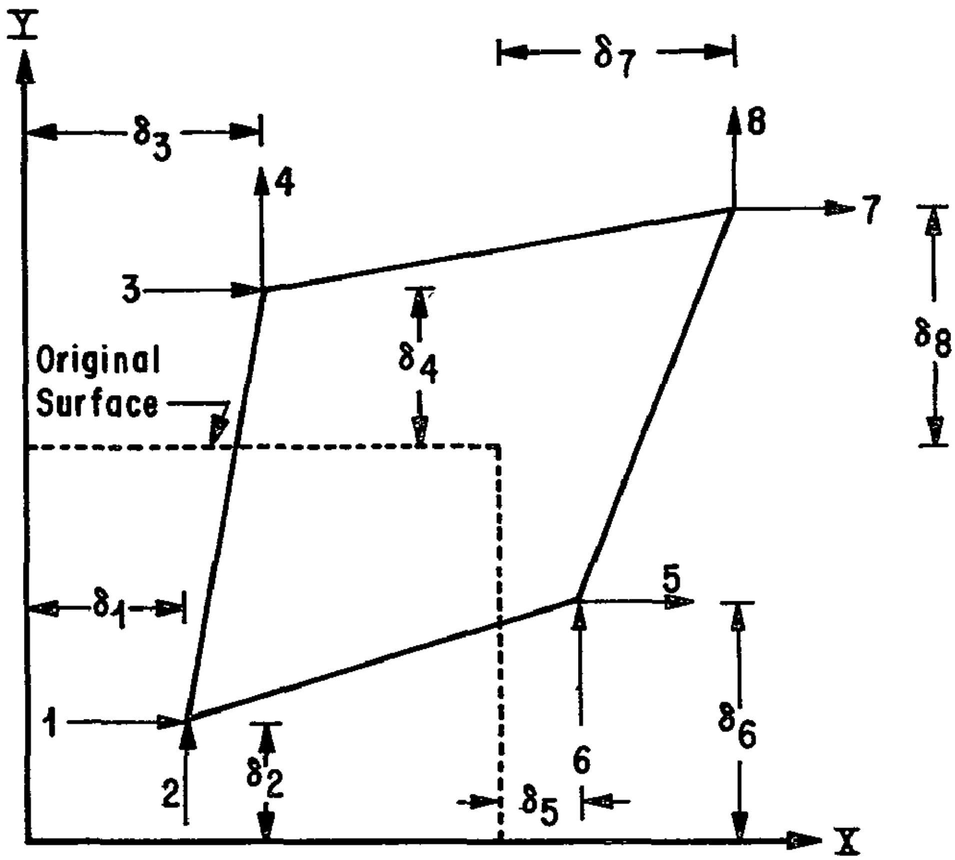

Figure 17. Completely Deformed Eight-Node Shear Panel

Thus, from figure 17,

$$\epsilon_x = \frac{\delta_5 - \delta_1}{A} + \left(\frac{\delta_7 - \delta_3}{A} - \frac{\delta_5 - \delta_1}{A}\right)\frac{y}{B} \qquad (71)$$

where $\frac{\delta_5 - \delta_1}{A}$ is the strain at the lower surface and $\frac{\delta_7 - \delta_3}{A}$ at the upper surface. Similarly,

$$\epsilon_y = \frac{\delta_4 - \delta_2}{B} + \left(\frac{\delta_8 - \delta_6}{B} - \frac{\delta_4 - \delta_2}{B}\right)\frac{x}{A} \tag{72}$$

$$\gamma = \frac{\delta_3 - \delta_1}{B} + \frac{\delta_6 - \delta_2}{A} + \left(\frac{\delta_7 - \delta_5}{B} - \frac{\delta_3 - \delta_1}{B}\right)\frac{x}{A} + \left(\frac{\delta_8 - \delta_4}{A} - \frac{\delta_6 - \delta_2}{A}\right)\frac{y}{B}\,, \tag{73}$$

where $\frac{\delta_3 - \delta_1}{B} + \frac{\delta_6 - \delta_2}{A}$ is the shear strain at the 1-2 node,

$\frac{\delta_7 - \delta_5}{B} - \frac{\delta_3 - \delta_1}{B}$ is the linear strain variation with x, and

$\frac{\delta_8 - \delta_4}{A} - \frac{\delta_6 - \delta_2}{A}$ is the linear variation with y.

Again to calculate a stiffness element, S_{ij}, set all δ_K contained in the expressions for ϵ_x, ϵ_y, and γ equal to zero, $K \neq i$ or j. Next, take the partials of ϵ_x, ϵ_y, and γ with respect to δ_i, then set $\delta_i = 0$ and $\delta_j = 1$ in the resulting expressions of ϵ_x, ϵ_y, γ, $\partial\epsilon_x/\partial\delta_i$, $\partial\epsilon_y/\partial\delta_i$, and $\partial\gamma/\partial\delta_i$, and finally substitute these expressions into equation (45) and integrate.

The first column of the stiffness matrix is

$$S_{1,1} = \frac{ET}{1 - \mu^2}\left[\frac{2\ B/A + A/B\ (1 - \mu)}{6}\right] \tag{74}$$

$$S_{2,1} = \frac{ET}{1 - \mu^2}\left(\frac{1 + \mu}{8}\right) \tag{75}$$

$$S_{3,1} = \frac{ET}{1 - \mu^2}\left[\frac{B/A - A/B\ (1 - \mu)}{6}\right] \tag{76}$$

$$S_{4,1} = \frac{ET}{1 - \mu^2}\left(\frac{1 - 3\mu}{8}\right) \tag{77}$$

$$S_{5,1} = \frac{-ET}{1 - \mu^2}\left[\frac{4\ B/A - A/B\ (1 - \mu)}{12}\right] \tag{78}$$

$$S_{6,1} = \frac{-ET}{1 - \mu^2}\left(\frac{1 - 3\mu}{8}\right) \tag{79}$$

$$S_{7,1} = \frac{-ET}{1 - \mu^2}\left[\frac{2\ B/A + A/B\ (1 - \mu)}{12}\right] \tag{80}$$

$$S_{8,1} = \frac{-ET}{1 - \mu^2}\left(\frac{1 + \mu}{8}\right) . \tag{81}$$

The geometry of substitution is identical to that of the 8 x 8 submatrix of figure 10 of the text

	j = 1	2	3	4	5	6	7	8
i = 1	1,1	2,1	3,1	-6,1	5,1	-4,1	7,1	8,1
2	2,1	$\overline{1,1}$	-4,1	$\overline{5,1}$	-6,1	$\overline{3,1}$	8,1	$\overline{7,1}$
3	3,1	6,1	1,1	-2,1	7,1	-8,1	5,1	4,1
4	4,1	$\overline{5,1}$	-2,1	$\overline{1,1}$	-8,1	$\overline{7,1}$	6,1	$\overline{3,1}$
5	5,1	4,1	7,1	-8,1	1,1	-2,1	3,1	6,1
6	6,1	$\overline{3,1}$	-8,1	$\overline{7,1}$	-2,1	$\overline{1,1}$	4,1	$\overline{5,1}$
7	7,1	8,1	5,1	-4,1	3,1	-6,1	1,1	2,1
8	8,1	$\overline{7,1}$	-6,1	$\overline{3,1}$	-4,1	$\overline{5,1}$	2,1	$\overline{1,1}$

Figure 18. Stiffness Matrix Substitutions for Eight-Node Shear Panel

APPENDIX B

The General Stiffness Equation

For a shear panel of length A, width B, and thickness T (figure 19a), apply a deforming stress σ_x and a restraining stress σ_y(figure 19b):

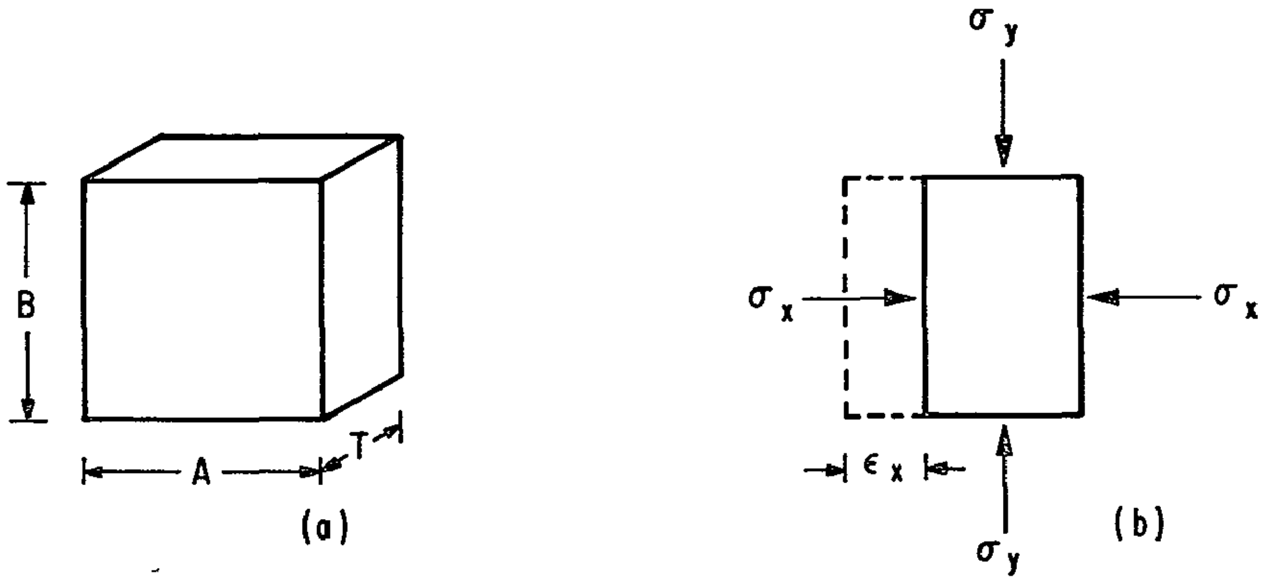

Figure 19. Unstressed Shear Panel with σ_x Added

The unit volume strain energy now stored in the panel is

$$u = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x = \frac{E}{1 - \mu^2} \int_0^{\epsilon_x} \epsilon_x \, d\epsilon_x = \frac{E\epsilon_x^2}{2(1 - \mu^2)} \, . \qquad (82a)$$

Next, add a deforming stress σ_y (figure 20a)

$$u = \frac{E\epsilon_x^2}{2(1 - \mu^2)} + \int_0^{\epsilon_y} \sigma_y \, d\epsilon_y$$

$$u = \frac{E\epsilon_x^2}{2(1 - \mu^2)} + \frac{E}{1 - \mu^2} \int_0^{\epsilon_y} (\epsilon_y + \mu\epsilon_x)\, d\epsilon_y$$

$$u = \frac{E}{2(1 - \mu^2)} (\epsilon_x^2 + 2\mu\epsilon_x \epsilon_y + \epsilon_y^2) \tag{82b}$$

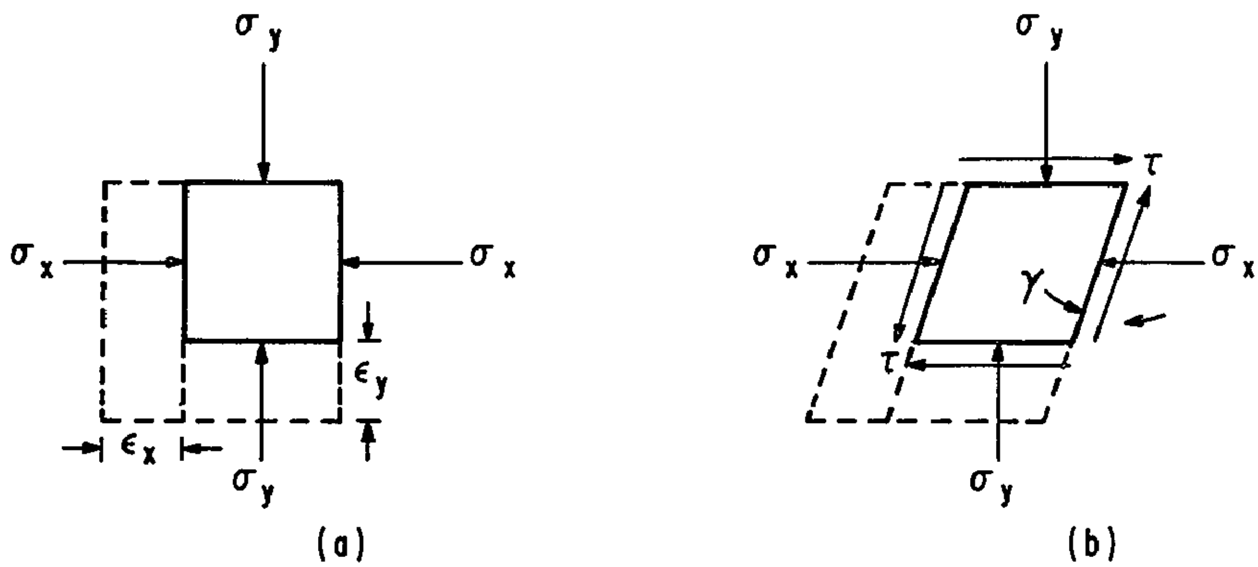

Figure 20. Addition of σ_y and τ, Respectively

Last, by adding a shear stress τ (figure 20b)·

$$u = \frac{E}{2(1 - \mu^2)} (\epsilon_x^2 + 2\mu\epsilon_x \epsilon_y + \epsilon_y^2) + \int_0^{\gamma} \tau\, d\gamma$$

$$u = \frac{E}{2(1 - \mu^2)} (\epsilon_x^2 + 2\mu\epsilon_x \epsilon_y + \epsilon_y^2) + G \int_0^{\gamma} \gamma\, d\gamma$$

$$u = \frac{E}{2(1 - \mu^2)} \left[\epsilon_x^2 + 2\mu\epsilon_x \epsilon_y + \epsilon_y^2 + \frac{1}{2} \gamma^2(1 - \mu)\right] \tag{82c}$$

Total strain energy is

$$U = \int_0^A \int_0^B \int_0^T u \, dz \, dy \, dx$$

$$= \frac{ET}{2(1 - \mu^2)} \int_0^A \int_0^B \left[\epsilon_x^2 + 2\mu\epsilon_x \epsilon_y + \epsilon_y^2 + \frac{1}{2} \gamma^2 (1 - \mu) \right] dy \, dx. \tag{83}$$

Finally, by using Castigliano's first theorem and taking partials under the integral signs,

$$S_{ij} = P_i = \frac{\partial U}{\partial \delta_i} = \frac{ET}{1 - \mu^2} \int_0^A \int_0^B \left[\epsilon_x \frac{\partial \epsilon_x}{\partial \delta_i} + \mu \left(\epsilon_x \frac{\partial \epsilon_y}{\partial \delta_i} + \epsilon_y \frac{\partial \epsilon_x}{\partial \delta_i} \right) \right.$$

$$\left. + \epsilon_y \frac{\partial \epsilon_y}{\partial \delta_i} + \frac{1}{2} \gamma \frac{\partial \gamma}{\partial \delta_i} (1 - \mu) \right] dy \, dx. \tag{45}$$

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APPENDIX C

Matrix Algebra

The stiffness matrix [S] of a composite structure, which is also positive definite, is inverted to find the flexibility matrix [F] by first defining a lower triangular matrix [A], i.e., $i \geq j$, and its transpose such that

$$[A] \times [A^T] = [S] \tag{84}$$

and

$$[F] = [A^T]^{-1} \times [A]^{-1}, \tag{85a}$$

or in algebraic notation

$$A_{\ell\ell} = \sqrt{S_{\ell\ell} - \sum_{i=m}^{\ell-1} A_{\ell i}^2} \tag{86a}$$

$$A_{K\ell} = \left(S_{\ell K} - \sum_{i=m}^{\ell-1} A_{Ki}\, A_{\ell i}\right) \Big/ A_{\ell\ell} \tag{86b}$$

$$A_{\ell\ell}^{-1} = 1/A_{\ell\ell} \tag{87a}$$

$$A_{K\ell}^{-1} = \left(\sum_{i=m}^{K-1} A_{Ki}\, A_{i\ell}^{-1}\right) \Big/ A_{KK} \tag{87b}$$

$$F_{ij} = \sum_{K=i}^{n} A_{Ki}^{-1}\, A_{Kj}^{-1}. \tag{85b}$$

For the given matrix [S] of order r, indices are such that $1 \leqq m \leqq i$, j, K, $\ell \leqq n \leqq r$, with those indices representing foundation nodes omitted If $m = 1$ and $n = r$, then the entire matrix is inverted, otherwise an arbitrary submatrix. By definition, an index of zero denotes an element of zero value, and $K \neq \ell$.

If nonzero deflections K are desired in addition to or instead of a set of loads, then the nodes K are treated as "foundations," and all values $S_{iK}\ \delta_K$ subtracted from each P_i, such that

$$(P_i)_{new} = (P_i)_{old} - S_{iK}\ \delta_K, \tag{88}$$

for i not a foundation node. The provision is especially useful in calculating the stiffness matrix of a complex composite structure of shear panels, beams, etc., by systematically assigning all loads zero, an arbitrary node a unit deflection, all other nodes zero deflections, and applying equations (88), (89), and (90) until the matrix is complete. Otherwise, once [S] is inverted and {P} is biased for "nonzero foundations," use

$$\{\delta\} = [F] \times \{P\} \tag{89}$$

and

$$\{P\} = [S] \times \{\delta\} \tag{90}$$

in that order.

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1. Tiedemann, J. B., "The Jayhawk Bender, A Structural Analysis Program," Report No. 22, CRES, University of Kansas, February 1966

2. Rubinstein, Moshe F., Matrix Computer Analysis of Structures, Prentice-Hall, Inc., 1966.

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DERIVATION OF TWELVE-BY-TWELVE STIFFNESS MATRIX
FOR SHEAR PANEL UNDERGOING PARABOLIC DEFORMATION

by Gary Muller

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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