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HEAT TRANSFER IN THE CONVECTIVE ENVELOPE OF THE SUN

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SUMMARY

If in radiative equilibrium due to adiabatic expansion a rising mass of gas becomes warmer than its surroundings, its motion will continue. Under this condition convection currents are stable. Therefore, convection currents can be produced only if the amount of the adiabatic temperature gradient is smaller than the radiative gradient. The convective heat flux can be computed from the energy balance during convection. It is also possible to derive the convective flux from the equation of radiative transfer, which applies to non-equilibrium photon transport processes, by substituting convective for the radiative parameters.

LIST OF SYMBOLS

Symbol	Definition
a	Radiation density constant
B	Source function
c	Velocity of light
c_p	Specific heat at constant pressure per unit mass
c_v	Specific heat at constant volume per unit mass
$E(r)$	Energy density (energy per unit volume)
$g(r)$	Acceleration due to gravity at distance r from the center
$H(r)$	Radiative power flux-density at distance r from the center
$I(r)$	Radiant power per unit area per unit solid angle at r
$j(r)$	Emitted power per unit mass per unit solid angle at r

LIST OF SYMBOLS (Concluded)

Symbol	Definition
ℓ	Mixing length
$L(r)$	Radiant power at the distance r from the center
$P(r)$	Hydrostatic pressure at the distance r from the center
$P(r)_{rad}$	Radiation pressure at the distance r from the center
r	Radius from the center of the Sun to some point of the interior
R^*	Universal gas constant
S	Scale height
$T(r)$	Temperature at the distance r from the center
U	Internal energy per unit of convective mass
v	Velocity of a stable convective element
γ	Ratio of specific heats, c_p/c_v
θ	Angle between the direction of a ray and the axis of a small cylinder
$\overline{\kappa}$	Rosseland's κ (corrected for induced emission)
μ	Mean molecular weight of the Sun's material
$\rho(r)$	Density at the distance r from the center
σ	Stefan-Boltzmann constant
τ	Optical depth
$d\omega$	Element of solid angle
∇	Logarithmic gradient, temperature to pressure
Subscripts	
ad	adiabatic
k	convective
rad	radiative
Superscript	
$'$	prime: refers to convecting element

I. INTRODUCTION

The physical properties of the Sun are explained by a composite model for which the energy transport mode is radiative in the core and radiative and convective in the envelope. If the condition of radiative equilibrium is maintained, no unsteady convective mass motions can occur. However, if the conditions for radiative equilibrium are not fulfilled, perturbed mass and energy transport may result. The basic conditions which affect stability are stated by the four differential equations for hydrostatic equilibrium, conservation of mass, luminosity, and temperature gradient (convective and radiative). These equations have to be complemented by three constitutive equations in order that the behavior of the gases may be characterized. The constitutive equations are the equation of state, the equation for the absorption coefficient, and the equation for energy generation by nuclear processes.

The temperature gradient has two distinct forms, adiabatic and radiative. Which form is applicable is determined by whether the stability condition,

$$- \frac{dT}{dr} < - \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr} ,$$

is met (adiabatic) or is not met (radiative).

By using only the convective energy transport condition, the entire outer region of the Sun is described by the adiabatic gradient. This is a rough procedure which yields erroneous results. Since both radiative and convective fluxes occur in convective equilibrium, both contributions are considered in each of the two derivations outlined in the next two sections.

II. DERIVATION OF THE CONVECTIVE FLUX FROM THERMODYNAMIC EQUILIBRIUM

For the calculation of energy transport by radiation and convection in the convective layer of the Sun, the pure adiabatic and radiative gradients are supplemented by an overall average gradient and a gradient describing a turbulent element in surroundings which are otherwise in radiative equilibrium. These four gradients, negative in the sense of decreasing temperature with increasing radius r, are in order of decreasing magnitude:

∇_{rad}	$= (d\ln T/d\ln P)_{rad}$	for radiative equilibrium
∇	$= (d\ln T/d\ln P)$	actual average gradient at radius r
∇'	$= (d\ln T/d\ln P)'$	for an average convective element
∇_{ad}	$= (d\ln T/d\ln P)_{ad}$	for adiabatic changes of state.

If as a result of adiabatic expansion a mass of gas rises through surroundings which are otherwise in radiative equilibrium, and if the negative adiabatic temperature gradient of the rising mass is less than the negative radiative gradient of the surroundings, then the rising mass will cool less than the surroundings and thus be relatively warmer. If expansion exactly compensates for relative warming, then stable mass motion will continue. These conditions, adiabatic expansion with temperature gradient of the rising mass less than the radiative gradient of the surroundings, are the necessary and only conditions for generation of stable convection currents. The gradient relationship can be expressed in the form:

$$\nabla_{ad} \leq \nabla' < \nabla < \nabla_{rad}.$$

The convective power density H_k can be computed by the following procedure. When a turbulent mass rises a distance Δr, its temperature drops less rapidly than that of its surroundings and thus the mass tends to be warmer than its surroundings:

$$\Delta T = [(dT/dr)' - (dT/dr)] \Delta r.$$

(The primed and unprimed terms have the same meaning as in the gradient relationships in the preceding paragraph.)

The convectively transported power density equals:

$$H_k = c_p \rho \overline{\Delta T} \, \overline{v} = c_p \rho T (\overline{\Delta T}/T) \overline{v}.$$

(The bars above the symbols indicate mean values over a layer of thickness dr.)

The formula for hydrostatic pressure is determined as follows:

$$dP = -g\rho dr$$

$$d\ln P = -(g\mu / R^* T) dr = -dr/S.$$

The mean ascent of an arbitrary turbulent element is set equal to half the mixing length:

$$\Delta r = dr = \frac{1}{2} \ell ,$$

and the final expression for the convective power-density transport is obtained:

$$H_k = c_p T \rho \overline{v} \frac{\ell}{2S} (\nabla - \nabla'). \qquad (1)$$

III. DERIVATION OF THE CONVECTIVE FLUX BY THE PHOTON TRANSPORT EQUATION

In order to compute the power density transported by radiation, H_{rad}, we apply the "gray spectrum" approximation with Rosseland's coefficient of opacity $\overline{\kappa}$. The source function is expressed as:

$$B(\overline{\tau}) = \frac{\sigma}{\pi} T^4(\overline{\tau}).$$

To express the dependence of B on the optical depth $\overline{\tau}$, we use the equation of radiative transfer:

$$L = -\frac{4\pi r^2}{3\overline{\kappa}\rho} \frac{d(acT^4)}{dr},$$

together with the geometrical relation:

$$L(r) = 4\pi r^2 H_{rad}.$$

The expression for optical depth is introduced:

$$d\overline{\tau} = \overline{\kappa}\rho dr,$$

and the Stefan-Boltzmann constant:

$$\sigma = ac/4.$$

These five relationships are combined:

$$H_{rad} = -\frac{ac}{3}\frac{dT^4}{d\overline{\tau}} = -\frac{4}{3}\sigma\frac{dT^4}{d\overline{\tau}} = -\frac{4}{3}\pi\frac{dB}{d\overline{\tau}},$$

and the formula for hydrostatic pressure is used:

$$d\overline{\tau} = \overline{\kappa}\rho S d(\ln P);$$

therefore,

$$H_{rad} = \frac{16}{3}\frac{\sigma T^4}{\overline{\kappa}\rho S}\nabla . \qquad (2)$$

The convective envelope is a fluid which possesses, in principle, all possible degrees of freedom. At large Reynolds numbers, very many of them come into play. The motion then becomes so complicated that simple answers can be given to only those questions which concern the average behavior of the moving particles. Therefore, the equilibrium between the different elements has to be described statistically. As in the kinetic theory of gases, the interest lies in the statistically most probable distribution of the elements.

Equation 2 for radiative power transport is derived from the equation of radiative transfer, which applies to all statistically controlled photon transport processes. Equation 1 for convective power transport is also based on statistical premises. The equation of radiative power transport will be used next as a base for deriving equations of convective power transport.

The derivation of the equilibrium condition, which takes the place of Equation 1, is the same as the derivation of Equation 2 except that convective parameters are substituted for the radiative parameters. The method of Schwarzschild is employed.

The basic equation of radiative transfer, which is valid at every point within the Sun, has the form:

$$\frac{\partial I}{\partial r} \cos \theta - \frac{\partial I}{\partial \theta} \frac{\sin \theta}{r} + \overline{K} \rho I - \frac{1}{4\pi} j\rho = 0 .$$

Instead of considering the function I, which represents the distribution of the radiation in all directions from a point, we consider the first three moments of this distribution function:

$E(r) = \frac{1}{c} \int I \, d\omega$	radiant energy density
$H(r) = \int I \cos \theta \, d\omega$	radiation power flux-density
$P_{rad}(r) = \frac{1}{c} \int I \cos^2 \theta \, d\omega$	radiation pressure.

Differential equations for these moments can be obtained by forming the corresponding moments of the equation of radiative transfer. The radiative transfer equation is multiplied by powers of cos θ and integrated over all directions; then multiplying it by 1 and cos θ , respectively, we obtain the first two moments:

$$\frac{dH}{dr} + \frac{2}{r} H + c\overline{\kappa}\rho E - j\rho = 0$$
$$\frac{dP}{dr} + \frac{1}{r} (3P_{rad}-E) + \frac{\overline{\kappa}\rho}{c} H = 0. \tag{3}$$

These are two equations for the three radiation functions E, H, and P. To secure a definite solution, we have to derive an additional relation between the moments. At a given point, the radiation field can be represented by the series:

$$I = I_0 + I_1 \cos\theta\, I_2 \cos^2\theta + \ldots\ldots$$

As a result of the large absorption coefficient, this series converges rapidly in the solar interior. Therefore, the series will be restricted to the first two terms; it cannot be restricted to the first term only, because the radiation field would then be isotropic, without any net flux. If the first two terms of the series are introduced into the moment equations, the following results are obtained:

$$E = \frac{4\pi}{c} I_0$$

$$H = \frac{4\pi}{3} I_1$$

$$P_{rad} = \frac{4\pi}{3c} I_0 \ ;$$

therefore,

$$P_{rad} = E/3 . \tag{4}$$

Equations 3 and 4 form the required set of three equations for the three moments.

The flux density H(r) may be represented in terms of L(r), the flux through the spherical area of radius r:

$$H = L/4\pi r^2 .$$

So far the derivation has been identical with that developed from the radiation equilibrium condition. However, instead of the Stefan-Boltzmann law being introduced, which is the next step in the radiative derivation, the analysis proceeds as follows:

In order that the adiabatic equilibrium condition may be derived the negligible contribution of adiabatic mass expansion is recognized in defining the emission coefficient for the convecting mass:

$j = \bar{\kappa}\rho c c_v T$; (c now denotes the velocity of sound).

Substituting these relations for H and j into the first Equation 3 yields the simple form:

$$E = c_v \rho T . \tag{5}$$

Equations 4 and 5 yield the following expression for the radiation pressure:

$$P_{rad} = c_v \rho T/3 . \tag{6}$$

When Equations 5 and 6 are substituted into the second of Equations 3, the new expression for power flux-density transport (convective) is obtained:

$$H = -\frac{cc_v}{3\overline{K}}\frac{dT}{dr} . \tag{7}$$

When H is replaced by $L/4\pi r^2$, the basic equilibrium condition is obtained in the form:

$$L = -\frac{4\pi r^2 \, cc_v}{3\overline{K}}\frac{dT}{dr} , \tag{8}$$

which is a new expression for the basic condition of convective equilibrium stated in the Introduction.

According to Equation 5, the total internal energy of the convecting unit mass equals:

$$U = c_v T ,$$

which yields for the specific heat:

$$\left(\frac{\partial U}{\partial T}\right)_v = c_v .$$

Since Equations 1 and 7 are both valid expressions for convective power-density transport, they must be equal to each other. When the scale height S is set equal to the mixing length ℓ in Equation 1, the following relation is derived:

$$\overline{\nabla}(\nabla - \nabla') = -\frac{2}{3}\frac{c}{\gamma\overline{K}\rho T}\frac{dT}{dr} .$$

IV. RESULTS AND CONCLUSIONS

The heat flux in the convective layer of the Sun is derived by two different methods. The first method is conventional, and starts with the condition of thermodynamic equilibrium. The second method is new, and is based upon the photon transport equation. The two formulas for the same quantity yield a relation between the temperature gradients occurring in the equations.

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