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AN ANALYTICAL DEVELOPMENT OF THE RELATIVE MOTION OF TWO CLOSE SATELLITES OF AN OBLATE PLANET

RICHARD W. BARBIERI

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GSFC

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ABSTRACT

The purpose of this study is to describe the relative planar motion of two satellites which are close ($\approx$ 5 km) in comparison to their respective distances ($\approx$ 20,000 km) from the center of an oblate central body. This research was undertaken to support current studies in orbiting long baseline interferometry systems like, for example, the proposed Radio Astronomy Explorer C & D satellite. The equations which govern the motion of one of the satellites with respect to the other were derived from a Lagrangian formulation and neglect the mutual attraction of the two satellites.

The equations have been expanded to first order in eccentricity of the orbit of the reference satellite and also to first order in a small parameter which is a function of the oblateness of the central body. The expansion to first order of eccentricity is compatible with the intended small eccentricity (< 0.01) of the proposed RAE C & D.

The results to first order eccentricity, neglecting oblateness, show that the motion is almost periodic. When the first order oblateness is included a secular term appears and the almost periodic behavior becomes superimposed upon a secular drift. Certain small divisors appear when the oblateness is included; in particular the rate of change of the argument of perigee appears in the form of linear combinations with multiples of the mean motion and the modified mean motion.

Glossary of Symbols

$\vec{r}_M$ radius vector of the M satellite with respect to the center of mass of the central body

$\vec{r}_{0'}$ radius vector of the 0′ satellite with respect to the center of mass of the central body

$\vec{R}$ position vector of M with respect to 0′

a the semi-major axis of the orbit of 0′

ι the inclination of the orbit of 0′

e the eccentricity of the orbit of 0′

Ω the longitude of the ascending node of the orbit of 0′

w the argument of perigee of 0′

v the true anomaly of 0′

u the sum $(w + v)$.

ω the angular velocity of 0′

T the kinetic energy of the M satellite

U the potential energy of the M satellite

L the Lagrangian, T-U.

$\{\dot{x}_{0'}, \dot{y}_{0'}, \dot{z}_{0'}\}$ the velocity components of 0′ in the rotating coordinate system.

$\{F_x, F_y, F_z\}$ the force components, in the rotating coordinate system, acting on the M satellite.

μ the gravitational constant of the central body

R_e the equatorial radius of the central body

J_{20} the first non zero coefficient in the spherical harmonic expansion of the gravity field of the central body

$\epsilon \equiv \frac{3}{2} J_{20}$

$n \equiv \sqrt{\mu a^{-3}}$

$\bar{n} \equiv n + \dot{\Omega} \cos \iota$

$M \equiv nt$

w_0 the secular rate of change of the argument of perigee

AN ANALYTICAL DEVELOPMENT OF THE RELATIVE MOTION OF TWO CLOSE SATELLITES OF AN OBLATE PLANET

INTRODUCTION

The solution of the problem of relative motion of two spacecraft has usually been constructed in the past with rendezvous and docking applications in mind. Since such applications were usually performed within several revolutions about the central body it was unnecessary to consider the oblateness of the central body. In this report the non-spherical nature of the central body is taken into account because problems of a different nature than those of rendezvous and docking have received considerable attention over the last few years. In particular, the feasibility of using orbiting long baseline interferometry systems has been extensively studied. Such a system will be orbiting the central body for a few years at least and it would therefore be advantageous to be able to approximate the accumulative effects of the oblateness on the motion. Fortunately the intended eccentricity of such interferometry systems is small, less than 0.01, and therefore an expansion to the first order of eccentricity will be presented.

The technique used here is rather well known among those working with perturbation theory although its application to this particular problem is novel. To be explicit, a double expansion has been used to isolate the first order effects due to eccentricity from the first order effects due to the presence of an oblate central body. Furthermore, even though a device first used by Poincare, that of expanding the frequency in terms of a small parameters, has been adopted in this problem to eliminate mixed secular terms, a small purely secular drift cannot be removed from this first order theory.

A complete derivation of the equations of relative motion has been presented even though only the case of planar motion will be considered in this report. The planar and non-planar motions are mathematically and physically distinct problems requiring decidedly different techniques for solution. It is felt then that a separate consideration of the non-planar motion and it's stability characteristics is justifed.

In 1960 Clohessy and Wiltshire [1] published a paper concerned with the relative motion of two close satellites. Their development concerned itself with circular orbits about a spherically symmetric central body. H. S. London [2] extended the results by including second order terms in various components of the relative distance and relative velocity normalized with respect to orbital radius and orbital velocity. His work is confined to circular orbits and spherically symmetric central bodies. In [3], J. de Vries made a generalization by allowing the nominal orbit to be slightly eccentric. His work assumes a spherically symmetric central body. In [4], Anthony and Sasaki obtain an approximate solution of the equations of relative motion including linear and quadratic terms in the relative distance for the case of nearly circular orbits about a spherical central body. In [5], Euler and Shulman made a slight extension of the work of Anthony and Sasaki by allowing orbits of arbitrary eccentricity. In [6], Schechter and Cole are concerned with the influence of air drag and oblateness. In their study the origin is assumed to be located in an infinitely heavy satellite moving in a circular orbit at an altitude of 100 n miles. The motion of each of the satellites, obtained by an application of a two variable expansion procedure, is referred to this ficticious satellite and then, by subtraction, is made relative to each other.

FORMULATION OF THE EQUATIONS OF RELATIVE MOTION

The equations of relative motion will be derived from the Lagrangian formulation where the Lagrangian L is represented by the difference between the kinetic energy T and the potential energy U of a spherically symmetric central body. The force per unit mass will be denoted by F.

Choose an inertial coordinate system X Y Z where X is in the plane of the equator of the central body and lies along some fixed direction in space, Y is also in the plane of the equator ninety degrees ahead of X and Z completes an orthogonal coordinate system whose origin is at the center of mass of the central body (see Figure 1).

Now consider the motion of one of the two satellites as completely known relative to this inertial system and define a frame of reference 0′, whose origin is at the center of mass of this satellite, in such a way that x is along the radius vector to 0′, y is in the orbit plane of 0′ ninety degrees ahead of x and z is normal to the orbit plane of 0′. The inclination of the orbit of 0′ is denoted by ι, it's longitude of the ascending node is Ω and the sum of it's true anomaly, v, and argument of perigee, w, is denoted by u.

The problem now is to formulate the equations of motion of the satellite M with respect to the moving reference frame 0′. It is easily seen that

$$\vec{r}_M = \vec{r}_{0'} + \vec{R} \tag{1}$$

and that

$$\frac{d\,\vec{r}_M}{d\,t} = \frac{d\,\vec{r}_{0'}}{d\,t} + \frac{d\vec{R}}{d\,t} \tag{2}$$

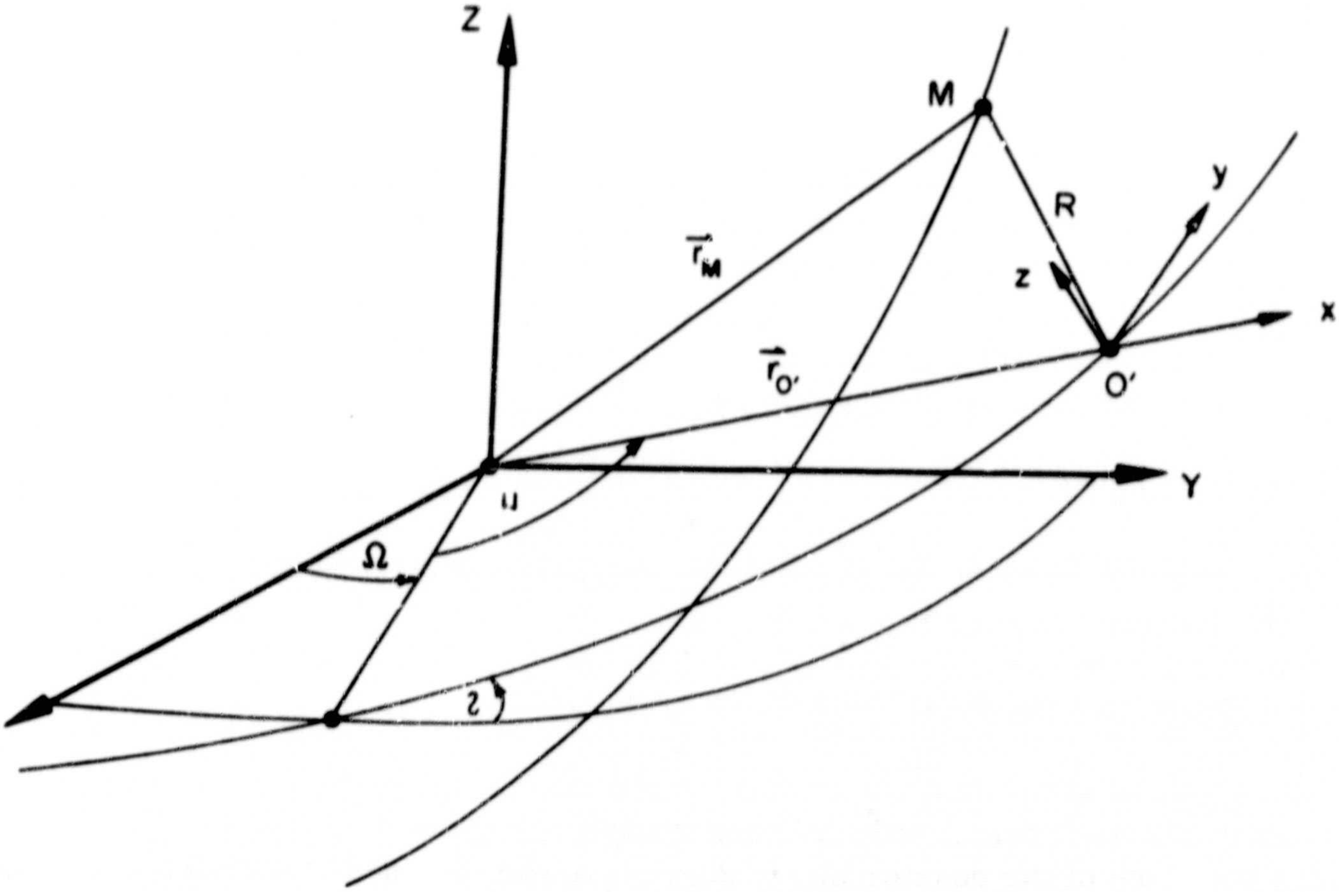

Figure 1. Geometry

where all terms in (2) are referred to the inertial coordinate system. Taking into account that 0′ is moving with respect to the fixed inertial system, the derivative $d\vec{R}/dt$ can be represented as

$$\left(\frac{d\vec{R}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{Moving}} + \vec{\omega} \times \vec{R} \tag{3}$$

where $\vec{\omega}$, the angular velocity vector of the moving reference frame 0′, is given by

$$\begin{aligned} \vec{\omega} = {} & (\dot{\iota} \cos u + \dot{\Omega} \cos u \sin \iota)\, \hat{x} \\ & + (-\dot{\iota} \sin u + \dot{\Omega} \cos u \sin \iota)\, \hat{y} \\ & + (\dot{u} + \dot{\Omega} \cos \iota)\, \hat{z} \end{aligned}$$

Here the dots indicate derivates with respect to time and the symbol ^ indicates a unit vector. Note that $\vec{\omega}$ is expressed in the moving reference frame. A derivation of the components of $\vec{\omega}$ in the moving coordinate system is presented in Appendix 1.

If the velocity of 0′ is resolved along the instantaneous directions of the moving axes, yielding the components $\{\dot{x}_{0'}, \dot{y}_{0'}, \dot{z}_{0'}\}$, then the kinetic energy T of the M satellite written in the inertial frame, takes the form

$$T = \frac{m}{2}\left[\frac{d\vec{r}_{0'}}{dt} + \left(\frac{d\vec{R}}{dt}\right)_{\text{Moving}} + \vec{\omega}\times\vec{R}\right]^2 \tag{4}$$

where m is the mass of the M satellite. Carrying out the necessary operations leads to

$$\begin{aligned} T = {} & \frac{m}{2}(\dot{x}_{0'}^2 + \dot{y}_{0'}^2 + \dot{z}_{0'}^2) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ & + \frac{m}{2}\{(y\,\omega_z - z\,\omega_y)^2 + (z\,\omega_x - x\,\omega_z)^2 + (x\,\omega_y - y\,\omega_x)^2\} \\ & + m\,\{\dot{x}_{0'}(\dot{x} - y\,\omega_z + z\,\omega_y) + \dot{y}_{0'}(\dot{y} - z\,\omega_x + x\,\omega_z) + \dot{z}_{0'}(\dot{z} - x\,\omega_y + y\,\omega_x)\} \\ & + m\,\{\dot{x}(-y\,\omega_z + z\,\omega_y) + \dot{y}(-z\,\omega_x + x\,\omega_z) + \dot{z}(-x\,\omega_y + y\,\omega_x)\} \end{aligned} \tag{5}$$

The first term in this expression is just the kinetic energy of the moving origin. The second term is the kinetic energy as would be calculated by an observer in the 0′ frame who thinks 0′ is fixed. The third term is related to the moment of inertia about the axis of rotation and may be written simply as $I\,\omega^2$ where

$$I = \frac{m}{2}\left[R^2 - \frac{(\vec{R}\cdot\vec{\omega})^2}{\omega^2}\right].$$

The fourth term represents the coupling of the motion of 0′ to the motion of M and may be rewritten in the form

$$m\frac{d}{dt}(\vec{R}\cdot\vec{V}) - m\,(\vec{R}\cdot\vec{A})$$

where $\vec{V}$ and $\vec{A}$ are respectively the velocity and acceleration of $0'$. The last term may be written in the form

$$\omega_x \left[m\,(y\,\dot z - z\,\dot y)\right] + \omega_y \left[m\,(z\,\dot x - x\,\dot z)\right] + \omega_z \left[m\,(x\,\dot y - y\,\dot x)\right]$$

which is recognized as the scalar product of the angular velocity vector and the apparent angular momentum vector, that is, the angular momentum as would be calculated by an observer in the $0'$ frame who thinks $0'$ is fixed.

The equations of relative motion expressed in generalized coordinates, $(q, \dot q)$, take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = F_q \tag{6}$$

where $L = T - U$. Allowing q to take the notation x, y, z respectively yields, from the Lagrangian formulation

$$\begin{aligned} &\ddot x + \ddot x_{0'} + 2\,(\omega_y\,\dot z - \omega_z\,\dot y) + z\,\dot\omega_y - y\,\dot\omega_z + \omega_y\,\dot z_{0'} - \omega_z\,\dot y_{0'} \\ &\quad + \omega_y\,(z\,\omega_x - x\,\omega_z) + \omega_y\,(y\,\omega_x - x\,\omega_y) + \frac{1}{m}\,\frac{\partial U}{\partial x} = F_x \end{aligned} \tag{7}$$

$$\begin{aligned} &\ddot y + \ddot y_{0'} + 2\,(\omega_z\,\dot x - \omega_x\,\dot z) + x\,\dot\omega_z - z\,\dot\omega_x + \omega_z\,\dot x_{0'} - \omega_x\,\dot z_{0'} \\ &\quad + \omega_x\,(x\,\omega_y - y\,\omega_x) + \omega_z\,(z\,\omega_y - y\,\omega_z) + \frac{1}{m}\,\frac{\partial U}{\partial y} = F_y \end{aligned} \tag{8}$$

$$\begin{aligned} &\ddot z + \ddot z_{0'} + 2\,(\omega_x\,\dot y - \omega_y\,\dot x) + y\,\dot\omega_y - x\,\dot\omega_y + \omega_x\,\dot y_{0'} - \omega_y\,\dot x_{0'} \\ &\quad + \omega_y\,(y\,\omega_z - z\,\omega_z) - \omega_x\,(z\,\omega_x - x\,\omega_z) + \frac{1}{m}\,\frac{\partial U}{\partial z} = F_z \end{aligned} \tag{9}$$

Neglecting for a moment the potential U and the components of force it is seen that the total acceleration of the M satellite written in the rotating frame is made up of three terms:

(i) acceleration relative to the rotating frame; this leads to terms like $\ddot{x}, \ddot{y}, \ddot{z}$

(ii) acceleration due to Coriolis effects; this leads to terms like

$$2\,(\omega_y\,\dot{z} - \omega_z\,\dot{y}),\ 2\,(\omega_z\,\dot{x} - \omega_x\,\dot{z}) \text{ and } 2\,(\omega_x\,\dot{y} - \omega_y\,\dot{x})$$

(iii) acceleration as would be calculated by an observer fixed with respect to the rotating reference frame; this leads to terms like

$$\ddot{x}_{0'} + (\omega_y\,\dot{z}_{0'} - \omega_z\,\dot{y}_{0'}) + (z\,\dot{\omega}_y - y\,\dot{\omega}_z) - x\,\omega^2 + \omega_x\,(\vec{R}\cdot\vec{\omega})$$

$$\ddot{y}_{0'} + (\omega_z\,\dot{x}_{0'} - \omega_x\,\dot{z}_{0'}) + (x\,\dot{\omega}_z - z\,\dot{\omega}_x) - y\,\omega^2 + \omega_y\,(\vec{R}\cdot\vec{\omega})$$

$$\ddot{z}_{0'} + (\omega_x\,\dot{y}_{0'} - \omega_y\,\dot{x}_{0'}) + (y\,\dot{\omega}_x - x\,\dot{\omega}_y) - z\,\omega^2 + \omega_z\,(\vec{R}\cdot\vec{\omega})$$

In order to cast the equations (7) thru (9) into a form which will yield a first order theory more readily certain simplifications will be made. To begin ω_x will be neglected so as to uncouple the planar motion and the non-planar motion. A constraint on a component of the total rotation of the orbital plane of 0′ is thereby imposed namely, the motion governed by the equation

$$\dot{\iota}\cos u + \dot{\Omega}\sin u \sin \iota$$

is being neglected. What this means physically is that motion of the orbital plane of 0′ about the radius vector of 0′ is being held fixed. Secondly the mutual attraction of the two satellites is being neglected. Thirdly it is noted that ω_y must be indentically zero because the xy-plane must contain, at every instant, the radius vector and velocity vector of 0′. The preceding comments concerning ω_x and ω_y reduce the equations of relative motion to the form

$$\ddot{x} + \ddot{x}_{0'} - 2\,\dot{y}\,\omega_z - y\,\dot{\omega}_z - \omega_z\,\dot{y}_{0'} - x\,\omega_z^2 + \frac{1}{m}\,\frac{\partial U}{\partial x} = F_x \tag{10}$$

$$\ddot{y} + \ddot{y}_{0'} + 2\,\dot{x}\,\omega_z + x\,\dot{\omega}_z + \omega_z\,\dot{x}_{0'} + y\,\omega_z^2 + \frac{1}{m}\,\frac{\partial U}{\partial y} = F_y \tag{11}$$

$$\ddot{z} + \ddot{z}_{0'} + \frac{1}{m}\,\frac{\partial U}{\partial z} = F_z \tag{12}$$

The formulation of the equations of motion is almost complete. To finish the objective certain representations must be used. In particular, the potential U takes the form

$$U = -\frac{\mu}{r_m} \tag{13}$$

where μ is the gravitational parameter of the central body and

$$r_m^2 = (x + x_{0'})^2 + y^2 + z^2;$$

furthermore the components of force per unit mass due to the oblateness of a central body of radius R_e take the form

$$F_x = \frac{3}{2} \mu J_{20} R_e^2 \frac{x + x_{0'}}{x_{0'}^5} (1 - 3 \sin^2 \iota \sin^2 u) \tag{14}$$

$$F_y = \frac{3}{2} \mu J_{20} R_e^2 \frac{\sin u \cos u \sin^2 \iota}{x_{0'}^4} \tag{15}$$

$$F_z = \frac{3}{2} \mu J_{20} R_e^2 \frac{\sin u \sin \iota \cos \iota}{x_{0'}^4} \tag{16}$$

Since the motion of $0'$ is being assumed to obey perturbed two body laws, it is clear that the governing equations in the rotating reference frame are

$$\ddot{x}_{0'} - x_{0'} \omega_z^2 + \frac{\mu}{x_{0'}^2} = (F_x)_{0'} \tag{17}$$

$$\frac{1}{x_{0'}} \frac{d}{dt} (x_{0'}^2 \omega_z) = (F_y)_{0'} \tag{18}$$

where $(F_x)_{0'}$ and $(F_y)_{0'}$ denote the components of the force on $0'$ due to oblateness. In addition, since ω_x is being neglected, $\dot{y}_{0'} = x_{0'} \omega_z$ and the angular momentum of the perturbed motion will be approximated by that of the two body problem.

Making use of (13) thru (18) and the preceeding comments, the system (10) thru (12) may be written in the form

$$\ddot{x} - 2\,\dot{y}\,\omega_z + 2\,y\,\omega_z^2 \frac{e \sin v}{1 + e \cos v} - x\,\omega_z^2 \frac{e \cos v}{1 + e \cos v}$$

$$= \frac{3}{2}\mu\, J_{20}\, R_e^2 \frac{x}{x_0^{5\,\prime}} (1 - 3 \sin^2 \iota \sin^2 u) \tag{19}$$

$$\ddot{y} + 2\,\dot{x}\,\omega_z - 2\,x\,\omega_z^2 \frac{e \sin v}{1 + e \cos v} - y\,\omega_z^2 \frac{e \cos v}{1 + e \cos v}$$

$$= \frac{3}{2}\mu\, J_{20}\, R_e^2 \frac{y}{x_0^{5\,\prime}} \cos 2u \sin^2 \iota \tag{20}$$

$$\ddot{z} + z\,\omega_z^2 \frac{1}{1 + e \cos v} = \frac{3}{2}\mu\, J_{20}\, R_e^2 \frac{\sin u \sin \iota \cos \iota}{x_0^{4\,\prime}} \tag{21}$$

These equations represent the complete formulation of the equations of relative motion where oblateness of the central body is considered and when the planar and non-planar motions are uncoupled.

SOLUTION OF EQUATIONS GOVERNING PLANAR MOTION

In this section attention will be devoted to the solution of equations (19) and (20). These equations are coupled linear second order differential equations containing a small parameter $\epsilon \equiv 3/2\ J_{20}$ and are amenable to a solution which takes the form of a double expansion in eccentricity and in the small parameter. As already mentioned the eccentricity is assumed to be small (< 0.01) and therefore a first order theory will be presented; consequently consideration will be given to the formal summations

$$x(t) = x_{00}(t) + e\,x_{01}(t) + \epsilon\,x_{10}(t) \tag{22}$$

$$y(t) = y_{00}(t) + e\,y_{01}(t) + \epsilon\,y_{10}(t) \tag{23}$$

which are to be substituted into (19) and (20). To derive the equations which are satisfied by x_{00}, x_{01}, . . . , y_{10} it is first necessary to expand ω_z in terms of e and ϵ; thus

$$\omega_z = \omega_{z_0} + e\,\omega_{z_1} + \epsilon\,\tilde{\omega}_z \tag{24}$$

where, in terms of the orbital elements of the rotating origin 0′ it is found that

$$\omega_{z_0} = n + \dot{\Omega}\cos\iota \equiv \bar{n} \tag{25}$$

$$\omega_{z_1} = 2n\cos v. \tag{26}$$

$\tilde{\omega}_z$ is unknown at this point of the development; it's representation, to be obtained later, will serve the purpose of eliminating certain mixed terms which appear in the solution of x_{10} (t). Furthermore the representation

$$(1 + e\cos v)^{-1} = \frac{b_{10}}{2} + \sum_{k=1}^{\infty} b_{1k}\cos k v \tag{27}$$

where

$$b_{1k} = \frac{2}{\sqrt{1-e^2}}\left(\frac{\sqrt{1-e^2}-1}{e}\right)^k \tag{28}$$

will be used in the expansion of the equations of motion. Finally the independent variable, time, must be introduce by way of the true anomaly. To accomplish this use is made of the Hansen coefficients which are coefficients in the Fourier series representation of $(r/a)^\alpha$ exp $(i\gamma v)$ in terms of the mean anomaly, M. Lanzano [7] has carried the expansion to second order of eccentricity; the expansion to first order is sufficient for this development and is given by

$$\begin{aligned}\left(\frac{r}{a}\right)^\alpha \exp(i\gamma v) = {} & \exp i\gamma M - \frac{e}{2}(\alpha - 2\gamma)\exp(i(\gamma+1)M) \\ & - \frac{e}{2}(\alpha + 2\gamma)\exp i(\gamma - 1)M\end{aligned} \tag{29}$$

where

$$\exp i\gamma v = \cos\gamma v + i\sin\gamma v$$

$$\alpha,\ \gamma = 0,\ \pm 1,\ \pm 2,\ \ldots .$$

The substitution of (22) thru (29) into (19) and (20) yields the following system of equations

$$\ddot{x}_{00} - 2\bar{n}\,\dot{y}_{00} = 0 \tag{30}$$

$$\ddot{x}_{01} - 2\bar{n}\,\dot{y}_{01} - 4n\,\dot{y}_{00}\cos nt + 2\bar{n}^2\,y_{00}\sin nt - \bar{n}^2\,x_{00}\cos nt = 0 \tag{31}$$

$$\ddot{x}_{10} - 2\bar{n}\,\dot{y}_{10} - 2\,\tilde{\omega}_z\,\dot{y}_{00} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2\,x_{00}\left(1 - \frac{3}{2}\sin^2\iota\right)$$

$$+ \frac{3}{2}\left(\frac{R_e}{a}\right)^2 \bar{n}^2\,x_{00}\sin^2\iota\,\cos\,(2\,w_0 + 2n)\,t \tag{32}$$

$$\ddot{y}_{00} + 2\bar{n}\,\dot{x}_{00} = 0 \tag{33}$$

$$\ddot{y}_{01} + 2\bar{n}\,\dot{x}_{01} + 4n\,\dot{x}_{00}\cos nt - 2\bar{n}^2\,x_{00}\sin nt - \bar{n}^2\,y_{00}\cos nt = 0 \tag{34}$$

$$\ddot{y}_{10} + 2\bar{n}\,\dot{x}_{10} + 2\,\tilde{\omega}_z\,\dot{x}_{00} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2\,y_{00}\sin^2\iota\,\cos\,(2\,w_0 + 2n)\,t \tag{35}$$

where the substitutions $M = nt$ and $w = w_0\ t$ have been made.

The circular orbit, spherical central body contribution to the total first order solution is governed by equations (30) and (33); the solution is represented as

$$x_{00}(t) = \alpha_1 \sin\,2\bar{n}t + \alpha_2\cos\,2\bar{n}t + \frac{\alpha_0}{2\bar{n}} \tag{36}$$

$$y_{00}(t) = \alpha_1\cos\,2\bar{n}t - \alpha_2\sin\,2\bar{n}t + \alpha_3 \tag{37}$$

where α_j $j = 0, 1, 2, 3$

are constants which will be evaluated after the x_{10} (t) and y_{10} (t) solutions are determined. This solution is almost periodic due to the presence of $\dot{\Omega} \cos i$ in the expression for the frequency.

Substitution of (36) and (37) into (34) yields the equation

$$\ddot{y}_{01} + 2\bar{n}\dot{x}_{01} = \frac{\bar{n}^2 + 8n\bar{n}}{2}\alpha_2 \sin(2\bar{n} + n)t$$

$$- \frac{3\bar{n}^2 - 8n\bar{n}}{2}\alpha_2 \sin(2\bar{n} - n)t - \frac{\bar{n}^2 + 8n\bar{n}}{2}\alpha_1 \cos(2\bar{n} + n)t \quad (38)$$

$$+ \frac{3\bar{n}^2 - 8n\bar{n}}{2}\alpha_1 \cos(2\bar{n} - n)t + \bar{n}^2 \alpha_3 \cos nt + \bar{n}\alpha_0 \sin nt$$

This equation is integrated once and then substituted into (31); carrying out the necessary operations leads to the equation for a forced harmonic oscillator which x_{01} must satisfy, specifically

$$\ddot{x} + 4\bar{n}^2 x_{01} = \Phi^{01}(t) \quad (39)$$

where Φ^{01} (t) is given in Appendix II. The solution is given by

$$x_{01}(t) = \beta^{01}_{1x} \sin nt + \beta^{01}_{2x} \cos nt + \beta^{01}_{3x} \sin(2\bar{n} + n)t$$

$$+ \beta^{01}_{4x} \sin(2\bar{n} - n)t + \beta^{01}_{5x} \cos(2\bar{n} + n)t \quad (40)$$

$$+ \beta^{01}_{6x} \cos(2\bar{n} - n)t$$

where the β^{01}_{jx} $j = 1, \ldots, 6$ are given in Appendix II. The solution of equation (38) now follows easily and is given by

$$y_{01}(t) = \beta^{01}_{1y} \sin nt + \beta^{01}_{2y} \cos nt + \beta^{01}_{3y} \sin(2\bar{n} + n)t$$

$$+ \beta^{01}_{4y} \sin(2\bar{n} - n)t + \beta^{01}_{5y} \cos(2\bar{n} + n)t \quad (41)$$

$$+ \beta^{01}_{6y} \cos(2\bar{n} - n)t$$

where the β^{01}_{jy} j = 1, . . . , 6 are also to be found in Appendix II. The solutions x_{01} (t) and y_{01} (t) represent the first order contribution due to the eccentricity of the orbit of 0'; this contribution is almost periodic.

Attention will now be turned to equations (32) and (35) which govern the dependence of the motion on the small parameter. It is noticed that the equations depend on the circular orbit spherical central body solutions. It is this dependence which leads to mixed terms in the solution; that is, terms of the form

$$t \sin \delta t, \quad t \cos \delta t.$$

Specificantly the term

$$\left(\frac{R_e}{a}\right)^2 \bar{n}^2 x_{00} \left(1 - \frac{3}{2} \sin^2 \iota\right)$$

leads to terms which oscillate at the same frequency as the solution of the homogeneous equation. Elimination of these mixed terms will be accomplished by solving for the small parameter dependence of the angular velocity.

As before (36) and (37) are substituted into (35) which is then integrated once to yield $\dot{y}_{10}$ (t). With an expression for $\dot{y}_{10}$ (t) available it is easy to derive the equation which governs the x_{10} motion namely,

$$\ddot{x}_{10}(t) + 4\bar{n}^2 x_{10}(t) = \Phi^{10}(t) \tag{42}$$

where Φ^{10} (t) is given in Appendix III. It is noticed that Φ^{10} (t) contains two terms each of which oscillate with a frequency of $2\bar{n}$. Setting each of the coefficients to zero to the dependence of ω_z on the small parameter; explicitly

$$\tilde{\omega}_z = \frac{1}{8}\left(\frac{R_e}{a}\right)^2 \bar{n} \left(1 - \frac{3}{2} \sin^2 \iota\right). \tag{43}$$

The solution of (42) is then given by

$$\begin{aligned} x_{10}(t) = {} & \beta^{10}_{1x} \sin 2(w_0 + n)t + \beta^{10}_{2x} \cos 2(w_0 + n)t \\ & + \beta^{10}_{3x} \sin 2(w_0 + n + \bar{n})t + \beta^{10}_{4x} \sin 2(w_0 + n - \bar{n})t \\ & + \beta^{10}_{5x} \cos 2(w_0 + n + \bar{n})t + \beta^{10}_{6x} \cos 2(w_0 + n - \bar{n})t + \beta^{10}_{7x} \end{aligned} \tag{44}$$

where the β^{10}_{jx} j = 1, . . . , 7 are to be found in Appendix III. The substitution of x_{10} (t) into the first integral of (35) yields the solution

$$
\begin{aligned}
y_{10}(t) = {} & \beta^{10}_{1y} \sin 2\bar{n} t + \beta^{10}_{2y} \cos 2\bar{n} t + \beta^{10}_{3y} \sin 2 (w_0 + n) t \\
& + \beta^{10}_{4y} \cos 2 (w_0 + n) t + \beta^{10}_{5y} \sin 2 (w_0 + n + \bar{n}) t \\
& + \beta^{10}_{6y} \sin 2 (w_0 + n - \bar{n}) t + \beta^{10}_{7y} \cos 2 (w_0 + n + \bar{n}) t \\
& + \beta^{10}_{8y} \cos 2 (w_0 + n - \bar{n}) t + \beta^{10}_{9y} t
\end{aligned}
\tag{45}
$$

where the β^{10}_{jy} j = 1, . . . , 9 are also found in Appendix III.

Equations (44) and (45) yield the dependence of the motion upon the small parameter. It is noticed that this motion is dependent upon the secular rate of change of the true anomaly and that, although the mixed terms have been eliminated, a purely secular term remains in the y_{10} contribution. The y_{10} contribution then exhibits an almost periodic behavior superimposed upon a secular drift.

The complete first order theory for the case of planar motion is represented by equations (36), (37), (40), (41), (44) and (45); these representations must be combined in the way shown in (22) and (23). The constants of integration α_j j= 0, 1, 2, 3 may now be easily evaluated; using x (0), $\dot{x}$ (0), y (0), $\dot{y}$ (0) to denote initial values, it is found that

$$\alpha_0 = \frac{D_2\, x(0) - B_1\, \dot{y}(0)}{A_1 D_2 - B_1 C_2}$$

$$\alpha_1 = \frac{B_2\, y(0) - D_1\, \dot{x}(0)}{B_2 C_1 - A_2 D_1}$$

$$\alpha_2 = \frac{A_1\, \dot{y}(0) - C_2\, x(0)}{A_1 D_2 - B_1 C_2}$$

$$\alpha_3 = \frac{C_1\, \dot{x}(0) - A_2\, y(0)}{B_2 C_1 - A_2 D_1}$$

where A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 are functions of the eccentricity and the small parameter and are given explicity in Appendix IV.

CONCLUSION AND COMMENTS

The first order development of the relative motion of two spacecraft exhibits an almost periodic behavior imposed upon a secular drift. The secular contribution arises when the oblateness of the central body is taken into account; in particular, when (32) and (35) are considered. The secular term fails to appear in the lower order contribution x_{00} (t) and y_{00} (t), because the components of the derivative of U are restricted to the first order only; this is justified in light of the fact that for the application to RAE C & D the ratio of x to x_0, is roughly 10^{-4}.

The dependence of the angular velocity on the small parameter disappears when the inclination is 54°.73. At such an inclination the secular contribution also disappears and the motion becomes strictly almost periodic.

It has been pointed out that the subclass of the total motion which depends on the small parameter exhibits a dependence upon the secular rate of change of perigee. This dependence in turn leads to a singularity at an inclination of 67°.8.

The treatment of the out-of-plane motion will be presented in a subsequent report. Furthermore studies are currently being conducted to obtain numerical results by comparing the first order analytic theory with the numerical integration of equations (7) and (9); these will be reported shortly. This comparison is expected to show the effect of the x component of the angular velocity, which, as it is recalled, was neglected to uncouple the planar and non-planar motion.

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APPENDIX I

DERIVATION OF THE COMPONENTS OF $\vec{\omega}$ IN THE MOVING COORDINATE SYSTEM

Let two position of the moving frame of reference 0′ be determined by the Euler angles ι, Ω, u and $\iota + d\iota$, $\Omega + d\Omega$ and u + du. The increment $d\iota$ corresponds to an infinitesimal rotation about the line of nodes 0N (see the accompanying figure)

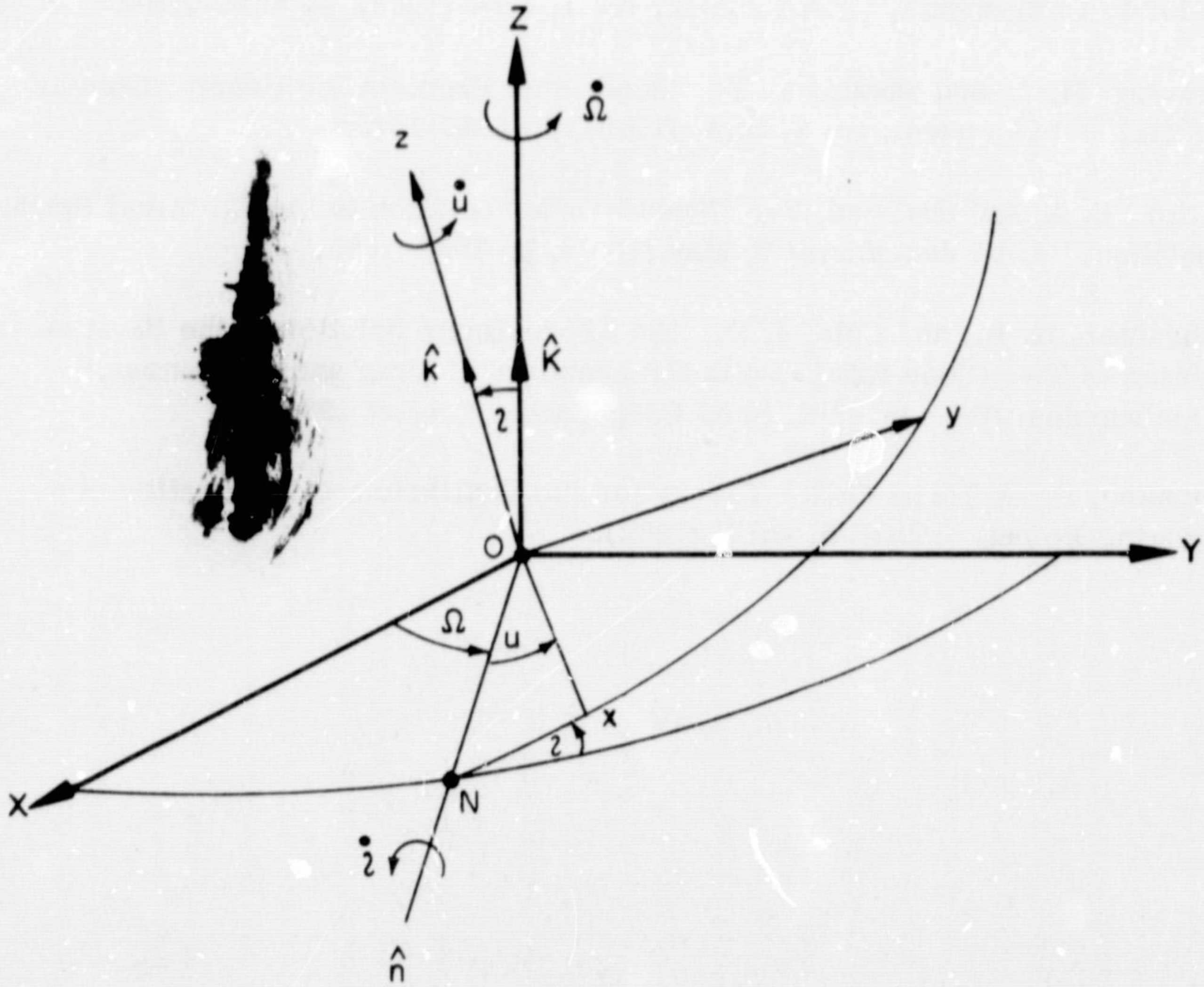

Figure 2.

Similarly $d\Omega$ and du correspond to infintesimal rotations about the 0Z and 0z respectively. The angular velocity $\vec{\omega}$ therefore has the components $\dot{\iota}$, $\dot{\Omega}$, $\dot{u}$ along the axes 0N, 0Z, and 0z respectively. The decomposition of $\hat{n}$, $\hat{K}$ and $\hat{k}$ into the 0xyz system is easily seen to be

$$\hat{n} = \hat{x}\cos u - \hat{y}\sin u$$

$$\hat{K} = \hat{x}\sin u \sin \iota + \hat{y}\cos u \sin \iota + \hat{z}\cos \iota$$

$$\hat{k} = \hat{z}$$

and therefore

$$\vec{\omega} = (\dot{\iota}\cos u + \dot{\Omega}\sin u \sin \iota)\,\hat{x}$$

$$+ (-\dot{\iota}\sin u + \dot{\Omega}\cos u \sin \iota)\,\hat{y}$$

$$+ (\dot{u} + \dot{\Omega}\cos \iota)\,\hat{z}$$

APPENDIX II

The forcing function of equation (39) is

$$\Phi^{01}(t) = \lambda_1^{01} \sin(2\bar{n} + n)t + \lambda_2^{01} \sin(2\bar{n} - n)t$$

$$+ \lambda_3^{01} \cos(2\bar{n} + n)t + \lambda_4^{01} \cos(2\bar{n} - n)t + \lambda_5^{01} \cos nt$$

$$+ \lambda_6^{01} \sin nt$$

where

$$\lambda_1^{01} = -\frac{\bar{n}(4\bar{n} + n)(\bar{n} + 8n)}{2(2\bar{n} + n)} \alpha_1$$

$$\lambda_2^{01} = \frac{\bar{n}(4\bar{n} - n)(3\bar{n} - 8n)}{2(2\bar{n} - n)} \alpha_1$$

$$\lambda_3^{01} = -\frac{\bar{n}(4\bar{n} + n)(\bar{n} + 8n)}{2(2\bar{n} + n)} \alpha_2$$

$$\lambda_4^{01} = \frac{\bar{n}(4\bar{n} - n)(3\bar{n} - 8n)}{2(2\bar{n} - n)} \alpha_2$$

$$\lambda_5^{01} = -\frac{\bar{n}}{2n}(4\bar{n} - n)\alpha_0$$

$$\lambda_6^{01} = 2\frac{\bar{n}^2}{n}(\bar{n} - n)\alpha_3$$

The coefficients appearing in equation (40) are

$$\beta_{1x}^{01} = \frac{\lambda_6^{01}}{4\bar{n}^2 - n^2} \qquad \beta_{4x}^{01} = \frac{\lambda_2^{01}}{n(4\bar{n} - n)}$$

$$\beta^{01}_{3x} = \frac{\lambda^{01}_5}{4\,\bar{n}^2 - n^2} \qquad \beta^{01}_{5x} = -\frac{\lambda^{01}_3}{n\,(4\,\bar{n} + n)}$$

$$\beta^{01}_{3x} = -\frac{\lambda^{01}_1}{n\,(4\,\bar{n} + n)} \qquad \beta^{01}_{6x} = \frac{\lambda^{01}_4}{n\,(4\,\bar{n} - n)}$$

The coefficients appearing in equation (41) are

$$\beta^{01}_{1y} = -\frac{\bar{n}}{n}\left(\frac{\alpha_0}{n} + \frac{2\,\lambda^{01}_5}{4\,\bar{n}^2 - n^2}\right)$$

$$\beta^{01}_{2y} = -\frac{\bar{n}}{n}\left(\frac{\bar{n}}{n}\,\alpha_3 - \frac{2\,\lambda^{01}_6}{4\,\bar{n}^2 - n^2}\right)$$

$$\beta^{01}_{3y} = \frac{1}{4\,\bar{n} + 2\,n}\left(\frac{4\,\bar{n}\,\lambda^{01}_3}{n\,(4\,\bar{n} + n)} - \frac{\bar{n}^2 + 8\,n\,\bar{n}}{2\,\bar{n} + n}\,\alpha_2\right)$$

$$\beta^{01}_{4y} = \frac{1}{4\,\bar{n} - 2\,n}\left(\frac{3\,\bar{n}^2 - 8\,n\,\bar{n}}{2\,\bar{n} - n}\,\alpha_2 - \frac{4\,\bar{n}\,\lambda^{01}_4}{n\,(4\,\bar{n} - n)}\right)$$

$$\beta^{01}_{5y} = -\frac{1}{4\,\bar{n} + 2\,n}\left(\frac{4\,\bar{n}\,\lambda^{01}_1}{n\,(4\,\bar{n} + n)} - \frac{\bar{n}^2 + 8\,n\,\bar{n}}{2\,\bar{n} + n}\,\alpha_1\right)$$

$$\beta^{01}_{6y} = -\frac{1}{4\,\bar{n} - 2\,n}\left(\frac{3\,\bar{n}^2 - 8\,n\,\bar{n}}{2\,\bar{n} - n}\,\alpha_1 - \frac{4\,\bar{n}\,\lambda^{01}_2}{n\,(4\,\bar{n} - n)}\right)$$

APPENDIX III

The forcing function of equation (42) is

$$\Phi^{10}(t) = \lambda_1^{10} \sin 2\bar{n} t + \lambda_2^{10} \cos 2\bar{n} t + \lambda_3^{10} \sin 2(w_0 + n) t$$

$$+ \lambda_4^{10} \cos 2(w_0 + n) t + \lambda_5^{10} \sin 2(w_0 + n + \bar{n}) t$$

$$+ \lambda_6^{10} \sin 2(w_0 + n - \bar{n}) t + \lambda_7^{10} \cos 2(w_0 + n + \bar{n}) t$$

$$+ \lambda_8^{10} \cos 2(w_0 + n - \bar{n}) t + \lambda_9^{10}$$

where

$$\lambda_1^{10} = \left[\left(\frac{R_e}{a}\right)^2 \bar{n}^2 \left(1 - \frac{3}{2}\sin^2 \iota\right) - 8\bar{n}\tilde{\omega}_z\right]\alpha_1$$

$$\lambda_2^{10} = \left[\left(\frac{R_e}{a}\right)^2 \bar{n}^2 \left(1 - \frac{3}{2}\sin^2 \iota\right) - 8\bar{n}\tilde{\omega}_z\right]\alpha_2$$

$$\lambda_3^{10} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \left(\frac{\bar{n}}{w_0 + n}\right)\alpha_3$$

$$\lambda_4^{10} = \frac{3}{4}\left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \left(\frac{\alpha_0}{\bar{n}}\right)$$

$$\lambda_5^{10} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \left[\frac{3(w_0 + n) + 5\bar{n}}{4(w_0 + n + \bar{n})}\right]\alpha_1$$

$$\lambda_6^{10} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \left[\frac{-3(w_0 + n) + 5\bar{n}}{4(w_0 + n - \bar{n})}\right]\alpha_1$$

$$\lambda_7^{10} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \left[\frac{3(w_0 + n) + 5\bar{n}}{4(w_0 + n + \bar{n})}\right]\alpha_2$$

$$\lambda_8^{10} = \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \left[\frac{3\,(w_0 + n) - 5\,\bar{n}}{4\,(w_0 + n - \bar{n})}\right] \alpha_2$$

$$\lambda_9^{10} = \frac{1}{4}\left(\frac{R_e}{a}\right)^2 \bar{n} \left(1 - \frac{3}{2}\sin^2 \iota\right) \alpha_0$$

The coefficients appearing in equation (44) are

$$\beta_{1x}^{10} = \frac{\lambda_3^{10}}{4\,[\bar{n}^2 - (w_0 + n)^2]}$$

$$\beta_{2x}^{10} = \frac{\lambda_4^{10}}{4\,[\bar{n}^2 - (w_0 + n)^2]}$$

$$\beta_{3x}^{10} = -\frac{\lambda_5^{10}}{4\,(w_0 + n)\,[2\,\bar{n} + (w_0 + n)]}$$

$$\beta_{4x}^{10} = \frac{\lambda_6^{10}}{4\,(w_0 + n)\,[2\,\bar{n} - (w_0 + n)]}$$

$$\beta_{5x}^{10} = -\frac{\lambda_7^{10}}{4\,(w_0 + n)\,[2\,\bar{n} + (w_0 + n)]}$$

$$\beta_{6x}^{10} = \frac{\lambda_8^{10}}{4\,(w_0 + n)\,[2\,\bar{n} - (w_0 + n)]}$$

$$\beta_{7x}^{10} = \frac{1}{16\,\bar{n}}\left(\frac{R_e}{a}\right)^2 \left(1 - \frac{3}{2}\sin^2 \iota\right) \alpha_0.$$

The coefficients appearing in equation (45) are

$$\beta_{1y}^{10} = -\frac{\tilde{\omega}_z}{\bar{n}}\,\alpha_2$$

$$\beta^{10}_{2y} = \frac{\tilde{\omega}_z}{\bar{n}} \alpha_1$$

$$\beta^{10}_{3y} = - \frac{\bar{n} (w_0 + n)^{-1}}{4 [\bar{n}^2 - (w_0 + n)^2]} \lambda^{10}_4$$

$$\beta^{10}_{4y} = \frac{2 (w_0 + n)^2 - \bar{n}^2}{4 \bar{n} [\bar{n}^2 - (w_0 + n)^2]} (w_0 + n)^{-1} \lambda^{10}_3$$

$$\beta^{10}_{5y} = \frac{\bar{n} (w_0 + n)^{-1}}{4 (w_0 + n + \bar{n}) (2 \bar{n} + w_0 + n)} \lambda^{10}_7 + \frac{1}{2} \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \frac{\alpha_2}{4 (w_0 + n + \bar{n})^2}$$

$$\beta^{10}_{6y} = - \frac{\bar{n} (w_0 + n)^{-1}}{4 (w_0 + n - \bar{n}) (2 \bar{n} - w_0 - n)} \lambda^{10}_8 - \frac{1}{2} \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \frac{\alpha_2}{4 (w_0 + n - \bar{n})^2}$$

$$\beta^{10}_{7y} = - \frac{\bar{n} (w_0 + n)^{-1}}{4 (w_0 + n + \bar{n}) (2 \bar{n} + w_0 + n)} \lambda^{10}_5 - \frac{1}{2} \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \frac{\alpha_1}{4 (w_0 + n + \bar{n})^2}$$

$$\beta^{10}_{8y} = \frac{\bar{n} (w_0 + n)^{-1}}{4 (w_0 + n - \bar{n}) (2 \bar{n} - w_0 - n)} \lambda^{10}_6 - \frac{1}{2} \left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota \frac{\alpha_1}{4 (w_0 + n - \bar{n})^2}$$

$$\beta^{10}_{9y} = - 2 \frac{\tilde{\omega}_z}{\bar{n}} \alpha_0 .$$

APPENDIX IV

Coefficients appearing in the constants of integration are

$$A_1 = \frac{1}{2\bar{n}} - e\,\frac{\bar{n}}{2n}\,\frac{4\bar{n} - n}{4\bar{n}^2 - n^2} + \epsilon\left[\frac{3}{16}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}}{\bar{n}^2 - (w_0 + n)^2}\ \sin^2 \iota + \frac{1}{16}\left(\frac{R_e}{a}\right)^2 (\bar{n})^{-1}\left(1 - \frac{3}{2}\sin^2 \iota\right)\right]$$

$$B_1 = 1 + e\,\frac{\bar{n}}{n}\,\frac{4\bar{n}^2 + n\bar{n} - 8n^2}{4\bar{n}^2 - n^2} + \epsilon\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^2}{16(w_0 + n)}\left[\frac{3(w_0 + n) - 5\bar{n}}{(w_0 + n - \bar{n})\,[2\bar{n} - (w_0 + n)]} - \frac{3(w_0 + n) + 5\bar{n}}{(w_0 + n + \bar{n})\,(2\bar{n} + w_0 + n)}\right]\sin^2 \iota$$

$$C_1 = 1 + e\,\frac{\bar{n}}{2n}\,\frac{8\bar{n}^2 - n\bar{n} - 16n^2}{(2\bar{n} + n)(2\bar{n} - n)} + \epsilon\left\{\frac{\tilde{\omega}_z}{\bar{n}} - \frac{1}{8}\left(\frac{R_e}{a}\right)^2 \bar{n}^2 \sin^2 \iota\left[\frac{1}{(w_0 + n + \bar{n})^2} + \frac{1}{(w_0 + n - \bar{n})^2}\right] - \frac{1}{16}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^2 \sin^2 \iota}{w_0 + n}\left[+\frac{3(w_0 + n)\bar{n} + 5\bar{n}^2}{(w_0 + n + \bar{n})^2\,(2\bar{n} + w_0 + n)} + \frac{3(w_0 + n)\bar{n} - 5\bar{n}^2}{(w_0 + n - \bar{n})^2\,[2\bar{n} - (w_0 + n)]}\right]\right\}$$

$$D_1 = 1 - e\,\frac{\bar{n}^2}{n}\,\frac{4\bar{n} - n}{4\bar{n}^2 - n^2} - \epsilon\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^2 \sin^2 \iota}{(w_0 + n)^2}\,\frac{\bar{n}^2 - 2(w_0 + n)^2}{4\,[\bar{n}^2 - (w_0 + n)^2]}$$

$$A_2 = 2\bar{n} + e\,\frac{2\bar{n}^2}{n} - \epsilon\,\frac{1}{8}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^2 \sin^2 \iota}{w_0 + n}\left[\frac{3(w_0 + n) + 5\bar{n}}{2\bar{n} + w_0 + n} + \frac{3(w_0 + n) - 5\bar{n}}{2\bar{n} - (w_0 + n)}\right]$$

$$B_2 = e\,\frac{2\,\bar{n}^2\,(\bar{n} - n)}{4\,\bar{n}^2 - n^2} + \epsilon\,\frac{1}{2}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^3 \sin^2 \iota}{\bar{n}^2 - (w_0 + n)^2}$$

$$C_2 = -\,e\,\frac{\bar{n}\,(\bar{n} - n)}{4\,\bar{n}^2 - n^2} - \epsilon\left[\frac{3}{8}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^2 \sin^2 \iota}{\bar{n}^2 - (w_0 + n)^2} + \frac{\tilde{\omega}_z}{\bar{n}}\right]$$

$$D_2 = -\,2\,\bar{n} - e\,\frac{2\,\bar{n}^2}{n} + \epsilon\left\{-\,2\,\tilde{\omega}_z + \frac{1}{2}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^3 \sin^2 \iota}{\bar{n}^2 - (w_0 + n)^2}\right.$$

$$+ \frac{1}{8}\left(\frac{R_e}{a}\right)^2 \frac{\bar{n}^3 \sin^2 \iota}{w_0 + n}\left[\frac{3\,(w_0 + n) + 5\,n}{(w_0 + n + \bar{n})\,(2\,\bar{n} + w_0 + n)}\right.$$

$$\left.\left. - \frac{3\,(w_0 + n) - 5\,\bar{n}}{(w_0 + n - \bar{n})\,(2\,\bar{n} - (w_0 + n))}\right]\right\}$$