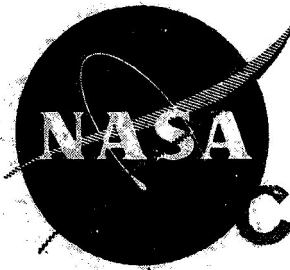


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**LOW GRAVITY DRAINING FROM
HEMISPHERICALLY BOTTOMED CYLINDRICAL TANKS**

by

G. D. Bizzell, P. Concus, G. E. Crane, and H. M. Satterlee

LOCKHEED MISSILES & SPACE COMPANY

prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

NASA Lewis Research Center

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S. G. Berenyi, Project Manager

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FINAL REPORT

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Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

June 1, 1970

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ABSTRACT

A finite-difference technique utilizing an irregular triangular mesh is used to study tank draining in low gravity. Assuming that the liquid is inviscid and incompressible, that its motion is irrotational and axisymmetric, and that the solid-liquid contact angle maintains a constant value (5°), the initial boundary-value problem is solved numerically for several values of Bond number, Weber number, initial fill level, and drain size. Velocity field data, the evolution of free surface shapes with time, liquid residual volume, and time to vapor ingestion are presented. Comparisons with experimental data are favorable.

SUMMARY

The draining of inviscid liquids from hemispherically bottomed cylindrical tanks in zero and low gravity environments is studied by analytical and numerical methods. The objectives of the study are to determine streamlines, equipotentials, and free surfaces as functions of time, tank geometry, liquid properties, tank fill level, and gravity level. A computer code determines these quantities to the time of vapor ingestion into the drain. Liquid residuals left in the tank upon vapor ingestion are calculated for a broad range of the Weber number W , the ratio of dynamic forces due to draining to surface tension forces, and the Bond number B , the ratio of gravity forces to surface tension forces.

The evolution of the free surface is followed in a Lagrangian manner by integrating a system of first-order differential equations. The condition on the potential at the drain is satisfied by introducing the auxiliary potential for a liquid draining from a flat bottomed infinite circular cylinder through an axial drain of the same size and at the same rate and by representing the potential as the sum of the auxiliary potential and a residual potential. This device eliminates the mesh refinement otherwise needed near the drain. The mixed boundary value problem satisfied by the residual potential is approximated on an irregular triangular mesh, part of which is revised, from time to time, to accommodate the motion of the free surface. The solution of the linear system for the potential is arranged so that the computation is, in general, restricted to a neighborhood of the free surface. Representation of the free surface between mesh points by cubic spline interpolation permits accurate calculation of the surface tension term in the Bernoulli equation. Successful simulations are obtained with 14 to 22 mesh points on the free surface. The development of the numerical method used in this study is among the principal results.

Results of exercising the simulation program over a broad range of the parameters indicate several conclusions: (1) Liquid residuals increase with Weber number and decrease with Bond number; (2) drain times must be long enough to permit

surface tension and body forces to reorient liquid from the region near the tank wall to the region at the tank centerline; (3) draining and liquid reorientation often result in axisymmetric slosh waves; (4) drain size does not influence residuals for high Weber numbers, but no conclusion has been reached for very low Weber numbers; and (5) vapor ingestion times depend on initial fill level and a parameter, $W/(1 + B)$, for a wide range of B and W .

INTRODUCTION

When a tank is drained under high gravitational conditions, the nearly flat liquid-gas interface can fall to very low levels before the dynamic effects of the drain cause vapor ingestion. Assuming that vortexing is absent, experiments [1,2,3] show that the interface falls steadily until a critical depth is reached, whereupon the free surface accelerates rapidly into the drain. Simple, but highly successful, analytical models [2,3] have been advanced and verified by experiments to define the critical vapor ingestion height as a function of the important problem parameters. Because, in high gravitational draining, the free surface is essentially flat and relatively undisturbed by the draining forces, knowledge of the vapor ingestion height allows estimation of the liquid residuals remaining in the tank at the time of vapor ingestion.

The Present Study

The draining of tanks subject to reduced gravity is substantially more complex, both experimentally and analytically. To conduct meaningful experiments is very difficult -- available data [1,4-7] have been obtained, however, for brief periods of weightlessness of up to five seconds. In this period, not only must the tank be drained; but before initiation of draining, the interface must first reorient from a flat configuration to the highly curved shape characteristic of reduced gravity, in the process introducing slowly damped transient oscillations of the free surface. Consequently, the ability to survey experimentally the full range of problem parameters is limited.

This report describes the results of an analytical study of the draining of tanks under reduced gravity. The mathematical description of the physical problem is formulated under the assumptions that the liquid is inviscid and incompressible and that its motion is axisymmetric and irrotational in a tank that is a right circular cylinder with a hemispherical bottom. The static and dynamic effects of surface tension with the solid-liquid contact angle taken as constant (here 5°) are included; nonlinear convective terms are retained as well. With

this generality of formulation, the results may be realistically compared with the available experimental data [1,4]. If a favorable comparison with experimental data is achieved, the mathematical model can be used with confidence to simulate draining under low gravity, thereby producing an alternative to difficult experiments.

Numerical methods are required to solve the analytical problem. The numerical techniques used herein are extensions of those used in earlier studies of interface shapes and lateral sloshing of liquids under low gravity conditions [8,9]. The results, here, include graphical presentation of free surface shapes as functions of time during the draining process; this type of numerical "experiment" offers the best comparison with the results of physical experiments, where motion pictures are used to follow the progress of the free surface from initiation of draining to vapor ingestion.

Related Experiments

The first study of tank draining under low gravity in a drop tower facility is [4]; additional experiments are reported in [5,7] for various tank geometries; and comprehensive experiments for hemispherically bottomed cylinders are reported in [1,6].

The experiments show the presence of a liquid layer which hangs on the wall as the bulk drains; the thickness of the layer increases with drain velocity. A distortion parameter, defined as $(V - V_m)/V$, where V is the interface velocity at the tank centerline and V_m is the mean velocity in the tank, is correlated with a suitable Weber number. This distortion parameter increases with fill level but is not influenced by the particular geometry of the tank bottom.

Vapor ingestion heights are correlated with a suitable Weber number in [1,7]. Knowledge of the vapor ingestion height is not as valuable in low gravity as in high gravity draining because the entire free surface shape is needed to calculate the liquid residuals. For this reason, residual data are also presented and correlated with Weber number in [1,7].

The influence of initial fill level on residuals is significant at high values

of Weber numbers, where the free surface distortion is severe and the liquid layer trapped on the tank wall does not have sufficient time to reorient to the liquid bulk; however, the effect of fill level is less at low Weber numbers, where more reorientation time is available and free surface distortion is less [6].

Related Analysis

The problem has also been approached using linearization techniques [10,11] but the results do not satisfactorily explain experimental data. As is well-known, low gravity free surface shapes are strongly dependent upon the non-linear terms in the Bernoulli equation, namely the curvature (surface tension) and the velocity-squared terms. Consequently, linearization and representation of the solution by the superposition of orthogonal functions limit the applicability of the results. The assumption of a near 90° contact angle [10] makes linearization more reasonable, but such solid-liquid combinations are rare. The comparison of these results with experimental data confirms the need for a numerical approach in solving the general problem.

During the course of this study two papers have appeared that describe the simulation of low gravity draining from flat bottomed tanks. One [12] uses the "marker-and-cell" technique (MAC) with a flat initial free surface very close to the vapor ingestion height. The problem formulation in the other [13] differs from the one presented in the next section in the treatment of the constant contact angle condition. The numerical treatment is based upon a rectangular mesh and uses successive overrelaxation to determine the potential within the liquid. Such methods were rejected for the present study in the light of earlier LMSC experience with them for the related reorientation problems.

PROBLEM FORMULATION

In this section the mathematical formulation of the draining problem is presented. All the variables that appear are dimensionless, unless otherwise designated, and they are defined in the Symbol List, Appendix B, where the relationships to the physical dimensioned variables are given.

Initial Configuration

Consider an axisymmetric vertical container consisting of a right circular cylinder with a hemispherical bottom that has an orifice of circular cross-section centered on the axis (see Figure 1(a)). Let the container be so oriented that the gravitational field is directed parallel to the axis. Prior to the initiation of draining at time $t = 0$, it is assumed that there is sufficient liquid in the container to cover the orifice and that the liquid is at rest.

The equilibrium free surface of the liquid is found by solving the two-point boundary value problem

$$\frac{1}{r} \frac{d}{dr} \left[\frac{r \, dF/dr}{(1 + (dF/dr)^2)^{1/2}} \right] - B(F - h_c) = 2H_0, \quad 0 < r < r_w \quad (1)$$

with

$$\frac{dF}{dr} = 0 \quad \text{at} \quad r = 0 \quad (2)$$

and

$$\frac{dF/dr}{(1 + (dF/dr)^2)^{1/2}} = r \cos \Theta - (1 - r)^{1/2} \sin \Theta \quad (3)$$

at $r = r_w$

Equation (1) is the requirement that the static liquid pressure at the free surface be in equilibrium with the constant atmospheric pressure there; (2) is the condition arising from the symmetry of the container; and (3) is the requirement that the free surface meet the container with contact angle Θ . (Computations are carried out in this study for the case where Θ has the value 5°). Circular cylindrical coordinates are used with origin at the

center of the orifice, and the Bond number B is chosen to be positive when gravity acts vertically downward (see Figure 1(a)).

The initial volume of the liquid is

$$V_o = 2\pi \int_0^{r_w} r F(r) dr - V_s \quad (4)$$

where, for an orifice of radius r_o ,

$$V_s = \frac{\pi}{3} \left[(1 - r_o^2)^{3/2} - 3(1 - r_w^2)(1 - r_o^2)^{1/2} + 2(1 - r_w^2)^{3/2} \right]$$

is the volume between the hemispherical bottom of the tank and a right circular cylinder of radius r_w with bottom touching the orifice (see Figure 1(b)). By specifying, along with (1), (2), and (3), the initial volume V_o , the height $F(r)$ of the initial equilibrium free surface is determined (and hence the values of h_c , the height of the surface at its center, H_o , the mean curvature there, and r_w , the value of r where the free surface intersects the wall)[14].

Draining

Dynamic Equations. At time $t = 0$, it is assumed that draining is initiated by pumping the fluid out in such a manner that the outflow velocity is directed vertically downward and is uniform over the orifice. This outflow velocity is attained instantaneously and remains constant until the time that the free surface first begins to enter the orifice. It is further assumed that the liquid is incompressible and inviscid and that vortexing and turbulence are absent so that the motion can be taken to be irrotational. In actual practice the effect of a small nonzero viscosity is evidenced by a layer of liquid that remains on the container wall during draining, but such a layer is generally thin enough for problems of interest here that it may be neglected (see Appendix A).

Under the above assumptions, one is led to the following initial boundary-value problem for axisymmetric motion from $t = 0$ until free surface

breakthrough.

Velocity Potential. The sign of the velocity potential φ is chosen so that the velocity is given by $\underline{v} = \underline{\nabla}\varphi$. The velocity potential satisfies Laplace's equation,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (5)$$

in the liquid.

Boundary Conditions. The conditions on φ over the fixed portion of the boundary are that on the symmetry axis and container wall the normal velocity be zero

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } r = 0 \text{ and on the wall} \quad (6)$$

and at the orifice the outward-directed normal velocity be $1/r_o^2$

$$\frac{\partial \varphi}{\partial n} = \frac{1}{r_o^2} \quad \text{for } z = 0, 0 < r < r_o \quad (7)$$

On the instantaneous free surface, one has the kinematic condition, which can be expressed either in the form

$$\frac{\partial f}{\partial t} = \frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial r} \frac{\partial f}{\partial r} \quad (8a)$$

or

$$\frac{dR}{dt} = \frac{\partial \varphi}{\partial r}, \quad \frac{dZ}{dt} = \frac{\partial \varphi}{\partial z} \quad (8b)$$

and the Bernoulli equation, which can be expressed as either

$$W \frac{\partial \varphi}{\partial t} = 2H - Bf - \frac{W}{2} |\underline{\nabla}\varphi|^2 + C(t) \quad (9a)$$

or

$$W \frac{d\varphi}{dt} = 2H - BZ + \frac{W}{2} |\nabla\varphi|^2 + C(t) \quad (9b)$$

where

$$2H \equiv \frac{1}{r} \frac{\partial}{\partial r} \left[\frac{r \partial f / \partial r}{(1 + (\partial f / \partial r)^2)^{1/2}} \right]$$

is twice the mean curvature of the free surface, $z = f(r, t)$. The first forms (8a) and (9a) are written in Eulerian (partial time derivative) form for points $z = f(r, t)$ on a single valued free surface, whereas the second forms (8b) and (9b) are written in Lagrangian (total time derivative) form for points $r = R(t)$, $z = Z(t)$ on a free surface which may be multiply-valued (with $2H$ appropriately expressed). The first forms will provide the basis for following the motion of the free surface along fixed values of r , whereas the second are natural for following the actual motion of points on the surface. The numerical computations are based primarily on the Lagrangian forms of the free surface equations, with supplemental utilization of the Eulerian forms.

The choice of the particular function of integration $C(t)$ that, from computational considerations, is the most convenient to use in (9a) and (9b) is the one corresponding to φ being independent of time when the velocity at the free surface is uniform and directed vertically downward. For this case of stationary flow, we have that $\varphi = \text{const.} - z$ near the free surface and $f(r, t) = F(r) - t$, so that setting $\partial\varphi/\partial t = 0$ in (9a) yields, with the use of (1), for $C(t)$ the value,

$$C(t) = -Bt + W/2 - (2H_0 - Bh_c) \quad (10)$$

Finally, the end condition for (9a) and (9b), that the contact angle remain fixed during the motion at its equilibrium value Θ , is

$$\frac{\partial f / \partial r}{(1 + (\partial f / \partial r)^2)^{1/2}} = r \cos \Theta - (1 - r^2)^{1/2} \sin \Theta \quad \text{at } r = r_w \quad (11)$$

Initial Conditions. Equations (5) - (11) are to be solved subject to the initial conditions at $t = 0$ that the free surface is the equilibrium shape $z = F(r)$ and that the velocity potential is the solution of (5), (6), (7), and the condition that the potential vanish on the free surface,

$$\varphi = 0 \quad \text{on } z = F(r) \quad (12)$$

That the free surface is initially an equipotential can be observed by examining the nature of the discontinuity in the Bernoulli equation (9) at $t = 0$ [10]. Take $C(t)$ equal to $-2H_0 + Bh_c$ for $t < 0$ and the value given by (10) for $t \geq 0$; then (9) holds for non-positive as well as for positive t with $\varphi \equiv \text{const.}$ for $t < 0$. The right-hand side is piecewise continuous with a step discontinuity at $t = 0$ corresponding to the instantaneous change of velocity. Thus the time derivative of φ at the free surface has only a step discontinuity at $t = 0$, and hence φ itself must be continuous there at $t = 0$ and equal to the same constant value it had for $t < 0$. The value of this arbitrary constant is taken, for convenience, to be zero. The continuity of φ on the free surface at $t = 0$ corresponds to the condition that the external atmospheric pressure remain fixed there as draining is initiated [15].

Auxiliary Potential

For purposes of numerical computation, it is convenient to introduce the residual velocity potential φ_2 ,

$$\varphi_2 = \varphi - \varphi_1 \quad (13)$$

where φ_1 is the auxiliary potential corresponding to the draining problem in a full infinitely-tall cylinder having a flat, rather than hemispherical, bottom. This, in essence, transfers the nonhomogeneous boundary condition on φ at the orifice to one on φ_2 at the free surface, thus obviating the need for an accurate numerical solution near the drain. The potential φ_1 can be expressed as the infinite series

$$\varphi_1 = c - z + \frac{2}{r_0} \sum_{m=1}^{\infty} \frac{J_1(j_{1,m} r_0)}{[j_{1,m} J_0(j_{1,m})]^2} J_0(j_{1,m} r) \exp(-j_{1,m} z) \quad (14)$$

where J_0 and J_1 are Bessel functions of the first kind and $j_{1,m}$ is the m -th zero of the latter (taking $j_{1,0} = 0$). For computational convenience the additive constant c is normally chosen so that $\varphi_1 = 0$ at the intersection of the equilibrium free surface and the container wall. Substitution of $\varphi_1 + \varphi_2$ for φ into the draining equations yields the desired problem with φ_2 , which does not vary rapidly near the orifice, as the unknown. See Figure 1(b)).

NUMERICAL ANALYSIS

Basic Ideas

The evolution of the free surface in time is simulated by numerically integrating the characteristics (8b) of the kinematic equation (8a) starting from an appropriate set of initial points on the equilibrium free surface (1), (2), (3). The evolution of the potential on the free surface is followed by simultaneously integrating the Bernoulli equation (9b), regarded as a differential equation for φ as a function of t , along the characteristics. The derivatives of the potential needed in evaluating the right sides of these equations are determined from the solution of the mixed boundary value problem (5), (6), and (7) with the solution to (9b) providing the boundary values on the free surface.

Introducing an irregular triangular mesh into the axial cross section of the liquid facilitates the construction of a finite difference approximation to the mixed boundary value problem because all boundaries can be approximated by polygonal mesh lines. At each time step, the movement of the free surface adjusts the top line of the mesh and, as the calculation proceeds, the mesh lines in the neighborhood of the free surface are progressively readjusted to allow for further movement. Mesh lines are dropped to accommodate the decrease in the volume of the liquid. The goal in readjusting is to make the mesh near the free surface conform to the shape of the free surface to minimize mesh changes and to facilitate accurate evaluation of finite difference approximations to the right sides of (8b) and (9b). Restricting the changes to a neighborhood of the free surface retains the advantage of a fixed mesh throughout a large part of the liquid-filled region.

Because the mesh is readjusted only near the free surface and because the finite difference approximations to the right sides of (8b) and (9b) involve only values near the free surface, the mesh rows are ordered from drain to free surface. Let ϕ be the finite difference approximation to φ and at

time t let

$$\underline{A}(t) \underline{\Phi}(t) = \underline{b}(t) \quad (15)$$

be the linear system which is the finite difference approximation to the mixed boundary value problem satisfied by φ with the matrix $\underline{A}(t)$ and the vector $\underline{b}(t)$ partitioned into blocks and sub-vectors corresponding to mesh rows ordered from drain to free surface.

The readjustment of the mesh near the free surface results in changes to $\underline{A}(t)$ in the lower right blocks and in changes to $\underline{b}(t)$ in the last components corresponding to the mesh line just below the free surface. Hence if the matrices $\underline{A}(t)$ and $\underline{A}(t + \Delta t)$ are factorized into left and right triangular factors by the same method,

$$\underline{A}(t) = \underline{L}(t) \underline{R}(t)$$

for example, then $\underline{L}(t + \Delta t)$ differs from $\underline{L}(t)$ only in the last columns (and $\underline{R}(t + \Delta t)$ differs from $\underline{R}(t)$ only in the last rows). Thus to construct the solution of $\underline{L}(t + \Delta t) \underline{x}(t + \Delta t) = \underline{b}(t + \Delta t)$ from the solution at the previous time step, only the final components of $\underline{x}(t)$ need be recomputed. Moreover, the solution of $\underline{R}(t + \Delta t) \underline{\Phi}(t + \Delta t) = \underline{x}(t + \Delta t)$ need only be continued until the rows used in approximating the derivatives of φ at the free surface have been computed. Consequently, with the upper parts of $\underline{L}(t)$ and $\underline{R}(t)$ saved to permit continuation, determination of the potential near the free surface by direct solution of (15), working only with the lower blocks, becomes a feasible numerical method.

Interpolating the free surface between mesh points by approximating the functions R and Z with cubic splines in terms of arc length, s , along the free surface gives a representation in which the approximation and its first two derivatives are continuous [16, p.51]. Thus the surface tension term in the Bernoulli equation (9) takes the form

$$2H = Z_{ss} R_s - Z_s R_{ss} + Z_s/R \quad (16)$$

and can be approximated smoothly by values of a continuous function. The contact angle condition (11) and the symmetry condition that the

free surface be horizontal at the axis are explicitly introduced into the computation as end conditions for the interpolation. The values used in rotating ϕ_n and ϕ_s , the normal and tangential derivatives on the free surface, into ϕ_r and ϕ_z , namely

$$R_s = \cos \chi \quad \text{and} \quad Z_s = \sin \chi \quad (17)$$

where χ is the angle between the horizontal and the tangent to the free surface, are smoothed as a by-product of the interpolation. The smoothness of the cubic spline approximation permits representation of the free surface by few mesh points, 14 to 22 points have proved satisfactory, which makes direct solution of (15) for the potential near the free surface even more attractive.

A smooth approximation to ϕ_s is calculated by interpolating $\bar{\phi}$ along the free surface by a cubic spline. The symmetry condition $\bar{\phi}_s = 0$ on the axis provides one end condition; the other is determined by extrapolating interior values of $\bar{\phi}_s$ to the wall. A matrix arising naturally from the approximation to $\underline{A}(t)$, (15), is used to calculate $\bar{\phi}_n$ on the free surface from the values of $\bar{\phi}$ on the top two mesh lines.

Relation to Other Work

Calculating the evolution of the free surface by following individual Lagrangian particles thereon has been used in studying reorientation problems [17,18]. In addition to the difficulty encountered in computing series of special functions or harmonic polynomials to represent the potential within the liquid, these studies report problems in the calculation of the surface tension term, $2H$. One proposal to overcome this difficulty is to smooth the surface by solving the Bernoulli equation (9), with appropriate boundary conditions, as a two-point boundary value problem for the shape of the free surface at the end of every time step [19]. The present use of cubic spline interpolation in both the predictor and corrector steps of the numerical integration provides an adequate, smooth representation for the surface tension term. The variable time-step integration package used here is

based upon one developed at LMSC for an experimental finite difference reorientation program with a rectangular mesh.

The generation and readjustment of the triangular mesh and the revision of the lower blocks of the matrix $\underline{A}(t)$, as well as the plotting of the mesh, potentials, and streamlines, are adapted from programs developed at Lawrence Radiation Laboratory, Livermore, California, which are described in [20]. The adaptation was begun in studying small amplitude sloshing [8,9]. The latter studies introduce the idea of ordering the mesh from drain to free surface to permit restricting the direct solution of the linear system representing the potential at each time step (iteration) to mesh lines near the free surface.

The LINC method for fluid dynamics problems, which uses a completely Lagrangian mesh, has been applied to sloshing and reorientation problems [21]. Mesh distortion causes difficulty in the later stages of the flow. The present method, with only a single Lagrangian line, namely the free surface, encounters similar difficulty when a mesh point moves too close to the wall or the axis. This can be avoided by forcing some points near the boundaries to move along fixed r-lines. Although neither the LINC method nor the present method are fully developed, it would seem that the latter, with a better representation for the surface tension which permits using fewer points on the free surface, is inherently faster.

A technique based upon a cubic spline fit of free surface markers has been developed to introduce surface tension effects into the "marker-and-cell" (MAC) method [22]. For sloshing problems, the MAC method produces a rougher free surface than does the LINC method [21]. Moreover, it would seem that the MAC programming for moving points through a rectangular mesh is considerably more complicated and time consuming than the revision of the fixed mesh against which the free surface moves in the present method.

Thus the method developed here appears to be potentially superior for reorientation, sloshing, and draining problems which are axially symmetric, inviscid and irrotational.

Details of the Computation

In this section some aspects of the computation are described to indicate the limitations and possible extensions of the method. The program is designed to simulate draining over a wide range of parameters. In general the limitations result from omitting desirable features to expedite experiments with a working program. Early versions were developed to simulate experimental results reported in [4] and [5], particularly those for low Bond number and high Weber number. Modifications to achieve the potential of the design have been progressively introduced as the range of parameters has been widened.

Items are in general discussed in the order of their appearance in the program. However, to facilitate explanation of the method and to demonstrate its feasibility, some details will be discussed early, out of narrative order.

Initial Surface. The initial free surface is taken as the equilibrium free surface satisfying (1), (2), and (3) with B , Θ , and either h_c or V_0 given. Currently, the initial free surface is assumed to intersect the cylindrical portion of the tank wall [8,p. 44]; however this restriction may be removed by reviving existing coding [9,p. 26]. Mesh points on the free surface may be distributed in a geometric series, with the mesh spacing adjacent to the contact point prescribed, or in two such series, with the mesh spacing adjacent to the axis and the arc length covered by one series prescribed, in addition.

Were axisymmetric eigenvalues and eigenvectors available, they could be used to construct initial shapes and velocity distributions more closely resembling those occurring in drop tower experiments and these could be incorporated into the program.

Initial Mesh Generation. In a regular triangular mesh, lay off along a horizontal line the number of mesh points to be used on the free surface. With this line segment as the top, construct from mesh lines of the regular triangular mesh a parallelogram sloping down to the left. This parallelogram is the logical diagram for the mesh. Mesh lines in the interior and on the

boundary of the logical diagram are mapped into curved lines in the interior and on the boundary of the axial cross section of the liquid filled region by a numerical algorithm [20,p. 168]. Mesh points are intersections of curved lines. Figure 2 shows an example of an irregular triangular mesh used in this study. The free surface and the tank wall, including the orifice, from the axis to the point where only two triangles meet on the wall correspond to the horizontal sides of the logical diagram; the axis and the remainder of the wall correspond to the slanting sides. By mentally making all triangles equilateral, the reader may recover the underlying logical diagram.

The mesh spacing on the initial free surface is controlled as described in the preceding subsection. The procedures used to control the mesh are described in a previous report [8,p. 53]. They include prescribing the location of certain control (mesh) points on the wall and axis, the number of mesh intervals in the group to be constructed between control points, and the length of the first interval of a group. When the data permits, the length of the mesh intervals form geometric progressions; otherwise they are equally spaced. To accommodate the eventual distortion of the draining free surface, the mapping of some horizontal logical mesh lines may be controlled by prescribing their images in the axial cross section. One simple way to prescribe a mesh control line is to distort the free surface $(r, f(r))$ into the curve $(r, g(r))$ by the mapping

$$g(r) = f(r) - (1 - r)[f(0) - g(0)] - r[f(1) - g(1)]$$

where $g(0)$ and $g(1)$ are prescribed and satisfy $g(1) - g(0) \geq f(1) - f(0)$. The obvious break in the center of the mesh in Figure 2 is produced by using a mesh control line. Small circles mark the mesh triangles that are obtuse.

Auxiliary Potential. Associated with each mesh point is a mesh region obtained by rotating a cross section around the axis of symmetry. Each mesh triangle is subdivided into three quadrilaterals by lines from an interior point to the midpoints of the sides. For obtuse triangles, the point is the centroid; for acute triangles, the intersection of the perpendicular bisectors. For

interior mesh points the associated cross section is the union of the six quadrilaterals meeting there; for a boundary point the cross section consists of the one, two, or three quadrilaterals meeting at the point and lying within the liquid filled region. Thus each mesh region is a solid bounded by frustums of cones. Rotation of the region cross sections is a convenient way to establish the proper weighting in the finite difference approximations to account for the circular symmetry in the problem.

The first step in the construction of the finite difference approximation to the mixed boundary value problem satisfied by $\varphi_2 = \varphi - \varphi_1$ is to integrate the Laplacian of φ_2 over a mesh region and to replace this volume integral by the integral of the exterior normal derivative of φ_2 over the boundary of the region. The second step is to replace the integrals of the normal derivative over frustums of cones by finite difference approximations [20,p. 151;23,p. 181]. Because (5) holds, the approximations at a mesh point must sum to zero. Thus each mesh point supplies a linear equation to the system approximating the mixed boundary value problem.

For mesh points on the axis or the vertical wall, where both $\partial\varphi/\partial n$ and $\partial\varphi_1/\partial n$ vanish, and for interior mesh points, the construction of the finite difference approximations is a natural extension and/or simplification of the techniques used in the earlier sloshing studies [8,p. 27,p. 51]. The same is true for quadrilaterals with sides in the orifice where $\partial\varphi_2/\partial n = 0$. For interior mesh points, the resulting difference equations connect the value of Φ_2 at a mesh point with the values of Φ_2 at the six neighboring mesh points. The coefficients at points on the exterior of this seven point star in general have sign opposite to that of the coefficient at the center.

Because the values of R , Z , and Φ are available from the numerical solution of (8b) and (9b), the values of $\Phi_2 = \Phi - \Phi_1$ are known on the free surface and are used in approximating the normal derivative across boundaries between the free surface and the first mesh line below it; thus the mesh points on the line below the free surface contribute a nonzero component to the vector $\underline{b}_2(t)$ in the linear system

$$\underline{A}(t) \Phi_2(t) = \underline{b}_2(t) \quad (18)$$

approximating the mixed boundary value problem satisfied by φ_2 , which is the numerical problem to be solved. (The matrix $\underline{A}(t)$ is appropriate because the problems for φ and φ_2 differ only in boundary conditions.) For numerical purposes it is convenient to normalize (18) so that the diagonal of $\underline{A}(t)$ is positive. Because $\partial\varphi/\partial n = 0$ and $\partial\varphi_1/\partial n \neq 0$ on the curved wall,

$$\frac{\partial\varphi_2}{\partial n} = -\frac{\partial\varphi_1}{\partial n} \neq 0 \quad (19)$$

holds there (see Figure 1(b)). Consequently mesh points on the curved wall contribute nonzero components to the vector $\underline{b}_2(t)$.

Both $\underline{b}(t)$ and $\underline{b}_2(t)$ have nonzero components associated with mesh points that lie on the line below the free surface. All other components of $\underline{b}(t)$ are zero except for large values corresponding to mesh points that lie in the orifice. A fine mesh would be required nearby to accommodate the jump of $\partial\varphi/\partial n$ from $1/r_0^2$ to zero along the wall. All other components of $\underline{b}_2(t)$, including those associated with points in the orifice, are zero except for nonzero values corresponding to points along the curved wall. The latter components are orders of magnitude smaller than the large components of $\underline{b}(t)$ and produce no requirement for a detailed mesh to represent the solution. Consequently, the numerical solution of the linear systems (15) and (18) differ in that the former would require a fine mesh near the drain and a very accurate solution to carry the effects of the drain to the free surface while the latter requires less accuracy in the solution and a revision of the nonzero components of $\underline{b}_2(t)$ whenever the area of mesh revision overlaps the curved wall. Thus the system (18) is preferable.

Let arc length w along the curved wall be measured from the edge of the drain. Let w_1 and w_2 be the arc lengths to the midpoints of two adjacent mesh intervals on the curved wall with $w_1 < w_2$. Because (19) holds on these intervals, the component of $\underline{b}_2(t)$ corresponding to the mesh point between w_1 and w_2 is

$$2\pi \int_{w_1}^{w_2} \frac{\partial\varphi_2}{\partial n} r dw = -2\pi \int_{w_1}^{w_2} \frac{\partial\varphi_1}{\partial n} r dw .$$

Using (14) and inverting the order of integration and summation lead to representing a component of $b_2(t)$ for a point on the curved wall as

$$2\pi \int_{w_1}^{w_2} \frac{\partial \varphi_2}{\partial n} r dw = 2\pi [I(w_1) - I(w_2)] \quad (20)$$

with

$$I(w) = - \int_w^{w_c} \frac{\partial \varphi_1}{\partial n} r dw = \frac{r_o^2 - 1}{2} + \frac{2r_o}{r_o} \sum_{m=1}^{\infty} C_m(r_o) J_1(j_{1,m} r) \exp(-j_{1,m} z) \quad (21)$$

where

$$C_m(r_o) = J_1(j_{1,m} r_o) / [j_{1,m} J_0(j_{1,m})]^2$$

and w_c is the total arc length along the curved wall. When $w = 0$, the integral in (21) represents the total flow into and out of the closed region bounded by the bottom and wall of the circular cylinder and by the hemispherical bottom of the tank (see Figure 1(b)). Consequently $I(0) = I(w_c) = 0$. Moreover, it can be shown that $dI/dw = 0$ at w_c .

The coefficients $C_m(r_o)$ are slowly oscillating functions of m with a slowly decreasing amplitude. They oscillate more slowly with m as r_o decreases. The two series (14) and (21) differ in that $J_0(j_{1,m} r)$ in the former is replaced by $J_1(j_{1,m} r)$ in the latter. Thus the terms in both series are the product of two oscillatory functions and a decreasing exponential. For most values of z , the exponential factor dominates and practical convergence requires few terms; for very small values of z , the practical convergence results from the cancellation between groups of terms with like sign.

A simple way to evaluate $I(w)$ is to prepare in an auxiliary calculation, for each r_o , a table of $I(w)$ versus w and to facilitate interpolation by adding the coefficients for a cubic spline fit to the table [16,p. 10]. In the auxiliary calculations the terms of like sign are summed separately in

double precision, the value of $I(w)$ is computed at the end of every complete oscillation of the terms, and the summation continued until the relative error of $I(w)$ has fallen below 10^{-6} at the end of two successive full oscillations or until the previously prepared table of several thousand values of $j_{1,m}$ and $C_m(r_o)$ is exhausted. The values of $I(w)$ used are always summed over complete oscillations. The end condition at $w = 0$ for the cubic spline interpolation is obtained by extrapolating numerical values of dI/dw near $w = 0$ to a value at $w = 0$.

Comparison of values of $I(w)$ computed via the single precision cubic spline fit with those computed directly shows that six or seven digits are accurate except near the end points where only three to five digits may be accurate. The difficulty in evaluating $I(w)$ accounts for the discrepancy at $w = 0$; the values are simply small at $w = w_c$, where $I(w_c) = dI/dw = 0$ holds. Experience suggests that these values are accurate enough to compute the components of $b_2(t)$ along the curved tank bottom for all practical purposes. However, had more time been available, experiments with accelerating the convergence of the series might have proved helpful.

The approximations Φ_1 to φ_1 (14) below the top two lines of the mesh are used only to construct the potential and streamline plots. The values of Φ_1 on the lower mesh lines are computed during the initial matrix generation and are recomputed whenever the lower part of the matrix $A(t)$ (mesh near the free surface) is revised. Thus the computation of Φ_1 throughout the mesh is a small part of the total calculation and is done by double precision routines designed to maintain accuracy near the free surface.

The constant c in the series representation φ_1 (14) is chosen so that $\Phi_1 = 0$ at $r = 1$, $z = F_w$, and $t = 0$. The coefficients $C_m(r_o)$ and zeros $j_{1,m}$, obtained as a by-product of the evaluation of $I(w)$, are stored in double precision tables with M values. For $r_o < 1/10$, M is 84 and the table includes 4 oscillations of $C_m(1/10)$. For $r = 1/30$ and $1/50$, M is 182 and the table includes $2\frac{3}{4}$ oscillations of $C_m(1/30)$ and $1\frac{1}{2}$ oscillations of $C_m(1/50)$. To avoid undue distortion of the plots, the evaluation of Φ_1 is suppressed when z falls below a fixed constant (.06

and .04 have been used).

Computing Φ_n and Ψ . Consider a general point on the free surface and the mesh region generated by rotating the three mesh quadrilaterals that meet there. This region has eight exterior sides; two on the free surface and six within the liquid-filled region. The integral of the exterior normal derivative over the boundary of this region must vanish. The integral over the two sides on the free surface can be approximated by the value of Φ_n at the mesh point within the region times the sum of the areas of the two bounding sides. Along the six interior sides, the integral can be approximated by a weighted sum of finite difference approximations to $\partial\varphi/\partial n$ on each of the sides, the weights being the areas of the bounding sides. The weighted sum is a finite difference expression involving the values of Φ at five mesh points, the one in the region and its four neighbors. Alternatively, the sum involves values of Φ at three points on the free surface and two points on the mesh line immediately below. Thus Φ_n can be found by dividing the negative of the weighted sum just described by the sum of the bounding areas on the free surface. For mesh points on the free surface at the axis and the wall, the same argument can be used over smaller mesh regions.

This argument is not restricted to the free surface; it can be used on any "parallel" mesh line to approximate $\partial\varphi/\partial n$ with the positive direction being toward the free surface. Comparing this construction with the one for the matrix $\underline{A}(t)$ in (18) and taking the normalization of the latter into account show that the matrix of coefficients needed to approximate $\partial\varphi/\partial n$ on a mesh line and the one immediately below is just one half of the block row of $\underline{A}(t)$ corresponding to the mesh line in question.

Thus, storing on drum (or other auxiliary storage), for each mesh line, the matrix needed to approximate $\partial\varphi/\partial n$ (and the distances between points on that line) make it possible to compute an approximation, Ψ , to the stream function

$$\psi(s) = \int_0^s r \frac{\partial\varphi}{\partial n} ds \quad (22)$$

as soon as Φ is known on the mesh line and the one below it. In (22), $\psi(0)$ has been taken as zero at the wall and s increases from wall to axis. Because $\partial\phi/\partial n$ is negative, $\psi(s)$ is negative and assumes the value -1 at the axis.

Initial Matrix Generation. As the free surface falls, it pushes against the fixed mesh line just below it. When these two lines become too close, it is necessary to revise the upper portion of the mesh between the free surface and some fixed, "parallel" line deeper in the mesh. After several revisions the mesh between the free surface and the fixed line may become too crowded. One remedy, that may be available, is to drop a "parallel" line from the mesh so that there are fewer mesh lines between the free surface and the fixed line. This corresponds to the intuitive feeling that the mesh must become smaller as the draining proceeds. Eventually, there will be too few mesh lines above the current fixed line. The remedy is to use a new fixed line lower in the mesh. Thus the input prescribes a set of lines that will be used as fixed lines in the course of the computation.

At each of these fixed lines, it is desirable to have stored on drum the matrices, vectors, and working arrays that will permit continuing from the fixed line the generation of the lower portions of the matrix $\underline{A}(t)$ and the vector $\underline{b}_2(t)$ (if necessary), the factorization of the former into $\underline{L}(t)$ and $\underline{R}(t)$, and the solution of $\underline{L}(t) \underline{x}(t) = \underline{b}_2(t)$.

The generation of the matrix $\underline{A}(t)$ and the vector $\underline{b}_2(t)$ proceeds mesh line by mesh line starting from the bottom line of the mesh (in Figure 2 from the axis to the point on the curved wall where only two mesh triangles meet). Corresponding to each mesh line there is a block row of $\underline{A}(t)$ and a block component of $\underline{b}_2(t)$. As soon as a new block row and component are available, the factorization of the matrix so far constructed into $\underline{L}(t)$ and $\underline{R}(t)$ is continued and the next block of component of the solution of

$$\underline{L}(t) \underline{x}(t) = \underline{b}_2(t) \quad (23)$$

is constructed. In general, the block component of $\underline{x}(t)$, the block row of

$R(t)$, and the values of Φ_1 on the mesh line are stored in a first drum area to be retrieved and used in determining Φ from

$$\underline{R}(t) \Phi_2(t) = \underline{x}(t)$$

and

(24)

$$\Phi = \Phi_1 + \Phi_2$$

when plots of potentials and streamlines are desired. To facilitate the transfer of information from drum to core, the "half-block-row" of $\underline{A}(t)$ needed to approximate $\partial\varphi/\partial n$ is stored in a second drum area. Whenever the current mesh line is one of the fixed lines, that data needed to continue the generation of $\underline{A}(t)$ and $\underline{b}_2(t)$ is stored in a third drum area.

For the mesh line just below the free surface, the procedure just described can be terminated with the construction of the final block component of $\underline{x}(t)$ via (23). The next step is to construct the first unknown block component of Φ via (24) so that Φ_n on the free surface can be computed from the values of Φ there and on the line below.

The approximation of Φ_n is the principal goal of approximating the solution to the mixed boundary value problem (5) and (6) with φ replaced by φ_2 , $\partial\varphi_2/\partial n = 0$ in the drain, (19) on the curved wall, and $\varphi_2 = \varphi - \varphi_1$ given on the free surface (initially $\varphi \equiv 0$ there).

A subsidiary goal may be to determine Φ and/or Ψ throughout the tank. In this case the values of Ψ on the free surface are determined by approximating (22), the values of Φ on the second line below the free surface are obtained from (24), the values of Ψ on the line below the free surface are computed, and so forth until the precomputed values of Φ_1 run out. At the end of this process, core must be restored to the state in which Φ_n was determined.

The plots of Φ and Ψ so obtained for $t = 0$ represent the instantaneous potential and streamlines set up at the instant at which the drain is opened. As pointed out in the Problem Formulation, the initial free surface is an equipotential and consequently the streamlines and velocities must be orthogonal to the free surface at $t = 0$.

Were the surface moving with velocity $\underline{v}^-$ at $t = 0^-$ and were the drain opened at $t = 0$, then the instantaneous velocity at $t = 0^+$ would be $\underline{v}^+ = \underline{v}^- + \varphi_n$ where φ_n represents the normal derivative of the impulsive potential satisfying (5), (6), and (7) with $\varphi = 0$ on the free surface at $t = 0$. The only difficulty in incorporating a moving free surface into the problem would lie in prescribing realistic shapes and velocities at the instant draining is initiated.

Other Derivatives and Treatment of the Contact Point. The free surface is followed in the calculation as the successive positions in time of a discrete set of mesh points which are frequently, but not always, Lagrangian particles with coordinates given by the functions R and Z . To calculate spatial derivatives for a fixed t , some sort of interpolation is necessary. Interpolating the points (R,Z) by a polygonal line gives R and Z as piecewise linear functions of u , the chord length along the polygon increasing from the axis. Interpolating R and Z , individually, by cubic splines with u as the independent variable represents the free surface between the mesh points by pairs of cubics $(R(u), Z(u))$ which have the property that the first two derivatives are continuous at the values of u corresponding to the mesh points [16,p. 51].

The symmetry condition at the axis implies that

$$R_s = 1 \quad \text{and} \quad Z_s = 0 \quad (25)$$

holds there, and the boundary condition (11) implies that at the wall

$$R_s = \cos(\alpha - \Theta) \quad \text{and} \quad Z_s = \sin(\alpha - \Theta) \quad (26)$$

where α is the angle between the horizontal and the tank wall at the intersection with the free surface. By using the columns of the array (25) and (26) as the end conditions, for example by setting $R_u = 1$ at the axis and $R_u = \cos(\alpha - \Theta)$ at the wall, the cubic spline interpolation of R and Z can be calculated from the algorithm in which R_{uu} and Z_{uu} at mesh points are the unknowns [16,p. 10]. The values of R_u and Z_u at mesh

points are obtained by separate calculation. This representation of the free surface as a function of u satisfies at the end points the conditions (25) and (26) that are identical with those that should obtain were arc length s along the interpolated curve the independent variable. The value of du/ds may be obtained from the relation

$$1 = R_s^2 + Z_s^2 = (R_u^2 + Z_u^2)(du/ds)^2 \quad (27)$$

and, in view of the end conditions, assumes the value 1 at the end points. Thus spatial derivatives with respect to s at the mesh points may be obtained by using the chain rule and (27). Hence, the surface tension contribution $2H$ (16) to the Bernoulli equation (9) and the sine and cosine of the angle χ (17) between the horizontal and the tangent to the free surface are now available at all points thereon.

When desired, the liquid volume under the cubic spline interpolation is computed from the expression

$$\begin{aligned} V &= 2\pi \int_0^{r_w} r f(r) dr - V_s \\ &= 2\pi \int_0^{u_w} R(u) Z(u) R_u(u) du - V_s, \end{aligned}$$

where V_s is given in (4), by sixth order Lobatto quadrature over each chord of the polygonal approximation with total chord length u_w .

To evaluate the dynamic term in (9), as well as the right sides in (8), an approximation, Φ_s , for $\partial\varphi/\partial s$ on the free surface is needed. Here it is convenient, and consistent with the convention used in computing ψ (22), to take the positive direction of Φ_s to the left of that for Φ_n so that arc length s increases from zero at the wall toward the axis. The symmetry condition implies that the axis $\Phi_s = 0$ holds. To obtain an end condition at the wall, the first four values of Φ_s at the midpoints of mesh intervals (finite difference quotients of adjacent values of Φ) are extrapolated to the wall by evaluating the cubic Lagrangian interpolating polynomial there.

With this ad hoc boundary condition and (28), the coding used to interpolate R and Z by cubic splines may be revised to obtain a cubic spline interpolation of Φ from which the values of Φ_s at the mesh points can be calculated. So, the dynamic term in the Bernoulli equation (9) may be calculated and Φ_n and Φ_s may be rotated into Φ_r and Φ_z to use in evaluating the right sides of (8).

Because there is room for only one mesh triangle between the free surface and the wall at the contact point (see Figure 2), the value of Φ_n assigned to the free surface at the contact point by the algorithm of the preceding subsection is derived from a mesh region with a single mesh quadrilateral as cross section. On the edge of the quadrilateral on the wall $\partial\phi/\partial n = 0$, (6); on the edge on the free surface $\partial\phi/\partial n$ is approximated by the unknown Φ_n ; on the edge that runs from the free surface to the interior point, $\partial\phi/\partial n$ is to first order approximated by the difference quotient Φ_f of the values of Φ at the first interior point on the free surface and the contact point; and along the edge that runs from the interior point to the wall, $\partial\phi/\partial n$ is to first order approximated by Φ_w , the difference quotient of the values of Φ at the contact point and the next mesh point below on the wall. Thus the value of Φ_n at the contact point is essentially a linear combination of Φ_f and Φ_w . As the former is one of the values used in extrapolating Φ_s to the wall, the rotation of Φ_n and Φ_s back into Φ_w and an orthogonal component will not, in general, make the latter zero, which is required for the contact point to move along the (vertical) wall. However, approximating the velocity of the contact point by a vector along the wall with magnitude Φ_w will satisfy the boundary condition $\partial\phi/\partial n = 0$, (6), in the neighborhood of the contact point and be consistent with the approximation used for Φ_n elsewhere on the free surface.

When the contact point and the next wall mesh point below both lie on the vertical wall, the use of Φ_w introduces no complications. When one or both of these points lie on the curved part of the wall, Φ_w is computed over a chord interior to the tank and the movement of the contact point along the tank tangent may (if the contact point is on the vertical wall) or will (if the contact point is on the curved wall) produce a point outside the

tank. For small time intervals and reasonable mesh spacing, the angle between the chord and the tangent will be negligible and the distance between the advanced point and the wall will be so small that the practical expedient is simply to pull the advanced point back onto the curved boundary before any derivatives are calculated in both the predictor and corrector of the numerical integration.

One limitation of the current version of the code is the assumption that, once the contact point leaves the vertical wall, it never returns.

Because Lagrangian points sometimes get too close to the wall or the axis, it is desirable to be able to fix the r-mesh on the free surface near the axis, near the wall, or on the entire free surface. In these cases, part or all of the system of differential equations for R may be dropped from the calculation. The vertical velocity of the surface is computed from the usual form of the kinematic equation (8a) and the Bernoulli (9b) is replaced by

$$W \left. \frac{\partial \varphi}{\partial t} \right|_{r \text{ fixed}} = W \left[\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial z} \frac{\partial f}{\partial t} \right] = 2H - Bf + C(t) - \frac{W}{2} \left[\left(\frac{\partial \varphi}{\partial r} \right)^2 + 2 \frac{\partial f}{\partial r} \frac{\partial \varphi}{\partial r} \frac{\partial \varphi}{\partial z} - \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] \quad (28)$$

in view of (8a) and (9a).

The function $C(t)$ (10) has been used for most of the runs to control the magnitude of the derivative of φ with respect to t . Another function has also been used in some runs, namely a $C(t)$ defined implicitly so that $d\varphi/dt \equiv 0$ at the wall. No essential differences between the results of the two functions have been observed except that with the implicitly defined $C(t)$ the value of Φ at the contact point is zero for all time so that the plotted potential lines move from plot to plot instead of staying essentially constant.

Solutions of Linear Systems. The linear system (18) must be solved twice in each time step, once to compute the predicted values of Φ_n and again to compute the corrected values. As pointed out in the section Basic Ideas and amplified in the subsection Initial Mesh Generation, such repeated

solution is only feasible because the mesh can be ordered so as to restrict the calculation to a small neighborhood of the free surface. The finite differencing used to generate the matrix $\underline{A}(t)$ has been described in the last cited subsection with additional material in the subsection Auxiliary Potential. Here the numerical methods used to accomplish the solution are sketched.

Because the basic finite difference star for the Laplacian on a triangular mesh in which six triangles meet at an interior point is a seven point one, each row of the matrix $\underline{A}(t)$ has seven nonzero entries which are arranged in three bands. The central one consists of the principal diagonal of the matrix plus the two adjacent diagonals and the matrix entries are the coefficients of the difference equations for values of Φ that correspond to points on a mesh row. The outer bands each consist of two adjacent diagonals with the matrix entries being the coefficients corresponding to the mesh row above and below the central row. Nonzero entries occur in symmetric positions; $\underline{A}(t)$ is normalized so that the entries on the principal diagonal are positive; and the off diagonal entries are, in general, negative. Because the values of Φ_2 are given on the free surface, it follows from the construction of the matrix that $\underline{A}(t)$ is nonsingular.

When the logical domain is the parallelogram of the subsection, Initial Mesh Generation, with N points on the free surface, the outer diagonals lie in $(N + 1)$ -th diagonal above and below the principal diagonal. In this case, the matrix $\underline{A}(t)$ has a block tridiagonal structure with $N \times N$ blocks.

Because the factorization and solution routines used here come from the earlier sloshing studies, they are designed to handle matrices of similar but irregular structure [8,p. 60]. In these routines all inner products are accumulated in double precision and then rounded to single precision [24,p. 116] and the sums are evaluated from the outer to central diagonal, that is, in general, from small to large values. Because $\underline{A}(t)$ may not be symmetric, the factorization of $\underline{A}(t)$ into left and right triangular factors tests for the possibility of an interchange of rows being desirable and notes in the list output the satisfaction of this test [23,p. 194;24,pp. 221-232]. (Whenever such a note has occurred, it has always followed a disaster in the

revision of the mesh.) The triangular factors, $\underline{L}(t)$ and $\underline{R}(t)$, consist, in general, of solid bands of nonzero entries from the corresponding outer diagonal of $\underline{A}(t)$ to the principal diagonal.

The matrix of the linear system used to construct the cubic spline interpolations for R , Z , and Φ is tridiagonal, symmetric and positive definite. Thus the numerical work amounts to forming one of the Cholesky factors of the matrix and using it to solve three linear systems with different right sides [24, pp. 229-230].

Integration of First Order Differential Equations. The evolution of the free surface is described by three systems of first order differential equations, two for the coordinate functions $R(t)$ and $Z(t)$ from (8b) and one for the potential (9b). Each system has an equation for each mesh point on the free surface. Because the evaluation of the right side of these equations takes up the major portion of the running time in this program, multistep methods, which reuse previously computed derivatives (right sides), are the methods of choice for the numerical integration of these systems.

For the runs reported here, the numerical integration method is a second order generalized Adams predictor-corrector method. The generalization consists of replacing the equal time steps of the usual Adams methods by sequences of time steps in which the lengths form a geometric progression with ratio τ . Once a new τ is given, the program (i) takes the next time step as τ times the preceding one, (ii) recomputes the coefficients of the integration method on the new set of time steps (which may no longer form a geometric progression), repeats (i) and (ii) until all time steps in the current recomputation form a geometric progression with the new ratio τ , and then uses the last set of coefficients without recomputation until the next change of τ . The generalized methods share with the usual Adams methods the advantage of having constant coefficients. In the new method, storage does not have to be rearranged as in doubling or halving the time step and the time step can be varied smoothly provided that τ is near 1. The recomputation involves accurately solving two linear systems of small order with matrices resembling Vandermondians. If several orders of

integration are to be used, the computation can be arranged to produce efficiently the coefficients for all orders less than the maximum order to be used. Thus for small initial mesh spacing the method can be made to be essentially self starting.

Although, because of the length of the calculation, the right sides of (8b) and (9b) are computed in single precision (except for the essential accumulation of inner products), double precision is used for the values of the integrated functions R , Z , and Φ and for the computation of the predictor and corrector formulas (essentially inner products) to minimize the accumulation of round-off errors. The parts of the predictor and corrector which involve common values of the derivatives are evaluated in the same loop to make iteration of the corrector easy, if desired. So far, only one corrector iteration has been used. Each system is dealt with independently to facilitate special testing of R and Z and dropping the R -system when the R -mesh is fixed.

These features are inherited from an integration package designed for use in an LMSC experimental reorientation code done in 1964, which required the integration of the same three systems. Among the inherited features, that have not been exploited yet, are storage for up to seven old derivatives (methods of up to seventh order may be used), the ability to use different orders of integration in parts of each of the three systems independently, and the ability to change orders and the parts of the system to which they apply at any time step. The variable mesh features were adapted from the routine used to distribute a variable mesh smoothly on the highly curved menisci occurring in spheroidal tanks [9, pp. 36-37].

Mesh Revision. The variety of shapes for which the program produces meshes is shown in Figures 2 through 12; the meshes in Figures 2 and 3 are not only examples of the configurations from which the mesh revision starts but also samples of the mesh throughout the tank. A nearly complete design for the mesh revision control is described in terms of input parameters, p_i , $i = 1, 2, \dots, 24$, to indicate the potential of the technique and to provide a uniform basis for discussing differences between runs. The controls have

been developed stepwise with a bias against over elaboration.

The subprograms that revise the mesh in a region between the free surface and a line lower in the mesh require as input that each point on the boundary be specified. Because points on the free surface and the fixed line cannot be moved, the control over the revision consists in specifying the mesh points on the axis and the wall which lie in the region to be revised. The first mesh interval along the axis is a specified fraction p_1 of the adjacent mesh interval on the free surface; a similar prescription with p_2 is used for the first interval along the wall. When there is more than one mesh line between the free surface and the fixed line, the remaining boundary points are determined so that the mesh intervals form a geometric series. To avoid meshes with lines that are unduly close or spread, the ratio of the last to first interval in each series is tested against a pair of specified bounds, p_3 and p_4 at the axis, p_5 and p_6 at the wall.

As the free surface moves, the areas of the mesh triangles in the row between the free surface and the fixed mesh line just below change. These areas are available at each step as a by-product of the revision of the last block row of the matrix $\underline{A}(t)$, and they provide a convenient way to test whether the surface has moved too close or too far away from the fixed line. Because the areas in general decrease from axis to wall, the free surface may be subdivided into parts (specified by the right end points $p_7, \dots, p_{10}$) and the areas in each part tested against lower and upper bounds for that part. At each mesh revision, the minimum and maximum area within each part are determined. The lower bound is taken as a fraction ($p_{11}, \dots, p_{14} < 1$) of the minimum; the upper bound as a fraction ($p_{15}, \dots, p_{18} > 1$) of the maximum. Whenever the area of any triangle passes over a bound for its part, mesh revision is initiated. If the proposed revision satisfies the ratio tests against $p_3, \dots, p_6$ or their analogs, the revised mesh is constructed and the factorization of $\underline{A}(t)$ and so forth continued from the fixed line as sketched in the subsection Initial Matrix Generation.

In addition to the fixed lines prescribed in the input, the second line below the free surface is an extra fixed line. This permits "minor" revisions of

the mesh by readjusting only the points on the first line below the free surface. When a test of the ratios of last to first intervals on the axis or wall fails for a minor revision, the next step is a major revision of the region between the free surface and the current fixed line specified in the input. (Minor revision can be eliminated by choosing $p_3, \dots, p_6$ so that the tests always fail.)

A major revision begins by distributing points along the axis and the wall in a geometric series beginning with the mesh intervals adjacent to the free surface which are prescribed as before. If the ratios of last to first intervals satisfy tests against pairs of prescribed bounds, $p_{19}, \dots, p_{22}$, analogs of $p_3, \dots, p_6$, the construction of the proposed revision proceeds.

Upon failure, dropping a line is permitted if $\dot{\phi}_w < p_{23}$. In general, p_{23} is negative, so that dropping occurs when the contact point is moving downward fast enough. A block transfer of the description of the free surface into the core locations previously occupied by the line below effects the dropping. The procedure of the preceding paragraph is carried out again. Upon a second failure to satisfy the bounds $p_{19}, \dots, p_{22}$, the region of mesh revision is enlarged by taking as new fixed line the next one, lower in the mesh, in the set prescribed in the input; and one more attempt is made to satisfy the bounds $p_{19}, \dots, p_{22}$. After a third failure the program quits. If dropping is not permitted, the program, upon the first failure of a major revision, enlarges the region to be revised and tries once more.

When the region of revision reaches the bottom of the tank, the free surface at the axis has generally been moving rapidly downward in the problems so far attempted. Thus provision must be made to contract the mesh along the axis, particularly if dropping is not permitted. To achieve this, the fraction p_1 is progressively reduced at each time step by multiplying a fraction p_{24} less than but close to 1.

Because the emphasis in running the program has been to study draining per se rather than wave motion on the free surface, the development of the mesh revision controls has been biased toward downward motion. Of the controls for upward motion, only the upper area bounds $p_{15}, \dots, p_{18}$ have actually

been implemented. In most runs, the free surface has been treated as a single part in testing triangle areas against upper and lower bounds, that is, only p_7 , p_{11} , and p_{15} among $p_7, \dots, p_{18}$ have been used. In all cases, the two sets of ratio bounds, $p_3, \dots, p_6$ and $p_{19}, \dots, p_{22}$, have been identified; and an additional test, based upon how far the axial point on the free surface has moved, has been used to insure that major revisions occur occasionally.

The computer time taken by a minor mesh revision is approximately the same as that taken by a time step; major mesh revisions take from three to ten time steps, depending on the depth of the revision and the complication of the shape involved.

In Figures 4 through 12 there are some lines crossing the tank from axis to wall which do not belong to the orthogonal set of potentials and streamlines. They are the mesh control lines used to contour the mesh. They first appeared by error but were retained when it was observed that they gave a clue to the shape of the mesh used without unduly cluttering the output. Because the fixed mesh lines tend to be "parallel" to the mesh control lines, the reader can get an idea of the complication of the shapes of the major revision regions by interpolating some fixed lines between the free surface and the first mesh control line.

Controlling the Mesh at the Axis. Instabilities may arise in numerically integrating a system of differential equations if too large a time step is used. If uncontrolled, changes in the mesh at the juncture of the axis and the free surface may be an additional source of instability. The finite difference equations used to compute $\bar{\Phi}$ and $\bar{\Phi}_n$ are essentially weighted sums of finite difference quotients of $\bar{\Phi}$ along the boundaries of mesh quadrilaterals. The weights are the areas of frustums of cones obtained by rotating the boundaries of the quadrilaterals around the axis. The weights are small for mesh points on or near the axis because the radii of the cones are small. Because all the quadrilaterals used for $\bar{\Phi}_n$, while only half of those used for $\bar{\Phi}$, lie between the moving free surface and the line below, which is fixed, $\bar{\Phi}_n$ is the more sensitive. For example, some entries in the last row of the matrix used to compute $\bar{\Phi}_n$ are smaller than the

corresponding entries in other rows; thus, compared with the general component of Φ_n , the one at the axis is relatively more affected by changes in the mesh and the solution. This is also true of Φ and is particularly important when Φ is nearly constant both in the r and z directions in the corner, so that loss of significance in subtraction becomes a problem in computing both Φ and Φ_n . See for example Figure 8(e) - (h), where the streamlines which were originally near the axis have moved very far to the right.

Upon revision of the mesh, the value of Φ_n at the axis may jump and then oscillate. For small jumps the oscillations are rapidly damped out and barely affect the nearby points. For large jumps, they tend to grow unacceptably. (The use of Φ_w at the wall, where the area weights are large, minimizes the problem at the wall.) Once the streamlines reached the configuration of Figure 8(e), no reduction of time step could produce a successful continuation of the run so long as infrequent major revisions were used.

By setting the fractions, $p_{11}, \dots, p_{18}$, controlling the area testing to values close to one, minor mesh revisions every five to seven time steps can be obtained. At each revision, the first mesh interval on the axis is reset to p_1 times the adjacent interval on the free surface. As the length of the latter varies slowly, the effect of frequent revision is to keep the mesh in the corner approximately the same shape over long periods of the computation. With frequent minor mesh revisions, the run was continued beyond the state shown in Figure 8(h), where a small wave near the axis is beginning to develop.

For Figure 12 only 14 mesh points are used, whereas 22 are used for Figure 8. Thus the success of the former can be partly explained by larger area weights near the axis. However, the run time for Figure 12 was long because the early version of the mesh control used there produces frequent major mesh revisions. Thus the results presented in Figure 12(d) - (h) suggest that a proper design for the mesh control will aid in minimizing the difficulty in computing Φ_n at the axis when the potential is flat in both the r - and z - directions there.

The device of keeping mesh points away from the axis as an aid to stabilizing the computation of Φ and Φ_n is not altogether satisfactory because it conflicts with the need to have mesh points near the axis to represent the s-curve that the free surface assumes as it nears the drain (see Figures 5(i) - 7(i), 9(i), and 10(i)). This conflict leads to a desire to redistribute points along the free surface from time to time.

This end could be achieved by modifying the paths that mesh points on the free surface follow in the (r,z) plane during the numerical integration of the system of first order differential equations describing the evolution of the free surface. The transition from the Lagrangian form (8b) and (9b) to the r-fixed form (8a) and (28) is an example of such a change of path. The transition can be regarded as a transformation of the first set of three equations at each mesh point into the second to accommodate a change of direction in the path of integration for that mesh point. The implementation of the proposed method for redistribution would involve making a similar transformation of the right side of the system for those mesh points that were to be moved at each time step. The desired direction of movement would have to be specified at each point. The principal advantage of such a variable-path method of integration would be to provide an additional means of controlling the structure of the finite difference equations satisfied by Φ_n and Φ at the juncture of the free surface and the axis by being able to move nearby mesh points on the free surface in the r-direction. It could also be used to move mesh points toward the axis just before vapor ingestion. (The principal reason for preferring the Lagrangian over the r-fixed form is that the former permits representing free surfaces which are multiple valued in r.)

Support of the Wall Layer. The development of a sheet of liquid hanging on the wall is shown in Figures 4(e) - (h) and 5(e) - (h). The distortions of the free surface near the wall in Figure 5(g) and (h) results from an experiment with using large triangles in the wall layer. In Figure 3, which shows examples of the meshes used to construct Figure 4, the triangles in the wall sheet are small, and there is a reasonably thick layer of triangles

throughout the sheet. The difficulty in handling the wall layer results from the basic parallelogram shape adopted for the logical diagram (see the subsection Initial Mesh Generation). The Figures 3 and 4 show a successful example of using the mesh control lines and other input mesh controls to map the logical domain into a physical domain which is more or less a parallelogram with one corner progressively pulled out like taffy. The goal is to keep enough mesh in the wall layer to represent the free surface smoothly. Subsidiary goals are to determine when and if the wall sheet starts to move downward and whether waves develop on the surface of the layer.

Observe that Figures 3(b) and (c) interpolate Figures 4(d), (e), and (f). The locations of the right end points of the mesh control lines that are obscured in Figure 3 can be read from the later time in Figure 4. The initial mesh, Figure 3(a), is essentially three replicas of an initial mesh for the same parameters with $h_c = 1$, with the mesh pulled up at the contact point. The replication is obtained by use of two mesh control lines parallel to the surface. In transiting between Figures 3(a) and 3(c), the lower mesh control line survives; at each step its right end point moves up; at the first step more mesh is shoved under it; and at the second the left end point is dropped. The upper mesh control line survives only the first step, more mesh is shoved under it (there are the same number of lines in Figures 3(a) and (b)), the left end point is dropped and the right one is moved up. The purpose of these maneuvers, which were accomplished by human intervention, is to insure that enough mesh is shoved up (continued raising of the right end point) into the developing wall layer so that no elongated triangles occur in the sheet or in the neck, where it joins the bulk of the fluid. The purpose is also to relieve the mesh in the center of the tank so that there will be room for the free surface to continue to develop easily. In the transition between Figures 3(b) and 3(c), six lines are dropped from the mesh. Because using a parallelogram as the logical domain requires that there be as many intervals on the axis as there are on the wall above the point where just two triangles meet, this is accommodated by using a slightly larger mesh, particularly in the top part of the wall sheet.

The success of this procedure in anticipating fixed lines which will be more or less parallel to the future shapes of the free surface can be judged by comparing a mesh line near 1.6 on the axis in Figure 3(b) with the free surface in Figure 4(f) and a mesh line near .9 in Figure 3(c) with Figure 4(g). The analysis of Figures 3 and 4 shows that even when a wall sheet is present, the parallelogram is a reasonable logical domain for all except the final stages of draining. Moreover the analysis shows that the procedure developed by human intervention can easily be automated.

In the final stages of draining (Figures 4(g) and 4(h)), it is impossible to cram enough mesh along the axis to give a desirable mesh along the wall when the logical diagram is a simple parallelogram. The mesh spacing at the top of the wall sheet is distorted.

An alternative logical diagram would consist of a stack of parallelograms with a common line on the left (the axis) and increasing extent to the right with the bottom and the jagged right side mapped onto the wall. The programming for such a domain is straightforward, but tedious, and time consuming. For draining problems it is applicable only to the later stages of draining with a wall sheet present; for reorientation problems, it would be essential.

Restarts. Periodically, under input control, the program dumps the essential computational and control arrays, at the end of a time step, onto a restart tape. With write orders switched to read orders, the dump program becomes the key part of a production restart which restores core to the state at the end of the time step dumped and takes off. This restart permits changing any of the control parameters before take off. However, it is inconvenient to use the continuation restart to change large arrays such as the mesh.

To permit limited production before the completion of the experiments with the mesh controls, a second restart was provided. This one reads the first part of the dump to obtain the current free surface, the derivatives from current and previous time steps, and the arrays controlling the continuation of the numerical integration; it reads card input, which is the original input less that for the equilibrium free surface and the numerical integration, to

permit changing the form of the mesh; and it begins with the procedure described in the subsection Initial Mesh Generation.

Both restarts read the summary dump, at the end of the restart tape, that contains the information used to construct the summary plots. Both restarts regenerate the drum storage by going through the procedure described in the subsection Initial Matrix Generation. This may result in a mesh discontinuity at the juncture of the axis and the free surface, particularly when the second form is used and p_1 is changed (see the subsections Mesh Revision and Controlling the Mesh at the Axis).

Using the second form of the dump and revising the mesh by hand to achieve the kind of results discussed in the subsection Support of the Wall Layer is not easy. So far, results obtained with human intervention suggest that the original automatic design of the program is sound.

DISCUSSION OF RESULTS

Computer Program

Relations to Other Problems. Among the major results of the study is a new method for calculating the irrotational motion of an inviscid, incompressible fluid with a free surface. The method is not only applicable to the present study of the draining of a liquid from a hemispherically bottomed cylindrical tank in zero and low gravity environments, but it can also be used for a large class of irrotational, axisymmetric, low gravity fluid dynamics problems in a tank with circular symmetry -- for example, reorientation and sloshing problems [8,9,17,18].

Because the moving free surface is treated in an essentially Lagrangian manner, the finite difference approximations directly represent the physics of the free surface both in the present and the LINC method [21]. Because a fixed mesh (albeit revised from time to time in the present method) is used throughout most of the domain, the simplicity of "Eulerian" programming is common to both the present and the MAC method [22]. By combining these features, the new method avoids the difficulties of distorted Lagrangian cells (LINC) and of representing the moving free surface in a fixed Eulerian mesh (MAC).

The success of this method in reproducing the features of draining phenomena found in experimental studies [1,4-7] gives credence to the belief that the assumptions made in the Problem Formulation are adequate for describing draining in the parameter ranges studied in this report. The introduction of the auxiliary potential permits modification of the spatial boundary value problem into one in which essentially all the action takes place on the free surface, a restricted part of the boundary -- a feature that characterized the preceding sloshing studies [8,9]. For such problems, solutions by direct methods restricted to a neighborhood of the active boundary is appropriate. Triangular meshes are used here, as well as in the predecessors, to approximate curved boundaries by polygonal ones made up of mesh lines to facilitate construction of finite difference approximations. In the present study, such a mesh is applied to a moving boundary problem.

The goal of each mesh revision is to construct a mesh that not only is adapted to the current cross section of the liquid filled region but also will continue to fit the problem for some time to come. During draining the free surface in general falls; however, because draining includes aspects of reorientation and sloshing, the motion may be irregular. The general falling of the free surface is accommodated by dropping lines from the mesh; the irregularities in the motion are dealt with by compressing or expanding the mesh.

Treatment of the Contact Point. Because the contact point must be constrained to move along the wall, its velocity is taken to be the difference quotient of the potential at the contact point and the mesh point just below on the wall. This choice is consistent with the calculation of the normal derivative at other points on the free surface. It also enforces on the vertical wall the boundary condition that the normal derivative of the potential vanish and provides a reasonable approximation to this condition on the curved wall. The value of $2H$ used in the Bernoulli equation (9a) is computed at the contact point just as at any other point. The condition that the contact angle remain fixed appears in the calculation as an end condition for the cubic spline interpolation. The extrapolated value of the tangential derivative at the contact point appears in the calculation only as an end condition for the cubic spline interpolation. All of the values used at the contact point can be viewed as interior limits along the free surface or wall.

Heuristic Error Analysis. A pleasant surprise in developing the code was to discover how few points on the free surface are needed to give satisfactory results (in the present calculations 22 points in general, once as few as 14). The design of the program was strongly influenced by the desire to simulate experiments at low Bond numbers and high Weber numbers. It was originally believed that as many as 50 mesh points would be needed to compute the shapes shown in Figures 4(h) and 5(h) and the program is so dimensioned. A second surprise was to discover that Bond number zero is not a special case. The emphasis during the design and development of the program has been on draining. The basic question was whether or not the proposed method would actually

simulate draining.

A global interpolation of the free surface by cubic splines between mesh points seems to account for success with so few points. The end conditions are constant contact angle at the wall and horizontalness at the axis. The interpolation is in terms of chord length, but the resulting derivatives at the mesh points can be transformed into arc length via (27). For example, in Figure 9(i), the cubic spline interpolant of the last computed mesh shape must be perpendicular to the axis; and it must pass through the mesh points, which can be identified as the points where the plotted linear interpolant changes slope. Thus adjacent to the axis, the cubic spline interpolant must lie below the linear interpolant.

The values of $2H$ (16) used at the mesh points are values of a continuous function and those for the components of the tangent vector along the free surface are values of a differentiable function. The value of the tangential derivative of the potential is determined by a similar cubic spline interpolation of the potential with the end conditions given by symmetry at the axis and by extrapolation of interior finite difference quotients to the wall. Thus the right side of (8a), (8b), and (9b) appear to be as smooth as the normal derivative of the potential. (In this argument, it is convenient to replace the r - and z -components of the velocity vector by components normal and tangent to the free surface.)

Because the tangent vector to the free surface is, in general, well determined, the system of differential equations describing the free surface may be integrated along fixed r -lines using (8a) and (28). For the curved free surfaces encountered here, such integration would fail were the free surface with so few points interpolated by a less accurate method because the slope would not be well determined. Even with slopes determined from cubic spline interpolation, too few points on a nearly vertical free surface may produce difficulty (see Figure 5(h) which was produced by integrating along fixed r -lines with too few points on the surface near the wall).

The integration is carried out by a multistep predictor-corrector method which

uses variable time steps in geometric progressions and allows the ratio of the progression to be varied at will. The order of integration is also under user control. To suppress accumulation of errors, the dependent variables are carried in double precision. Thus it would appear that the values of the dependent variables are reasonably accurate.

The normal derivative of the potential is calculated from a finite difference approximation to a mixed boundary value problem which has the potential on the free surface as the essential changing boundary value. The approximation is made on a triangular mesh which interpolates the mesh points on the free surface by straight lines (the domains shown in Figure 3, for example).

Thus the program, as it now stands, has an imbalance: all the terms on the right side of (8b) and (9b) are computed from the cubic spline interpolation of the free surface except for the normal derivative of the potential. The latter is computed from the potential determined within the domain bounded by the linear interpolant. Although the discrepancy between the domains bounded by the linear and the cubic spline interpolants is probably only important in the later stages of draining, it should be noted that some available information about the free surface is not being used in the computation of the potential and the normal derivative.

What is more important is an imbalance in accuracy. It appears probable that a detailed error analysis would show that the truncation errors of the terms in (9a) coming from the cubic spline interpolation are of higher order than that coming from the normal derivative.

An ideal finite differencing scheme would take into account the curved nature of the boundary. A desirable compromise would be to choose schemes for evaluating the normal derivative and for numerically integrating the system of differential equations describing the free surface that had truncation errors of about the same magnitude as that of the cubic spline.

At present the basic control on the error is the choice of the time step. The number of points on the free surface is another control affecting the

error. The spatial mesh spacing normal to the free surface can be controlled by prescribing the ratio of the first mesh interval on the wall and axis to the adjacent intervals on the free surface. Introducing an additional variable into the numerical integration scheme, namely a variable path for a point on the free surface, would give some control over the spatial mesh on the free surface. At present this is controlled only by the initial distribution of points on the free surface and the specification of Lagrangian or r- fixed integration.

Mesh Control. To remove the imbalance in accuracy from the program would require an investigation of higher order finite difference approximations for use along the entire free surface. Such an investigation should eventually be carried out with the goal of seeking a better estimate of the normal derivative at the axis; that is, one giving satisfactory values when the potential has small variation near the juncture of the axis and the free surface, or one weakening the requirements on control of the mesh near that point and on control of the time step (see the subsection Controlling the Mesh on the Axis). Computing the normal derivative by interpolating up the axis is easy and feasible. Were the calculation to be continued, the next potential to be determined would be at the second point on the axis below the free surface. (Mesh rows are ordered from bottom to free surface, mesh points from wall to axis.) In general, the two dimensional interpolation, based upon a finite difference approximation is preferable to a one dimensional interpolation.

Consequently mesh control seems the method of choice. Perhaps the most promising device to avoid this problem would be to add variable paths to the variable time steps and variable orders in the scheme for numerically integrating the system of differential equations describing the free surface with the intent of keeping mesh points away from the axis until the free surface is so low in the tank that the normal derivative on the axis has and will continue to have appreciable size. The programming for this last alternate is simple.

Although the procedure for mesh control described in the subsection Mesh Revision may seem elaborate, it involves, in general, only testing of quanti-

ties already computed. Therefore it could be fully implemented, and even expanded, without affecting run times except favorably. The time taken by mesh revision depends somewhat upon the shape of the region to be revised. Some of the shapes which require long times could be avoided by generalizing the specification of the mesh control lines which contour the mesh in advance of the evolving free surface. Both improvements involve simple programming.

A more difficult programming problem would be to remove the restriction that the logical domain be a parallelogram. By carrying out the program described in the subsection Supporting the Wall Layer, the black mass of small obtuse triangles (marked with a central "o") in Figure 3 could be reduced, thereby reducing mesh revision times; and more mesh could be introduced into the wall layer, thereby increasing confidence in the computed results in that region.

Time Step. For a problem and numerical method with the complexity of the present one, the practical way to determine the time step is to base the choice upon experience and a few common-sense principles. Throughout the initial transient the time step should be small. After perhaps 50 time steps, it may be increased more rapidly. The higher the fill level or the longer the expected run, the smaller the initial time step should be. The time step should be cut down when the free surface at the center line is decelerating into a wave or when the wall point begins to move. It appears that, when no wave motion occurs, the time step is an increasing function of W .

When the mesh revision controls are more fully developed, an upper bound for the time step in regions of difficulty could be experimentally determined. This would involve several restarts of a problem, for example over a wave, with an increasing set of time steps.

Because it has a short run, Case A3, where $B=0$, $W=36.06$, and $h_c=1$, was run to vapor ingestion over a decreasing series of initial time steps. The results showed that an initial time step of .005 increasing with a ratio of 1.01 to .0074 at the end was satisfactory for this study. In running case 44, where $B=0$, $W=100$, and $h_c=3$, an initial time step of .001 was increased with various ratios to a final step of .008. Cases 53, where $B=0$, and Case 98, where $B=3$,

both with $W=3$ and $h_c=1$, were run with an initial time step .001 increasing to .0024 at time step 50 and constant thereafter.

The time steps reported here and the average time steps deducible from the labeling of the frames in Figures 4-12 are unduly conservative. They were chosen small to maximize the probability of a long run and thereby minimize the need for human intervention.

Output. The computer program produces at any desired time step a print-out that includes the position of the free surface, the value of the potential thereon, and the derivatives of the potential in the form currently being used for the numerical integration. The integrated volume between the free surface and the tank wall and orifice is also given. The values of V_i , initial volume, and V_r , residual volume, presented in Table I are the integrated volumes normalized by dividing by the volume of a unit hemisphere. For the cases reported here the relative error in V_r has been as large as 2 percent, generally below 1 percent.

The program produces plots of meshes, free surface shapes, streamlines, and equipotentials. The plots give the linear interpolation of the free surface mesh points and not the interpolation by cubic splines. A plot is output at each mesh revision to provide evidence as to the effect of input parameters and mesh controls (circles within triangles mark those that are obtuse). Other plots occur periodically when requested via user input. For all these plots, scales are uniform in each direction. Each plot is labeled with the Bond number B , the Weber number W , initial height at the center line h_c , and the drain radius r_o . The latter two quantities have the labels HCL and DR on the plots. The notation $DR=.0333$ is to be interpreted as $r_o=1/30$. Each plot is labeled with the dimensionless time and the number (decimal integer) of the time step at which it was produced.

Free surface plots are put on the same frame until another kind of plot intervenes or until the current free surface would run off the frame. Each free surface is labeled with its time step and dimensionless time. The labels ap-

pear opposite the left end point of the free surface whenever possible; otherwise they are listed consecutively down the page in alternate columns. Examples are the initial and final computed free surface shapes in Figures 4 through 10 and frames in Figures 6, 9, and 11, where there were no streamline and equipotential plots sufficiently near the desired times.

Streamline and equipotential plots can be produced independently (see the initial frames of Figures 4 through 12); however they are usually requested together. Because streamlines may be defined as lines across which no fluid flows, it is convenient to use them to subdivide the liquid filled region into given volumes. Here ten subvolumes are used. Because the nature of the flow near the wall and axial streamlines is important to the understanding of both the dynamics and numerics of the problem, two additional streamlines have been introduced in the outer tenths to produce volumes adjacent to the axis and wall from which two percent of the flow comes. (The zero of the stream function is taken at the wall, arc length increases from wall to axis, and the normal derivative is negative in general; consequently the computed values of the stream function are negative.) The values of the stream function on the streamlines plotted are given on every plot.

The plotted equipotentials are equally spaced orthogonal trajectories of the streamlines. The spacing is one tenth of the range of the potential as computed by the program in the time interval since the last restart or since the initiation of draining. The extremes and the nine intermediate values are printed on each plot. Because the range of the potential at a time step may be smaller than the total range, sometimes only eight equipotentials appear. (The range changes with time in different ways depending upon whether the arbitrary function $C(t)$ in the Bernoulli equation (9a) is defined by (10) or implicitly by making the potential vanish at the contact point.)

The lines that extend from axis to wall and are not part of the orthogonal net of streamlines and equipotentials are current positions of the mesh control lines. As pointed out earlier, they give a useful indication of the shape of the computing mesh.

Summary plots follow the completion of every run. They consist of the initial and final free surface shapes, mentioned above, and a plot of the center line and wall heights as a function of time. A uniform scale is used for each of the initial fill heights h_c used in the study.

Solution of Draining Problems

Putting a program involving a new numerical method into operation is not routine, particularly when it is used, as in this draining study, to collect physical data over a number of parameters with wide ranges. As new values of B , W , h_c , and r_o were tried, ability to specify the controls for operating the code improved significantly.

Parameter Selection. Specifications of the initial mesh, mesh revision criteria, and time step are closely related to the physical problem and as a result very much influenced by the choice of the problem parameters. In this study Bond numbers are in the range of 0 to 20 and emphasize the smaller values; Weber numbers vary from 0.3 to 100 and emphasize the larger values; initial fill levels, h_c , are usually 1, 2, or 3, most often 1; and r_o is either 1/50, 1/30, or 1/10, principally $r_o=1/30$. These choices have been made in order that the solutions of the draining problem afford a broad review of the fluid mechanics of draining under low gravity. Bond numbers in this range are small enough so that the curvature of the interface is significant for all cases; the larger values of Weber number amplifies the importance of the velocity term in the Bernoulli equation; and the choices of h_c and r_o are reasonable for practical tank geometries. Collection of draining solutions for this range of parameters can provide useful physical data while establishing the techniques needed for operating the program.

Table I lists the specific cases computed and shows a more complete survey for $h_c=1$. The controls for $h_c=1$ have had more study and permit quite economical operation of the program while the controls for deeper cases, lower values of Weber number, and higher Bond number require additional study; consequently, fewer cases are reported for the latter conditions.

Development of the techniques required for handling free surface slosh induced by draining, liquid reorientation during draining, and thin wall sheets of liquid has not permitted a definitive parametric study which would provide the completeness needed in design calculations. However, the survey does reveal several important features of the physical problem. The following subsections describe the essential features of the fluid mechanics of tank draining and compare several of these features with experiment.

Comparison with Experimental Data. The computer program has evolved in a series of steps to its present form and contains a high degree of flexibility for further extension and improvement. Initially the first working code was used for justification of the basic problem formulation and choice of numerical methods by comparison of computed results with experimental data. Cases A3, B3, C3, and D3 given in Table I were used to develop the early versions of the code because these particular cases duplicate the parameters of or have similarities to experimental data supplied by NASA-Lewis Research Center.

A comparison of computed and experimental data for the variation with time of free surface height at the tank centerline is given in Figure 13. (These data pertain to case D3 of Table I.) The two curves are essentially parallel with the separation due to a difference in initial conditions. In the computed solutions the liquid is quiescent before draining is initiated, while the experiment has an oscillatory initial free surface resulting from low gravity reorientation. These oscillations do not have time to be damped during the brief experimental time so that draining must take place in the presence of the oscillations. For consistency in the experiments, draining has usually been initiated on the low point in the first cycle.

The shapes of the computed free surfaces are qualitatively similar to those observed in experiment. The presence of a liquid layer clinging to the tank wall for $B=0$ and high Weber numbers is common to computed results, such as those in Figure 4(a)-(h), and to experimental results, such as in Figure 7 of [5]. In fact, the shapes of both kinds of free surfaces are similar during

the entire draining sequence.

Prediction of these liquid layers is an important result of this study because the inviscid analysis reproduces behavior exhibited by the fluids in the experiments. While real liquids have viscosity and can be expected to leave a thin viscous layer on the wall, the observed wall layers are likely due primarily to the inviscid dynamics of draining low contact-angle liquids under reduced gravity at appreciable Weber numbers. The liquid layers shown in Figures 4-6 can be explained as follows: The velocity potential for draining at the contact point is substantially less than that at the centerline, which implies the wall layer lags considerably the faster moving center core. The small gravity and surface tension forces do not have time to reorient the wall layer prior to vapor ingestion. Certainly viscosity can cause wall layers to some extent, but the large thicknesses of the layers for cases when $h_c = 1$ and diminishing thicknesses when h_c is larger (low Bond number, large Weber number) suggest the relative importance of the dynamical mechanism.

Free Surface Behavior. Figures 4-12 have been chosen because these particular cases are representative of the fluid mechanical features computed for the various values of B , W , h_c , and r_0 summarized in Table I. The wave motion induced by draining is discussed in the following subsection; here the general behavior of the free surface is reviewed by relating observed phenomena on or near the free surface as events occurring before or after the downward acceleration of the contact point.

An important feature of the solution is the time delay observed before the contact point begins to fall. This time delay generally decreases with the Bond number, but in the case of $B=0$ the contact point actually rises noticeably as shown in Figures 4(j), 5(j), and 6(j). In those cases where the Bond number is larger, the contact point initially rises, as for example in Figure 8(j) where $B=1$, but eventually starts to fall if the Bond number and/or h_c are large enough or if the Weber number is small enough. Figures 9 and 10 show results for $B=1$ and $B=3$, respectively, where the contact points are very

sluggish, rise slightly, but never really fall prior to vapor ingestion because the Weber number is too large. For case 119 ($B=3$, $W=100$, $h_c=2$, and $r_o=1/30$) the extra initial depth provides sufficient drain time for the wall point to begin to move downward. In such cases as B3 ($B=19.32$, $W=3.41$, $h_o=1$, and $r_o=1/10$), Figure 11(g), and C3 ($B=19.32$, $W=3.41$, $h_c=2$, and $r_o=1/10$), Figure 12(q), the wall point begins to move rapidly after the initial delay and waves appear. Examination of streamlines reveals that the downward movement of the wall point occurs when the streamlines straighten to vertical near the contact point.

Once the wall point begins to drop, specification of controls is easier. In such cases as Figure 4(i) and 5(i) the failure of the contact point to move leaves the wall layer which does not reorient prior to vapor ingestion. Providing mesh support for these wall layers is difficult particularly in cases where $h_c=3$, but as a result of computing case 44 better techniques have emerged using the present version of the code. The restart meshes in Figure 3(b)-(d) for case 44 display the coarseness of the mesh in the wall sheet.

The downward movement of the wall point supports the idea that draining under low gravity is also a reorientation problem. Release of the wall point can be followed by wave motion such as that in Figure 12 or more tranquil reorientation such as that in Figure 8. For very small Weber numbers one anticipates that free surface reorientation dominates the liquid motion and that the wall point freely relaxes during the reorientation. At the other extreme of very high Weber number where the wall point moves very little, reorientation is minimal; the liquid is dominated by the drain; and residuals are high. Absence of reorientation of wall liquid to the centerline is verified by comparing the centerline plots of Figure 4(j), 5(j), and to a lesser degree 6(j) where these curves are essentially parallel and straight for each case even though starting at different fill levels. In Figure 8, where considerable reorientation is observed, the centerline plot flattens sharply (Figure 8(j)) just after the wall point releases.

The intermediate cases between the extremes of reorientation-dominated and

draining-dominated regions have wave motion on the free surface.

Wave Motion. For particular choices of the Weber number, Bond number, and initial fill level wave motion caused by draining is observed on the free surface. In some cases such as that shown in Figure 12, where the Bond number is sufficiently large, visible slosh is the case; but in other cases the wave motion is of such small amplitude that evidence is found only in the "vibrating reed" action of the streamlines. An example of the latter type is given in Figure 8(g) and (h) where significant streamline distortion near the free surface is present, just as in Figure 12, but the centerline point merely slows (Figure 8(j)) with visible waves on the free surface being absent.

In the present case the action of draining forces results in distortion of the free surface from the equilibrium shape produced by capillary and gravity forces acting alone. The introduction of this distortion triggers the wave motion by introducing a nonuniform acceleration field along the free surface. Coupling of this distribution of nonuniform acceleration with the inertia of the liquid results in the wave action. However, sufficient time must be available to allow the weaker gravity and surface tension forces to act. In Figure 11, where $B=19.32$, $W=3.41$, and $h_c=1$, the appearance of the wave occurs just prior to vapor ingestion, but in Figure 12, where B and W are identical but $h_c=2$, a full cycle of the oscillation has time to develop. (It is interesting to compare residuals for these two cases: In the latter slosh reorientation of liquid improves the draining efficiency and the residual is less than in the shallower case. Because the initial height determines the nature of the wave at vapor ingestion, residuals depend on starting height.)

In all computed cases observation of free surface slosh requires a Bond number of one or greater. For example, significant slosh is observed in case 137, where $B=10$, $W=0.3$, and $h_c=1$, and in preliminary calculations for a case where $B=100$, $W=3$, and $h_c=1$.

Generally, for cases where $h_c=1$, the concentration of streamlines close to the drain suppresses the rise of the wave at the centerline. This suppression

is expected as a result of the velocity term in the Bernoulli equation becoming important near the drain. However, in case C3, ($B=19.32$, $W=3.41$, and $h_c=2$), where the wave develops in the cylindrical section of the tank, interaction with the drain region is minimal because the streamlines deep in the tank are undistorted by the wave action.

The presence of waves on the free surface induced by draining has been revealed here for the studied highly curved free surfaces. A study of small amplitude waves on the free surface of a draining liquid is reported in [10] for the case of a flat-bottomed tank and a nearly flat free surface.

The presence of slosh induced by draining indicates that precautions may be required to control propellants during the period of time required to transfer liquids from one spacecraft to another under low gravity. Choice of Bond number and Weber number for the draining may be dependent on the need to maintain control of the liquid during vehicle attitude-control-system operation to prevent the slosh and control system frequency being too close in value.

Liquid Residuals

Table I summarizes the computed data for liquid residuals; Figure 14 also presents this information for $h_c=1$ and shows the variation of residuals with the Bond and Weber numbers. As indicated by Figure 14, the volume of liquid remaining in the tank is substantial for the ranges of Bond and Weber number covered by this survey. The lack of residual data for small Weber numbers limits the application of the results for design purposes.

The influence of the parameter, h_c , on residuals is greatest for large Weber numbers and low Bond numbers. This is due to the additional liquid found in the wall sheets (see Figures 4 and 5); however, in case 20, where $W=1$, $B=0$, and $h_c=2$, the wall sheet becomes quite thin by the time the free surface at the centerline reaches $z=1$, which suggests the residual is only slightly greater than that for case 5, where $W=1$, $B=0$, and $h_c=1$. Aside from the impact of draining very rapidly, which allows insufficient time for the capillary-gravity forces to act, the shape of the streamlines in the cylindrical

section suggest that efficient draining is the case when the free surface is in this part of the tank. One observes from Figures 4 and 5, for example, that the streamlines are essentially parallel, in the cylindrical section except near the free surface. Because concentration of streamlines is confined to the hemispherical section, even for $W=100$, h_c should have less significance when greater than one.

The data in Table I shows that drain size does not significantly affect liquid residuals for low Bond numbers and high Weber numbers. For example, both at $B=0$ (cases 7-9 and 13-15 in Table I) and at $B=3$ (cases 97-99 and 103-105), the variation of the residuals with Weber number from 3 to 100 does not exceed 4%. Contributing to this variation is the change in initial volume with drain radius (less than 0.5% for $h_c=1$) and the error in extrapolating plots of centerline height vs. time to vapor ingestion. For these cases the influence of drain radius is small.

The data for large Weber numbers and low Bond numbers result from vapor ingestion occurring while the free surface is high in the tank. For smaller Weber numbers drain size may be more important because the interface reaches a lower level in the tank before vapor ingestion. Intuitively, the tank should drain for a significant period of time with the free surface near the drain before any parametric spread of the residual data with r_o can be expected to occur.

The magnitude of liquid residuals is strongly dependent on Bond number because the initial volume for a given value of h_c is largely determined by this parameter. In retrospect, volume instead of h_c as a parameter would have suppressed much of the variation of residuals with Bond number.

Figure 14 shows only a slight increase in residuals with Weber number for Weber number greater than one at each value of Bond number; a change in slope in the curves near $W=1$ is apparent and expected because surface tension and/or gravity forces can have time to reorient liquid before vapor ingestion. To clarify this point an examination of specific cases is useful. For large Weber numbers and low Bond numbers the streamlines are dominated by the drain

which causes the interface in the vicinity of the tank axis to be brought to vapor ingestion before any action of surface tension can help appreciably in draining additional liquid (see Figures 4-6). This feature can be compared with the results in Figure 7, where $B=0$, and $W=1$, and Figure 8, where $B=1$ and $W=0.3$. In these two cases, and particularly in the latter, the streamlines gradually undergo a transition from a configuration suggesting domination by the sink (initial transient) to the configuration of Figure 8(g) and (h) where the streamlines cant sharply to the right near the free surface indicating the influence of capillary and/or gravity forces. Enough time appears to be available before vapor ingestion in the case of low values of Weber number to allow the weaker forces of surface tension and gravity (low Bond number) to restore liquid to the centerline region thus replacing the drained liquid that comes primarily from this region during the initial transient. The location of streamlines in Figure 8(g) and (h) show a large percentage of the draining liquid coming from the vicinity of the wall. Examination of Figure 8(j) shows that the centerline point has slowed considerably for a significant portion of the drain time, while the wall point has finally begun to accelerate downward. These features indicate that the capillary and gravity forces now have sufficient dynamical influence to delay significantly vapor ingestion.

In those cases, such as case C3 ($B=19.32$, $W=3.41$, $h_c=2$, and $r_o=1/10$), Figure 12, and case 137 ($B=10$, $W=0.3$, $h_c=1$, and $r_o=1/30$), the Bond number is large enough to cause visible slosh. As discussed in the section on wave motion, the gravity dominated wave is triggered by the nonuniform downward acceleration of the free surface during the initial transient. Comparison of the actual magnitude of the residuals for case C3 and B3 (same as C3, except $h_c=1$) shows the deeper case to have lower residuals. The difference is due to the reorientation by slosh. Figure 11 shows results for case B3 and Figure 12 for case C3. In the deeper case (C3) the slosh develops and undergoes a complete cycle during the course of draining whereas the shallower case (B3) has only time for the slosh to start building before vapor ingestion. The flow from the wall in C3 replenishes the centerline region thereby delaying vapor ingestion and reducing residuals.

Because the liquid volume remaining in a tank following vapor ingestion may ultimately become a weight penalty for a spacecraft [25] , a method for predicting residuals for arbitrary values of the pertinent parameters is desirable. As pointed out in the Introduction, a method is available for high Bond number applications (see [1-3]). These papers predict the height of the essentially flat interface corresponding to the incipience of vapor ingestion. Because the interface is flat, residuals are easily calculated using the predicted value of the critical height.

The complex shape of the interface for draining in low gravity does not allow a direct determination of residuals by knowing only critical height at the centerline. However, because residuals can be calculated using the expression, $V_r = V_i - 3/2 t_{vi}$, a method of correlating suitably normalized vapor ingestion times could serve the desired purpose. At first glance this appears equivalent to what is shown in Figure 14 for V_r ; however, a parameter is introduced in place of the Weber number and Bond number which (i) interpolates over the Bond number range from zero to infinity, and (ii) suppresses as much as possible the parametric variation of t_{vi} with Bond number. The parameter, $W/(1+B)$, introduced in [25] serves this purpose because it reduces to W when the Bond number decreases to zero and to W/B , the Froude number, when the Bond number increases to infinity. (The Froude number is the basic parameter for high Bond number draining correlations [1-3] .) The computed vapor ingestion times given in Table I for $h_c=1$ and $h_c=2$ (also cases E3 and 44 for completeness) are plotted as functions of $W/(1+B)$ in Figure 15; the curves indicate only minor parametric dependence on Bond number. In view of the sparseness of data for $W/(1+B)$ less than one, the correlation suggested by Figure 15 is tentative pending acquisition of additional data.

SUMMARY OF RESULTS

The low gravity draining of an inviscid, incompressible, and irrotational liquid from a cylindrical tank with a hemispherical bottom has been studied by formulating and solving numerically the appropriate initial boundary-value problem for the velocity potential. A new numerical method for fluid dynamics problems was developed for use in this study. The essential features of the method are:

1. The motion of the free surface, from an initial equilibrium shape, is followed by numerically integrating a system of first order differential equations to determine the coordinates of a set of points and the potential thereon.
2. Both the free surface and the potential thereon are interpolated by cubic splines to obtain smooth estimates of the surface tension term in the Bernoulli equation.
3. The potential of an infinite circular cylinder with the same drain condition is used as an auxiliary potential to transfer, in essence, the effect of the drain to the free surface.
4. The mixed boundary value problem satisfied by the residual potential (real less auxiliary) is approximated on an irregular triangular mesh which is revised from time to time to accommodate the motion of the free surface.
5. The solution of the linear system for the residual potential is confined to the mesh in the neighborhood of the moving free surface.

Having studied tank draining by repeated exercise of the program for several values of the important problem parameters; namely, Bond number, Weber number, initial fill level, and drain size, the following conclusions are reached:

1. Liquid residuals increase with the Weber number and decrease with the Bond number.

2. To achieve low residuals, drain times must be large enough to permit surface tension and gravity forces to reorient liquid from the region near the wall to the centerline.
3. Axisymmetric slosh can be induced by draining for certain ranges of the Bond number and Weber number.
4. Drain size does not influence residuals at high Weber numbers, but no conclusion has been reached for low Weber numbers.
5. Vapor ingestion times tend to correlate with a particular combination of the Bond number B and the Weber number W ; namely, $W/(1+B)$, for wide ranges of B and W .

Favorable comparison of computed results with experiments support the assumptions made in the problem formulation and the solution method. The computer code, once completely developed, has potential as a good simulation tool for this problem. Possible extensions to other axisymmetric problems, such as those of reorientation and sloshing, appear to be promising.

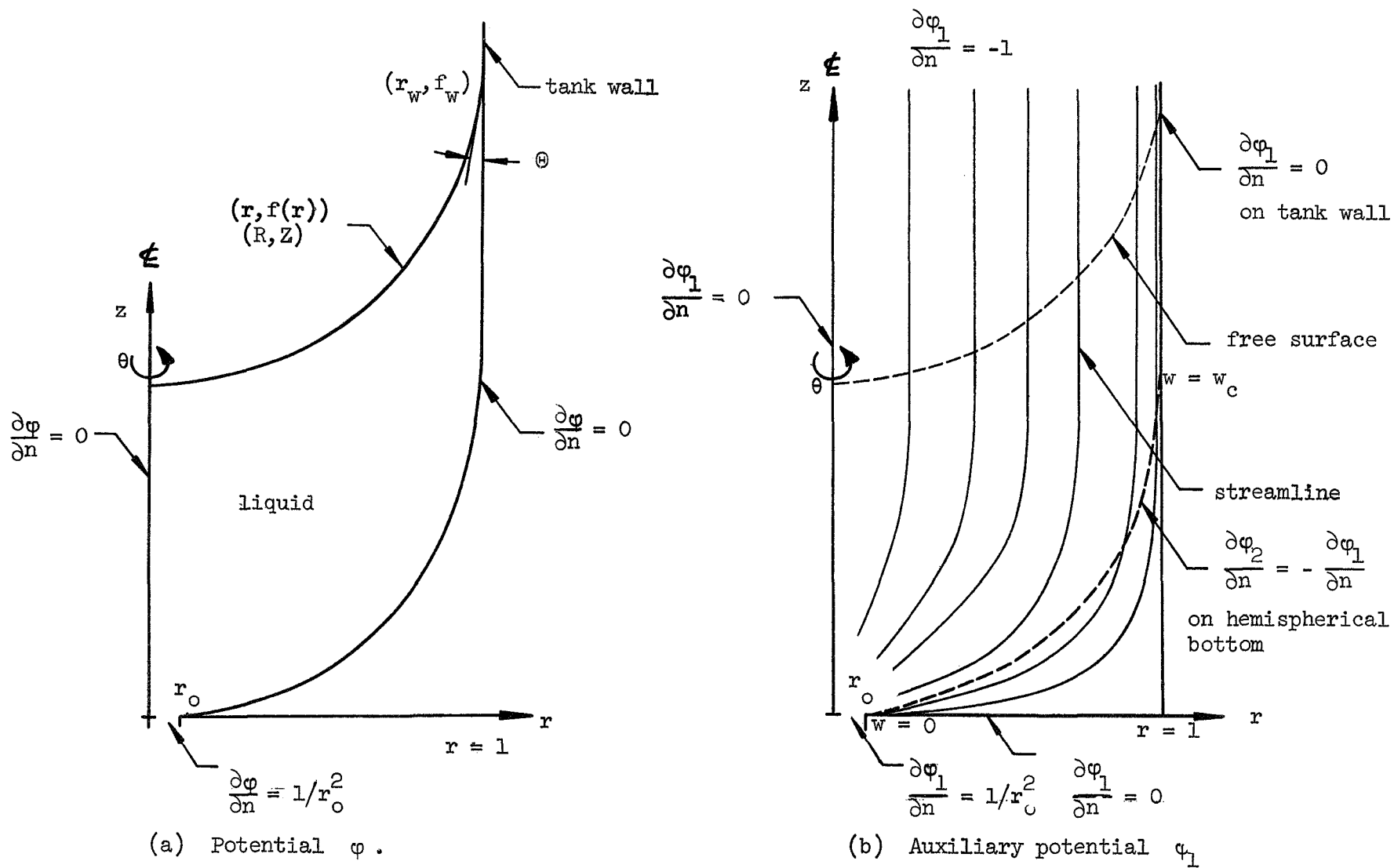


Figure 1. Definition of geometry and coordinate system for the draining problem.

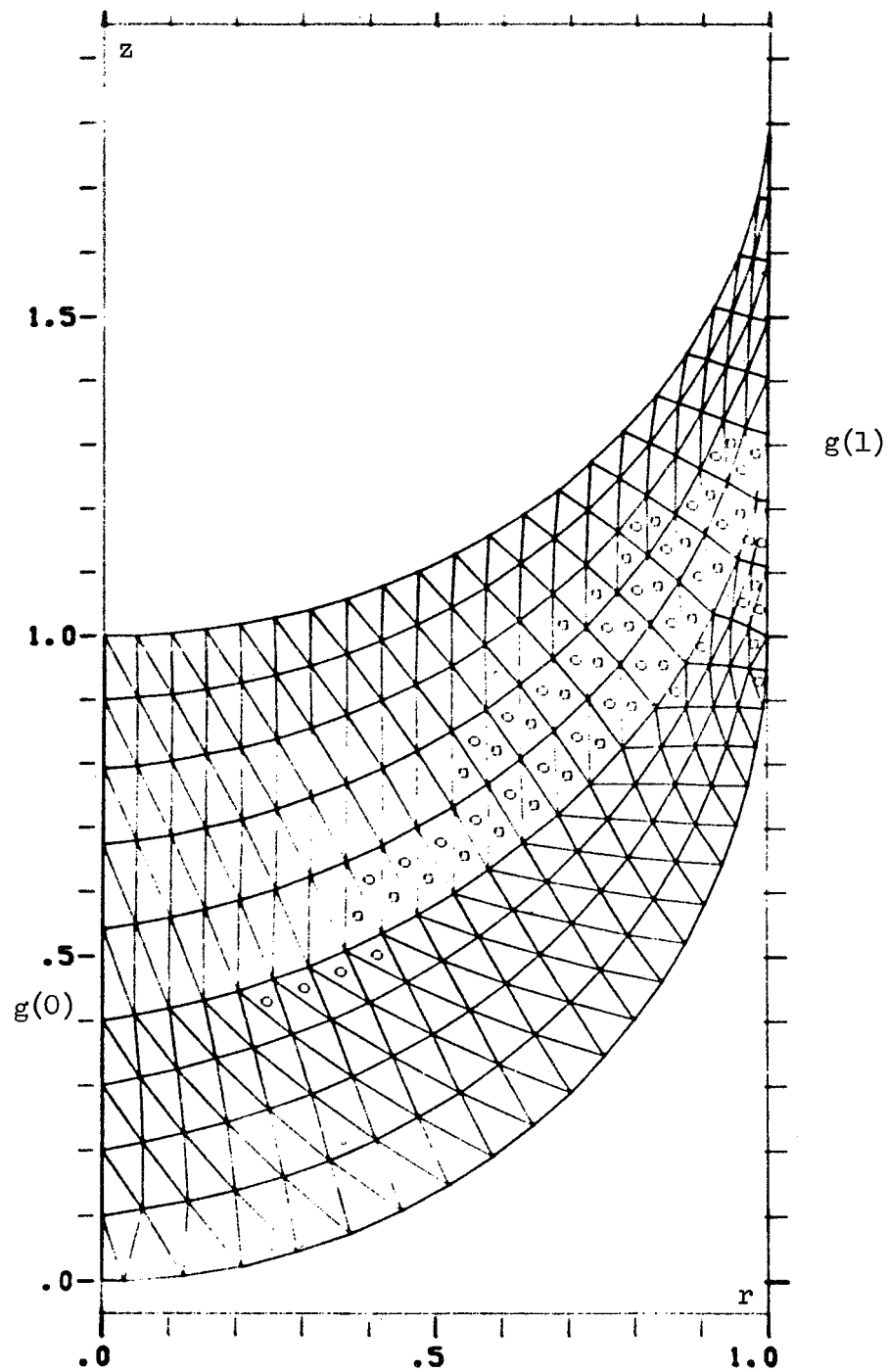
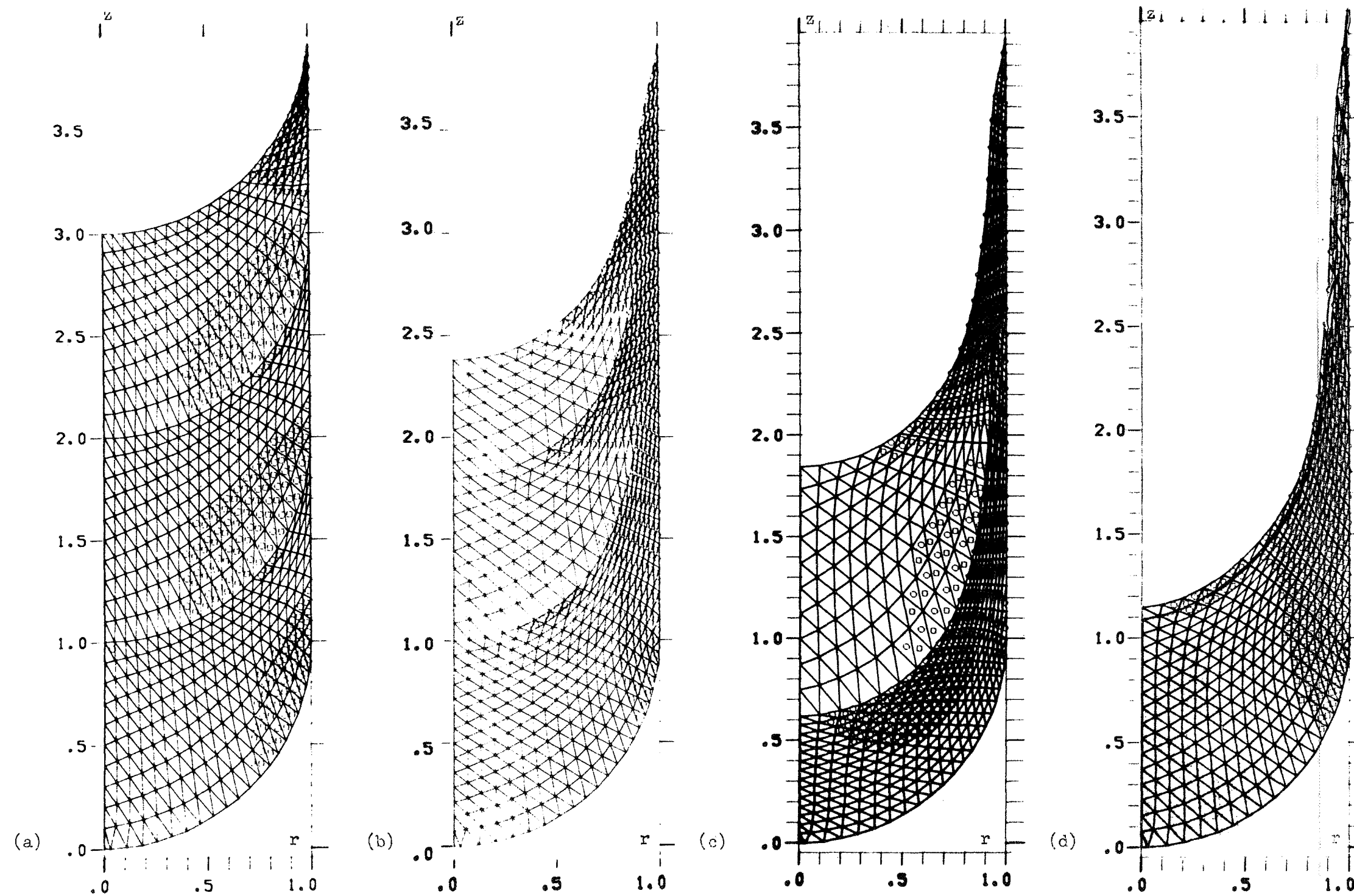
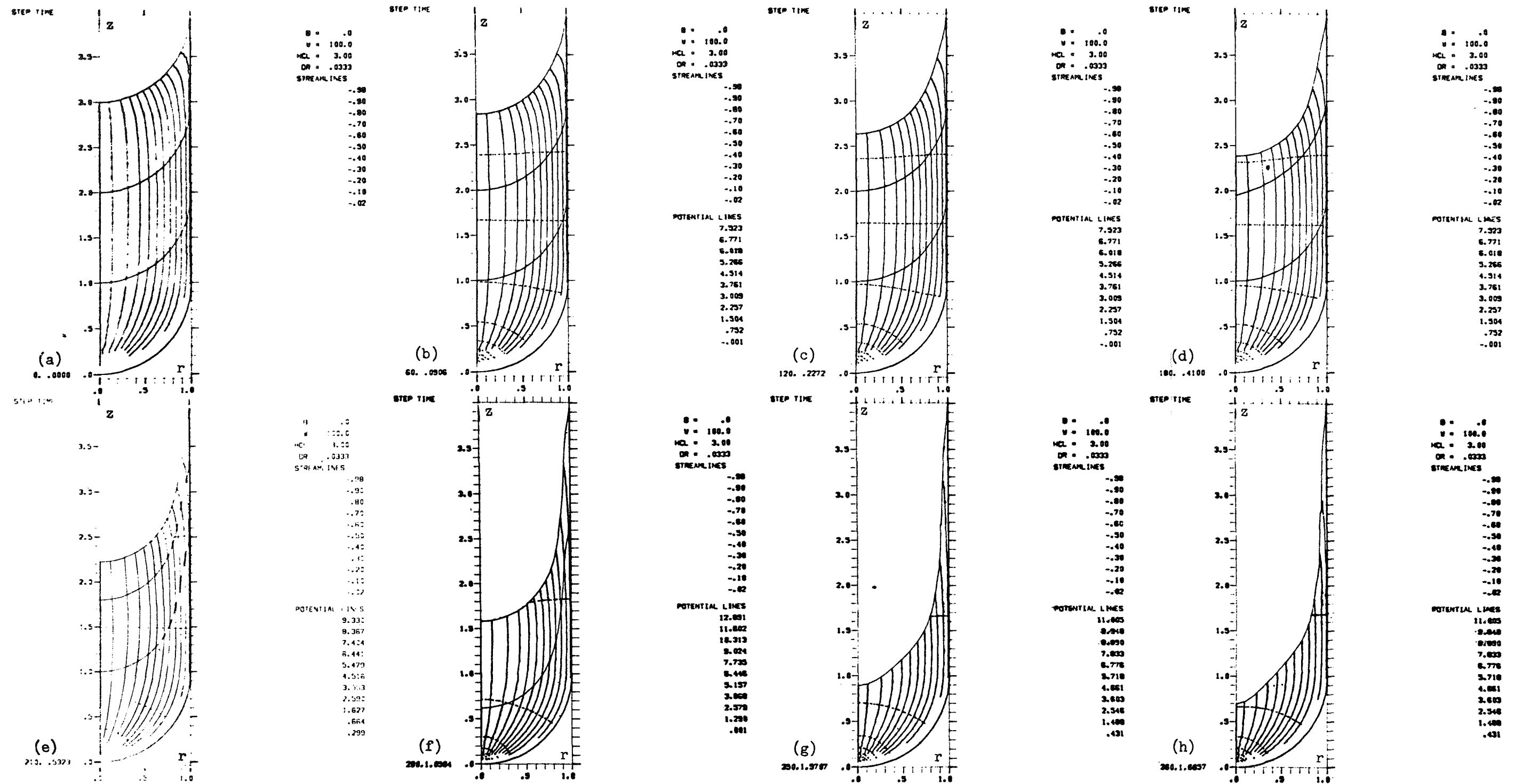


Figure 2. Example of the irregular, triangular mesh; mesh used for computing case 47: $B=0$, $W=0.3$, $h=1$, $r=1/30$. (An interior mesh control line runs between $g(0)$ and $g(1)$).



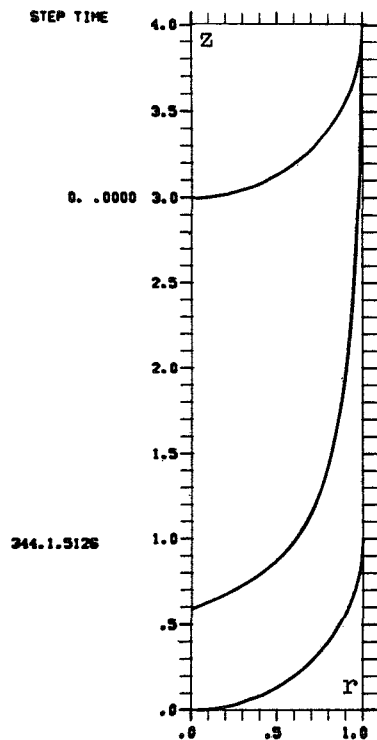
$B = .0$
 $W = 100.0$
 $HCL = 3.00$
 $OR = .0333$

Figure 3. Initial mesh and the three restart meshes used in computing case 44: $B=0$, $W=100$, $h_c=3$, $r_0=1/30$. (Multiple interior lines for mesh control are used in the deep fill case).

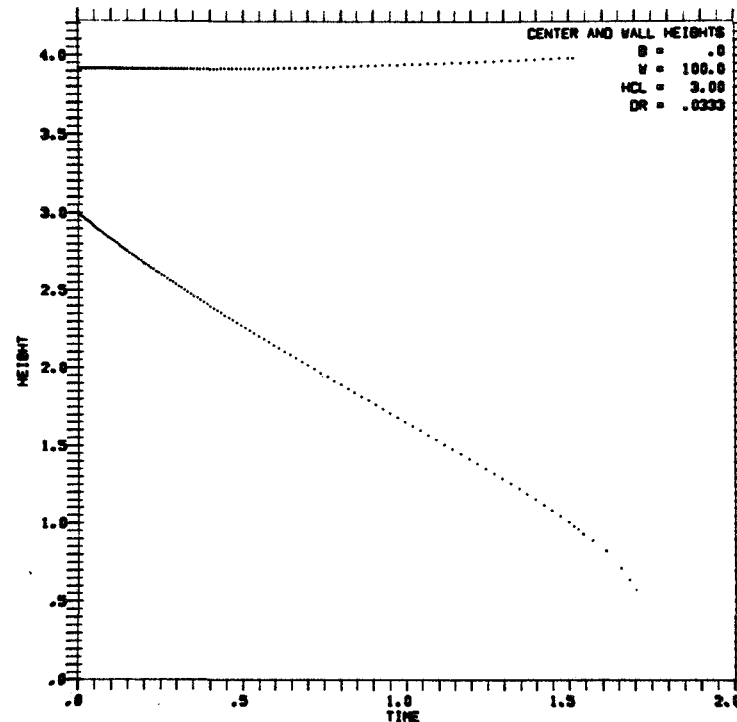


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 4. Draining characteristics computed for case 44: $B=0$, $W=100$, $h=3$, $r=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).



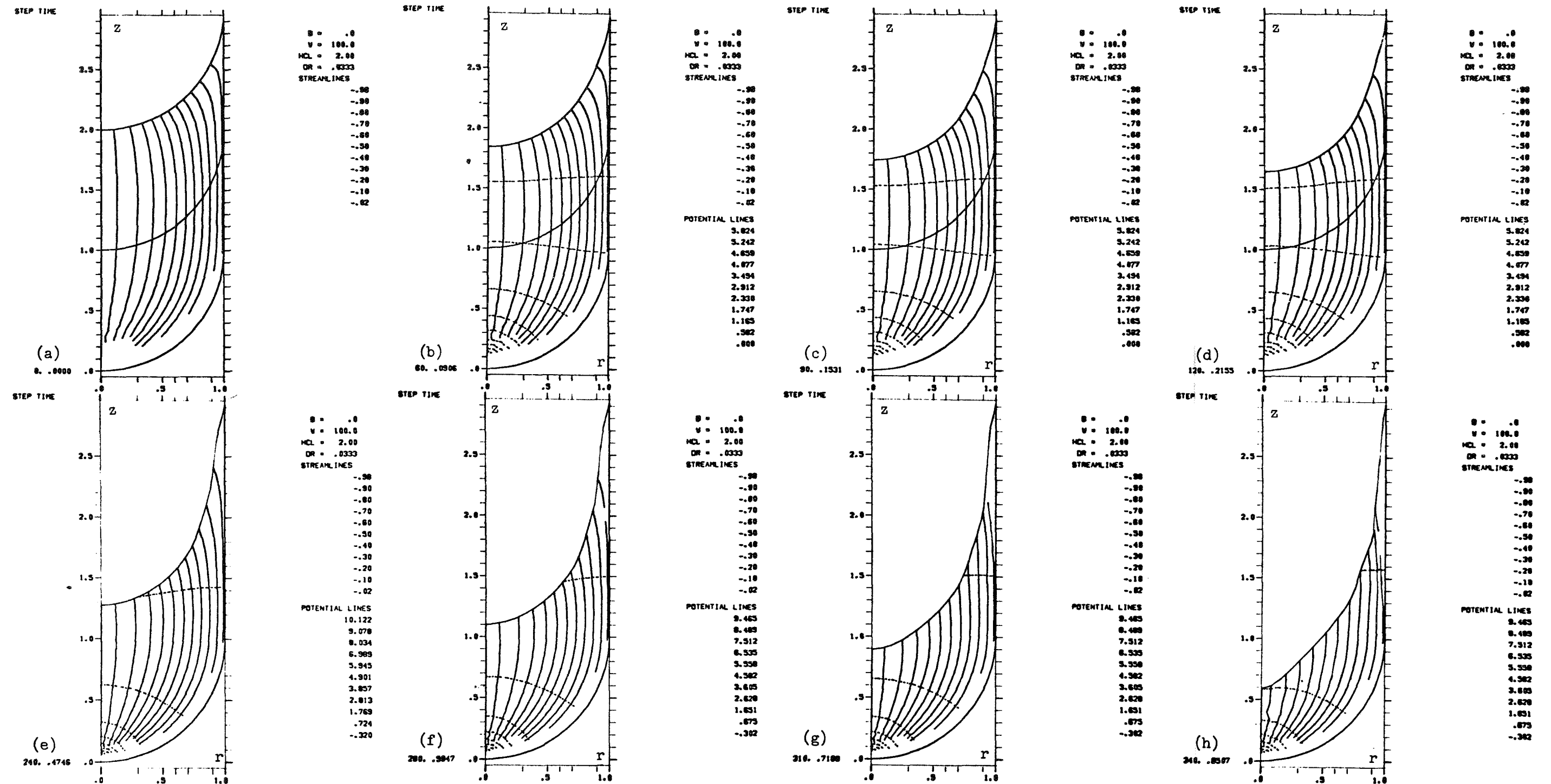
B = .0
V = 100.0
HCL = 3.00
DR = .0333



(i) Initial and last computed free surfaces.

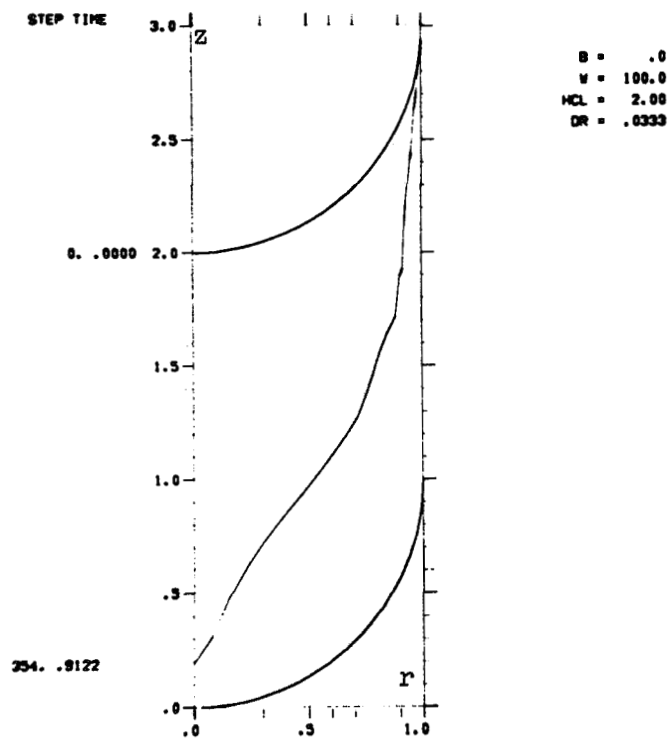
(j) Position of free surface points at the tank centerline and wall during draining.

Figure 4 (cont). Draining characteristics computed for case 44: $B=0$, $W=100$, $h_c=3$, $r_o=1/30$.

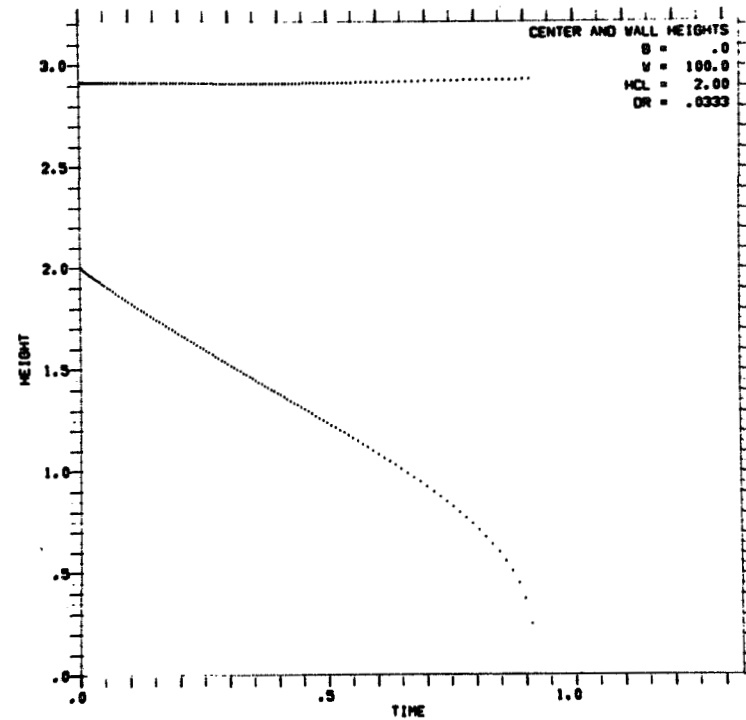


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 5. Draining characteristics computed for case 29: $B=0$, $W=100$, $h=2$, $r_0=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).

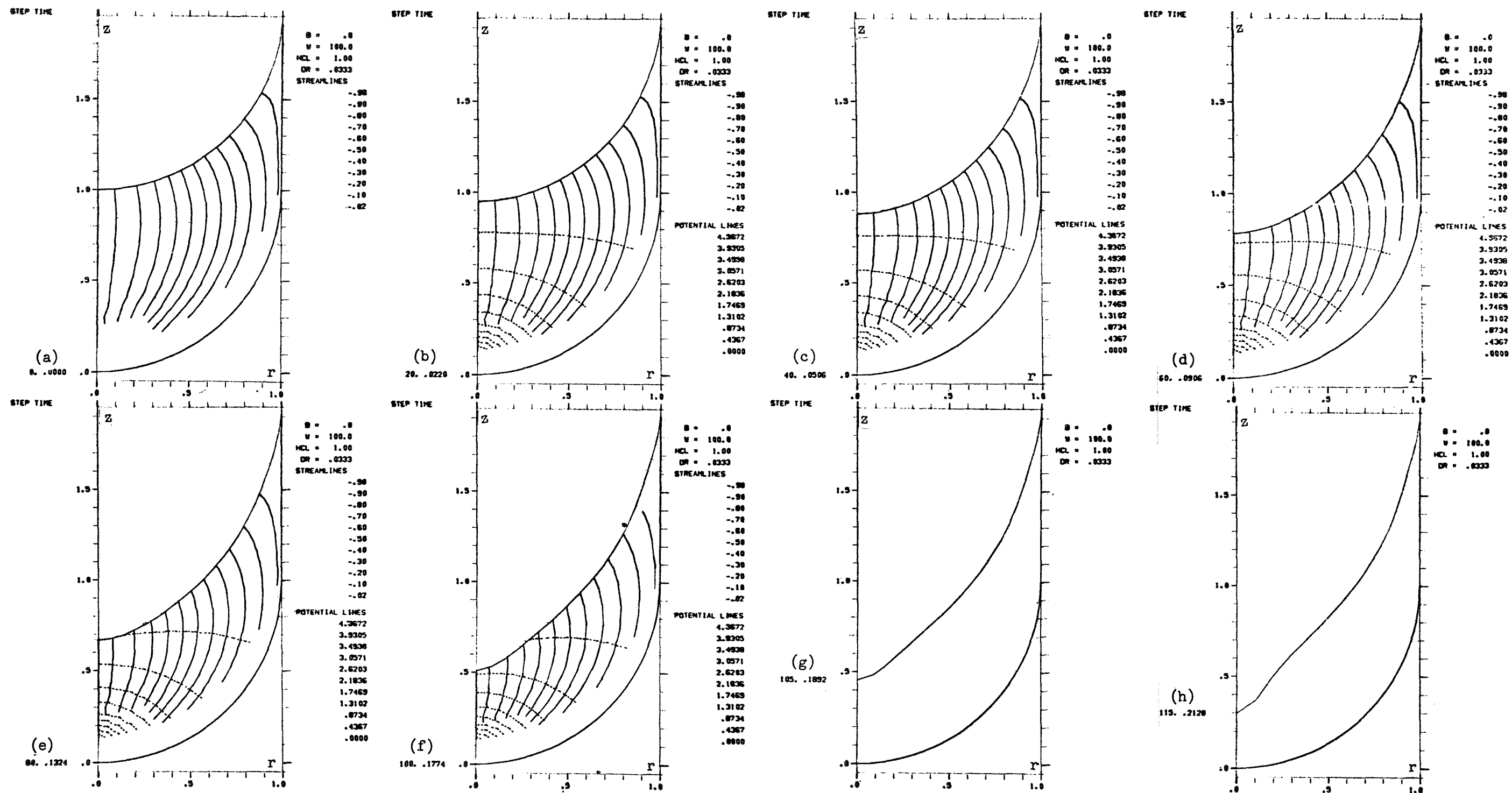


(i) Initial and last computed free surfaces.



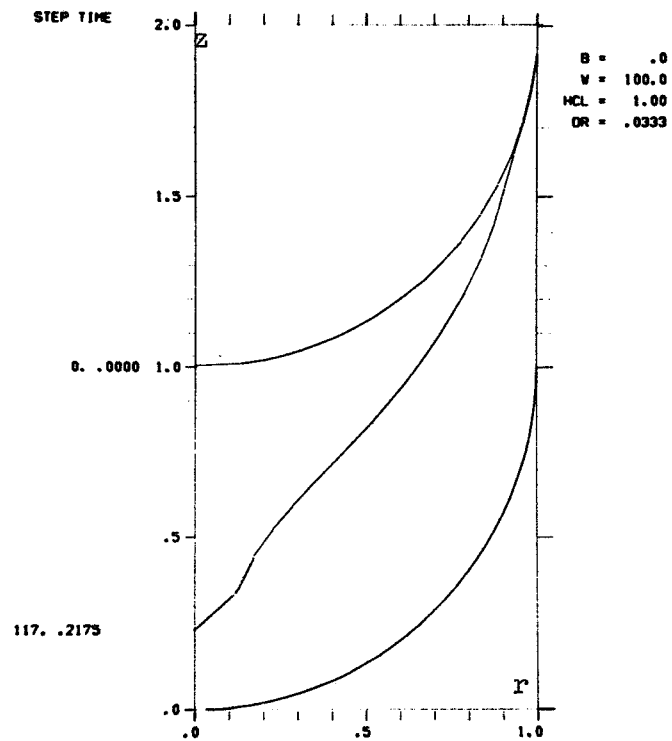
(j) Position of free surface points at the tank centerline and wall during draining.

Figure 5. Draining characteristics computed for case 29: $B=0$, $W=100$, $h_c=2$, $r_o=1/30$.

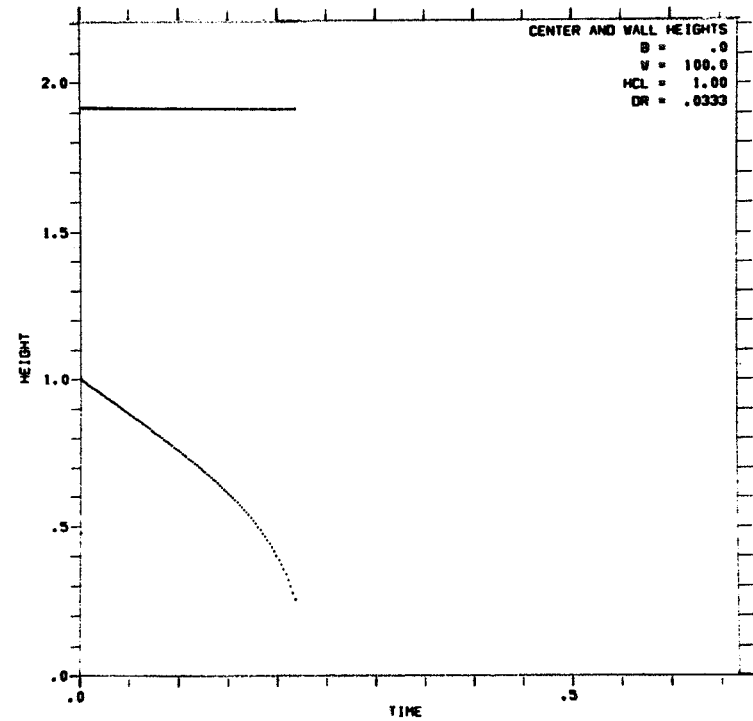


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 6. Draining characteristics computed for case 14: $B=0$, $W=100$, $h=1$, $r=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).

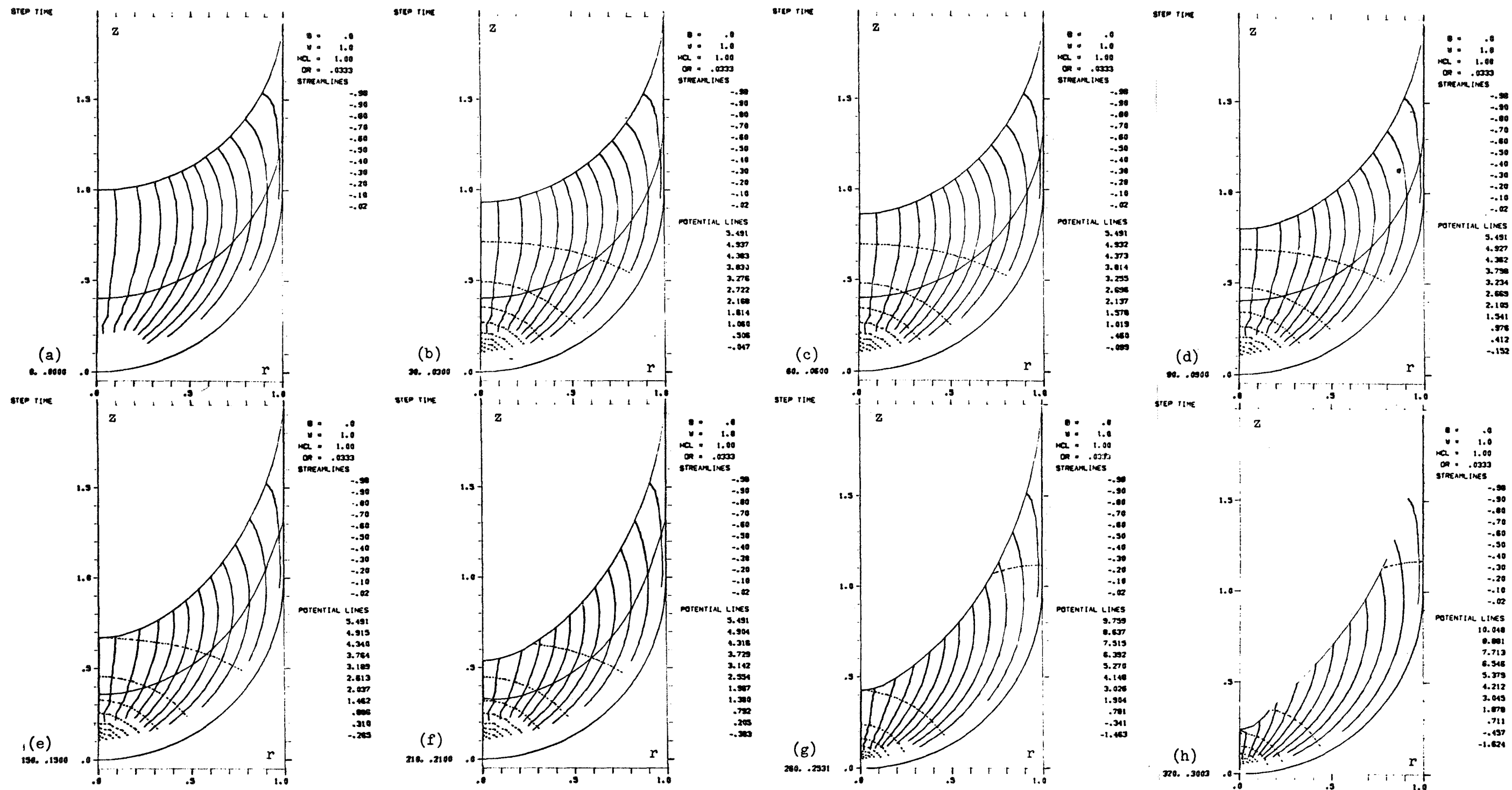


(i) Initial and last computed free surfaces.



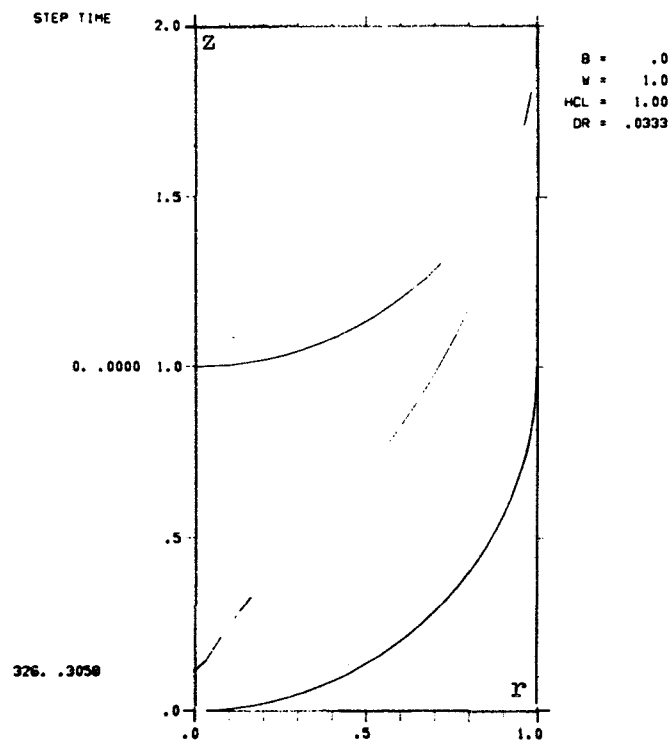
(j) Position of free surface points at the tank centerline and wall during draining.

Figure 6. Draining characteristics computed for case 14: $B=0$, $W=100$, $h_c=1$, $r_o = 1/30$.

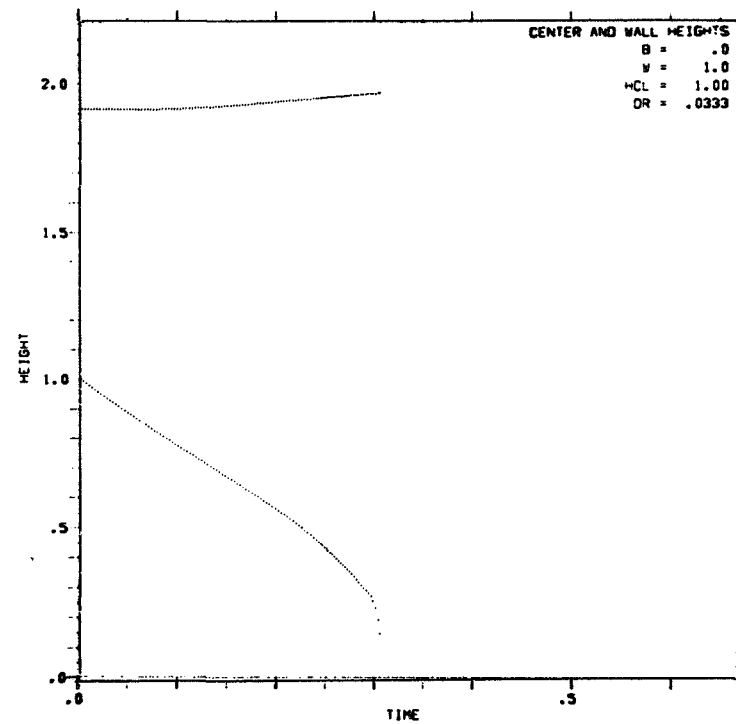


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 7. Draining characteristics computed for case 5: $B=0$, $W=1$, $h=1$, $r=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).

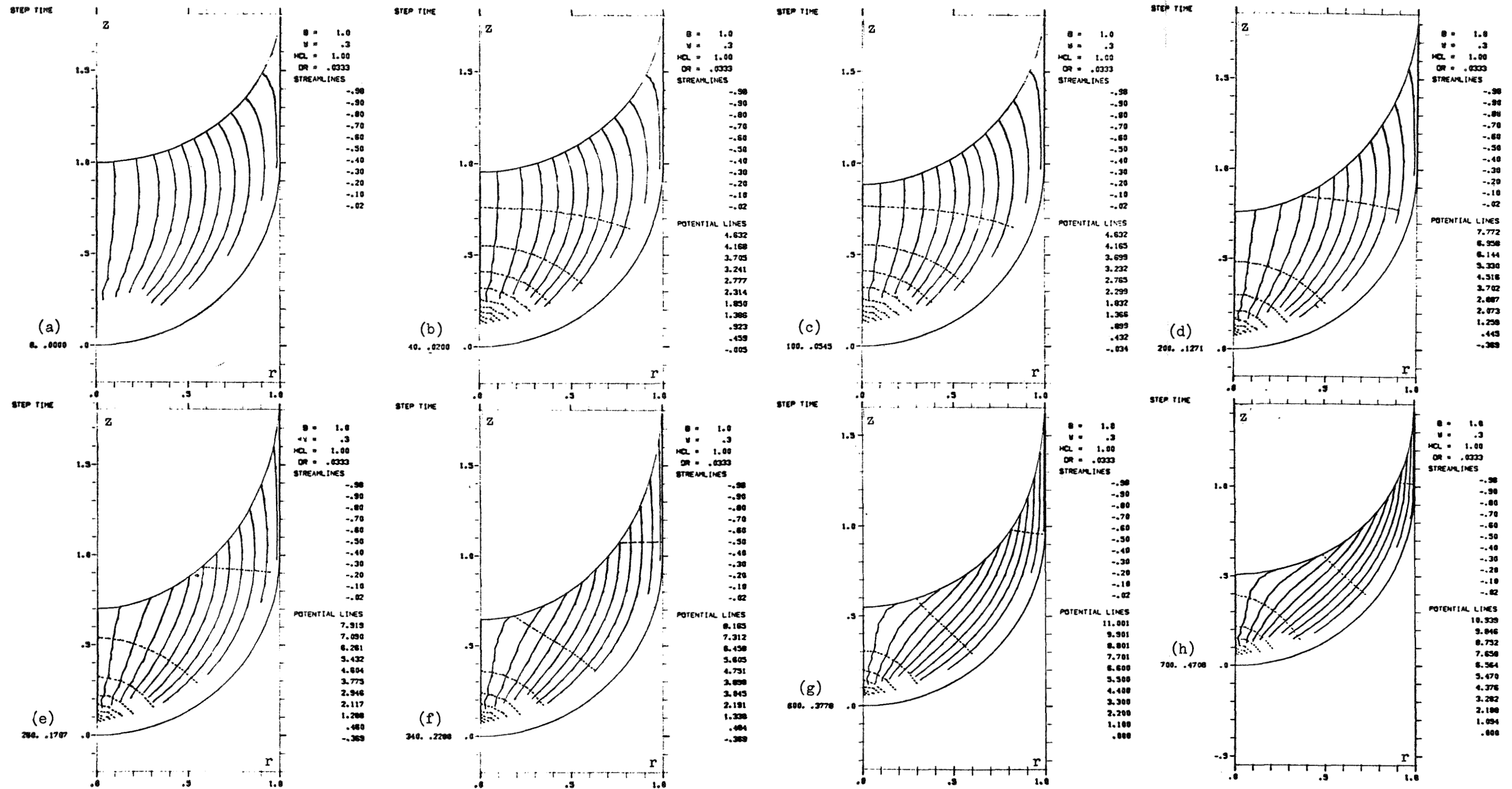


(i) Initial and last computed free surfaces.



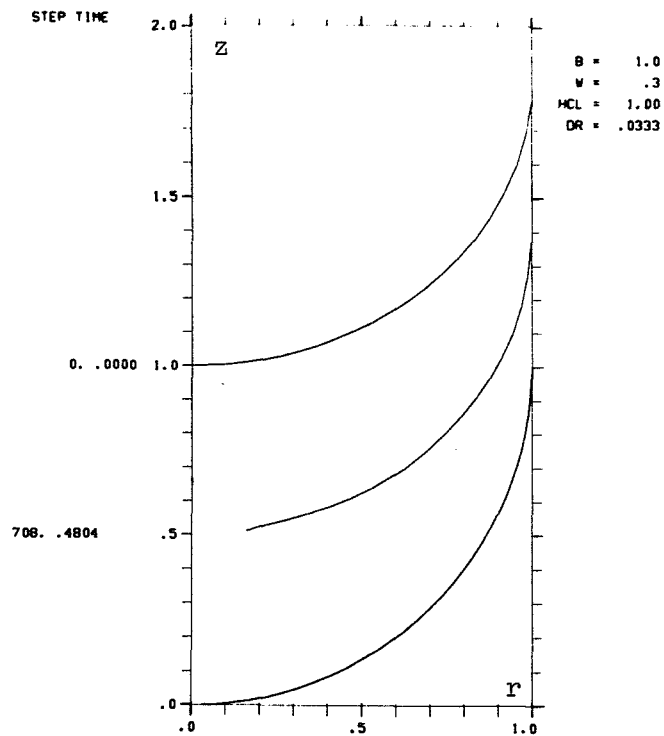
(j) Position of free surface points at the tank centerline and wall during draining.

Figure 7. Draining characteristics computed for case 5: $B=0$, $W=1$, $h_c=1$, $r_o=1/30$.

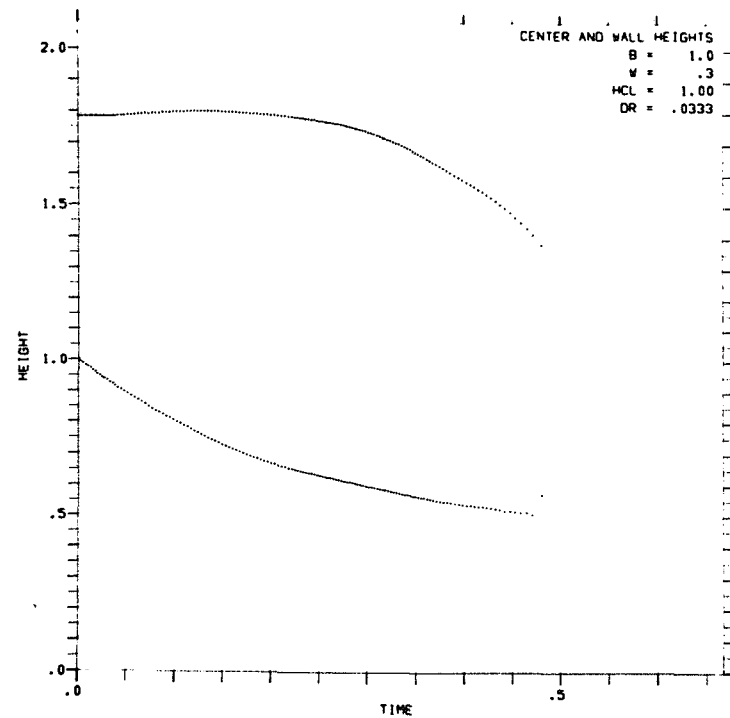


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

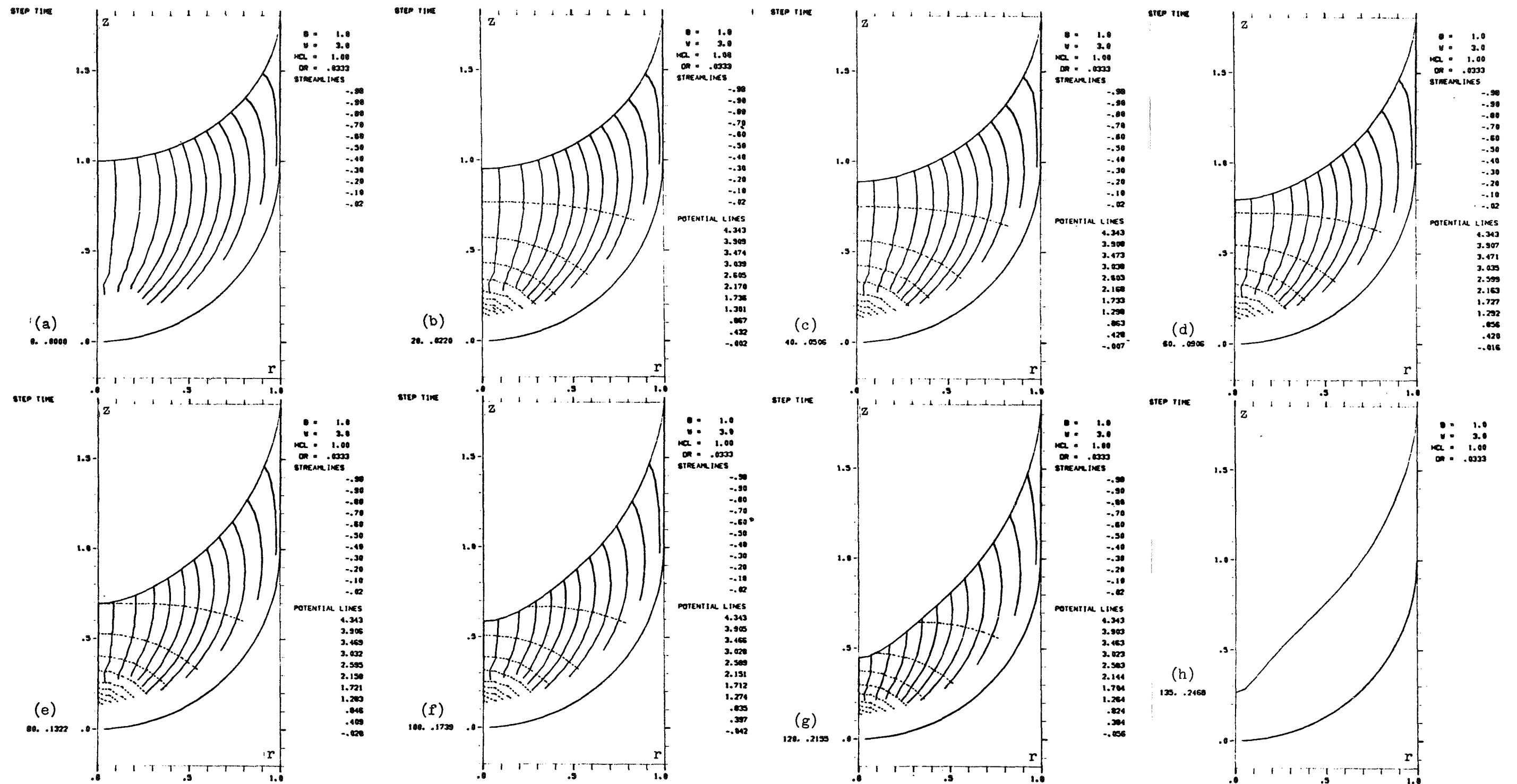
Figure 8. Draining characteristics computed for case 47: $B=1$, $W=0.3$, $h=1$, $r=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).



(i) Initial and last computed free surfaces. (Numerical instability toward the end of the run caused the gap at the centerline on step 708)

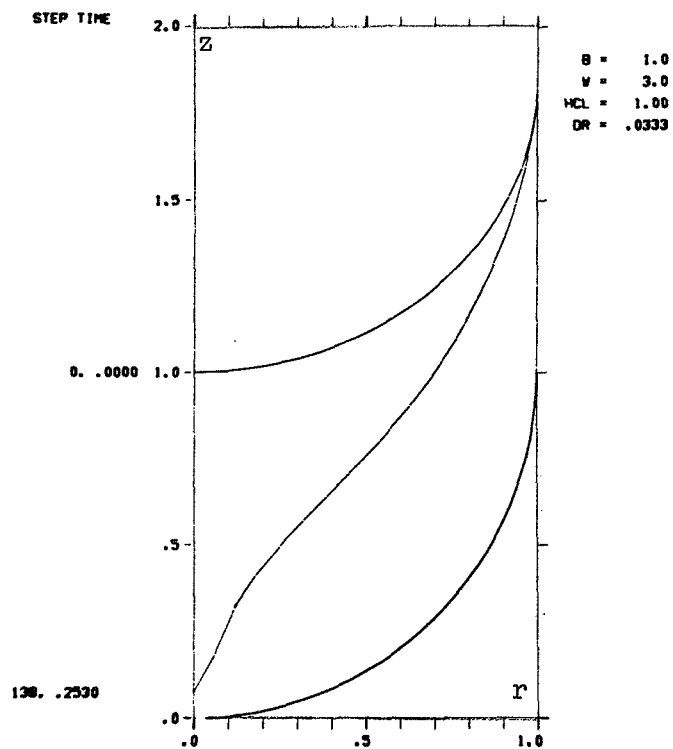


(j) Position of free surface points at the tank centerline and wall during draining.

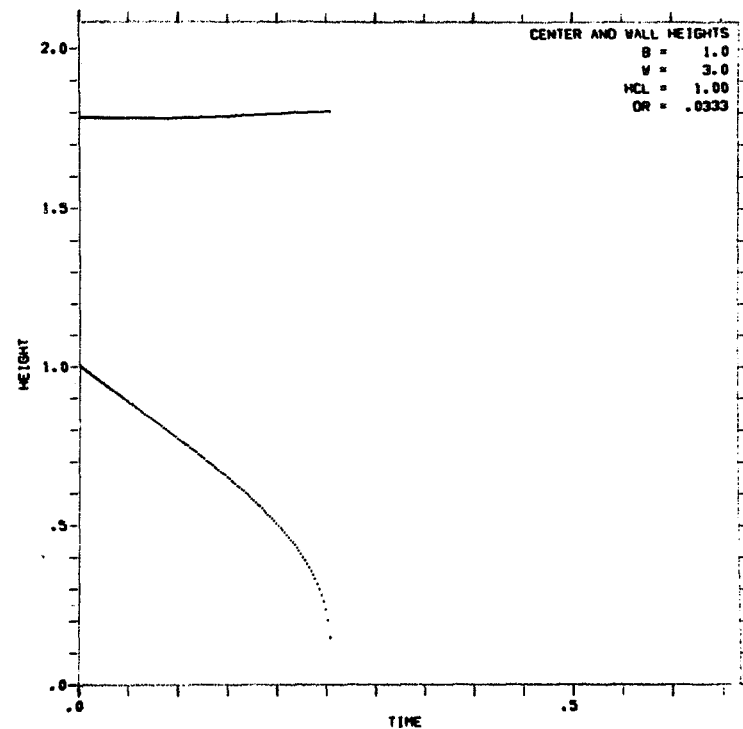


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 9. Draining characteristics computed for case 53: $B=1$, $W=3$, $h=1$, $r=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).

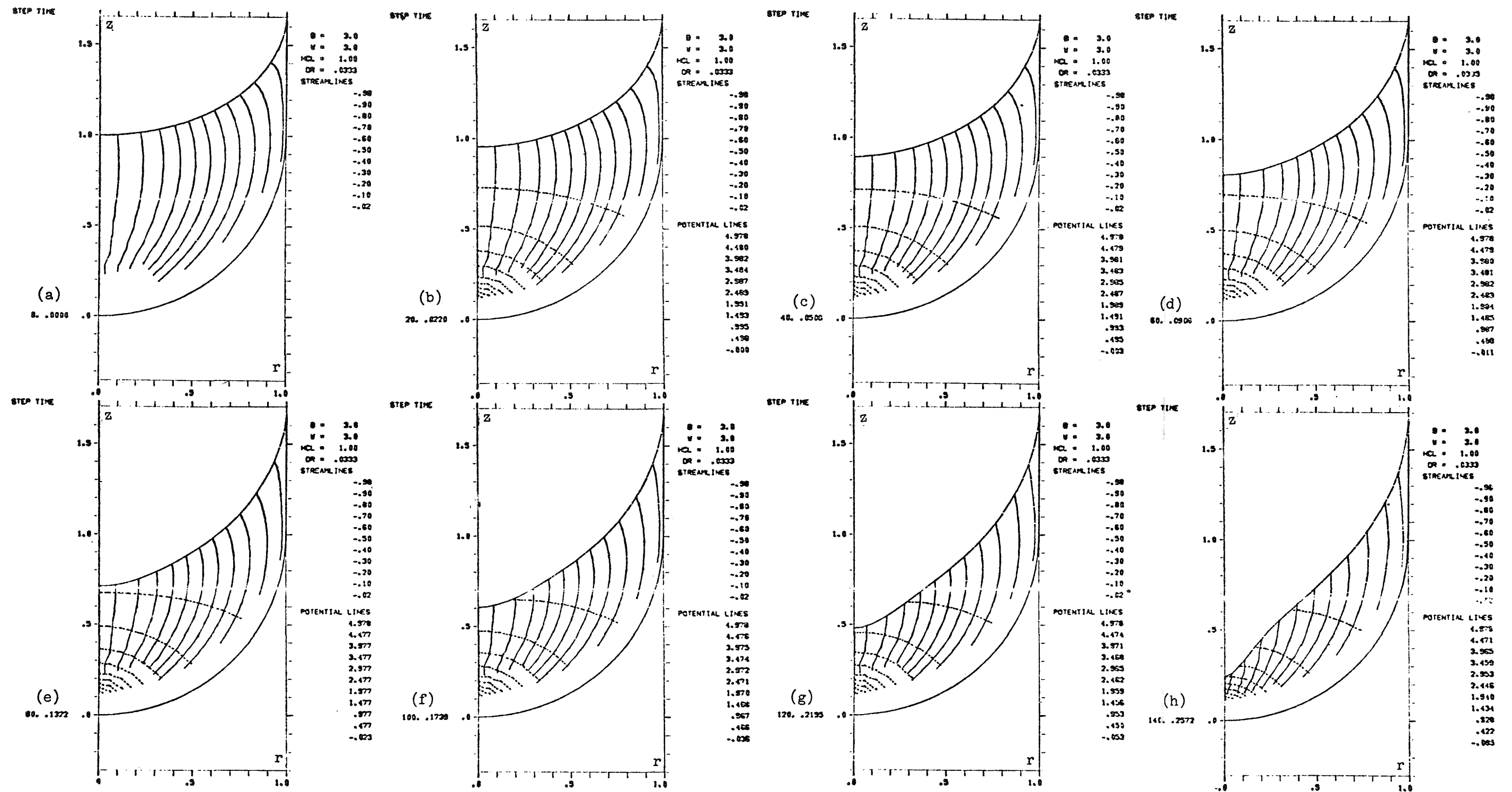


(i) Initial and last computed free surfaces.



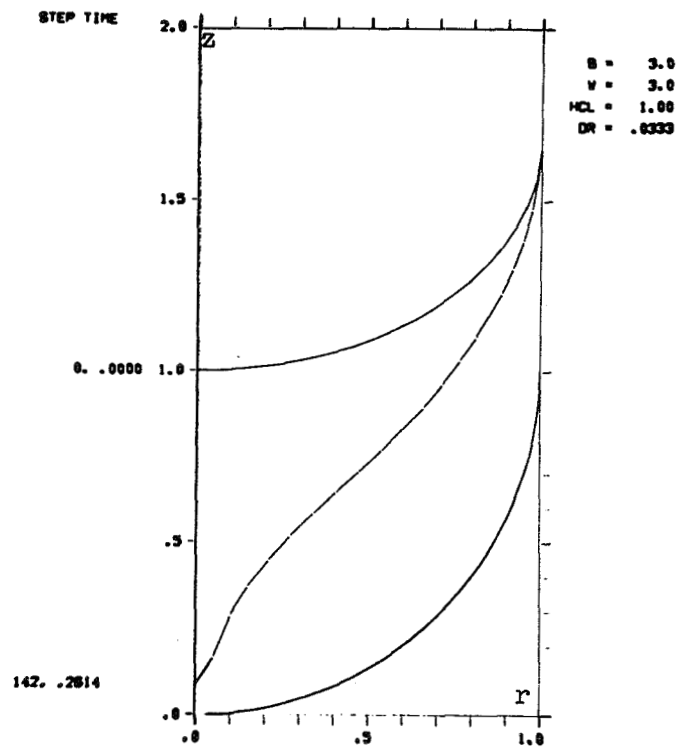
(j) Position of free surface points at the tank centerline and wall during draining.

Figure 9. Draining characteristics computed for case 53: $B=1$, $W=3$, $h_c=1$, $r_o=1/30$.

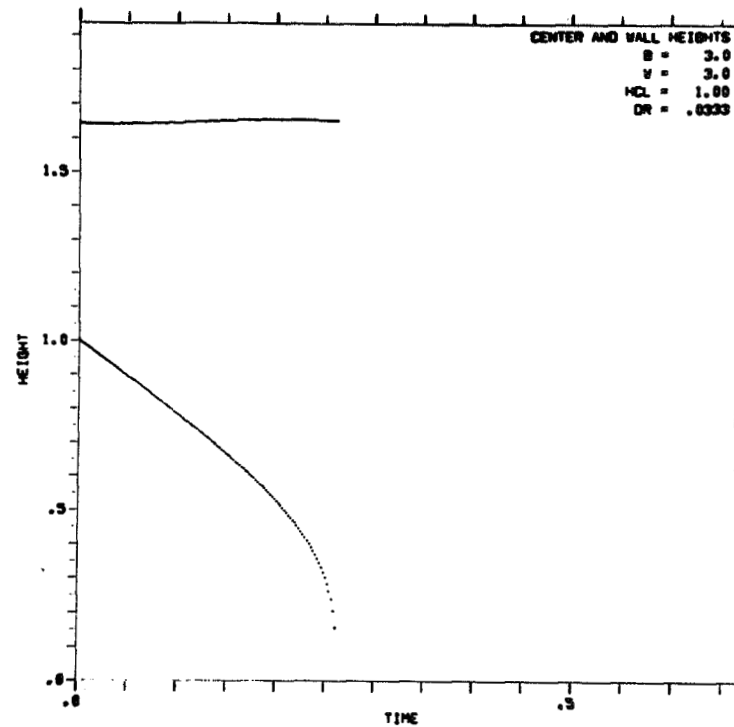


(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 10. Draining characteristics computed for case 98: $B=3$, $W=3$, $h=1$, $r=1/30$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).

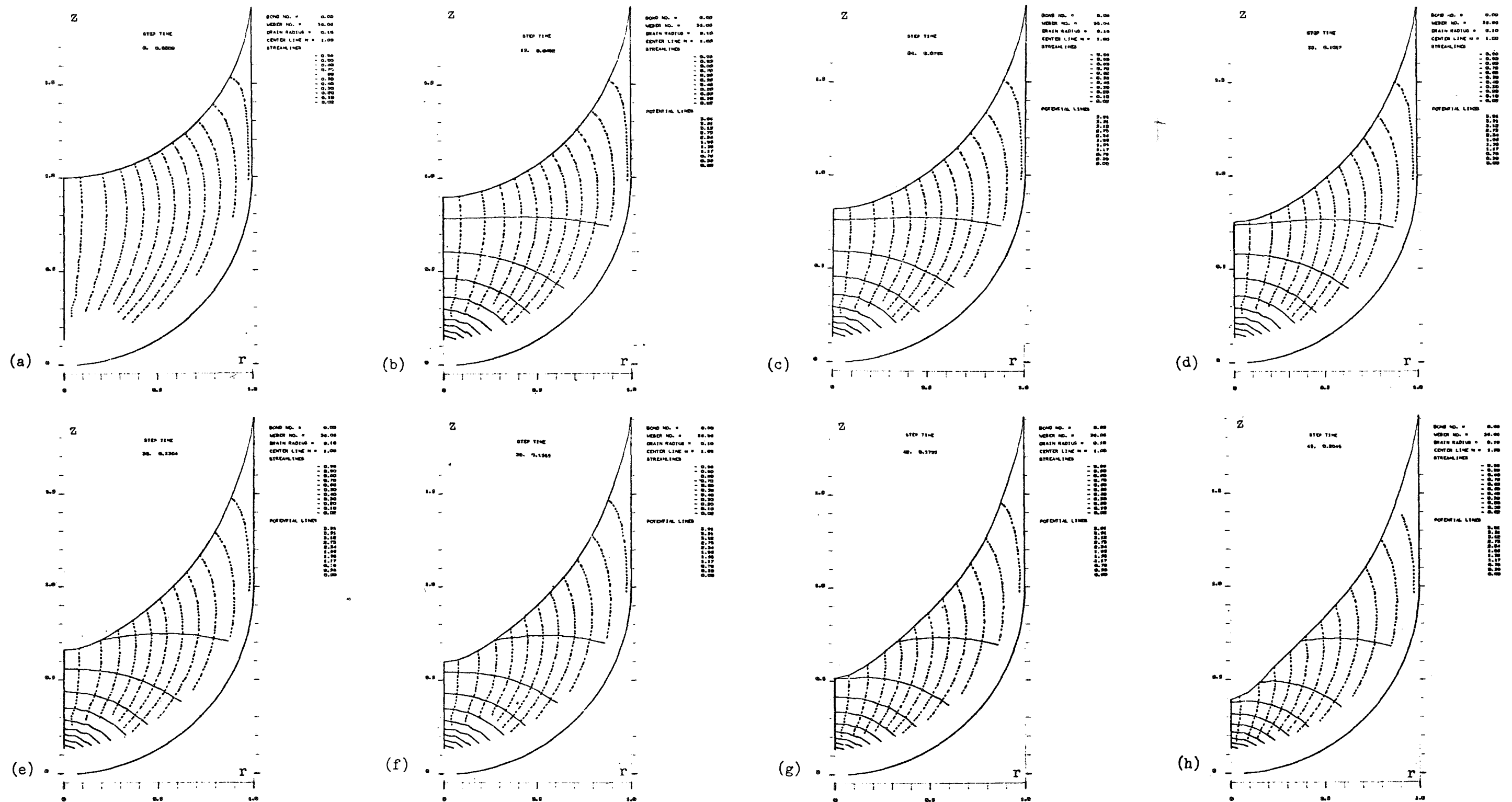


(i) Initial and last computed free surfaces.



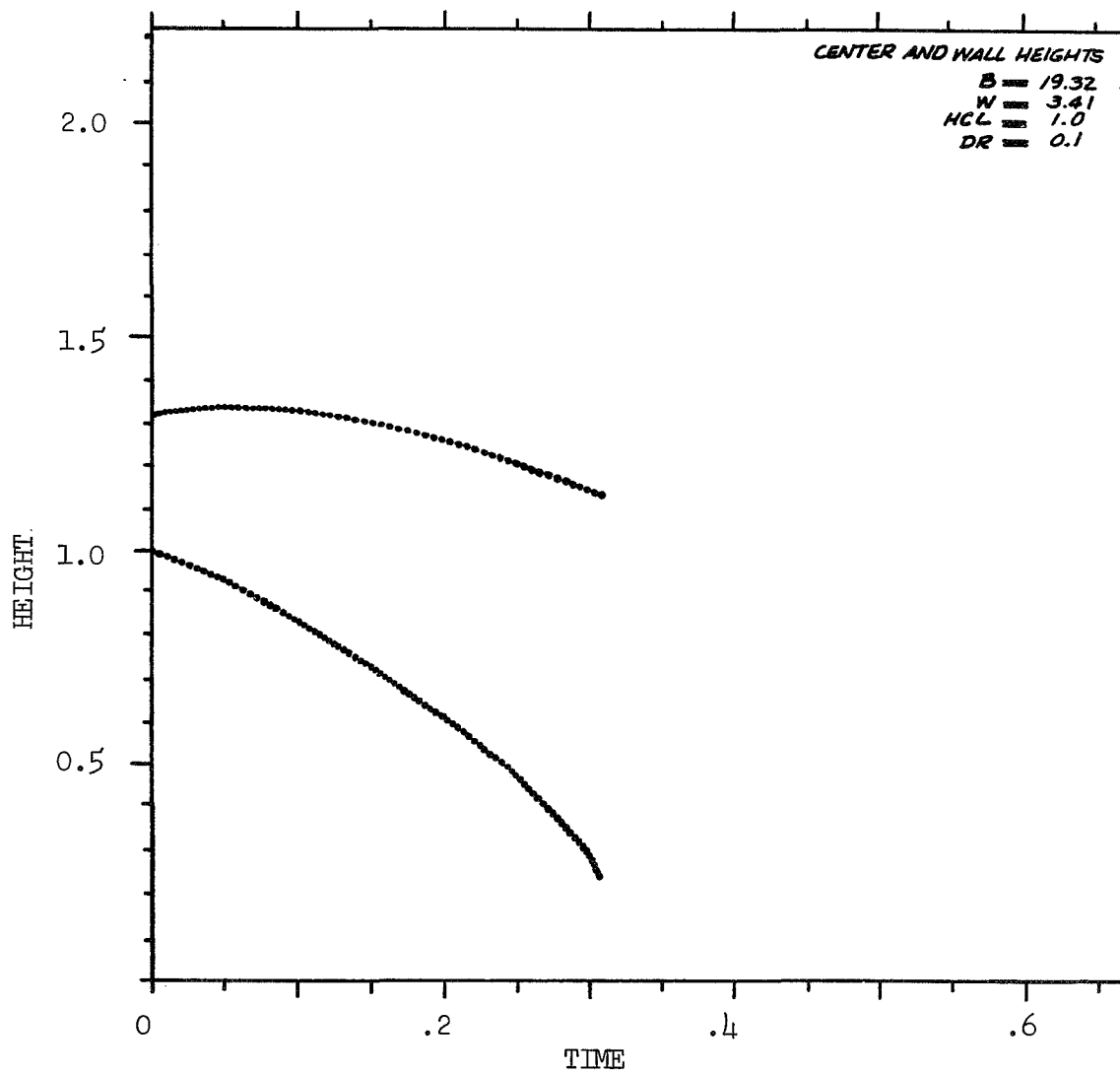
(j) Position of free surface points at the tank centerline and wall during draining.

Figure 10. Draining characteristics computed for case 98: $B=3$, $W=3$, $h_c=1$, $r_o=1/30$.



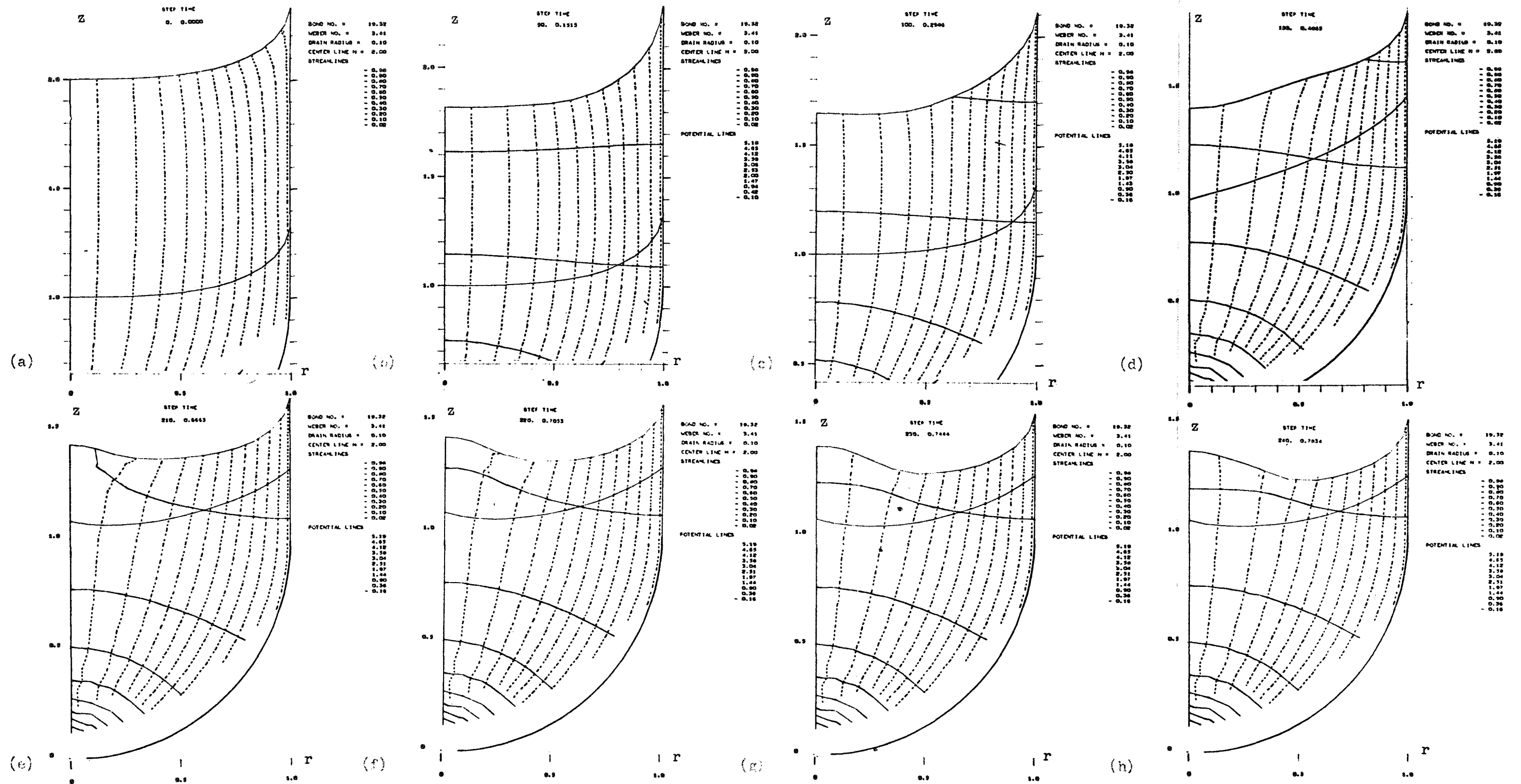
(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 11. Draining characteristics computed for case B3: $B=19.32$, $W=3.41$, $h=1$, $r=1/10$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).



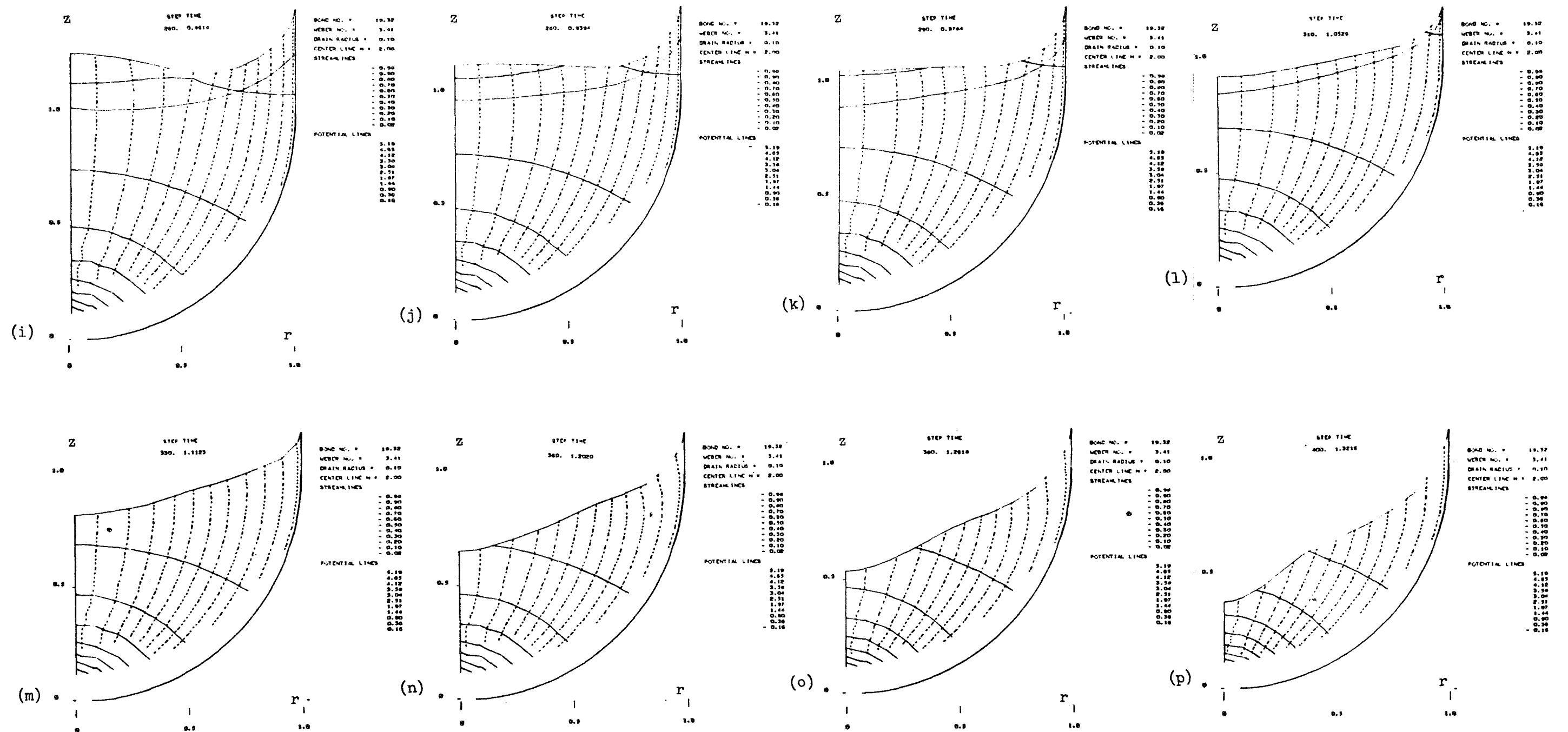
(g) Position of free surface points at the tank centerline and wall during draining.

Figure 11 (cont). Draining characteristics computed for case B3: $B=19.32$, $W=3.41$, $h_c=1$, $r_o=1/30$.



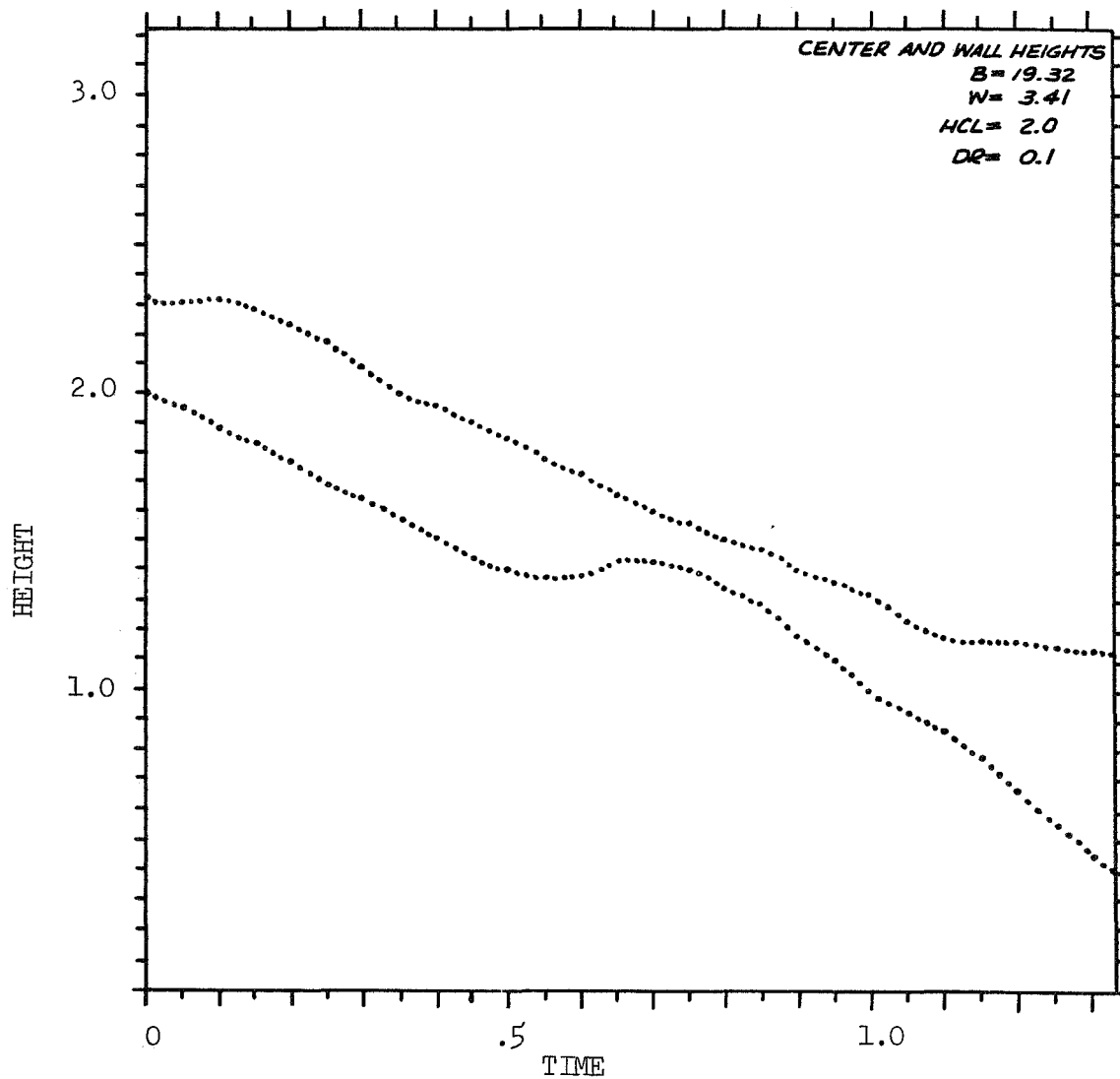
(a)-(h) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 12. Draining characteristics computed for case C3: $B=19.32$, $W=3.41$, $h=2$, $r=1/10$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).



(i)-(p) Streamlines, equipotentials, and free surface shapes as functions of the spatial coordinates.

Figure 12, (cont). Draining characteristics computed for case C3: $B=19.32$, $W=3.41$, $h_c=2$, $r=1/10$. (Problem parameters, elapsed time from the initiation of draining, and the computing step are shown in each frame).



(q) Position of free surface points at the tank centerline and wall during draining.

Figure 12 (cont). Draining characteristics computed for case C3: $B=19.32$, $W=3.41$, $h_c=2$, $r_o=1/10$.

TABLE I SUMMARY OF COMPUTED DATA FOR RESIDUAL VOLUME
(All quantities are dimensionless)

Case identification	Bond number(B)	Weber number(W)	Drain size (r_o)	Initial centerline height (h_c)	Initial volume(V_i)	Residual volume(V_r)	Time to vapor ingestion (t_{vi})
5 ^a	0	1	1/30	1	1.4949	1.030	.310
7	↓	3	1/10	↓	1.5016	1.119	.255
8	↓	3	1/30	↓	1.4949	1.118	.251
9	↓	3	1/50	↓	1.4949	1.119	.250
11	↓	10	1/30	↓	1.4944	1.146	.233
A3	↓	36.06	1/10	↓	1.5016	1.142	.240
13	↓	100	1/10	↓	1.5016	1.147	.237
14 ^a	↓	100	1/30	↓	1.4949	1.146	.234
15	↓	100	1/50	↓	1.4944	1.150	.230
47 ^a	1	0.3	1/30	↓	1.4268		
53 ^a	1	3	1/30	↓	1.4268	1.038	.257
59	1	100	1/30	↓	1.4268	1.075	.232
97	3	3	1/10	↓	1.3441	.935	.272
98 ^a	↓	3	1/30	↓	1.3374	.944	.262
99	↓	3	1/50	↓	1.3369	.944	.262
103	↓	100	1/10	↓	1.3341	.981	.242
104	↓	100	1/30	↓	1.3374	.979	.239
105	↓	100	1/50	↓	1.3369	.981	.237
137 ^b	10	0.3	1/30	↓	1.2040		
143	10	3	1/30	↓	1.2040	.771	.289
149	10	100	1/30	↓	1.2040	.823	.254
B3 ^a	19.32	3.41	1/10	↓	1.1391	.679	.307
20 ^b	0	1	1/30	2	2.9945		
23	↓	3	↓	↓		1.386	1.073
26	↓	10	↓	↓		1.533	.974
29 ^a	↓	100	↓	↓		1.606	.926
119	3	100	↓	↓	2.8369	1.388	.966
C3 ^a	19.32	3.41	1/10	↓	2.6368	.582	1.37
44 ^a	0	100	1/30	3	4.4941	1.85	1.76
D3 ^b	0	36.06	1/10	2.6704 ^c	4.0066		
E3	1	36.06	1/10	2.7185 ^c	4.0061	1.58	1.62

^aSimulation plots are presented for this case

^bImprovements in computing controls needed in order to reach vapor ingestion for this case

^cCorresponds to a flat free surface at $h_c = 3$

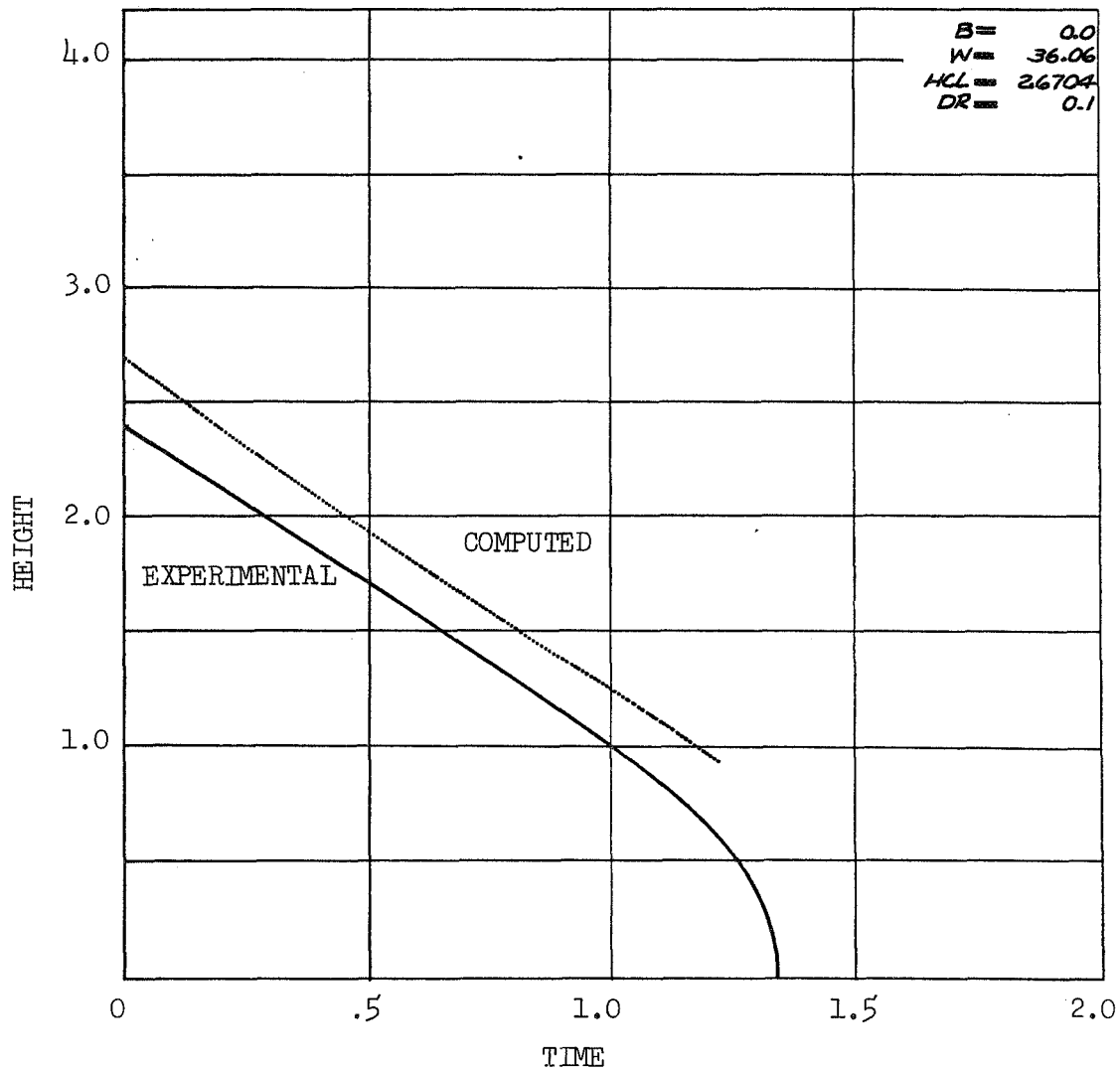


Figure 13. Comparison of computed and experimental values of centerline height during draining for case D3: $B=0$, $W=36.06$, $h_c=2.6704$, $r_o=1/10$; experimental data furnished by NASA-Lewis Research Center.

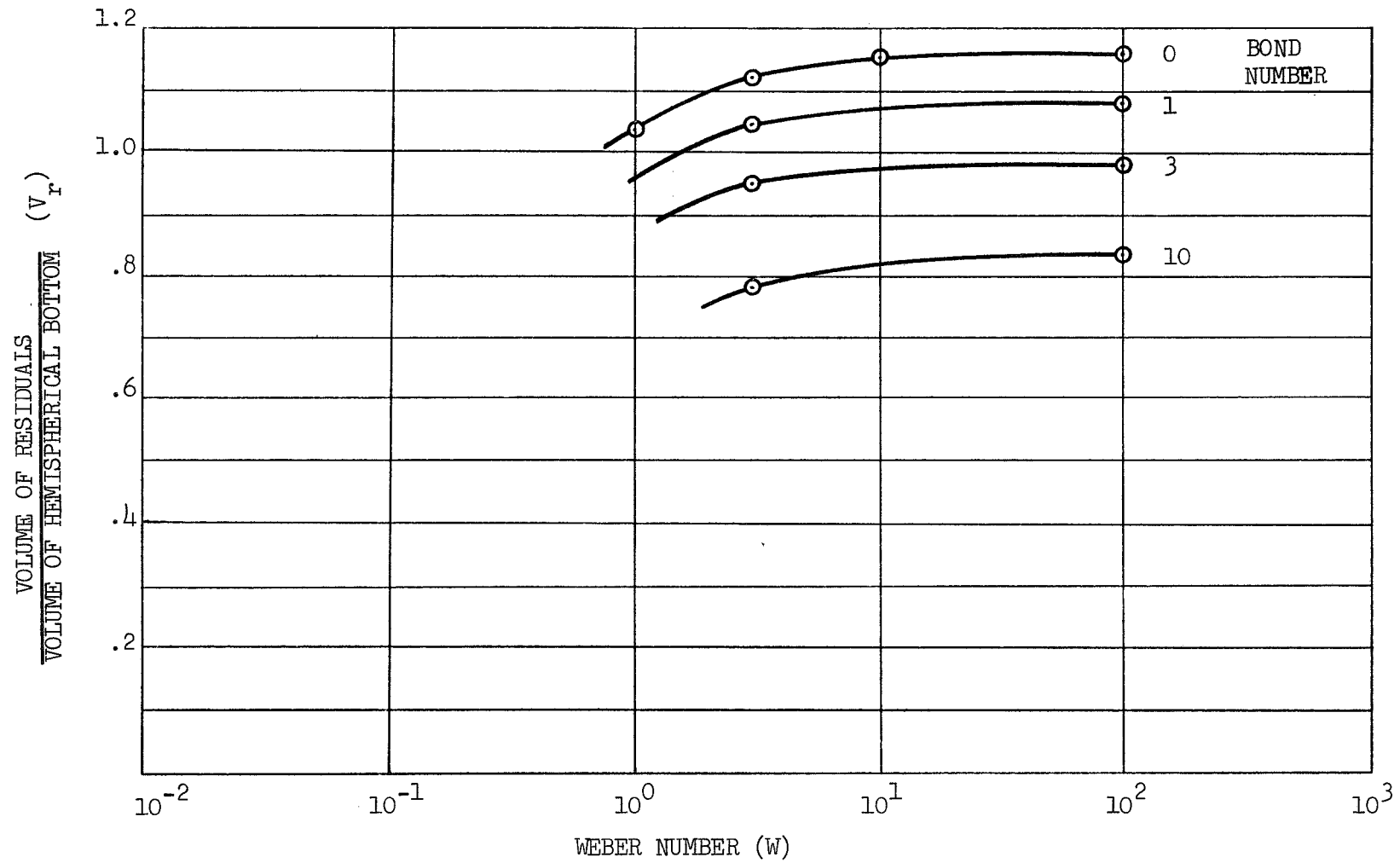


Figure 14. Liquid residuals for low gravity draining of a hemispherically bottomed, cylindrical tank: $r_o=1/30$, $h_c=1$.

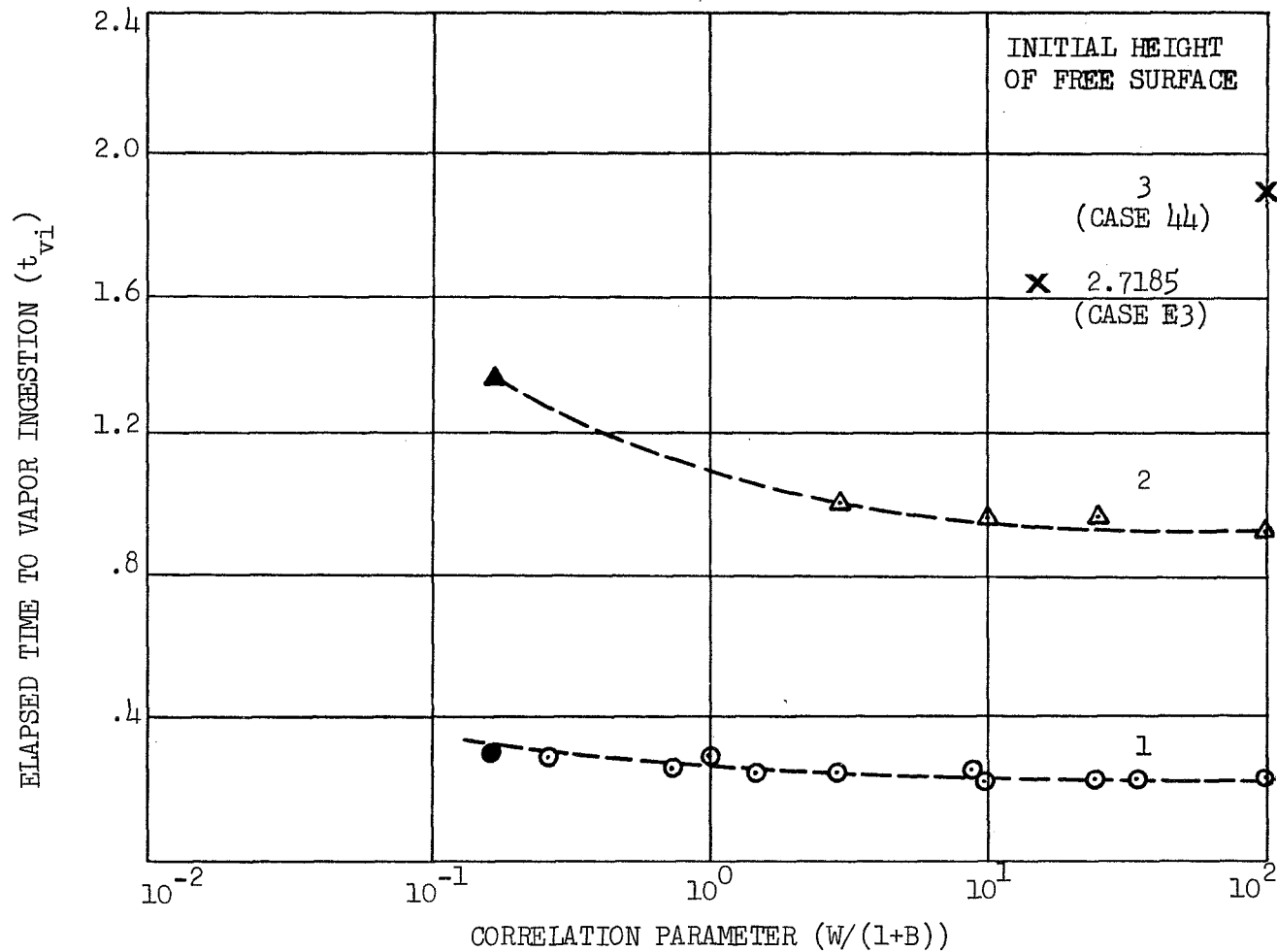


Figure 15. Computed vapor ingestion times for low gravity draining of a hemispherically bottomed, cylindrical tank. (Open symbols denote points calculated at $h_c = 1$ and 2 for $r = 1/30$; closed symbols are the special points $B = 19.32$, $W = 3.41$, $r_o = 1/10$, $h_c = 1.0$ and 2).

APPENDIX A

VISCOUS FILM THICKNESS

The thickness of the film, due to viscosity, that remains on the wall of a right circular cylindrical vessel during draining can be estimated by extending the analysis for the flat plate [26, pp. 678-680]. Using the assumption that the film thickness is small compared with the radius, one obtains for the constant asymptotic (dimensionless) film thickness h_o , for a draining velocity that is not too large,

$$h_o = 1.34(\mu \bar{v}_o/\sigma)^{2/3}/\kappa$$

where $\bar{v}_o = Q/\pi a^2$ is the (dimensional) bulk velocity of the liquid and κ is the curvature $d/dr \left[(dF/dr) / \sqrt{1 + (dF/dr)^2} \right]$ at the cylinder wall of a meridian of the equilibrium free surface.

From (1) one obtains that the appropriate expression for κ for a contact angle of 5° is

$$\kappa = B(F_w - h_c) + 2H_o - \sin 85^\circ,$$

where F_w is the value of F at the wall $r = 1$. The values of $F_w - h_c$ and H_o for any value of B can be found in [27]. For the limiting case of $B = 0$, (9) in [27] gives that $H_o = \sin 85^\circ$, so that $\kappa = \sin 85^\circ$ and

$$h_o = 1.35(\mu \bar{v}_o/\sigma)^{2/3}$$

for this case. For the limiting case of large B , on the other hand, (15a), (18), and (23) of [27] yield $\kappa \approx 2\sqrt{B} \sin 42 \frac{1}{2}^\circ$, so that

$$h_o \approx \frac{(\mu \bar{v}_o)^{2/3}}{(\rho g)^{1/2} \sigma^{1/6} a}.$$

For intermediate values of B , the values of $F_w - h_o$ and $2H_o$ can be

read directly from Figures 2 and 5 of [27], respectively.

For all cases considered in this study, the value of h_o is small enough so that an inviscid model can be expected to represent adequately the liquid behavior.

APPENDIX B

SYMBOL LIST

Quantities are nondimensional unless otherwise designated. When appropriate, the relation between a dimensionless variable and the physical dimensional one, which is topped by a bar, is given. Underlined variables are generally vectors. Spatial and temporal variables appearing as subscripts to R , Z , φ , and Φ indicate partial differentiation.

English Alphabet

a	-Dimensional radius of container. The characteristic length with respect to which the variables are made nondimensional.
$\underline{A}(t)$	-Matrix in finite difference approximation to the mixed boundary value problem satisfied by φ .
$\underline{b}(t)$	-Right side vector in finite difference approximation to the mixed boundary value problem satisfied by φ .
$\underline{b}_2(t)$	-Same as $b(t)$ for φ_2 .
B	-Bond number = $\rho g a^2 / \sigma$.
c	-Additive constant.
$C(t)$	-Additive function of time.
$C_m(r_o)$	-Coefficient in series representation of φ_1 .
DR	- r_o on plotted output.
f	-Height of point on instantaneous free surface meridian = $\bar{f}/a$.
f_w	-Value of f on cylinder wall = $\bar{f}_w/a$.
F	-Height of point on equilibrium free surface meridian = $\bar{F}/a$.
F_w	-Value of F at cylinder wall = $\bar{F}_w/a$.
g	-Dimensional acceleration due to gravity.
$g(r)$	-Distortion of free surface, $f(r)$.
h_c	-Height of equilibrium free surface at the container axis = $\bar{h}_c/a$.

h_o	-Thickness of viscous film = $\bar{h}_o/a$.
H	-Mean curvature of free surface = $a\bar{H}$.
H_o	-Mean curvature of equilibrium free surface at the container axis = $a\bar{H}_o$.
HCL	- h_c on plotted output .
$I(w)$	-Integral used to evaluate components of $b_2(t)$ on curved tank wall.
$j_{l,m}$	- m^{th} zero of J_l .
J_o, J_l	-Bessel functions of first kind.
$\underline{L}(t)$	-Left triangular factor of $\underline{A}(t)$.
m	-Summation index.
M	-Length of table of $C_m(r_o)$ and $j_{l,m}$.
n	-Denotes exterior normal direction.
p_i	-Input parameters for mesh revision, $i = 1, \dots, 24$.
Q	-Dimensional volumetric drain rate.
r	-Radial coordinate = $\bar{r}/a$.
r_o	-Drain radius = $\bar{r}_o/a$.
r_w	-Value of r at intersection of free surface and container wall = $\bar{r}_w/a$.
R	-Radial coordinate of point on free surface meridian = $\bar{R}/a$.
$\underline{R}(t)$	-Right triangular factor of $\underline{A}(t)$.
s	-Arc length along free surface or a mesh line "parallel" to the free surface.
s_w	-Total length of free surface.
t	-Time = $Q\bar{t}/\pi a^3$.
$\bar{t}$	-Dimensional time.
t_{vi}	-Time at vapor ingestion.
u	-Chord length along the free surface.
u_w	-Total chord length of free surface.
$\underline{v}$	-Liquid velocity = $\nabla\varphi = \pi a^2 \underline{\bar{v}}/Q$.
$\underline{v}^-, \underline{v}^+$	-Liquid velocity immediately before, after opening drain at $t = 0$.
$\bar{v}_o$	-Dimensional average bulk speed of liquid = $Q/\pi a^2$.

V	-Liquid volume in tank = $\bar{V}/a^3$; -Dimensional interface velocity at axis from [5].
V_i	-Initial liquid volume in tank measured in hemispherical volumes.
V_m	-Dimensional mean velocity in tank from [5].
V_r	-Residual volume measured in hemispherical volumes.
V_s	-Volume of right cylinder of radius l below the hemispherical bottom of tank with drain of radius r_o .
V_o	-Initial liquid volume in tank $\bar{V}_o/a^3$
w	-Arc length on curved tank wall.
w_1, w_2	-Arc lengths of midpoints of adjacent intervals on curved tank wall.
w_c	-Total arc length of curved tank wall.
W	-Weber number = $\rho Q^2 / \pi^2 \sigma a^3$.
$x(t)$	-Solution vector in equations of form, $\underline{L}(t) x(t) = \underline{b}(t)$.
z	-Axial coordinate = $\bar{z}/a$.
Z	-Axial coordinate of point on free surface meridian = $\bar{Z}/a$.

Greek Alphabet

α	-Angle between tank wall and horizontal at the intersection with the free surface.
Δt	-Time increment.
θ	-Azimuthal angle in right cylindrical coordinates.
Θ	-Contact angle (= 5° for numerical calculations).
κ	-Curvature at tank wall of equilibrium free surface meridian = $\bar{\kappa}$.
μ	-Dimensional liquid viscosity.
ρ	-Dimensional liquid density.
σ	-Dimensional surface tension.
τ	-Ratio of geometric progression of time steps.
φ	-Velocity potential = $\pi a \bar{\varphi}/Q$.
φ_1	-Auxiliary velocity potential.
φ_2	- $\varphi_2 = \varphi - \varphi_1$.

φ_n	-Exterior normal derivative of φ on f .
φ_s	-Tangential derivative of φ on f , positive to left of φ_n .
Φ, Φ_1, Φ_2	-Finite difference approximation to $\varphi, \varphi_1, \varphi_2$.
Φ_f	-Approximation to Φ_s at midpoint of interval adjacent to contact point.
Φ_n	-Approximation to φ_n on f .
Φ_s	-Approximation to φ_s on f .
Φ_w	-Finite difference quotient of Φ between the contact point and the mesh point below on the wall.
χ	-Angle between tangent to free surface and horizontal.
ψ	-Streamfunction = $\pi a \bar{\psi} / Q$.
Ψ	-Finite difference approximation to ψ .
Miscellaneous	
$\underline{\nabla}$	-Gradient operator = $a \underline{\nabla}$.

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