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ESTIMATION IN TRUNCATED POISSON DISTRIBUTIONS WITH CONCOMITANT EXPOSURE INTERVALS AND TRUNCATION POINTS

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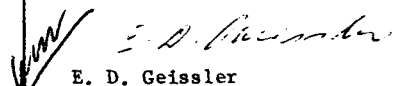University of Georgia
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16. Abstract

Let $\{x_i\}$, $i = 1, \ldots, n$, denote random observations of a truncated Poisson variable with parameter t_i, where the t_i are known positive numbers, and where truncation points c_i are specified such that $x_i \geq c_i$. Using x_i plus concomitant information provided by t_i and c_i, the likelihood estimating equation for λ becomes

$$\sum_1^n x_i = \lambda \left\{ \sum_1^n t_i + \sum_1^n [f(c_i-1)/\bar{F}(c_i)] \right\}$$

where

$$f(c_i - 1) = e^{-\lambda t_i}(\lambda t_i)^{c_i-1}/(c_i - 1)! \quad \text{and} \quad \bar{F}(c_i) = \sum_{x=c_i}^{\infty} e^{-\lambda t_i}(\lambda t_i)^x/x!$$

Solutions (estimates) are readily found using trial and error procedures with linear interpolation, but any standard iterative procedure would be applicable. The asymptotic variance of $\hat{\lambda}$ is obtained from the second partial of the likelihood function. An illustrative example is included.

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ESTIMATION IN TRUNCATED POISSON DISTRIBUTIONS

WITH CONCOMITANT EXPOSURE INTERVALS AND TRUNCATION POINTS

by

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1. INTRODUCTION

Estimation in the truncated Poisson distribution has previously been considered by numerous writers including Bliss [1], David and Johnson [4], Moore [6], Plackett [7], Subrahmanian [8], the writer [2,3], and others. Our concern here is with estimation of the Poisson parameter when the ith observation of the discrete random variable X extends over a specified exposure interval of magnitude t_i, subject to the restriction that $x_i \geq c_i$, where c_i is a specified truncation point. More specifically, our attention is directed toward maximum likelihood estimation based on a random sample consisting of n independent observations $\{x_i, t_i, c_i\}$, where x_i is the number of occurrences of the event of interest observed during the ith interval of exposure, while t_i and c_i are corresponding concomitant values. Of course, x_i and c_i must be non-negative integers, but t_i is permitted to assume any positive value. Accident data in which reports are rendered for time intervals of varying size with truncation as stipulated fall into the category of samples under consideration here. Inspection data in which defects per unit are reported for units of differing sizes might also result in samples of the type considered here.

With concomitant values t_i and c_i given, the applicable Poisson probability function may be written as

$$
\Pr\{X=x_i \,|\, t_i, c_i\} = \begin{cases} \dfrac{e^{-\lambda t_i}\,(\lambda t_i)^{x_i}}{x_i!\,\overline{F}(c_i)} & , \ x_i = c_i, c_i+1, \ldots, \\[20pt] 0 & , \ \text{elsewhere}, \end{cases} \tag{1}
$$

where

$$
\overline{F}(c_i) = \sum_{x_i=c_i}^{\infty} \frac{e^{-\lambda t_i}\,(\lambda t_i)^{x_i}}{x_i!} = 1 - F(c_i - 1) . \tag{2}
$$

2. MAXIMUM LIKELIHOOD ESTIMATION OF λ

The likelihood function for a sample consisting of the n observed values $\{x_i, t_i, c_i\}$ for which the probability function (1) is applicable, follows as

$$
L = \prod_{i=1}^{n} \frac{e^{-\lambda t_i}\,(\lambda t_i)^{x_i}}{x_i!\,\overline{F}(c_i)} . \tag{3}
$$

On taking logarithms of both sides of (3) and differentiating with respect to λ, we obtain

$$\frac{\partial \ln L}{\partial \lambda} = -\sum_1^n t_i + \frac{1}{\lambda}\sum_1^n x_i - \sum_1^n \frac{1}{\overline{F}(c_i)} \frac{\partial \overline{F}(c_i)}{\partial \lambda} . \tag{4}$$

It follows from (2) that

$$\frac{\partial \overline{F}(c_i)}{\partial \lambda} = \frac{e^{-\lambda t_i}(\lambda t_i)^{c_i-1}}{(c_i-1)!} = f(c_i-1) . \tag{5}$$

When (5) is substituted into (4) and the resulting expression is equated to zero and simplified, we obtain the estimating equation

$$\sum_1^n x_i = \lambda[\sum_1^n t_i + \sum_1^n f(c_i-1)/\overline{F}(c_i)] , \tag{6}$$

in which $f(c_i-1)$ and $\overline{F}(c_i)$ are defined by (5) and (2) respectively.

With the aid of an ordinary table of individual and cumulative values of the Poisson function such as those of Molina [5], equation (6) can be solved with relative ease for the required estimate $\hat{\lambda}$, using trial and error techniques. Once two values λ_1 and λ_2 have been found in a sufficiently narrow interval such that $G(\lambda_1) < \sum_1^n x_i < G(\lambda_2)$, where $G(\lambda)$ has been written for the right side of (6), the required estimate $\hat{\lambda}$ can be obtained by interpolating linearly between λ_1 and λ_2. Of course, standard iterative procedures

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might also be employed in the calculation of $\hat{\lambda}$.

3. SOME SPECIAL CASES

By appropriately restricting the values of t_i and c_i in (6), we obtain results that apply in various cases of special interest. Certain of these specialized results are, of course, quite well known.

No Truncation — Unequal Intervals

In this case, $c_i = 0$ for all i, and equation (6) leads to the estimator

$$\hat{\lambda} = \sum_1^n x_i / \sum_1^n t_i \; . \tag{7}$$

No Truncation — Equal Intervals

If in addition to the restriction, $c_i = 0$ for all i, we impose the further restriction that $t_i = t$ for all i, the applicable estimator reduces to

$$\hat{\lambda} = \bar{x}/t \; , \tag{8}$$

where $\bar{x} = \sum_1^n x_i/n$. When $t = 1$, as in the usual case of estimation in the Poisson distribution, then (7) leads to the familiar estimator $\hat{\lambda} = \bar{x}$.

Only Zero Class Truncated In Making Each Observation — Unequal Intervals

In this case $c_i = 1$ for each i, and the estimating equation (6) reduces to

$$\sum_1^n x_i = \lambda \sum_1^n [t_i/(1-e^{-\lambda t_i})] \; , \tag{9}$$

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a result which was obtained independently by one of the writer's graduate assistants, Mr. S. S. Srivastava.

Only Zero Class Truncated In Making Each Observation — Equal Intervals

When we further specialize the preceeding result by requiring that $t_i = t$ for all i, then (9) becomes

$$\bar{x} = \lambda t / [1 - e^{-\lambda t}] \ . \tag{10}$$

If we specialize still further by requiring that $t = 1$, then (10) reduces to

$$\bar{x} = \lambda / [1 - e^{-\lambda}] \ , \tag{11}$$

as previously given in [3], where a table of the function on the right side of (11) is provided in order to facilitate solution for the required estimate $\hat{\lambda}$.

4. VARIANCE OF ESTIMATES

The asymptotic variance of $\hat{\lambda}$ can be expressed as

$$V(\hat{\lambda}) = - \left[E \left(\frac{\partial^2 \ln L}{\partial \lambda^2} \right) \right]^{-1} \doteq - \left[\frac{\partial^2 \ln L}{\partial \lambda^2} \right]^{-1}_{\lambda = \hat{\lambda}} \tag{12}$$

for each of the cases considered here. The second derivative in the most general case under consideration follows from (4) as

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = - \frac{\sum\limits_{1}^{n} x_i}{\lambda^2} - \sum\limits_{1}^{n} \left[\frac{f(c_i - 2) - f(c_i - 1)}{\bar{F}(c_i)} \right] + \sum\limits_{1}^{n} \left[\frac{f(c_i - 1)}{\bar{F}(c_i)} \right]^2 \ , \tag{13}$$

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a result which was previously given in [2]. With the aid of tables

such as those of Molina [5], this derivative can easily be evaluated

once (6) has been solved for $\hat{\lambda}$. The variance then follows from (12).

5. AN ILLUSTRATIVE EXAMPLE

In order to illustrate the practical application of estimators

derived here, we consider inspection results on the number of surface

defects per unit found in products of differing surface areas but

which were manufactured on the same production line. In some of the

inspections, no truncation takes place. In others, the zeros are

eliminated, and in still others, both zeros and ones are eliminated.

Based on the total sample which results from combining the several

sub-samples involved, we proceed to estimate the Poisson parameter λ

which characterizes the underlying common production process, and to

determine the asymptotic variance of this estimate. The total sample

available for this purpose consists of inspection results on 422

separate manufactured units. These data are presented in Table 1, with

the total sample broken down into eight sub-samples having values of

(t,c) as follows: (1,0),(1.5,1),(2,1),(2.5,1),(3,1),(1,2),(2,2), and

(3,2). The value t = 1 indicates a surface area of one unit, while

t = 2 indicates a surface area of two units, etc. For the total

number of defects found in the 422 units inspected, we have

$\sum_{1}^{422} x_i = 1597$. For the sum of the t's, we have $\sum_{1}^{422} t_i = 100(1)$

+ 94(1.5) + 50(2) + 32(2.5) +28(3) + 30(1) + 34(2) + 54(3) = 765.

These values are substituted into (6) which must then be solved for

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TABLE 1

NUMBER OF SURFACE DEFECTS FOUND IN

FINAL INSPECTION OF 422 MANUFACTURED UNITS

No. Defects	Frequencies n_x								Row Totals
	c=0		c=1				c=2		
x	t=1	t=1.5	t=2	t=2.5	t=3	t=1	t=2	t=3	
0	13								13
1	28	16	5	1	0				50
2	27	21	6	3	1	14	6	3	81
3	17	22	9	5	3	9	5	3	73
4	10	16	11	6	5	5	7	7	67
5	3	11	9	5	4	1	6	7	46
6	2	6	4	5	5	1	5	9	37
7		1	3	4	4		3	8	23
8		1	2	2	3		1	6	15
9			1	1	1		1	5	9
10					2			3	5
11								2	2
12								1	1
Column Totals	100	94	50	32	28	30	34	54	422
$\Sigma x n_x$	200	294	203	154	162	86	153	345	1597

the required estimate $\hat{\lambda}$. As a first approximation, we try $\lambda_1 = 2.0000$, which comes from the sub-sample of size 100 with $t = 1$ and $c = 0$. In general, satisfactory first approximations might be obtained from any one or a combination of the available sub-samples. In some circumstances, a judicious guess might even suffice. We subsequently arrive at the final estimate $\hat{\lambda} = 2.027$, by linear interpolation as follows

λ		$G(\lambda)$
2.100		1652.91
	422	
2.027	$\sum\limits_{1} x_i = 1597.00$	
2.000		1576.75

where $G(\lambda)$ has been written for the right side of the estimating equation (6). For the variance of the above estimate, we employ (12) and (13) to calculate $V(\hat{\lambda}) = 0.0027$.

REFERENCES

[1] Bliss, C. I., (1948). Estimation of the mean and its error from incomplete Poisson distributions, Connecticut Agricultural Experiment Station Bulletin 513, 12 p.

[2] Cohen, A. C., Jr. (1954). Estimation of the Poisson parameter from truncated samples and from censored samples, J. Am. Stat. Assn. 49, 158-68.

[3] Cohen, A. C., Jr. (1960). Estimating the parameter in a conditional Poisson distribution, Biometrics 16, 203-11.

[4] David, F. N., and Johnson, N. L., (1952). The truncated Poisson, Biometrics 8, 275-85.

[5] Molina, E. C., (1942). Poisson's Exponential Binomial Limit, D. Van Nostrand Co., Inc.

[6] Moore, P. G., (1952). The estimation of the Poisson parameter from a truncated distribution, Biometrika, 39, 247-51.

[7] Plackett, R. L., (1953). The truncated Poisson distribution, Biometrics 9, 485-88.

[8] Rider, Paul R., (1953). Truncated Poisson distributions, J. Am. Stat. Assn. 48, 826-30.

[9] Subrahmanian, K., (1965). A note on estimation in the truncated Poisson, Biometrika 52, 279-82.

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