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AN ALGORITHM FOR COMPUTING SPHERICAL PARTIALS

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Program Systems Branch

Mission & Trajectory Analysis Division

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ABSTRACT

This paper presents a rapid method for computing satellite accelerations, position partials, and partials with respect to harmonic coefficients in the earth's geopotential using spherical recurrences. Some timing estimates and accuracy comparisons are given as a function of order of the harmonics for two different satellites.

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INTRODUCTION

The nonlinear equations of motion of a satellite moving in the gravitational field of the earth are presently being solved by high order predictor-corrector methods of numerical integration. The advent of extremely accurate electronic and optical measurements makes it both desirable and necessary to include in the force model terms of high degree and order as well as other forces such as drag, solar radiation, etc. when estimating geodetic parameters, tracking station coordinates, etc.

This paper describes a fast method for computing accelerations, partial derivatives with respect to position, and partial derivatives with respect to harmonic coefficients in the earth's geopotential using recurrences in spherical coordinates. This method [1], [3] was compared with De Witt's method [2] which uses recurrences in cartesian coordinates. The spherical recursion algorithm was found to be as accurate and at least twice as fast in the calculation of accelerations alone [1]. The author reached the same conclusion and, in addition found the spherical version much simpler from both analysis and programming points of view. The method described here computes the total acceleration (Keplerian plus perturbative) due to the geopotential field for an artificial satellite orbiting the earth. The principal features of this method have been applied in a program which estimates geodetic parameters, tracking station coordinates, etc. using multiple arcs of various satellites.

GRAVITATIONAL POTENTIAL OF THE EARTH

The potential function of the earth (adopted at the meeting of Commission 7 on celestial mechanics at the Berkeley meeting of the International Astronomical Union in August 1961) is given by

$$U = \frac{GM}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_n^m(\mu)\left[C_n^m \cos m\lambda + S_n^m \sin m\lambda\right]\right\}$$

where,

$P_n^m(\mu)$ = associated Legendre functions of degree n and order m,

$\mu = z/r$ = sine of the geocentric latitude,

G = Universal gravitational constant,

M = mass of the earth,

R = mean equatorial radius of the earth,

λ = east longitude of the satellite,

r = geocentric distance to satellite,

C_n^m, S_n^m = unnormalized harmonic coefficients.

If the center of gravity of the earth is chosen as the center of the coordinate system, the terms for $n = 1$ do not exist, i.e., $c_1^0 = c_1^1 = s_1^0 = s_1^1 = 0$. The form of the potential then becomes

$$U = \frac{GM}{r}\left\{1 + \sum_{n=2}^{N}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_n^m\left[C_n^m \cos m\lambda + S_n^m \sin m\lambda\right]\right\} \tag{1}$$

where

N = maximum order of harmonic expansion selected.

It is well known that the potential function U is a solution of Laplace's equation, which in rectangular coordinates is given by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 , \tag{2}$$

and in spherical coordinates by

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \psi^2} - \frac{1}{r^2}\tan\psi\frac{\partial U}{\partial \psi}\frac{1}{r^2\cos^2\psi}\frac{\partial^2 U}{\partial \lambda^2} = 0 . \tag{3}$$

COORDINATE SYSTEMS

Two different orthogonal right handed coordinate systems shown in Figure 1 are used. These systems are

(1) An inertial coordinate system referenced with respect to the first point of Aries, ♈, and

(2) A geocentric coordinate system referenced with respect to the Greenwich meridian,

where

$\underline{r}$ = satellite position vector in either coordinate system,

♈ = first point of Aries,

x, y, z = inertial cartesian coordinates of position vector $\underline{r}$,

x_G, y_G, z_G = geocentric coordinates referenced to Greenwich meridian,

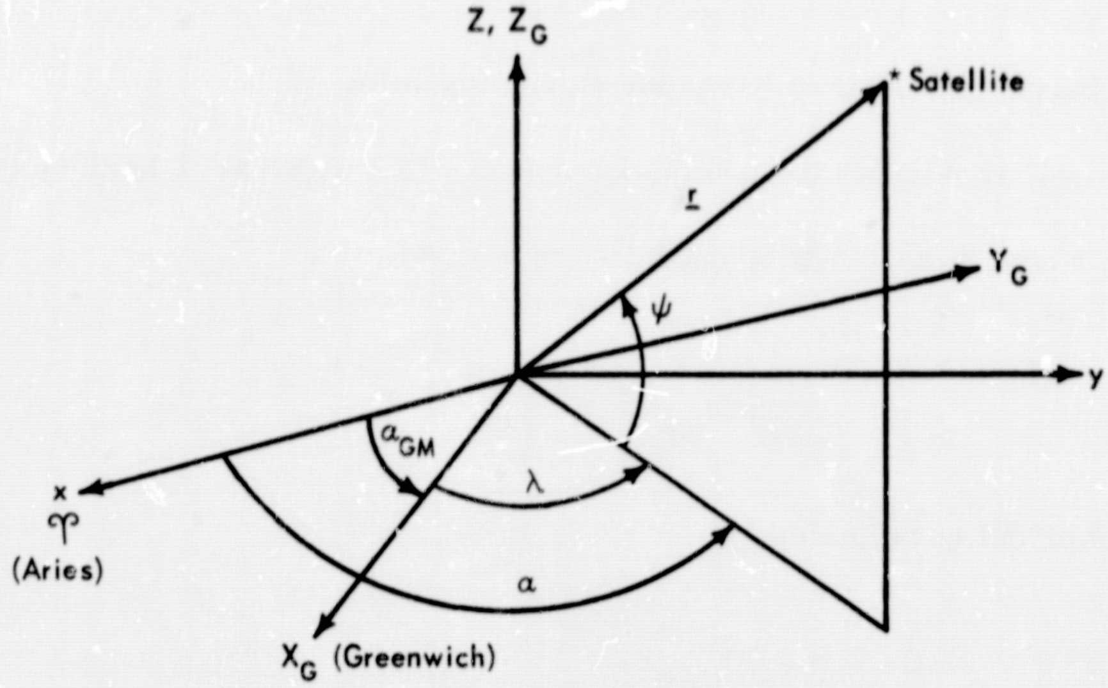

Figure 1.

α = right ascension of the satellite,

α_{GM} = right ascension of the Greenwich meridian,

λ = east longitude of the satellite,

ψ = geocentric latitude of the satellite.

We obtain the angle λ and the sine of ψ from the following equations

$$\alpha = \tan^{-1}\frac{y}{x}, \qquad \lambda = \alpha - \alpha_G, \qquad \sin\psi = \frac{z}{r}.$$

The partials of r, ψ, λ with respect to x, y, z can be obtained from differentiating the following expressions:

$$r^2 = x^2 + y^2 + z^2, \qquad \psi = \tan^{-1}\frac{z}{\left(x^2 + y^2\right)^{1/2}}, \qquad \lambda = -\alpha_G + \tan^{-1}\frac{y}{x}.$$

We obtain after differentiation the following partial derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial r}{\partial z} = \frac{z}{r},$$

$$\frac{\partial \psi}{\partial x} = \frac{-xz}{r^2\left(x^2+y^2\right)^{1/2}}, \qquad \frac{\partial \psi}{\partial y} = \frac{-yz}{r^2\left(x^2+y^2\right)^{1/2}}, \qquad \frac{\partial \psi}{\partial z} = \frac{\left(x^2+y^2\right)^{1/2}}{r^2},$$

$$\frac{\partial \lambda}{\partial x} = \frac{-y}{x^2+y^2}, \qquad \frac{\partial \lambda}{\partial y} = \frac{x}{x^2+y^2}, \qquad \frac{\partial \lambda}{\partial z} = 0.$$

The equations for the total accelerations then become

$$\ddot{x} = \frac{x}{r}\frac{\partial U}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial U}{\partial \lambda} - \frac{xz}{r^2\left(x^2+y^2\right)^{1/2}}\frac{\partial U}{\partial \psi}$$

$$\ddot{y} = \frac{y}{r}\frac{\partial U}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial U}{\partial \lambda} - \frac{yz}{r^2\left(x^2+y^2\right)^{1/2}}\frac{\partial U}{\partial \psi}$$

$$\ddot{z} = \frac{z}{r}\frac{\partial U}{\partial r} + \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial U}{\partial \psi}.$$

It should be noticed that in the equations for the partials of U with respect to the spherical coordinates r, ψ, and λ certain quantities such as P_n^m, $C_n^m \cos m\lambda + S_n^m \sin m\lambda$, and GM/r are common to these partials and need be computed only once. Furthermore, recursive relations are used for generating expressions such as $\sin m\lambda$, $\cos m\lambda$, $m \tan \psi$, and P_n^m so as to increase the speed of computation.

The formulas used in recursion are:

$$P_n^0(\mu) = \left[(2n-1)\sin\psi\ P_{n-1}^0(\mu) - (n-1)P_{n-2}^0(\mu)\right]\Big/ n$$

$$P_n^m(\mu) = P_{n-2}^m + (2n-1)\cos\psi P_{n-1}^{m-1}(\mu)\ m \neq 0,\ \ m < n$$

For sectorials $m = n$ and we have

$$P_n^n(\mu) = (2n-1)\cos\psi P_{n-1}^{n-1}(\mu)\ m \neq 0,\ \ m = n .$$

$$\sin m\lambda = 2\cos\lambda\sin(m-1)\lambda - \sin(m-2)\lambda$$

$$\cos m\lambda = 2\cos\lambda\cos(m-1)\lambda - \cos(m-2)\lambda$$

$$m\tan\psi = \left[(m-1)\tan\psi\right] + \tan\psi .$$

ACCELERATIONS

The total gravitational acceleration vector $\underline{F}_G$ in inertial coordinates (x, y, z) is given by

$$\underline{F}_G = \underline{i}\frac{\partial U}{\partial x} + \underline{j}\frac{\partial U}{\partial y} + \underline{k}\frac{\partial U}{\partial z},$$

where $\underline{i}$, $\underline{j}$, $\underline{k}$ form a triad of mutually orthogonal unit vectors in the inertial coordinate system, and the partial derivatives of the potential function U are given with respect to inertial coordinates x, y, z. Applying the chain rule of calculus to the function

$$U = U(x, y, z) = U(r, \psi, \lambda),$$

we obtain the following equations for the accelerations:

$$\ddot{x} = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial x},$$

$$\ddot{y} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \psi}\frac{\partial \psi}{\partial y} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial y},$$

$$\ddot{z} = \frac{\partial U}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U}{\partial \psi}\frac{\partial \psi}{\partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial z}. \tag{4}$$

The partials of U with respect to r, ψ, λ, i.e., $\partial U/\partial r$, $\partial U/\partial \psi$, $\partial U/\partial \lambda$ are obtained by explicit differentiation of the potential function U and are expressed by the following equations:

$$\frac{\partial U}{\partial r} = -\frac{1}{r}\left(\frac{GM}{r}\right)\left[1 + \sum_{n=2}^{N}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) P_n^m\right]$$

$$\frac{\partial U}{\partial \psi} = \left(\frac{GM}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)\left[P_n^{m+1} - m \tan\psi\, P_n^m\right]$$

$$\frac{\partial U}{\partial \lambda} = \left(\frac{GM}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n} m\left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right) P_n^m. \tag{5}$$

POSITION PARTIALS

The desired partial derivatives with respect to inertial cartesian coordinates can be written in the following matrix form

$$\begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial x \partial z} \\ \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y \partial z} \\ \frac{\partial^2 U}{\partial z \partial x} & \frac{\partial^2 U}{\partial z \partial y} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \frac{\partial \ddot{x}}{\partial z} \\ \frac{\partial \ddot{y}}{\partial x} & \frac{\partial \ddot{y}}{\partial y} & \frac{\partial \ddot{y}}{\partial z} \\ \frac{\partial \ddot{z}}{\partial x} & \frac{\partial \ddot{z}}{\partial y} & \frac{\partial \ddot{z}}{\partial z} \end{bmatrix} .$$

From Laplace's Equation 2 we can solve for $\partial^2 U/\partial x^2$, i.e.,

$$\frac{\partial^2 U}{\partial x^2} = -\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) .$$

Since the above matrix is symmetric, we need only compute the elements above the principal diagonal and two remaining elements on the principal diagonal yielding a total of 5 elements.

The elements in the above matrix can be obtained by differentiating the expressions (4) with respect to x, y, and z and are given as follows:

$$\frac{\partial \ddot{x}}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial U}{\partial \psi}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial^2 U}{\partial \psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\partial^2 \dot{U}}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2$$

$$+ 2\left(\frac{\partial^2 U}{\partial r \partial \psi}\frac{\partial r}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial^2 U}{\partial r \partial \lambda}\frac{\partial r}{\partial x}\frac{\partial \lambda}{\partial x} + \frac{\partial^2 U}{\partial \psi \partial \lambda}\frac{\partial \psi}{\partial x}\frac{\partial \lambda}{\partial x}\right)$$

$$
\begin{aligned}
\frac{\partial \ddot{x}}{\partial y} = {} & \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y\,\partial x} + \frac{\partial U}{\partial \psi}\frac{\partial^2 \psi}{\partial y\,\partial x} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial y\,\partial x} \\
& + \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial y}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial \psi^2}\frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} + \frac{\partial^2 U}{\partial \lambda^2}\frac{\partial \lambda}{\partial y}\frac{\partial \lambda}{\partial x} \\
& + \frac{\partial^2 U}{\partial r\,\partial \psi}\left(\frac{\partial r}{\partial y}\frac{\partial \psi}{\partial x} + \frac{\partial r}{\partial x}\frac{\partial \psi}{\partial y}\right) + \frac{\partial^2 U}{\partial r\,\partial \lambda}\left(\frac{\partial r}{\partial y}\frac{\partial \lambda}{\partial x} + \frac{\partial r}{\partial x}\frac{\partial \lambda}{\partial y}\right) \\
& + \frac{\partial^2 U}{\partial \psi\,\partial \lambda}\left(\frac{\partial \psi}{\partial y}\frac{\partial \lambda}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \lambda}{\partial y}\right) .
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ddot{x}}{\partial z} = {} & \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x\,\partial z} + \frac{\partial U}{\partial \psi}\frac{\partial^2 \psi}{\partial x\,\partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial x\,\partial z} \\
& + \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial z}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial \psi^2}\frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial x} + \frac{\partial^2 U}{\partial \lambda^2}\frac{\partial \lambda}{\partial z}\frac{\partial \lambda}{\partial x} \\
& + \frac{\partial^2 U}{\partial r\,\partial \psi}\left(\frac{\partial r}{\partial z}\frac{\partial \psi}{\partial x} + \frac{\partial r}{\partial x}\frac{\partial \psi}{\partial z}\right) + \frac{\partial^2 U}{\partial r\,\partial \lambda}\left(\frac{\partial r}{\partial z}\frac{\partial \lambda}{\partial x} + \frac{\partial r}{\partial x}\frac{\partial \lambda}{\partial z}\right) \\
& + \frac{\partial^2 U}{\partial \psi\,\partial \lambda}\left(\frac{\partial \psi}{\partial z}\frac{\partial \lambda}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \lambda}{\partial z}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ddot{y}}{\partial y} = {} & \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial U}{\partial \psi}\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial y^2} + \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial^2 U}{\partial \psi^2}\left(\frac{\partial \psi}{\partial y}\right)^2 + \frac{\partial^2 U}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial y}\right)^2 \\
& + 2\left(\frac{\partial^2 U}{\partial r\,\partial \psi}\frac{\partial r}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial^2 r}{\partial r\,\partial \lambda}\frac{\partial r}{\partial y}\frac{\partial \lambda}{\partial y} + \frac{\partial^2 U}{\partial \psi\,\partial \lambda}\frac{\partial \psi}{\partial y}\frac{\partial \lambda}{\partial y}\right)
\end{aligned}
$$

$$\frac{\partial \ddot{y}}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y \partial z} + \frac{\partial U}{\partial \psi}\frac{\partial^2 \psi}{\partial y \partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial y \partial z}$$

$$+ \frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial y}\frac{\partial r}{\partial z} + \frac{\partial^2 U}{\partial \psi^2}\frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial z} + \frac{\partial^2 U}{\partial \lambda^2}\frac{\partial \lambda}{\partial y}\frac{\partial \lambda}{\partial z}$$

$$+ \frac{\partial^2 U}{\partial r \partial \psi}\left(\frac{\partial r}{\partial y}\frac{\partial \psi}{\partial z} + \frac{\partial r}{\partial z}\frac{\partial \psi}{\partial y}\right) + \frac{\partial^2 U}{\partial r \partial \lambda}\left(\frac{\partial r}{\partial y}\frac{\partial \lambda}{\partial z} + \frac{\partial r}{\partial z}\frac{\partial \lambda}{\partial y}\right)$$

$$+ \frac{\partial^2 U}{\partial \psi \partial \lambda}\left(\frac{\partial \psi}{\partial y}\frac{\partial \lambda}{\partial z} + \frac{\partial \psi}{\partial z}\frac{\partial \lambda}{\partial y}\right)$$

$$\frac{\partial \ddot{z}}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial z^2} + \frac{\partial U}{\partial \psi}\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial U}{\partial \lambda}\frac{\partial^2 \lambda}{\partial z^2} + \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial z}\right)^2 + \frac{\partial^2 U}{\partial \psi^2}\left(\frac{\partial \psi}{\partial z}\right)^2 + \frac{\partial^2 U}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial z}\right)^2$$

$$+ 2\left(\frac{\partial^2 U}{\partial r \partial \psi}\frac{\partial r}{\partial z}\frac{\partial \psi}{\partial z} + \frac{\partial^2 U}{\partial r \partial \lambda}\frac{\partial r}{\partial z}\frac{\partial \lambda}{\partial z} + \frac{\partial^2 U}{\partial \psi \partial \lambda}\frac{\partial \psi}{\partial z}\frac{\partial \lambda}{\partial z}\right).$$

We need the expression for the double gradient of U given by

$$\nabla(\nabla U) = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \psi} \\ \frac{\partial}{\partial \lambda} \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial r}, & \frac{\partial U}{\partial \psi}, & \frac{\partial U}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 U}{\partial r^2} & \frac{\partial^2 U}{\partial r \partial \psi} & \frac{\partial^2 U}{\partial r \partial \lambda} \\ \frac{\partial^2 U}{\partial \psi \partial r} & \frac{\partial^2 U}{\partial \psi^2} & \frac{\partial^2 U}{\partial \psi \partial \lambda} \\ \frac{\partial^2 U}{\partial \lambda \partial r} & \frac{\partial^2 U}{\partial \lambda \partial \psi} & \frac{\partial^2 U}{\partial \lambda^2} \end{bmatrix}.$$

We notice that the 3×3 matrix is symmetric and hence need only to compute the elements on and above the principal diagonal. But we can also obtain one of the elements on the principal diagonal from Laplace's Equation 3. This means that only a total of 5 partials are required in this matrix. The second partials of U with respect to r, ψ, and λ are obtained by differentiating the expressions (5) for the first partials and are given by

$$\frac{\partial^2 U}{\partial r^2} = \frac{GM}{r^3}\left[2 + \sum_{n=2}^{N}\left(\frac{R}{r}\right)^n (n+2)(n+1) \sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) P_n^m\right]$$

$$\frac{\partial^2 U}{\partial r\,\partial\psi} = -\frac{GM}{r^3}\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)\left(P_n^{m+1} - m\tan\psi P_n^m\right)$$

$$\frac{\partial^2 U}{\partial r\,\partial\lambda} = -\frac{GM}{r^3}\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n} m\left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right) P_n^m$$

$$\frac{\partial^2 U}{\partial\psi^2} = \left(\frac{GM}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\Big(C_n^m \cos m\lambda$$

$$+ S_n^m \sin m\lambda\Big)\left\{\tan\psi P_n^{m+1} + \left[m\sec^2\psi - m\tan^2\psi - n(n+1)\right] P_n^m\right\}$$

$$\frac{\partial^2 U}{\partial\psi\,\partial\lambda} = \left(\frac{GM}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n} m\left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right)\left(P_n^{m+1} - m\tan\psi P_n^m\right)$$

$$\frac{\partial^2 U}{\partial\lambda^2} = \left(-\frac{GM}{r}\right)\sum_{n=2}^{N}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n} m^2\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) P_n^m .$$

We mentioned above that we needed only 5 partials. Hence, we arbitrarily chose to compute $\partial^2 U/\partial\psi^2$ from Laplace's Equation 3, i.e.,

$$\frac{\partial^2 U}{\partial \psi^2} = \tan\psi \frac{\partial U}{\partial \psi} - \left[r^2 \frac{\partial^2 U}{\partial r^2} + 2r \frac{\partial U}{\partial r} + \frac{1}{\cos^2\psi} \frac{\partial^2 U}{\partial \lambda^2} \right].$$

The second partial derivatives of r, ψ, and λ with respect to x, y, and z can be represented in the symmetric matrix form

$$\begin{bmatrix} \frac{\partial^2 r}{\partial x^2} & \frac{\partial^2 r}{\partial x \partial y} & \frac{\partial^2 r}{\partial x \partial z} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial y^2} & \frac{\partial^2 r}{\partial y \partial z} \\ \frac{\partial^2 r}{\partial z \partial x} & \frac{\partial^2 r}{\partial z \partial y} & \frac{\partial^2 r}{\partial z^2} \end{bmatrix}$$

where the elements of the r matrix are given by

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3}, \qquad \frac{\partial^2 r}{\partial y^2} = -\frac{y^2}{r^3}, \qquad \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{z^2}{r^3},$$

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial x} = -\frac{xy}{r^3}, \quad \frac{\partial^2 r}{\partial x \partial z} = \frac{\partial^2 r}{\partial z \partial x} = -\frac{xz}{r^3}, \quad \frac{\partial^2 r}{\partial y \partial z} = \frac{\partial^2 r}{\partial z \partial y} = -\frac{yz}{r^3}.$$

The partial derivatives of ψ with respect to x, y, and z can be represented by the symmetric matrix

$$\begin{bmatrix} \frac{\partial^2 \psi}{\partial x^2} & \frac{\partial^2 \psi}{\partial x \partial y} & \frac{\partial^2 \psi}{\partial x \partial z} \\ \frac{\partial^2 \psi}{\partial y \partial x} & \frac{\partial^2 \psi}{\partial y^2} & \frac{\partial^2 \psi}{\partial y \partial z} \\ \frac{\partial^2 \psi}{\partial z \partial x} & \frac{\partial^2 \psi}{\partial z \partial y} & \frac{\partial^2 \psi}{\partial z^2} \end{bmatrix}$$

where the elements of the ψ matrix are given by

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{z}{r^4 \left(x^2 + y^2\right)^{3/2}} \left[2x^2 \left(x^2 + y^2\right) - r^2 y^2\right]$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = \frac{xyz}{r^4 \left(x^2 + y^2\right)^{3/2}} \left[3\left(x^2 + y^2\right) + z^2\right]$$

$$\frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial^2 \psi}{\partial z \partial x} = \frac{x}{r^4 \left(x^2 + y^2\right)^{1/2}} \left[z^2 - \left(x^2 + y^2\right)\right]$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{z}{r^4 \left(x^2 + y^2\right)^{3/2}} \left[2y^2 \left(x^2 + y^2\right) - r^2 x^2\right]$$

$$\frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial^2 \psi}{\partial z \partial y} = \frac{y}{r^4 \left(x^2 + y^2\right)^{1/2}} \left[z^2 - \left(x^2 + y^2\right)\right]$$

$$\frac{\partial^2 \psi}{\partial z^2} = -\frac{2z}{r^4} \left(x^2 + y^2\right)^{1/2} .$$

The partial derivatives of λ with respect to x, y, and z can be represented by the symmetric matrix

$$\begin{bmatrix} \frac{\partial^2 \lambda}{\partial x^2} & \frac{\partial^2 \lambda}{\partial x \partial y} & \frac{\partial^2 \lambda}{\partial x \partial z} \\ \frac{\partial^2 \lambda}{\partial y \partial x} & \frac{\partial^2 \lambda}{\partial y^2} & \frac{\partial^2 \lambda}{\partial y \partial z} \\ \frac{\partial^2 \lambda}{\partial z \partial x} & \frac{\partial^2 \lambda}{\partial z \partial y} & \frac{\partial^2 \lambda}{\partial z^2} \end{bmatrix},$$

where the elements of the λ matrix are given by

$$\frac{\partial^2 \lambda}{\partial x^2} = -\frac{\partial^2 \lambda}{\partial y^2} = \frac{2xy}{\left(x^2 + y^2\right)^2}, \qquad \frac{\partial^2 \lambda}{\partial x \partial y} = \frac{\partial^2 \lambda}{\partial y \partial x} = \frac{1}{\left(x^2 + y^2\right)^2}\left(y^2 - x^2\right),$$

$$\frac{\partial^2 \lambda}{\partial z^2} = \frac{\partial^2 \lambda}{\partial x \partial z} = \frac{\partial^2 \lambda}{\partial z \partial x} = \frac{\partial^2 \lambda}{\partial y \partial z} = \frac{\partial^2 \lambda}{\partial z \partial y} = 0.$$

The higher accuracy that is required in computing accelerations is not needed in the calculation of the position partials so that fewer terms are included in the geopotential.

HARMONIC COEFFICIENT PARTIALS

The equations for computing partial derivatives of acceleration with respect to harmonic coefficients C_n^m or S_n^m obtained by explicit differentiation of the equations for the accelerations (4) are given by

$$\frac{\partial \ddot{x}}{\partial C_n^m} = \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial r}\right)\frac{\partial r}{\partial x} + \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial \psi}\right)\frac{\partial \psi}{\partial x} + \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial x}$$

$$\frac{\partial \ddot{x}}{\partial S_n^m} = \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial r}\right)\frac{\partial r}{\partial x} + \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial \psi}\right)\frac{\partial \psi}{\partial x} + \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial x}$$

$$\frac{\partial \ddot{y}}{\partial C_n^m} = \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial r}\right)\frac{\partial r}{\partial y} + \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial \psi}\right)\frac{\partial \psi}{\partial y} + \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial y}$$

$$\frac{\partial \ddot{y}}{\partial S_n^m} = \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial r}\right)\frac{\partial r}{\partial y} + \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial \psi}\right)\frac{\partial \psi}{\partial y} + \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial y}$$

$$\frac{\partial \ddot{z}}{\partial C_n^m} = \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial r}\right)\frac{\partial r}{\partial z} + \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial \psi}\right)\frac{\partial \psi}{\partial z} + \left(\frac{\partial}{\partial C_n^m}\frac{\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial z}$$

$$\frac{\partial \ddot{z}}{\partial S_n^m} = \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial r}\right)\frac{\partial r}{\partial z} + \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial \psi}\right)\frac{\partial \psi}{\partial z} + \left(\frac{\partial}{\partial S_n^m}\frac{\partial U}{\partial \lambda}\right)\frac{\partial \lambda}{\partial z}$$

where

$$\frac{\partial}{\partial C_n^m}\left(\frac{\partial U}{\partial r}\right) = -\frac{1}{r}\left(\frac{GM}{r}\right)\left(\frac{R}{r}\right)^n (n+1)\cos m\lambda\, P_n^m$$

$$\frac{\partial}{\partial C_n^m}\left(\frac{\partial U}{\partial \psi}\right) = \left(\frac{GM}{r}\right)\left(\frac{R}{r}\right)^n \cos m\lambda \left(P_n^{m+1} - m\tan\psi\, P_n^m\right)$$

$$\frac{\partial}{\partial C_n^m}\left(\frac{\partial U}{\partial \lambda}\right) = -\left(\frac{GM}{r}\right)\left(\frac{R}{r}\right)^n m\sin m\lambda\, P_n^m$$

$$\frac{\partial}{\partial S_n^m}\left(\frac{\partial U}{\partial r}\right) = -\frac{1}{r}\left(\frac{GM}{r}\right)\left(\frac{R}{r}\right)^n (n+1)\sin m\lambda\, P_n^m$$

$$\frac{\partial}{\partial S_n^m}\left(\frac{\partial U}{\partial \psi}\right) = \left(\frac{GM}{r}\right)\left(\frac{R}{r}\right)^n \sin m\lambda \left(P_n^{m+1} - m \tan \psi P_n^m\right)$$

$$\frac{\partial}{\partial S_n^m}\left(\frac{\partial U}{\partial \lambda}\right) = \left(\frac{GM}{r}\right)\left(\frac{R}{r}\right)^n m \cos m\lambda \, P_n^m \; .$$

TIMING ESTIMATES

Several runs were made on the IBM 360 model 91 which uses the OS MVT multiprogramming system. Some estimates of running time were obtained by a pair of routines called ETIMIN, ETIMOT (TIMEIN, TIMOUT). The first routine reads the system clock but returns no output and the second routine returns the time elapsed since the first routine was called. This actual elapsed time may include other users' time as well. Consequently, the estimates do vary somewhat and are usually greater than they should be.

The program was called upon repeatedly for 10,000 cases to simulate the calculations required for the same number of time points t. Some timing estimates are presented for accelerations plus position partials, and position partials only in Tables 1 and 3. The number of "significant" digits indicated in Tables 2 and 4 represents the number of digits agreeing with the standard case (degree = 15, order = 15). The first and second columns in each table show the order and degree of the harmonic coefficients respectively for the computation of accelerations. In addition the third column provides the degree of the expansion used in computing the position partials.

Table 1

Time Estimates

Order	Degree (accelerations)	Degree (position partials)	Running Time in Seconds	
			Accelerations	Accelerations Plus Position Partials
15	15	2	57.0	59.9
15	15	4	57.0	60.9
15	15	6	57.0	62.0
15	15	8	57.0	63.3
15	15	10	57.1	64.1
15	15	12	57.1	64.7
15	15	15	57.2	65.5
10	10	10	32.3	38.2
5	5	5	11.7	14.5
2	2	2	3.4	4.7

TETR-C DATA

x = −0.197 171 190 (06) Meters

y = −0.646 084 338 (07) Meters

z = +0.250 067 586 (07) Meters

Table 2

Significant Digits

Order	Degree (accelerations)	Degree (position partials)	No. of Significant Digits	
			Accelerations	Accelerations Plus Position Partials
15	15	2	16	3
15	15	4	16	3
15	15	6	16	4
15	15	8	16	4
15	15	10	16	5
15	15	12	16	5
15	15	15	16	16
10	10	10	6	5
5	5	5	4	3
2	2	2	4	3

TETR-C DATA

x = −0.197 171 190 (06) Meters

y = −0.646 084 338 (07) Meters

z = +0.250 067 586 (07) Meters

Table 3

Time Estimates

Order	Degree (accelerations)	Degree (position partials)	Running Time in Seconds		
			Accelerations	Accelerations Plus Position Partials	Position Partials
15	15	2	45.5	48.4	46.6
15	15	4	45.5	48.9	48.0
15	15	6	45.5	51.5	48.4
15	15	8	45.5	54.5	49.5
15	15	10	45.5	56.3	49.9
15	15	12	45.5	55.3	50.4
15	15	15	45.5	53.1	52.7
10	10	10	26.2	32.8	31.1
5	5	5	—	12.9	12.3
2	2	2	—	4.6	4.4

GEOS DATA

x = +0.569 053 863 879 2412 (07) Meters

y = +0.147 453 452 873 1973 (07) Meters

z = +0.601 344 521 360 5027 (07) Meters

Table 4

Significant Digits

Order	Degree (accelerations)	Degree (position partials)	No. of Significant Digits	
			Accelerations	Accelerations Plus Position Partials
15	15	2	16	4
15	15	4	16	5
15	15	6	16	5
15	15	8	16	5
15	15	10	16	5
15	15	12	16	6
15	15	15	16	16
10	10	10	6	5
5	5	5	6	5
2	2	2	5	4

GEOS DATA

x = +0.569 053 863 879 2412 (07) Meters

y = +0.147 453 452 873 1973 (07) Meters

z = +0.601 344 521 360 5027 (07) Meters

Hence, it is possible to compute position partials of the same or lower order than the accelerations. This option is available when position partials need not be computed as accurately as the accelerations.

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