NASA TECHNICAL MEMORANDUM

NASA TM X-52883

NASA TM X-52883

DECAY OF HOMOGENEOUS TURBULENCE FROM A GIVEN INITIAL STATE

by Robert G. Deissler
Lewis Research Center
Cleveland, Ohio

TECHNICAL PAPER proposed for presentation at
Twenty-third Meeting of Division Fluid Dynamics
sponsored by the American Physical Society
Charlottesville, Virginia, November 23-25, 1970

DECAY OF HOMOGENEOUS TURBULENCE FROM A GIVEN INITIAL STATE

by Robert G. Deissler

National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio

E-5761

ABSTRACT

The homogeneous turbulence problem is formulated by first specifying the multipoint velocity correlations or their spectral equivalents at an initial time. Those quantities, together with the correlation or spectral equations are then used to calculate initial time derivatives of correlations or spectra. The derivatives in turn are used in time series to calculate the evolution of turbulence quantities with time. When the problem is treated in this way the correlation equations are closed by the initial specification of the turbulence and no closure assumption is necessary. An exponential series which is an iterative solution of the Navier-Stokes equations gave much better results than a Taylor power series when used with the limited available initial data. In general, the agreement between theory and experiment was good.

INTRODUCTION

A basic difficulty in the usual analyses of homogeneous turbulence is the closure problem; that is, the set of correlation or moment equations contains more unknowns than equations. The problem occurs, of course, because of the nonlinearity of the Navier-Stokes equations from which the correlation equations are obtained.[1]

Although many approximations have been introduced into the correlation equations (or equivalent spectral equations) in attempts to obtain closure, those suggestions have varying degrees of arbitrariness. The analyses in Refs. 2 and 3, although based on definite physical ideas, contain dimensionless constants which must be determined by experiment. Those in Refs. 4 to 7, although somewhat more deductive, have other difficulties. That in Ref. 4, at least for the restricted initial condition for which it has been worked out, sometimes gives negative energies.[8] The analyses in Refs. 5 and 6 give reasonable results for moderately weak turbulence but become unduly complex for high Reynolds numbers. That in Ref. 7, although it has yielded some realistic deductions, also has computational difficulties because of its complexity.

There is another way of looking at the problem of homogeneous turbulence. In order not to lose sight of our goal we will first give a statement of that problem. The statement given by Batchelor is essentially this: given the statistical state of a homogeneous turbulent field at an initial instant, to predict the evolution of the turbulence (in probability) as a function of time. In order to completely specify a turbulent field at an initial time, it is necessary to give all of the multipoint velocity correlations or their spectral equivalents at that time.[1] It is not hard to show that, given these multipoint correlations and the correlation equations, all of the time derivatives of the turbulent energy tensor and of other pertinent turbulence quantities can be calculated. These time derivatives can then be

used in a series, for instance a Taylor series, to calculate the evolution of the turbulent energy tensor (or of the equivalent energy spectrum tensor) and of other turbulence quantities.

It is noted that when the turbulence is treated in this way, there is no longer a closure problem. The correlation equations are used only to relate the correlations at an initial time to their time derivatives, and those correlations must be given in order to have a complete specification of the turbulence at that time. Of course, in practice only a small number of the correlations, and thus of their time derivatives will ordinarily be available, but a sufficient number may be known to give a reasonably good representation. It might be pointed out that even in those analyses which require a closure assumption, the turbulence should be specified initially by its correlations or spectra since the correlation equations require initial conditions.

Kraichnan[9] has very recently studied the convergence properties of series such as those considered here. As mentioned in another paper by that author[10], it is not necessary that an expansion be convergent in order to be useful, since divergent series can provide excellent asymptotic approximations.[11]

In the present paper we will not concern ourselves primarily with convergence questions but will use as a test the agreement with experiment of the results. Although a Taylor series would no doubt give good results if sufficient statistical information were available at the initial time, it will be seen that an exponential series which arises in a study of the nonlinear decay of a disturbance in a fluid[12] is much

more satisfactory when a limited amount of initial information is available. This is not surprising since the exponential series is an iterative solution of the Navier-Stokes equations and thus contains information which is not contained in the Taylor series. The resulting formulation gives results which are in quite good agreement with the available experimental data.

INITIAL TIME DERIVATIVES AND SIMPLE EXPANSIONS

As mentioned in the introduction, if the multipoint correlations are known at an initial instant, as they must be for a complete specification of the turbulence at that instant, then the time derivatives of the correlations can be calculated from the correlation equations. For illustrative purposes we will consider the derivatives of the turbulent energy tensor $\overline{u_i u'_j}$, where u_i and u'_j are respectively velocity components at the points P and P' separated by the vector $\vec{r}$, and the overbar indicates an averaged value. Then the first time derivative of $\overline{u_i u'_j}$ at $t = t_1$ is given directly by the two-point correlation equations[5] evaluated at $t = t_1$:

$$\left(\frac{\partial \overline{u_i u'_j}}{\partial t}\right)_{t=t_1} = -\frac{\partial}{\partial r_k}\left[\left(\overline{u_i u'_j u'_k}\right)_{t=t_1} - \left(\overline{u_i u'_j u_k}\right)_{t=t_1}\right]$$

$$-\left[\frac{1}{\rho}\frac{\partial}{\partial r_j}\left(\overline{u_i p'}\right)_{t=t_1} - \frac{\partial}{\partial r_i}\left(\overline{p u'_j}\right)_{t=t_1}\right] + 2\nu \frac{\partial^2 \left(\overline{u_i u'_j}\right)_{t=t_1}}{\partial r_k \partial r_k} \qquad (1)$$

where the pressure-velocity correlations are given by

$$\frac{1}{\rho}\frac{\partial^2(\overline{pu_j'})_{t=t_1}}{\partial r_k \partial r_k} = -\frac{\partial^2(\overline{u_\ell u_k u_j'})_{t=t_1}}{\partial r_k \partial r_\ell} \qquad (2)$$

and a similar equation for $(\overline{u_i p'})_{t=t_1}$. The pertinent solution of Eq. (2) is[1]

$$\frac{1}{\rho}(\overline{pu_j'})_{t=t_1} = \frac{1}{4\pi}\int \frac{1}{|\vec{r}-\vec{s}|}\frac{\partial^2(\overline{u_i'' u_k'' u_j'})_{t=t_1}}{\partial s_i \partial s_k}\, d\vec{s},$$

where u_j'' is the velocity at the point $\vec{x}'' = \vec{x}' - \vec{s}$, and the integration is over all $\vec{s}$ space. This solution is for an infinite fluid, for which case the boundary conditions are that $\overline{pu_j'}$ is bounded for $\vec{r} = 0$ and zero for $\vec{r} = \infty$. The quantity ρ is the density, ν is the kinematic viscosity and p is the pressure. A repeated subscript in a term indicates a summation, with the subscript successively taking on the values 1, 2, and 3. The correlation equations are, of course, derived from the Navier-Stokes equations. The quantity $\partial\overline{u_i u_j'}/\partial t$ at $t=t_1$ can be calculated from Eqs. (1) and (2) if $\overline{u_i u_j'}$ and the two-point triple correlations are known at $t=t_1$.

The second time derivative of $\overline{u_i u_j'}$ is obtained by differentiating the two=point correlation equations and evaluating the result at t_1. This gives

$$\left(\frac{\partial^2 \overline{u_i u'_j}}{\partial t^2}\right)_{t=t_1} = -\frac{\partial}{\partial r_k}\left[\left(\frac{\partial}{\partial t}\,\overline{u_i u'_j u'_k}\right)_{t=t_1} - \left(\frac{\partial}{\partial t}\,\overline{u_i u'_j u_k}\right)_{t=t_1}\right]$$

$$-\frac{1}{\rho}\left[\frac{\partial}{\partial r_j}\left(\frac{\partial \overline{u_i p'}}{\partial t}\right)_{t=t_1} - \frac{\partial}{\partial r_i}\left(\frac{\partial \overline{p u'_j}}{\partial t}\right)_{t=t_1}\right]$$

$$+ 2\nu \frac{\partial^2}{\partial r_k \partial r_k}\left(\frac{\partial \overline{u_i u'_j}}{\partial t}\right)_{t=t_1} \qquad (3)$$

and

$$\frac{1}{\rho}\frac{\partial^2}{\partial r_k \partial r_k}\left(\frac{\partial \overline{p u'_j}}{\partial t}\right)_{t=t_1} = -\frac{\partial^2}{\partial r_k \partial r_\ell}\left(\frac{\partial \overline{u_\ell u_k u'_j}}{\partial t}\right)_{t=t_1} \qquad (4)$$

The quantity $[(\partial/\partial t)(\overline{u_i u'_j u'_k})]_{t=t_1}$ in Eq. (3) is obtained from the three-point correlation equations[5] written for $t=t_1$ and $\vec{r}' = \vec{r}$. (The vector $\vec{r}'$ separates the points P and P".) Thus

$$\left(\frac{\partial}{\partial t}\ \overline{u_i u_j' u_k'}\right)_{t=t_1} = \left\{ \frac{\partial}{\partial r_\ell}\left(\overline{u_i u_j' u_k'' u_\ell}\right)_{t=t_1}\right.$$

$$+ \frac{\partial}{\partial r_\ell'}\left(\overline{u_i u_j' u_k'' u_\ell}\right)_{t=t_1} - \frac{\partial}{\partial r_\ell}\left(\overline{u_i u_j' u_k'' u_\ell'}\right)_{t=t_1}$$

$$- \frac{\partial}{\partial r_\ell'}\left(\overline{u_i u_j' u_k'' u_\ell''}\right)_{t=t_1} - \frac{1}{\rho}\left[- \frac{\partial}{\partial r_i}\left(\overline{p u_j' u_k''}\right)_{t=t_1}\right.$$

$$- \frac{\partial}{\partial r_i'}\left(\overline{p u_j' u_k''}\right)_{t=t_1} + \frac{\partial}{\partial r_j}\left(\overline{p' u_i u_k''}\right)_{t=t_1}$$

$$\left. + \frac{\partial}{\partial r_k'}\left(\overline{p'' u_i u_j'}\right)_{t=t_1}\right] + 2\nu\left[\frac{\partial^2\left(\overline{u_i u_j' u_k''}\right)_{t=t_1}}{\partial r_\ell \partial r_\ell}\right.$$

$$\left.\left. + \frac{\partial^2\left(\overline{u_i u_j' u_k''}\right)_{t=t_1}}{\partial r_\ell \partial r_\ell'} + \frac{\partial^2\left(\overline{u_i u_j' u_k''}\right)_{t=t_1}}{\partial r_\ell' \partial r_\ell'}\right]\right\}_{\vec{r}' = \vec{r}} \qquad (5)$$

where $\left(\overline{p u_j' u_k''}\right)_{t=t_1}$ is given by

$$\frac{1}{\rho}\left[\frac{\partial^2(\overline{pu_j'u_k''})_{t=t_1}}{\partial r_\ell \partial r_\ell} + 2\,\frac{\partial^2(\overline{pu_j'u_k''})_{t=t_1}}{\partial r_\ell \partial r_\ell'}\right.$$

$$\left. + \frac{\partial^2(\overline{pu_j'u_k''})_{t=t_1}}{\partial r_\ell' \partial r_\ell'}\right] = -\frac{\partial^2(\overline{u_\ell u_m u_j' u_k''})_{t=t_1}}{\partial r_m \partial r_\ell}$$

$$- \frac{\partial^2(\overline{u_\ell u_m u_j' u_k''})_{t=t_1}}{\partial r_m \partial r_\ell'} - \frac{\partial^2(\overline{u_\ell u_m u_j' u_k''})_{t=t_1}}{\partial r_m' \partial r_\ell}$$

$$- \frac{\partial^2(\overline{u_\ell u_m u_j' u_k''})_{t=t_1}}{\partial r_m' \partial r_\ell'} \qquad (6)$$

Similar equations are obtained for the other pressure-velocity correlations. The boundary conditions for Eq. (6) are similar to those for Eq. (2); that is, $\overline{pu_j'u_k''}$ is bounded for $\vec{r}$ or $\vec{r}' = 0$ and zero for $\vec{r}$ or $\vec{r}' = \infty$. Also, an expression for $\left[(\partial/\partial t)(\overline{u_i u_j' u_k})\right]_{t=t_1}$ in Eq. (3) is obtained by letting $\vec{r}' = 0$ instead of $\vec{r}' = \vec{r}$ in Eq. (5). Thus, if the turbulence is specified sufficiently well at $t=t_1$ that the double, triple and quadruple velocity correlations are known, $(\partial^2\overline{u_i u_j'}/\partial t^2)_{t=t_1}$ can be calculated. Similarly higher-order derivatives are obtained by considering four or more point correlations in the turbulent field[6]. With the time derivatives of $\overline{u_i u_j'}$ known at $t=t_1$, a Taylor series gives $\overline{u_i u_j'}$ as a function of time as

$$\overline{u_i u_j'} = (\overline{u_i u_j'})_{t=t_1} + \left(\frac{\partial \overline{u_i u_j'}}{\partial t}\right)_{t=t_1} (t - t_1)$$

$$+ \frac{1}{2!} \left(\frac{\partial^2 \overline{u_i u_j'}}{\partial t^2}\right)_{t=t_1} (t - t_1)^2 + \ldots \qquad (7)$$

A similar analysis can be carried out in wave number space. For instance, the energy spectrum function E, which shows the contributions at various wave numbers to $\overline{u_i u_1'}/2$, can be written as

$$E = (E)_{t=t_1} + \left(\frac{\partial E}{\partial t}\right)_{t=t_1} (t - t_1) + \frac{1}{2!} \left(\frac{\partial^2 E}{\partial t^2}\right)_{t=t_1} (t - t_1)^2 + \ldots$$

where $\partial E/\partial t$ is obtained from the Fourier transform of the two-point correlation equation (Eq. (9) in ref. 5) as

$$\frac{\partial E(\kappa)}{\partial t} = \int_A \frac{1}{2}\left\{ - 2\nu\kappa^2 \phi_{ii}(\vec{\kappa}) + i\kappa_k \left[\phi_{iki}(\vec{\kappa}) - \phi_{iki}(-\vec{\kappa})\right]\right\} dA(\kappa)$$

where dA is an element of surface area of a sphere of radius κ, $\vec{\kappa}$ is the wave number vector corresponding to the spacial vector $\vec{r}$, and φ_{ii} and φ_{iki} are respectively the Fourier transforms of $\overline{u_i u_i'}$ and $\overline{u_i u_k u_i'}$. Extracting from the integral that portion which can be written in terms of E, and setting the rest of the integral equal to T, gives

$$\frac{\partial E}{\partial t} = T - 2\nu\kappa^2 E \tag{10}$$

Equation (10) is the well known scalar form of the two-point spectral equation. The transfer term T produces energy transfer between wave numbers and arises from the triple correlation term in Eq. (1) (with $i=j$).[1] (Note that the pressure-velocity correlation terms in Eq. (1) drop out for $i=j$.) The second time derivative of E is

$$\left(\frac{\partial^2 E}{\partial t^2}\right)_{t=t_1} = \left(\frac{\partial T}{\partial t}\right)_{t=t_1} - 2\nu\kappa^2\left(\frac{\partial E}{\partial t}\right)_{t=t_1}$$

$$= \left(\frac{\partial T}{\partial t}\right)_{t=t_1} - 2\nu\kappa^2(T)_{t=t_1} + (2\nu\kappa^2)^2(E)_{t=t_1}$$

The quantity $(\partial T/\partial t)_{t=t_1}$ can be calculated from the two- and three-point spectral equations if the two- and three-point spectral quantities in those equations are known at $t=t_1$. From Eqs. (20, (23), and (24) in reference 5 we obtain

$$\frac{\partial T}{\partial t} = \int_A \int_{-\infty}^{\infty} \frac{1}{2} \left\{ -2\nu\kappa^2 \left[i\kappa_k \left(\beta_{iik}(\vec{\kappa}) - \beta_{iik}(-\vec{\kappa}) \right) \right] \right.$$

$$\left. + f(\beta_{ijk}, \beta_{ijk\ell}) \right\} d\vec{\kappa}' \, dA(\kappa)$$

where $\vec{\kappa}'$ is the wave number vector corresponding to $\vec{r}'$, $d\kappa = d\kappa_1 \, d\kappa_2 \, d\kappa_3$, and β_{ijk} and $\beta_{ijk\ell}$ are respectively the Fourier

transforms of $\overline{u_i u_j' u_k''}$ and $\overline{u_i u_j u_k' u_\ell''}$. If by analogy with the procedure used for obtaining Eq. (10), we extract from the integral that portion which can be written in terms of spectral quantities already defined (E and T), we have

$$\frac{\partial T}{\partial t} + 2\nu\kappa^2 T = \int_A \int_{-\infty}^{\infty} \frac{1}{2} f(\beta_{ijk}, \beta_{ijk\ell}) d\vec{\kappa}' \, dA(\kappa)$$

$$= V(\beta_{ijk}, \beta_{ijk\ell}) \tag{11}$$

where V is a quantity related to the three-point spectral tensors β_{ijk} and $\beta_{ijk\ell}$. More precisely we can say that V is a functional of β_{ijk} and $\beta_{ijk\ell}$, since each value of V depends on values of β_{ijk} and $\beta_{ijk\ell}$ at all points of $\vec{\kappa}'$ space. With Eq. (11), the expression for $(\partial^2 E/\partial t^2)_{t=t_1}$ becomes

$$\left(\frac{\partial^2 E}{\partial t^2}\right)_{t=t_1} = (V)_{t=t_1} - 4\nu\kappa^2 (T)_{t=t_1} + (2\nu\kappa^2)^2 (E)_{t=t_1} \tag{12}$$

The Taylor series for E then becomes

$$E = (E)_{t=t_1} + \left[(T)_{t=t_1} - 2\nu\kappa^2 (E)_{t=t_1}\right](t - t_1)$$

$$+ \frac{1}{2!}\left[(V)_{t=t_1} - 4\nu\kappa^2 (T)_{t=t_1} + (2\nu\kappa^2)^2 (E)_{t=t_1}\right](t - t_1)^2$$

$$+ \ldots \tag{13}$$

Equation (13) was used in conjunction with available experimental data at an initial time[13] in an attempt to calculate the variation with time of E and thus of $\overline{u_i u_i}$. However, with the available initial data ($(E)_{t=t_1}$, $(T)_{t=t_1}$, and $(V)_{t=t_1}$), reasonable results were not obtained except at small times (Fig. 2). It thus appears that in order to obtain good results by using a simple Taylor series, initial statistical information of much higher order than that which is available would have to be given. Thus, an alternative approach which makes more efficient use of the initial statistical information and also incorporates additional information from the equations of motion will be considered.

A WORKABLE FORMULATION FOR THE DEVELOPMENT OF TURBULENCE FROM A GIVEN INITIAL STATE

In order to obtain a more efficient means for calculating the evolution of turbulence than by a Taylor series in time, we consider an iterative solution of the Navier-Stokes equations similar to that in Ref. 12. In addition to the initial statistical information and calculated time derivatives we will then have information about the form of the decay law from the equations of motion.

Although attention was confined to determinate initial conditions in Ref. 12, for the present purposes we can just as well assume the initial velocity fluctuations to be random or turbulent. Thus, we consider a field of homogeneous turbulence to be made up of a very high density of eddies or harmonic disturbances in wave number space. For all practical purposes then, since the density of disturbances is very

high, the spectrum of the turbulence can be considered continuous. The velocity and pressure at any point in the field are given by

$$\frac{\partial u_i}{\partial t} - \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial (u_i u_k)}{\partial x_k} \tag{16}$$

and

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_k \partial x_k} = - \frac{\partial^2 (u_k u_\ell)}{\partial x_k \partial x_\ell} \tag{17}$$

The latter equation is obtained by taking the divergence of Eq. (16) and applying the continuity equation.

From the spectrum of harmonic disturbances we arbitrarily select two cosine terms with wave number vectors $\vec{q}$ and $\vec{r}$. Then, the velocity associated with those disturbances will be

$$u_i^{cc} = a_i \cos \vec{q} \cdot \vec{x} + b_i \cos \vec{r} \cdot \vec{x} \tag{18}$$

where the superscript cc on the velocity indicates that it depends on two cosine terms. The results that follow would be the same if two sine terms or a sine and a cosine term were considered. If u_i^{cc} is substituted for u_i in the right sides of Eqs. (16) and (17), the time variations of a_i and b_i plus additional harmonic terms are obtained. If we then substitute that new expression into Eqs. (16) and (17), another expression containing still more harmonic terms is obtained. In each approximation, the linear terms of the Navier-Stokes

equations are considered as unknown and the nonlinear terms as known from the preceding approximation. As shown in Ref. 12, continuation of this process leads to

$$u_i^{cc} = \sum_{\vec{K}} \left(A_{i,\vec{K}}^{c'} \cos \vec{K} \cdot \vec{x} + A_{i,\vec{K}}^{s'} \sin \vec{K} \cdot \vec{x} \right) \tag{19}$$

where

$$A_{i,\vec{K}}^{c'} = \sum_{q} a_{i,K,q}^{c} \, e^{-b_{\vec{K},q}^{c} (t-t_1)} \tag{20}$$

and

$$A_{i,\vec{K}}^{s'} = \sum_{r} a_{i,\vec{K},r}^{s} \, e^{-b_{\vec{K},r}^{s} (t-t_1)} \tag{21}$$

Comparison of Eqs. (19) to (21) with the first and second approximations in Ref. 12 shows that $b_{\vec{K},1}^{c} = \nu K^2$ and $b_{\vec{K},1}^{s} = \nu K^2$. Also, we note that since the two harmonic components in Eq. (18) were selected arbitrarily, expressions similar to Eqs. (19) to (21) will be obtained for any other two components. But the nonlinear interaction of any number of harmonic components can be expressed as the sum of the interactions of pairs of components (Eqs. (37) and (38), Ref. 12). Thus u_i, the velocity resulting from all the harmonic components will be of the form of equations (19) to (21), and can be written as

$$u_i = \sum_{\vec{\kappa}} \left(A^c_{i,\vec{\kappa}} \cos \vec{\kappa} \cdot \vec{x} + A^s_{i,\vec{\kappa}} \sin \vec{\kappa} \cdot \vec{x} \right) \tag{22}$$

where

$$A^{(\,)}_{i,\vec{\kappa}} = a^{(\,)}_{i,\vec{\kappa},1} \, e^{-\nu\kappa^2(t-t_1)} + \sum_{\substack{q \\ q\neq 1}} a^{(\,)}_{i,\vec{\kappa},q} \, e^{-b^{(\,)}_{\vec{\kappa},q}(t-t_1)} \tag{23}$$

The summations in Eqs. (22) and (23) will of course contain many orders of magnitude more terms than those in Eqs. (19) to (21). Since the initial conditions are random, the quantities $A^{(\,)}_{i,\vec{\kappa}}$, $a^{(\,)}_{i,\vec{\kappa},q}$, and $b^{(\,)}_{\vec{\kappa},q}$ are assumed to be random variables. The space-averaged value of u_i^2 (no sum on i) is obtained from Eq. (22) by squaring, integrating over a cycle, and using the orthogonality property of sines and cosines. This gives

$$\overline{u_i^2} = \sum_{\vec{\kappa}} \frac{1}{2} \left[\left(A^c_{i,\vec{\kappa}} \right)^2 + \left(A^s_{i,\vec{\kappa}} \right) \right]^2 \tag{24}$$

where

$$\left(A^{(\,)}_{i,\vec{\kappa}} \right)^2 = \left(a^{(\,)}_{i,\vec{\kappa},1} \right)^2 e^{-2\nu\kappa^2(t-t_1)} + \sum_{\substack{q \\ q\neq 1}} \left(a^{(\,)}_{i,\vec{\kappa},q} \right)^2 e^{-2b^{(\,)}_{\vec{\kappa},q}(t-t_1)} \tag{25}$$

In obtaining Eq. (25) the various $a^{(\ )}_{i,\vec{\kappa},q}$ were assumed uncorrelated. According to Eq. (25), $\langle(A^{c}_{i,\vec{\kappa}})^2\rangle$ and $\langle(A^{s}_{i,\vec{\kappa}})^2\rangle$ in Eq. (24) have the same form, so that we do not need to carry along the superscripts c and s.

We want to obtain an averaged form of Eq. (25) which is a smoothed function of the magnitude of the vector $\vec{\kappa}$ (but not of its direction). In order to do that, we divide the interval of $\kappa = (\kappa_i\kappa_i)^{1/2}$ over which disturbances occur into a large number of small increments $\Delta\kappa$. The terms in $\sum_{\vec{\kappa}}$ in Eqs. (24) and (25) are divided into groups each of which corresponds to a particular $\Delta\kappa$. (Note that while the magnitudes of the various vectors lying in a particular $\Delta\kappa$ are approximately equal, their directions can of course vary.) The group of terms corresponding to each $\Delta\kappa$ is then subdivided into groups in each of which the values of the $b_{i,\vec{\kappa},q}$ in $\sum_{\substack{q\\ q\neq 1}}$ do not vary appreciably from a value of $b_s(\kappa)$. The index s designates a particular increment in the values of the $b_{i,\vec{\kappa},q}$. Also, for each s, $a^2_{i,\vec{\kappa},q}$ will have an average value which we designate by $\langle a^2_{i,\vec{\kappa},}\rangle_s$. The summation $\sum_{\substack{q\\ q\neq 1}}$ in eq. (25), which applies to a particular $\vec{\kappa}$, is then replaced by

$$\sum_s n_{s,(i)} \left\langle a^2_{i,\vec{\kappa}} \right\rangle_{s(\kappa)} e^{-2b_s(\kappa)(t-t_1)}$$

which applies to a particular $\Delta\kappa$, and where $n_{s,(i)}$ is the number of terms in $\sum_{\substack{q \\ q\neq 1}}$ which are assigned to the group s for the component i. The parenthesis on i indicates that there is no summation on that subscript. Then the average value of $A^2_{i,\vec{\kappa}}$ in the increment $\Delta\kappa$ becomes (see Eq. (25)).

$$\left\langle A^2_{i,\vec{\kappa}} \right\rangle (\kappa) = \left\langle a^2_{i,\vec{\kappa},1} \right\rangle (\kappa)\, e^{-2\nu\kappa^2(t-t_1)}$$

$$+ \sum_s \left(\frac{n_s}{n_\kappa}\right)_{(i)} \left\langle a^2_{i,\vec{\kappa}} \right\rangle_s (\kappa) e^{-2b_s(\kappa)(t-t_1)} \qquad (26)$$

where n_κ is the number of terms in $\sum_{\substack{q \\ q\neq 1}}$ that lie in $\Delta\kappa$.

The expression for $\overline{u_i^2}$ (Eq. (24)) then becomes

$$\overline{u_i^2} = \sum_\kappa \left[\left\langle a^2_{i,\vec{\kappa},1} \right\rangle (\kappa)\, e^{-2\nu\kappa^2(t-t_1)} \right.$$

$$\left. + \sum_s \left(\frac{n_s}{n_\kappa}\right)_{(i)} \left\langle a^2_{i,\vec{\kappa}} \right\rangle_s (\kappa)\, e^{-2b_s(\kappa)(t-t_1)} \right] \qquad (27)$$

To obtain an expression for the energy spectrum function E, we note that[1]

$$\frac{1}{2}\overline{u_i u_i} = \int_0^\infty E \, d\kappa \tag{28}$$

where $\overline{u_i u_i} = \overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}$

Equations (27) and (28) then give

$$\int_0^\infty E \, d\kappa = \sum_\kappa \frac{1}{2} \left[\frac{\left\langle a_{i,\vec{\kappa},1} a_{i,\vec{\kappa},1} \right\rangle}{\Delta\kappa} e^{-2\nu\kappa^2(t-t_1)} \right.$$

$$\left. + \sum_s \left(\frac{n_s}{n_\kappa}\right)_i \frac{\left\langle a^2_{i,\vec{\kappa}\,s} \right\rangle}{\Delta\kappa} e^{2b_s(t-t_1)} \Delta\kappa \right] \tag{29}$$

where there is now a summation on i. If $\Delta\kappa$ is very small, we can write, to a very good approximation,

$$E(\kappa) = B^2(\kappa) \, e^{-2\nu\kappa^2(t-t_1)} + \sum_s B_s^2(\kappa) \, e^{-2b_s(\kappa)(t-t_1)} \tag{30}$$

Equation (30) gives the evolution in time of the energy spectrum function from an initial state which is specified by the B's and b's in the equation.

As shown in the last section, if the turbulence is specified at an initial instant, the time derivatives of E can be calculated at

that instant by using the Fourier transformed correlation equations. Thus, it is desirable to write the B's and b's in Eq. (30) in terms of E and its derivatives at the initial time. That can be done by evaluating Eq. (30) and its time derivatives at $t = t_1$ and solving the resulting system of equations for the B's and b's.

In what follows, we will first retain only two terms of Eq. (30), since we evidently do not have initial experimental data available to evaluate the unknown functions in additional terms. (The equation resulting from the retention of three terms will be considered later.) Equation (30) can then be written conveniently as

$$E = (E)_{t=t_1}\left[C(\kappa)\, e^{-2\nu\kappa^2(t-t_1)} + (1 - C)\, e^{-2b(\kappa)(t-t_1)}\right] \quad (31)$$

where $0 \leq C \leq 1$.

For $C = 1$ Eq. (31) reduces to the well-known expression for the final period of decay.[1] For the general case ($C \neq 1$) we could determine C and b in terms of the first and second derivatives of Eq. (31) for $t=t_1$ and then evaluate those derivatives by using the two-point spectral equations (see Eqs. (10) to (12)). The following procedure turns out to be simpler however. By substituting Eq. (31) into the spectral Eq. (10) we get for the energy transfer term

$$T = 2(1 - C)(\nu\kappa^2 - b)(E)_{t=t_1}\, e^{-2b(t-t_1)}$$

$$= (T)_{t=t_1}\, e^{-2b(t-t_1)} \quad (32)$$

Then

$$\frac{\partial T}{\partial t} = -\,2b(T)_{t=t_1} = \left(\frac{\partial T}{\partial t}\right)_{t=t_1} e^{-2b(t-t_1)} \tag{33}$$

Comparing the last two members of Eq. (33) and using Eq. (11) gives

$$b = \nu\kappa^2 - \frac{(V)_{t=t_1}}{2(T)_{t=t_1}} \tag{34}$$

From Eqs. (32) and (34) we have

$$C = 1 - \frac{(T)^2_{t=t_1}}{(V)_{t=t_1}\,(E)_{t=t_1}}. \tag{35}$$

Equations (31) and (32) then become

$$E = (E)_{t=t_1}\left(C \exp\left[-2\nu\kappa^2(t - t_1)\right] + (1 - C)\exp\left\{-2\left[\nu\kappa^2 - \frac{(V)_{t=t_1}}{2(T)_{t=t_1}}\right](t - t_1)\right\}\right) \tag{36}$$

and

$$T = (T)_{t=t_1} \exp\left\{-2\left[\nu\kappa^2 - \frac{(V)_{t=t_1}}{2(T)_{t=t_1}}\right](t - t_1)\right\} \tag{37}$$

From Eq. (11)

$$V = (V)_{t=t_1} \exp\left\{-2\left[\nu\kappa^2 - \frac{(V)_{t=t_1}}{2(T)_{t=t_1}}\right](t - t_1)\right\} \tag{38}$$

where C is given by Eq. (35).

Equations (36) and (37) were obtained by retaining two terms on the right side of Eq. (30). We consider next a higher order approximation in which three terms are retained in that equation. If Eq. (30), with three terms retained, is substituted into Eq. (10), we get for T

$$T = 2B_1^2(\kappa^2 - b_1)\, e^{-2b_1(t-t_1)} + 2B_2^2(\kappa^2 - b_2)\, e^{-2b_2(t-t_1)} \tag{39}$$

Equation (39) contains four unknown functions which are to be determined by the initial conditions. For that purpose we use Eq. (39) and its first three derivatives evaluated at $t=t_1$. Thus we obtain

$$b_1 = -\frac{T_1T_2 - TT_3}{4(T_1^2 - TT_2)} + \left\{\left[\frac{T_1T_2 - TT_3}{4(T_1^2 - TT_2)}\right]^2 - \frac{T_2^2 - T_1T_3}{4(T_1^2 - TT_2)}\right\}^{1/2} \tag{40}$$

$$b_2 = -\frac{T_1T_2 - TT_3}{4(T_1^2 - TT_2)} - \left\{\left[\frac{T_1T_2 - TT_3}{4(T_1^2 - TT_2)}\right]^2 - \frac{T_2^2 - T_1T_3}{4(T_1^2 - TT_2)}\right\}^{1/2} \tag{41}$$

$$B_1^2 = \frac{2b_2T_2 + T_3}{16b_1^2(\kappa^2 - b_1)(b_2 - b_1)} \tag{42}$$

and

$$B_2^2 = \frac{2b_1T_2 + T_3}{16b_2^2(\kappa^2 - b_2)(b_1 - b_2)} \tag{43}$$

where T_1, T_2 and T_3 are the first, second, and third time derivatives of T at $t=t_1$. The first derivative T_1 can be written in terms of the functional $(V)_{t=t_1}$, which gives a representation of three point spectral quantities (Eq. (11)). Equations for higher order functionals can be obtained by the procedure used for obtaining Eq. (11) for V. Thus by using the four-point spectral equations in Ref. 6 (Eqs. (11) and (12)) we get

$$\frac{\partial V}{\partial t} = - 2\nu\kappa^2 V + R \tag{44}$$

where R is a functional of three- and four-point spectral quantities. Similarly

$$\frac{\partial R}{\partial t} = - 2\nu\kappa^2 R + S \tag{45}$$

where S is a functional of three, four, and five-point spectral quantities. By using Eqs. (11), (44) and (45), the first, second,

and third time derivatives of T at $t=t_1$ in Eqs. (40) to (43) can be written in terms of higher order spectral quantities as

$$T_1 = -2\nu\kappa^2(T)_{t=t_1} + (V)_{t=t_1} \tag{46}$$

$$T_2 = (2\nu\kappa^2)^2(T)_{t=t_1} - 4\nu\kappa^2(V)_{t=t_1} + (R)_{t=t_1} \tag{47}$$

and

$$T_3 = -(2\nu\kappa^2)^3(T)_{t=t_1} + 3(2\nu\kappa^2)^2(V)_{t=t_1} - 6\nu\kappa^2(R)_{t=t_1} + (S)_{t=t_1} \tag{48}$$

RESULTS AND DISCUSSION

A comparison between the experimental data of Uberoi[13] and the present theory (Eqs. (36) to (38)) is given in Figs. 1 to 4.[14] The comparison is made for an initial time corresponding to $X/M = 48$ in the experiment ($t_1^* = (\nu/M^2)t = 0.001818$). ($X$ is the distance downstream from the grid and M is the mesh size of the grid.) For the initial specification of the turbulence values of E and T were obtained from Figs. 5, 9, and 10 in Ref. 13. Initial values of V were not given directly in Ref. 13 but were estimated from the decay data for T and Eq. (11). Except for experimental error those values will be the same as those that might have been measured directly.

The agreement between the predicted and experimental energy spectra

for the same initial conditions (Fig. 1) appears to be quite good, considering the difficulty of the measurements. The calculation of the experimental values of E required the differentiation of measured one-dimensional spectra and an assumption of isotropy.

Predicted and experimental values for the decay of $\overline{u_i u_i}$ are plotted in Fig. 2. The agreement between theory and experiment is excellent for values of t^* up to about 0.006. (Note that spectra were measured only for values of t^* between 0.00182 and 0.00417). Elimination of the moderate deviation for $t^* > 0.006$ might require a higher order theory (more terms in Eq. (30)), together with additional initial statistical information. Alternatively the deviation might be due to the amplification at large times of slight inaccuracies in the measured initial spectra. The theoretical values for t^* less than 0.00182 were calculated by working backwards from the measured initial spectra. Also included in Fig. 2 is a Taylor series solution which uses the same initial information as the exponential series, and the curve for the weak turbulence approximation. It might be pointed out that the curve for the weak turbulence approximation is not the -5/2 power decay law usually given for the final period,[1] but is the curve obtained by using the measured initial energy spectrum and Eq. (31) with $C = 1$.

Spectra for the energy transfer term T are plotted in Fig. 3. The experimental and theoretical curves are in good agreement except near the value of κ where $(T)_{t=t_1}$ changes sign. The deviation there results from a mathematical singularity in Eq. (37) when $(T)_{t=t_1} = 0$. However, that deviation does not seem to be serious, because the real

physical curve in that region can easily be estimated. This is particularly true since it is known that the total area enclosed by the T spectrum should be zero.[1] It appears likely that the difficulty could be eliminated if another term were retained in Eq. (30). (More will be said about that possibility in the next paragraph.) The deviation also carries over to some extent into the results for E and $\overline{u_i u_i}$. However, if one does not use values of κ close to the point where $(T)_{t=t_1}$ changes sign for calculating E and $\overline{u_i u_i}$, the inaccuracies in those quantities will be small. It appears that the overall agreement between theory and experiment obtained by using equations (35) to (37) should be considered encouraging.

For the sake of completeness, spectra of the functional V (Eqs. (11) and (38)), the third initial condition specified for the turbulence, are plotted in Fig. 4. The agreement between theory and experiment is probably within the uncertainty in estimating V from the decay data in Ref. 13, except in the vicinity of the point where $(T)_{t=t_1}$ changes sign. Thus the theory predicts the evolution in time of E, T, and V, when those quantities are specified at an initial time.

We have not been able to apply a higher order theory to Uberoi's data, that is, to evaluate three instead of two terms in Eq. (30) by using the initial data given in his paper. However, we can apply a higher order theory to an analysis in Ref. 5, since for that analysis we can, in effect, calculate as much initial information as is desired. That analysis neglects quadruple correlation terms in the three-point correlation equations and should apply, for a particular set of initial conditions, at times somewhat before the final period of decay. The

initial conditions, as well as values at later times, are given by closed form equations in that analysis and thus are better defined than may be possible in an experiment. For the present purposes, the analytical results from Ref. 5 might in fact be thought of as experimental results in which the initial conditions are specified exactly. This is because for the model chosen, the analysis is exact, and the initial conditions used in both that analysis and the present theory correspond to (or are a part of) that model.

The case considered here corresponds to Fig. 6 of Ref. 5. Values of dimensionless E, T, and time derivatives of T for the initial specification of the turbulence ($t_1^* = 0.002$) are obtained from Eqs. (40) and (39) in Ref. 5. We can eliminate the time derivatives of T by introducing V (Eq. (11)) and the higher order functionals R and S (Eqs. (44) and (45)). In the present case, those quantities will all be representations of correlations of order no higher than the third, since terms involving correlations of higher order than the third are assumed negligible in the analysis of Ref. 5.

Figure 5 gives a comparison between results for T calculated from the present analysis and those from Ref. 5. The quantity J_0 is a constant related to conditions at $t_0^* = -0.00633$ in the equations of Ref. 5. The starred quantities in Figs. 5 to 7 are the same as those in Figs. 1 to 4 if we let $J_0 = M^3\nu^2$. As expected, when T is calculated from Eq. (37), the agreement with Ref. 5 is good except in the region where $(T)_{t=t_1}$ changes sign. However, when a higher order theory is used by retaining three terms in the expression for E (two terms in expression for T) (Eq. (39)) the agreement is excellent at all values of κ. It might be expected

that a similar improvement would be obtained in Figs. 3 and 4 if a higher order theory could be used for comparison with the experimental data of Uberoi.

Because of the good agreement obtained for T in Fig. 5, one would expect the calculated energy spectra E to also be in good agreement with those from Ref. 5. Figure 6 shows that that is indeed the case. In order to show the effects of energy transfer between wave numbers, curves for the final period of decay (first term of Eq. (40) of Ref. 5) are also included in Fig. 6.

Figures 7 to 9 show plots for the decay of the higher-order spectral quantities V, R, and S. The agreement between the present higher order theory and the results of Ref. 5 is very good. Thus by specifying the initial conditions for E, T, V, R, and S, we can predict the evolution in time of those quantities by using the present higher order theory. That is, the required number of initial conditions is no greater than the number of quantities whose decay we can predict.

CONCLUDING REMARKS

If a homogeneous turbulent field is specified at an initial instant by its multipoint-velocity correlations (or their spectral equivalents), the initial time derivatives of those quantities can be calculated from the correlation or spectral equations. The development of the turbulence in time can then be obtained by using those derivatives in a series such as a Taylor power series. When the problem is formulated in this way, an assumption for closing the system of correlation equations is not required, since those equations are closed by the initially specified correlations or spectral quantities. A Taylor series expansion, however, did not give

realistic results (except for small times) when the limited initial experimental data were used. An exponential series (Eq. (30)), which is an iterative solution of the Navier-Stokes equations worked much better with the limited initial information.

In general, when the energy and transfer spectra and a quantity related to three-point spectra were specified at an initial time, the predicted changes with time of those spectra, as well as the turbulent energy, were in good agreement with experiment. Since the prediction of the changes of those spectra with time is evidently an essential part of the homogeneous turbulence problem, the results are encouraging.

A higher-order theory was given in which the above quantities, as well as two additional higher order spectral quantities, were specified initially. Very good agreement was obtained between the predicted decay of all of those quantities and the results for a previous analytical model (Ref. 5). For the present purposes the results for the previous model might be thought of as experimental results in which the initial conditions are specified exactly. Thus when the results from the present theory are compared with either experimental results or the results of an "analytical experiment," the agreement is good, and the number of specified initial conditions need be no greater than the number of quantities whose decay we can predict.

FOOTNOTES

1. G. K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, New York, 1953).

2. W. Heisenberg, Zeit. f. Phys., 124, 628 (1948).

3. L. S. G. Kovasznay, J. Aeron. Sci., 15, 745 (1948).

4. I. Proudman and W. H. Reid, Phil. Trans. Roy. Soc. (London) 247A, 163 (1954).

5. R. G. Deissler, Phys. Fluids 1, 111 (1958).

6. R. G. Deissler, Phys. Fluids 3, 176 (1960).

7. R. H. Kraichman, Phys. Fluids 8, 575 (1965).

8. Y. Ogura, J. Fluid Mech. 16, 33 (1963).

9. R. H. Kraichnan, J. Fluid Mech. 41, 189 (1970).

10. S. I. Pai, Dynamics of Fluids and Plasmas (Academic Press, New York, 1966), p. 239.

11. A. Erdelyi, Asymptotic Expansions (Dover Publications, New York, 1956).

12. R. G. Deissler, Applied Sci. Res. 21, 393 (1970). (See also NASA TN D-4947).

13. M. S. Uberoi, Phys. Fluids, 6, 1048 (1963).

14. Another pertinent experimental investigation is that of C. W. Van Atta, who recently measured directly the individual terms in the two-point spectral equation (preliminary report from University of California, San Diego). However, data are given at only one time in that work.

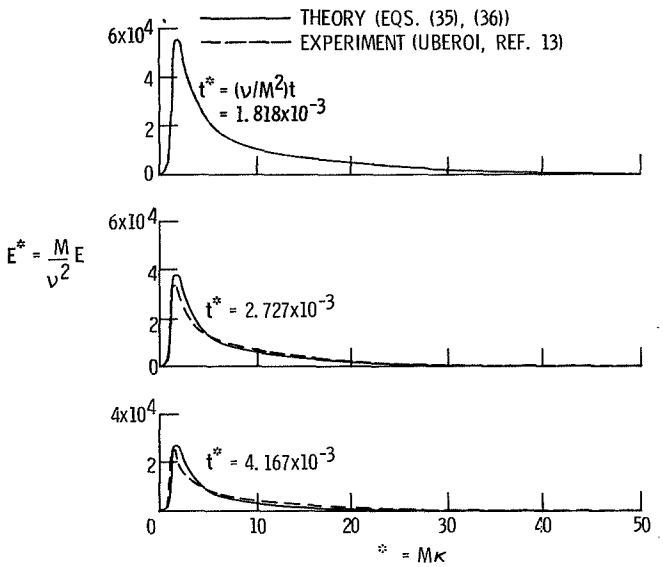

Figure 1. - Comparison of theory and experiment for decay of turbulent energy spectra.

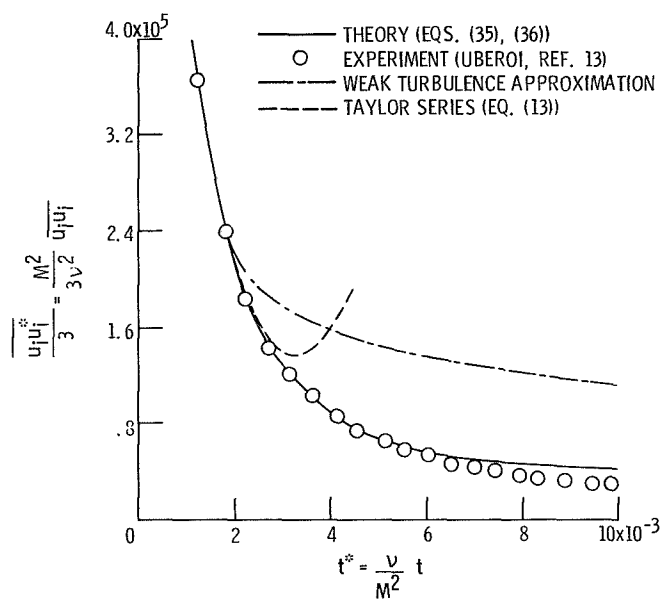

Figure 2. - Comparison of theory and experiment for decay of average component of velocity variance.

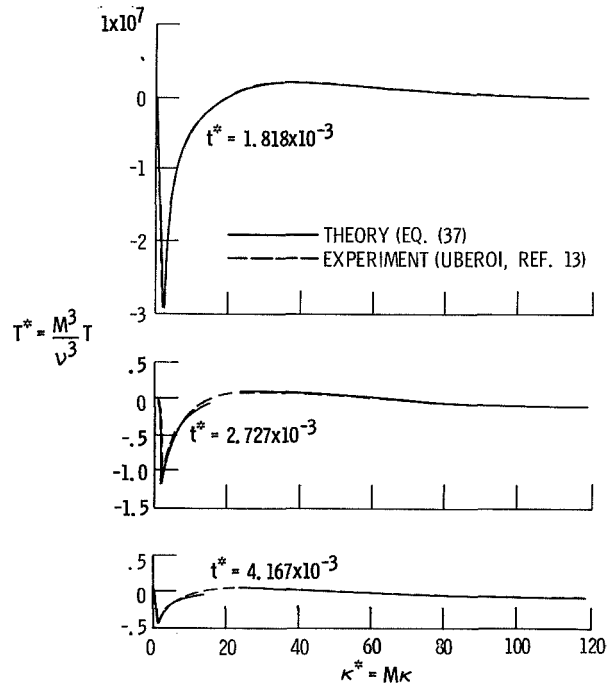

Figure 3. - Comparison of theory and experiment for decay of energy transfer spectra.

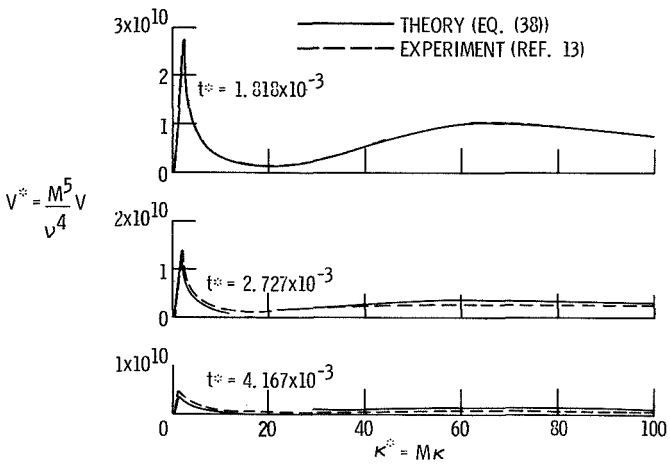

Figure 4. - Comparison of theory and experiment for decay of higher order spectral quantity V (Eq. (11)).

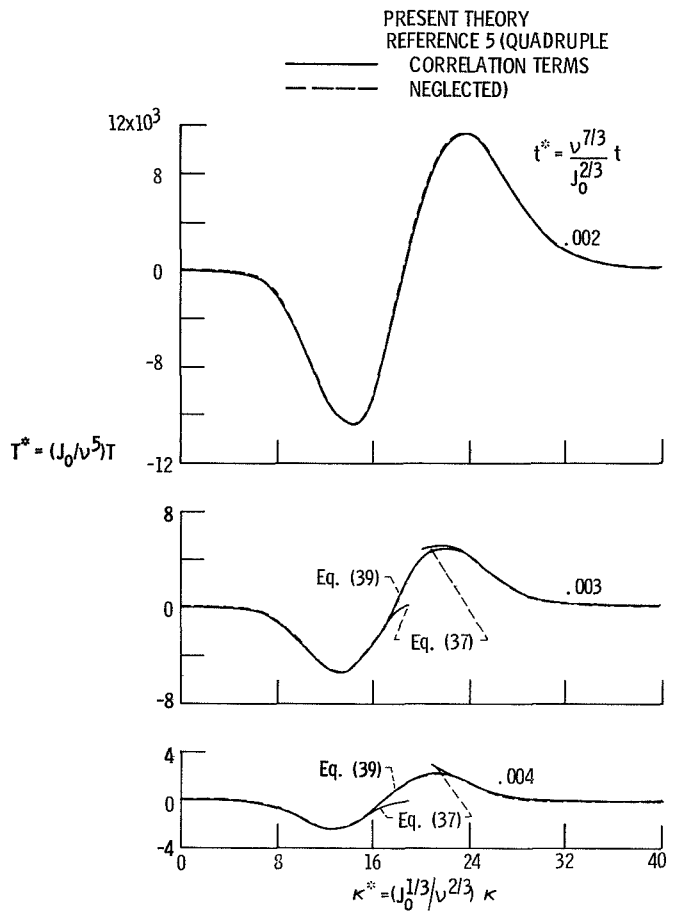

Figure 5. - Comparison of present theory with that of reference 5 for decay of energy transfer spectrum.

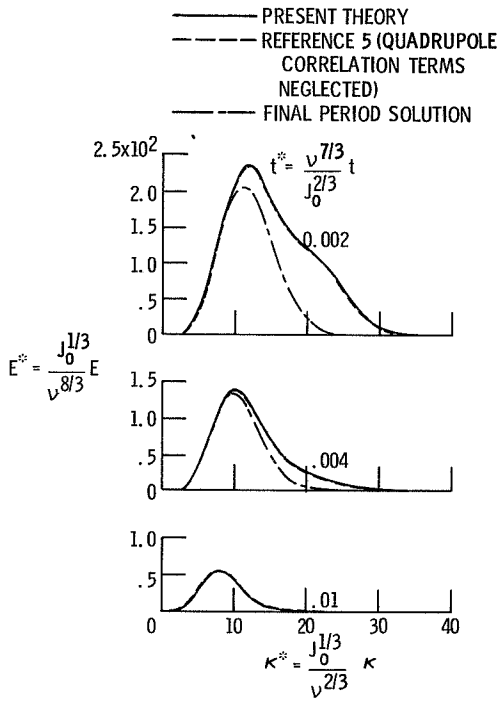

Figure 6. - Comparison of present theory with that of reference 5 for decay of turbulent energy spectrum.

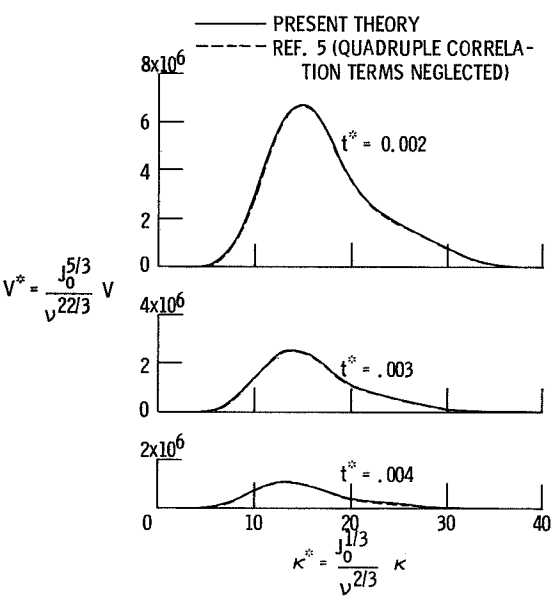

Figure 7. - Comparison of present (higher order) theory with that of reference 5 for decay of the higher order spectral quantity V (Eq. (11)).

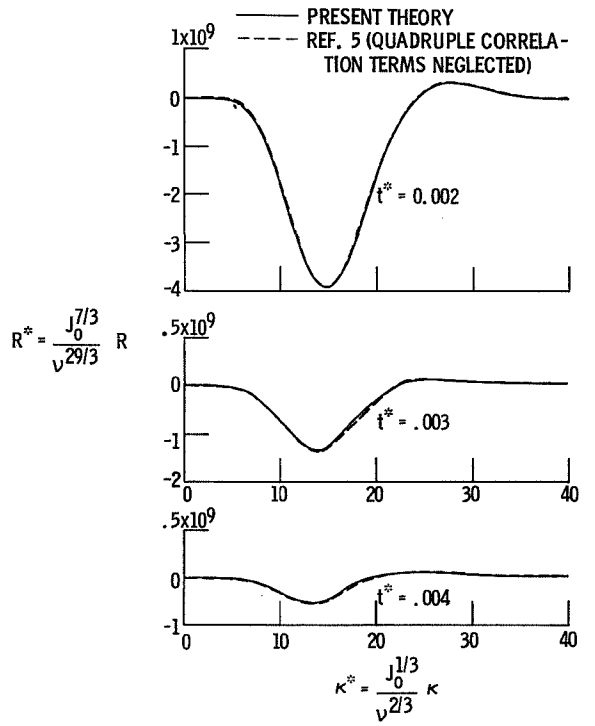

Figure 8. - Comparison of present (higher order) theory with that of reference 5 for decay of the higher order spectral quantity R (Eq. (44)).

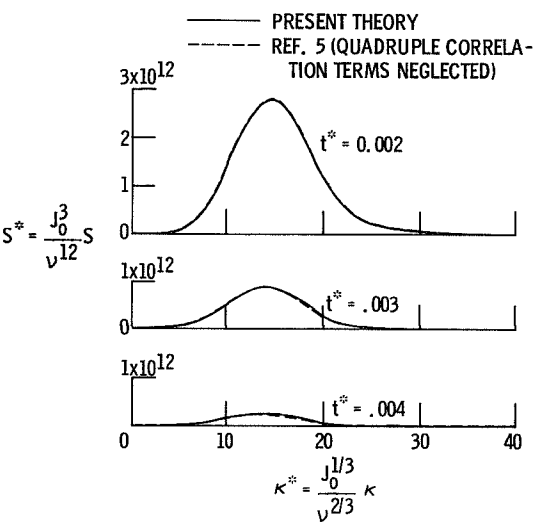

Figure 9. - Comparison of present (higher order) theory with that of reference 5 for decay of the higher order spectral quantity S (Eq. (45)).