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THE FLUCTUATION OF COSMIC RAY ANISOTROPY AND THE DIMENSIONALITY OF PROPAGATION

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The relationship between the dimensionality of the cosmic ray propagation function and the statistical distribution of anisotropies is demonstrated and an argument is presented in favor of "essentially" one dimensional propagation. This implies that a fluctuation explanation of low observed anisotropy can not be ruled out as has been stated by previous authors.

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In a recent letter Ramaty et al[1] report results of a Monte Carlo calculation in which the injection of cosmic rays into the galaxy is considered to be a sequence of random discrete events in space time. In their letter they state that their results are in conflict with a suggestion[2] of the present author that small values of the cosmic ray anisotropy could result from the statistical nature of the injection mechanism. This remark is based on the result[3] that in the distribution of anisotropies small values are suppressed and the maximum liklihood value is of the order of the R.M.S. value. Ramaty et al also state[1] that a fluctuation origin of small anisotropy is only possible in the case of "strictly one dimensional" propagation of cosmic rays in the galaxy. This is demonstrated[3] by the result that when two of the dimensions were suppressed in the Monte Carlo calculation the suppression of small values was not observed and a distribution was obtained that was flat down to zero.

It is the purpose of this note to point out the reason for this dependence on dimensionality, to demonstrate that small values of anisotropy are possible if the propagation is "essentially" one dimensional on a scale of the order of the Larmor radius of a cosmic ray particle, and to argue that this, in fact, is the case in our galaxy. In the following we shall consider the bulk cosmic ray flux density $\underset{\sim}{J}$ rather than the anisotropy δ since the former has simpler statistical properties and the latter quantity is simply related to it by $\delta \propto |\underset{\sim}{J}|/\rho$ where ρ is the cosmic ray particle density.

In the one dimensional case we note that the flux is the sum of many contributions from different injection events (supernovae explosions, pulsars etc.). Therefore, by the central limit theorem the distribution of J will be approximately Gaussian

$$P(J)\,dJ \simeq \left(2\pi\langle J^2\rangle_{Av}\right)^{-1/2} \exp\left(-J^2/2\langle J^2\rangle_{Av}\right)\, dJ \tag{1}$$

We shall assume a Gaussian form in the following argument since the results should be qualitatively the same for reasonable deviations from this form.

This distribution is seen to be flat in the vicinity of $J = 0$. However, in the three dimensional case $\underset{\sim}{J}$ is a vector with a distribution

$$P(\underset{\sim}{J})\,d^3J = (2\pi)^{-3/2}\left(\langle J_x^2\rangle_{Av}\langle J_y^2\rangle_{Av}\langle J_z^2\rangle_{Av}\right)^{-1/2}$$

$$\times \exp\left(-J_x^2/2\langle J_x^2\rangle_{Av} - J_y^2/2\langle J_y^2\rangle_{Av} - J_z^2/2\langle J_z^2\rangle_{Av}\right) d^3J \tag{2}$$

and if $\langle J_x^2\rangle_{Av} = \langle J_y^2\rangle_{Av} = \langle J_z^2\rangle_{Av} = M$ the quantity $J = |\underset{\sim}{J}|$ is distributed as

$$P(J)\,dJ = (2/\pi)^{1/2} M^{-3/2} \exp\left(-J^2/2M\right)\, J^2\, dJ \tag{3}$$

The origin of the suppression near $J = 0$ is now evident, it is the factor J^2 in the phase space volume element $4\pi J^2 dJ$.

If on the other hand, instead of the R.M.S. values of the components being equal we have $\langle J_x^2 \rangle_{Av} = \langle J_y^2 \rangle_{Av} \equiv \alpha < \beta \equiv \langle J_z^2 \rangle_{Av}$ straightforward integration gives a distribution for $J = |\underset{\sim}{J}|$ of

$$P(J)dJ = (2/\pi)^{1/2}(\beta\alpha^2)^{-1/2} \exp(-J^2/2\alpha)\, B^{-1} Erfi\,(JB)\, J\, dJ \tag{4}$$

where $Erfi\,(x) = -i\,Erf\,(ix)$ and $B = [(2\alpha)^{-1} - (2\beta)^{-1}]^{1/2}$.

Using the asymptotic expression[4], $Erfi(x) \rightarrow x\exp(x^2)$ for $x \to 0$ and $Erfi\,(x) \rightarrow \exp(x^2)/2x$ as $x \to \infty$ we have

$$P(J)dJ \simeq (2/\pi\beta\alpha^2)^{1/2} \exp(-J^2/2\beta)\, J^2 dJ \quad , JB \ll 1 \tag{5}$$

and

$$P(J)dJ \simeq (2/\pi\beta)^{1/2}(1-\alpha/\beta)^{-1} \exp(-J^2/2\beta)\, dJ \quad , \quad JB \gg 1 \tag{5'}$$

Clearly if $\alpha = \beta$, $B = 0$ and the first case is always applicable; expression (5) becomes identical to expression (3). On the other hand if $\alpha \ll \beta$, $B \simeq (2\alpha)^{-1/2}$ and the distribution is essentially equal to expression (1) (i.e. one dimensional) as long as $J \gg (2\alpha)^{1/2}$

We must now inquire as to what type of propagation of cosmic rays can bring about a distribution for the bulk flux of the form of expression (2) with the condition $\langle J_x^2 \rangle_{Av} \simeq \langle J_y^2 \rangle_{Av} \ll \langle J_z^2 \rangle_{Av}$. If the current $\underset{\sim}{J}$ is the sum of many small currents from individual injection events i.e. $\underset{\sim}{J} = \sum_{i=1}^{N} \underset{\sim}{j}_i$ with N large it can be shown[5] that the distribution of $\underset{\sim}{J}$ approaches the form

$$P(\underset{\sim}{J})\, d^3 J = (2\pi)^{-3/2} |M|^{-1/2} \exp\left[-\frac{1}{2}\sum_{\ell,m=1}^{3} (M^{-1})_{\ell m}\, \tilde{J}_\ell\, \tilde{J}_m\right] d^3 J \qquad (6)$$

where $\tilde{\underset{\sim}{J}} = \underset{\sim}{J} - \langle \underset{\sim}{J} \rangle_{Av} = \underset{\sim}{J} - N\langle \underset{\sim}{j} \rangle_{Av}$ and $(M^{-1})_{\ell m}$ is the inverse of the covariance matrix $M_{\ell m} = N\langle j_\ell j_m \rangle_{Av}$, $|M|$ being its determinate. In the above expressions $\langle\ \rangle_{Av}$ denote averages over the statistical variables[2]. Expression (6) is not yet in the form of (2) but one should note that M (and hence M^{-1}) is a real, symmetric matrix and that it may always be diagonalized. If we can now find a propagation function that will give average values of j_x and j_y that are small $(\mathcal{O}(\epsilon))$ compared to j_z we can have

$$\langle j_x^2 \rangle_{Av} \simeq \langle j_y^2 \rangle_{Av} \simeq \langle j_x j_y \rangle_{Av} = \mathcal{O}(\epsilon^2)\ ,\ \langle j_x j_z \rangle_{Av} \simeq \langle j_y j_z \rangle_{Av} = \mathcal{O}(\epsilon)$$

and $\langle j_z^2 \rangle_{Av} = \mathcal{O}(1)$ then it is straightforward to verify that the values of $M_{\ell\ell}$ for the diagonalized matrix (the eigenvalues) will have the proper ordering, one of $\mathcal{O}(1)$ and two of $\mathcal{O}(\epsilon^2)$.

If we first consider three dimensional diffusion in a homogeneous magnetic field we may note while the mean free path along the field can be essentailly any value depending on the density of small irregularities in the field the diffusion mean free path in a direction perpendicular to the field is limited to be of the order of the Larmor radius R_L. If we therefore choose $\sigma_x \simeq \sigma_y = \mathcal{O}(\epsilon)$ and $\sigma_z = \mathcal{O}(1)$ we have upon performing the suitable averages[1,2] (we here ignore the complication of the time dependence which while having problems of its own[2] adds essentially nothing to the ordering problem)

$$\langle j_x^2 \rangle_{AV} \propto \frac{\sigma_x}{\sigma_y \sigma_z} \quad ; \langle j_x j_z \rangle_{AV} \propto 1/\sigma_z \qquad \text{, etc.}$$

with obvious permutation of x, y and z.

This gives the right ratio of the various averages but absolutely it gives $\langle j_x^2 \rangle_{AV} \simeq \langle j_y^2 \rangle_{AV} = \mathcal{O}(1)$ and $\langle j_z^2 \rangle = \mathcal{O}(1/\epsilon^2)$ a very large $\langle j_z^2 \rangle_{AV}$ rather than a small value for $\langle j_x^2 \rangle_{AV}$ and $\langle j_y^2 \rangle$. In fact we have $\langle \rho^2 \rangle_{AV} / \langle \rho \rangle_{AV}^2 \propto (\sigma_x \sigma_y \sigma_z)^{-1} = \mathcal{O}(1/\epsilon^2)$ where ρ denotes the cosmic ray density produced by an injection event and we note that this comes about because in this model the cosmic rays diffuse along a thin flux tube never getting more than a few times R_L away from the line of force threading the injection event. If the cosmic rays are very lumpy (or stringy) we would expect this large fluctuation in the density. However, on the basis of meteorite data[6] we may rule out this model.

On the other hand, if we consider the effect of the ergodic nature of the galactic magnetic field[7] we see that cosmic rays will be transported across the mean field in a random or diffusive manner. The correct transport equation for this situation would describe diffusion in z along the mean field in time with diffusion in the x and y directions with the position along the z axis playing the role of time[7]. The diffusion coefficients are of the form $\langle \Delta x^2 \rangle_{AV} / \Delta z$ and $\langle \Delta y^2 \rangle_{AV} / \Delta z$. As a rough approximation to the more correct situation one can consider simply three dimensional diffusion with equal diffusion coefficients as was done by Ramaty et al. However, it should be noted that, like all diffusion theories, it only applies on a large scale. The characteristic length in this case is the correlation length L of the random part of galactic field and $L \approx 100\ pc$. In striking contrast the distance scale that is applicable in any measurement of cosmic ray flux is the Larmor radius of a typical cosmic ray particle $R_L \approx 3\times10^{-5}\, pc$. Therefore in computing local current densities we should consider the field to be smooth and homogeneous. If one treats the large scale transport of cosmic rays as an isotropic diffusion process with a mean free path λ the observed currents will have one component along the mean field with a R.M.S. value $\langle j_z^2\rangle_{AV}^{1/2}$ typical of λ but $\langle j_x^2\rangle_{AV}^{1/2}$ and $\langle j_y^2\rangle_{AV}^{1/2}$ will be smaller by a factor R_L/λ.

In this situation our previous analysis applies and the observed flux will be distributed with a quasi Gausian with $\langle J^2\rangle_{AV} \simeq N\langle j_i^2\rangle_{AV}$ (we assume $\langle j\rangle_{AV} = 0$) for values of $J > (R_L/\lambda)\langle J^2\rangle_{AV}^{1/2}$ and with a rather sharp cutoff for values of J below this value.

For any type of propagation function there will be a characteristic length L_c and a characteristic time T_c. We have[2] by simple dimensional analysis $N = \bar{n}(L_c)^3 T_c$ where $\bar{n}$ is the injection event rate per unit volume and $\langle j^2\rangle_{AV} \simeq L_c^{-4} T_c^{-2}$. This gives $\langle J^2\rangle_{AV} \simeq \bar{n} L_c^{-1} T_c^{-1}$ and for diffusion $L_c = (\frac{1}{3}\lambda c\, T_c)^{1/2}$. If we assume with Ramaty et al that $T_c = 3d^2/2\lambda c$ then $L_c = 2^{-1/2} d \simeq 500\, pc$ and $\langle J^2\rangle \propto \lambda$

It will be shown in a future publication that the streaming velocity is distributed with a singular distribution [$P(V) \propto V^{-2}$ for large V, hence there are no moments] that has a characteristic scale $\sigma_V =$
$\left(\langle J^2\rangle_{AV}/\langle\delta\rho^2\rangle_{AV}\right)^{1/2}$ where $\langle\delta\rho^2\rangle_{AV}$ is the mean square fluctuation of the cosmic ray density. Since[2] $\langle\delta\rho^2\rangle = N\langle\rho_i^2\rangle_{AV}$

$$\sigma_V = \left(\langle j_i^2\rangle_{AV}/\langle\rho_i^2\rangle_{AV}\right)^{1/2} \simeq L_c T_c \propto \lambda$$

We note that the median and 1σ and 2σ levels of the anisotropy reported by Ramaty et al are proportional to λ for $\lambda \gtrsim 10^{-1} pc$ (large enough for nuclear collisions to be neglected). If we take their median value m

to be a good measure of σ_v our analysis indicates that the distribution of δ should be quite flat down to values of about $\delta = [(R_c/\lambda)\,m]_{\lambda = 1pc}$ $\simeq 10^{-7}$ independent of λ or far smaller than anything observed.

The point of all this is simply that although a value of $\delta < 10^{-4}$ may be very unlikely with a flat distribution of this type it is no more unlikely than any other value measured with a precision of $\approx 10^{-4}$. From this we see that a small value is no more inconsistent with $\lambda > 1pc$ than any other value. Furthermore it could be pointed out that if one chooses a constant $\tau_c \simeq 10^6$ years as do Jokipii and Parker[7] one obtains $\sigma_v \propto \lambda^{1/2}$ and a median δ of $\approx 10^{-4}$ would not be obtained until $\lambda \simeq 10^{-2}\,pc$

In addition one could argue against simple diffusion from a point source by noting that the estimated[8] energy released in a typical injection event $\sim 10^{51}$ ergs is of the order of the ambient energy (a few eV per cc) contained in $\sim 10^9\,(pc)^3$. This would indicate that each event would violently disrupt the galaxy at least over distances of the order of its thickness $\simeq 200$ pc. Furthermore if the non-linear propagation equation proposed by Skilling[9] is correct it would appear that the proper transport equation would more closely resemble the simple function originally employed[2] by the present author than a diffusion function.

Perhaps the most serious argument against using the ergodic field line concept for justifying the three dimensional transport of cosmic rays is the fact that it will not, in fact, rapidly disperse particles that are injected on neighboring field lines[7]. If a given field line is dispersed as

$\langle \Delta x^2 \rangle_{Av} / \Delta z \simeq (2L/B_0^2)\, C_{xx}(0)$ where $C_{xx}(\chi)$ is

the correlation function for the random part of the magnetic field

$\langle B_x(z) B_x(z+\lambda) \rangle_{Av}$. A simple extension of the argument of Jokipii and Parker[7] gives for two field lines separated by a distance η at $z = 0$,

$$\langle (\Delta x_1 - \Delta x_2)^2 \rangle_{Av} / \Delta z \simeq (2L/B_0^2)\,[C_{xx}(0) - C_{xx}(\eta)].$$

For a Gaussian correlation function $C_{xx}(\eta) \propto exp(-\eta^2/2L^2)$ and we have $\langle(\Delta x_1 - \Delta x_2)^2\rangle_{Av} / \Delta z \simeq (\eta/L)^2 \langle \Delta x^2 \rangle_{Av} / \Delta z$
or the relative dispersion of two neighboring lines is a factor $(\eta/L)^2$ smaller than the total random wandering of a given field line. Since $L \simeq 100\,pc$ or of the order of the galactic disk thickness, a source of size η much smaller than this will produce a tangled thread of cosmic rays but not a diffuse cloud.

All in all it would appear much to early to rule out the possibility that a low value for the observed cosmic ray anisotropy might be just an accident of our particular position in space and time.

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