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ROTATION INVARIANT PROBABILITY DISTRIBUTIONS IN GEODESY

by

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Submitted to the Department of Aeronautics and Astronautics on June 4, 1970 in partial fulfillment of the requirement for the degree of Master of Science.

ABSTRACT

This thesis treats gravity anomalies as a stationary random process defined on the surface of a sphere. Certain linear transforms of gravity anomalies on the surface (Poisson's Integral and Stokes' Formula) which give gravity anomalies and anomalous potential respectively above the surface are examined. It is shown how certain statistics (covariance functions) for gravity anomalies and anomalous potential above the surface of the sphere can be obtained from the covariance function of gravity anomalies on the surface, and these statistics are then calculated for various models of covariance functions of gravity anomalies on the surface. The covariance function for anomalous potential obtained (using Stokes' Formula) from a covariance function derived from actual measurements of gravity anomalies on the surface is compared with the covariance function for anomalous potential resulting from observations of satellite motion. The close agreement between these two covariance functions derived from independent sources is used to justify treating gravity anomalies as a rotation invariant random process.

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NOTATION

Underlined lower case letters are position vectors in R^3. Underlined upper case letters are 3 by 3 matrices.

t = time

$x(t)$ = random process in time

$\underline{e}$ = position vector in R^3

$x(\underline{e})$ = a random process in space

Θ = angle between two vectors

$\tilde{L}$ = linear integral operator

$L(\underline{e}_1, \underline{e}_2)$ = kernel of $\tilde{L}$

$\Delta g(\underline{e})$ = gravity anonomaly at point $\underline{e}$

S = surface of a sphere

R = radius of a sphere with surface S

r = distance from center of the sphere

Δ^2 = laplacian

$C(x)$ = covariance function of Δg on S

$Q(x)$ = covariance function of Δg above S

$T(\underline{e})$ = anomalous potential at $\underline{e}$

$R(x)$ = covariance function of T above and on S

$\{C_k\}$ = discrete spectrum of a function $C(x)$ defined on $[-1,1]$

$[a,b]$ = closed interval in R^1

$|.|$ = absolute value

$P_n(x)$ = legendre polynomial of degree n

ε = upper bound on truncation error

Chapter 1

INTRODUCTION

Some of the central formulas in physical geodesy take the form of integrals over a surface approximating the earth's surface. For certain of these formulas the integrand is the product of some function (or kernel) and gravity anomalies on the surface. Gravity anomalies are related to the earth's gravity field and can be measured with a high degree of accuracy on the surface of the land masses. Due to measurement difficulties on the oceans' surfaces and the lack of accurate data over most of the inaccessable areas on the continents, accurate measurements of gravity anomalies are presently unavailable over much of the earth's surface. The investigation carried out in this thesis takes as its basic data gravity anomalies which are to be treated as a random process on the surface of a sphere[1]. In practice some assumptions must be made about the statistical structure of the anomaly field to eliminate the problem of incomplete coverage. The assumption of rotation invariance of the statistical properties of the anomaly field is used here.

In this thesis some of the mathematical implications of rotation invariance are examined, and their significance to physical geodesy indicated. In particular, it is shown that statistical information about certain geophysical quantities can be easily derived from statistical information about gravity anomalies on the surface of a sphere. The theory is applied to the determination of the covariance function of gravity anomalies (using Poisson's Integral) and anomalous potential (using Stokes' Formula) at any altitude above the surface of a sphere when the covariance function of gravity anomalies on the surface of the

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1. The formulas which are considered in this paper (Poisson's and Stokes') assume that the reference surface for the earth's gravity potential is a sphere.

sphere is given. The resulting covariance function for the anomalous potential is then compared with results derived from experimental data obtained in part from the observation of satellite motion (the Köhnlein model[1]). The close agreement between the two covariance functions derived from independent sources is used to justify the statistical assumptions (Rotation invariance, and later Ergodicity) which were made to extrapolate known data on gravity anomalies.

1. See reference (11)

Chapter 2

ROTATION INVARIANT STATISTICS AND OPERATORS

This chapter will be, for the most part, a restatement of part of a paper by J.E. POTTER and E.J. FREY [1967] (1). In what follows it is assumed that the reader has some familiarity with stochastic processes defined over the real line with time as the independent variable. Some familiarity with communications or control systems theory is also useful, although it is not essential in what follows. Reference (2) is one source for this material.

2.1 Rotation Invariant Statistics

A stochastic process $x(t)$ is said to be stationary if its statistics are unaffected by a shift in the time origin. This means that $x(t_1)$ has the same statistics as $x(t_2)$ for all t_1 and t_2. In particular, if $x(t)$ is a stationary random process, then its covariance, which is defined as the statistical average[1] of the product $x(t_1)\ x(t_2)$, depends only on the separation of the two points in time, $(t_2 - t_1)$. If a random process is defined over the surface of a sphere, stationarity is called rotation invariance, and means that the statitics of the process are invariant under rotations about the center of the sphere. Suppose $x(\underline{e}_1)$ is a stationary random process defined over the surface of a unit sphere, and $\underline{e}_1$ is a unit vector indicating position on the surface. Rotation invariance (stationarity) means that the statistics of $x(\underline{e}_1)$ are independent of the orientation of the position vector $\underline{e}_1$, and thus are invariant under any rotation of $\underline{e}_1$ about the center of the sphere. This implies that the covariance function, which is the statistical average of $x(\underline{e}_1)\ x(\underline{e}_2)$, will be a function of the

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1. The ensemble average, or in case of an ergodic process, the time average.

angle (denoted by Θ) separating $\underline{e}_1$ and $\underline{e}_2$ in the plane containing both unit vectors. Another way of stating this is to say that the covariance function depends only on the angular distance between $\underline{e}_1$ and $\underline{e}_2$, or on $\underline{e}_1 \cdot \underline{e}_2 = \cos\Theta$ where Θ is the angle between $\underline{e}_1$ and $\underline{e}_2$ as defined above. Thus $E[x(\underline{e}_1)\ x(\underline{e}_2)]$ is a function of the dot (or inner) product of $\underline{e}_1$ and $\underline{e}_2$. Treating gravity anomalies as a rotation invariant random process over the surface of a sphere permits the extrapolation of available gravity anomaly data over all of the surface of the sphere. This follows since the covariance function can be accurately determined over limited regions (and thus for limited variations in Θ) of the earth's surface from the accurate gravity measurements which are presently available (3). If gravity anomalies can be modelled as a rotation invariant random process then this covariance function will be valid everywhere on the earth's surface, regardless of the fact that it was derived from measurements made over a very small portion of the globe.

2.2 Rotation Invariant Operators

The integral formulas of geodesy which are of interest in this thesis can all be written in a single form

$$g(\underline{e}_1) = \int_S L(\underline{e}_1,\underline{e}_2)\, f(\underline{e}_2)\, dS(\underline{e}_2), \qquad 2\text{-}1$$

where the integration is carried out over the surface of a sphere, S, approximating the earth's surface. The vector $\underline{e}_2$ denotes points on S, and the function g is a function of $\underline{e}_1$ which denotes points on or outside of S. For example, with Stokes' formula, f will represent gravity anomalies and g will represent anomalous potential. We will call $g(\underline{e}_1)$ the transformation of $f(\underline{e}_2)$. Let the symbol $\tilde{L}$, an upper case letter with a tilde, denote a linear integral operator, so that equation 2-1 can be written as

$$\tilde{L}f(\underline{e}_1) = \int_S L(\underline{e}_1,\underline{e}_2)\, f(\underline{e}_2)\, dS(\underline{e}_2) \qquad 2\text{-}2$$

where f and L are functions continuous on S. The function $L(\underline{e}_1,\underline{e}_2)$ is called the kernel of the operator $\tilde{L}$. We define a rotation invariant operator in the following way. If F is a rotation invariant process and $\tilde{L}$ a rotation invariant operator then $\tilde{L}f$, the transformed process, will also be a rotation invariant process. The Funk-Hecke Formula (4) provides necessary and sufficient conditions for an integral operator $\tilde{L}$ to be rotation invariant. Let $\underline{A}$ be an *orthogonal* (rotation) matrix. Then the Funk-Hecke Formula shows that if $\tilde{L}f(\underline{e}) = g(\underline{e})$, then the rotation invariance of $\tilde{L}$ implies that $g(\underline{e})$ is rotation invariant, or

$$g(\underline{A}\ \underline{e}) = g(\underline{e}) \quad . \qquad 2\text{-}3$$

From equation (2-1), rotation invariance implies that

$$\int_S L(\underline{A}\ \underline{e}_1,\underline{e}_2)\, f(\underline{e}_2)\, dS(\underline{e}_2) = \int_S L(\underline{e}_1,\underline{e}_2)\, f(\underline{e}_2)\, dS(\underline{e}_2)$$

and hence it gives as a sufficient condition for rotation invariance that

$$L(\underline{e}_1,\underline{e}_2) = L(\underline{e}_1\cdot\underline{e}_2) \quad . \qquad 2\text{-}4$$

Thus the kernel of a rotation invariant operator is a function $\underline{e}_1\cdot\underline{e}_2$ only, just as the covariance function of a rotation invariant random process is a function of $\underline{e}_1\cdot\underline{e}_2$ only.

2.3 Poisson's Integral and Stokes' Formula

In this section we consider Poisson's and Stokes' formulas to examine their rotation invariance. We also present some results of future use. An important formula in physical geodesy is the upward continuation formula, often called Poisson's integral formula. It represents the solution to Dirichlet's Problem[1], and it generates a function V, harmonic[2] outside of a sphere, and equal to a prescribed function on the surface of the sphere. Thus if the product of gravity anomalies and the radius, $r\Delta g(\underline{e})$, forms the prescribed function on the surface of a sphere, then the upward continuation formula expresses $r\Delta g$ at some point above the surface of the sphere a distance r from the center of the sphere. Let S, as before, represent the surface of the sphere which is approximating the surface of the earth. If R is the radius of the sphere and r is the radius at which Δg is to be calculated, $(r>R)$, then the upward continuation formula[3] is

$$\Delta g(r\,\underline{e}_1) = \tilde{M}\Delta g(\underline{e}_1) = \int_S \frac{R(r^2 - R^2)}{4\pi[r^2+R^2-2rR(\underline{e}_1 \cdot \underline{e}_2)]^{\frac{3}{2}}} \Delta g(\underline{e}_2)\, dS(\underline{e}_2) \qquad 2\text{-}5$$

where $\underline{e}_1$ and $\underline{e}_2$ are unit vectors.

Thus $\tilde{M}$ is a <u>rotation invariant operator</u> since its kernel is a function of $\underline{e}_1 \cdot \underline{e}_2 = \cos\Theta$, where Θ is the angle separating $\underline{e}_1$ and $\underline{e}_2$,

$$M(\underline{e}_1 \cdot \underline{e}_2) = M(x) = \frac{R(r^2 - R^2)}{4\pi[r^2+R^2-2rRx]^{\frac{3}{2}}} \qquad 2\text{-}6$$

The kernel M(x) can also be expressed as a series of Legendre polynomials

$$M(x) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)\left(\frac{R}{r}\right)^{k+1} P_n(x). \qquad 2\text{-}7$$

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1. See page 34, reference (7)
2. A function satisfying Laplace's equation, $\Delta^2 V = 0$, see section 3.2.
3. Page 35, reference (7)

The coefficients in this expansion will play an important role in later developments.

The Stokes' formula solves a different geodetic problem, it gives a function harmonic on and outside S, subject to the boundary condition on S,

$$\frac{\partial T}{\partial r} + \frac{2T}{R} + \Delta g = 0 \qquad 2\text{-}8$$

where 2-8 is a spherical approximation of the fundamental boundary condition of physical geodesy[1]. Here T represents the anomalous potential[2], a harmonic function. The solution to Stokes' problem is given in reference (10) as

$$T(r\underline{e}_1) = \tilde{G}\ \Delta g(r\underline{e}_1) = \int_S G(\underline{e}_1,\underline{e}_2)\Delta g(\underline{e}_2)dS(\underline{e}_2) \qquad 2\text{-}9$$

with the kernel function

$$G(\underline{e}_1,\underline{e}_2) = \frac{1}{4\pi}\left\{\frac{2}{\ell} - \frac{5R}{r^2}\cos\Theta - \frac{3\ell}{r^2} - \frac{3R}{r^2}\cos\Theta\ \ell n\left(\frac{r-R\cos\Theta+\ell}{2r}\right)\right\} \qquad 2\text{-}10$$

where Θ is the angle between $\underline{e}_1$ and $\underline{e}_2$ in the plane containing $\underline{e}_1$ and $\underline{e}_2$ and

$$\ell = \sqrt{r^2 - 2rR\cos\Theta + R^2} \quad .$$

Thus $\tilde{G}$ is also a rotation invariant operator, since $\underline{e}_1 \cdot \underline{e}_2 = \cos\Theta$, and $G(\underline{e}_1,\underline{e}_2) = G(\underline{e}_1 \cdot \underline{e}_2)$.

1. See page 88, reference (7)

2. See page 82, reference (7)

The Legendre series representation for $G(x)$ is[1]

$$G(x) = \frac{1}{4\pi R} \sum_{n=0}^{\infty}{}^{1} \left(\frac{2n+1}{n-1}\right) \left(\frac{R}{r}\right)^{n+1} P_n(x) \qquad 2\text{-}11$$

where $\sum^{1}$ means that the term with n=1 is omitted.

Both of these formula will later be used to derive the covariance functions for anomalous potential and gravity anomalies at any altitude from the covariance function for gravity anomalies on the surface of the sphere.

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1. Reference (10), page 14.

Chapter 3

THE SIGNIFICANCE OF ROTATION INVARIANCE

3.1 Introduction

The purpose of this chapter is to explain how to determine the covariance function for a random process which is expressed as a rotation invariant operator operating on gravity anomalies over the surface of a sphere when the covariance function of gravity anomalies is assumed to be known. This chapter will be, for the most part, a reiteration of the material covered in reference (1).

3.2 Harmonic Functions

A scalar valued function on R^3 (a function defined on a three dimensional Euclidean space)is said to be harmonic if it satisfies Laplace's equation[1],

$$\Delta^2 V = 0 \quad . \tag{3-1}$$

If r is the distance from the center of a sphere S to a point, $r\Delta g$ satisfies Laplace's equation on and outside of the surface of the sphere, and so is harmonic on or outside of S. The gravity anomaly at the point $\underline{e}$ on S is denoted by $\Delta g(\underline{e})$, and if $\underline{e}$ represents a point (x,y,z) in cartesian coordinates

where

$$r = \sqrt{x^2 + y^2 + z^2} \quad ,$$

then

$$\frac{\partial^2}{\partial x^2} r\Delta g(x,y,z) + \frac{\partial^2}{\partial y^2} r\Delta g(x,y,z) + \frac{\partial^2}{\partial z^2} r\Delta g(x,y,z) = 0 \tag{3-2}$$

and equation 3-2 is valid everywhere on or outside of S.

---

1. $\Delta^2 V(x,y,z) = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V$

Let $q(\underline{e}_1)$ be some harmonic function obtained by integrating the product of a function of $\underline{e}_1$ and $\underline{e}_2$ (a kernel function) and $\Delta g(e_2)$, where the integration is over a unit sphere S and the variable of integration is $\underline{e}_2$.

Thus

$$q(\underline{e}_1) = \int_S F(\underline{e}_1,\underline{e}_2)\ \Delta g(\underline{e}_2)\,dS(\underline{e}_2) \tag{3-3}$$

and in the operator notation previously introduced,

$$\tilde{F}\Delta g(\underline{e}_1) = q(\underline{e}_1) \tag{3-4}$$

If $\tilde{F}$ is a rotation invariant operator,

$$F(\underline{e}_1,\underline{e}_2) = F(\underline{e}_1\cdot\underline{e}_2) = F(x)$$

where

$$x = \underline{e}_1\cdot\underline{e}_2 = \cos\theta\ , \tag{3-5}$$

then F(x) can be expanded in a series of Legendre Polynomials(5), valid over the closed interval [-1,1],

$$F(x) = \sum_{n=0}^{\infty}\left[\frac{2n+1}{2}\int_{-1}^{1} P_n(\mu)F(\mu)\,d\mu\right] P_n(x). \tag{3-6}$$

We are interested in those formulas of physical geodesy which transform the harmonic function $r\Delta g$ into another harmonic function, and so $q(\underline{e}_1)$ from equation 3-4 may be assumed to be harmonic on and outside of S. Thus $q(\underline{e}_1)$ will satisfy Laplace's Equation,

$$\Delta^2 q = 0 \tag{3-7}$$

with some boundary condition specified on S.
Equation 3-7 expresses $q(\underline{e}_1)$ as the solution to a second order homogeneous partial differential equation with appropriate boundary conditions on S.

3.3 Comparing the Statistics of Random Processes in Time with Rotation Invariant Processes on a Sphere

The covariance function of a stationary random process defined over the real line with time as the independent variable is called, in the terminology of control theory, the autocorrelation function. The Fourier transform of the autocorrelation function is called the power spectral density of the process, and it is non-negative. The Fourier transform can be used for the spectral representation of the autocorrelation function because the covariance is defined over the real line. A rotation invariant random process on S is a function of $\underline{e}_1 \cdot \underline{e}_2$, and if $x = \underline{e}_1 \cdot \underline{e}_2 = \cos\theta$, then the domain of the covariance function is [-1,1].

A function defined on this bounded interval may be represented by the coefficients of its Fourier or Legendre series representation. Because a Legendre series representation is valied over [-1,1], it is used to represent functions defined on a sphere, and the coefficients of the Legendre polynomials in the Legendre series representation of the covariance function will be called the discrete spectral representation. Analogous to the non-negative power spectral density, this discrete spectrum is non-negative[1]. Define $x(\underline{e})$ to be a rotation invariant random process defined on S, the surface of a unit sphere. The covariance function is

$$C(\underline{e}_1,\underline{e}_2) = C(\underline{e}_1 \cdot \underline{e}_2) = C(x) = E[x(\underline{e}_1)x(\underline{e}_2)] \qquad 3\text{-}8$$

since $x(\underline{e})$ is rotation invariant. An expression for the discrete

1. See appendix D for a proof of this.

spectrum of C(x) will now be derived. Define $\tilde{C}$ to be a rotation invariant operator with kernel $C(\underline{e}_1 \cdot \underline{e}_2)$,

$$\tilde{C}f(\underline{e}_1) = \int_S C(\underline{e}_1 \cdot \underline{e}_2)\ f(\underline{e}_2)\, dS(\underline{e}_2). \tag{3-9}$$

The Legendre series expansion for C(x) is, from equation 3-6,

$$C(x) = \sum_{n=0}^{\infty} \left[\frac{2n+1}{2} \int_{-1}^{1} P_n(\mu) C(\mu)\, d\mu\right] P_n(x) \tag{3-10}$$

From the theory of linear operators there exists a complex number λ , called an eigenvalue of the linear operator $\tilde{C}$, if there exists a non-zero vector f such that

$$\tilde{C}f = \lambda f. \tag{3-11}$$

Each such vector f is called on eigenfunction of $\tilde{C}$, and the constants λ corresponding to each f are called eigenvalues. If $y_m(\underline{e})$ is a spherical harmonic polynomial of degree m and $P_k(\mu)$ is a Legendre polynomial of degree k, the follwoing relationship holds[1],

$$\int_0^{2\pi} \int_{-1}^{1} y_m(\Theta,\phi)\ P_n(\cos\Theta)\, d(\cos\Theta)\, d\phi = \frac{4\pi}{2n+1}\, C\ \delta_{nm} \tag{3-12}$$

where C is the value of $y_n(\Theta,\phi)$ at the pole of the coordinate system for $P_n(\cos\Theta)$ and δ_{nm} is the Kronecker delta function.

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1. See page 126, reference (6).

Equation 3-12 can be written as

$$\int_S P_n(\underline{e}\cdot\underline{e}_1)\ y_m(\underline{e}_1)\,dS(e_1) = \frac{4\pi}{2n+1}\ \delta_{nm}\ y_m(\underline{e}), \qquad 3\text{-}13$$

if $\underline{e}$ is taken to be the pole (z axis) of the coordinate system.

Thus from equations 3-9, 3-10, the continuity of C(x) and $y_m(\underline{e})$, and the relationship from equation 3-13,

$$\tilde{C}\ y_m(\underline{e}) = \int_S C(\underline{e}\cdot\underline{e}_1)\ y_m(\underline{e}_1)\,dS(e_1)$$

$$= \sum_{n=0}^{\infty}\left[\frac{2n+1}{2}\int_{-1}^{1} P_n(\mu)C(\mu)\,d\mu\right]\int_S P_n(\underline{e}\cdot\underline{e}_1)y_m(\underline{e}_1)\,dS(\underline{e}_1)$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2}\int_{-1}^{1} P_n(\mu)C(\mu)\,d\mu\left[\frac{4\pi}{2n+1}\ \delta_{nm}\ y_m(\underline{e})\right]$$

$$= \left(2\pi\int_{-1}^{1} P_m(\mu)C(\mu)\,d\mu\right)\ y_m(\underline{e})$$

$$= C_m\ y_m(\underline{e})\ . \qquad 3\text{-}14$$

Thus the eigenvalues of $\tilde{C}$ will be denoted by C_k and from 3-14

$$C_k = 2\pi\int_{-1}^{1} C(\mu)P_k(\mu)\,d\mu\ , \qquad 3\text{-}15$$

and the eigenvectors (eigenfunctions) are the spherical harmonic polynomials $y_m(\underline{e})$.

Thus equation 3-10 can be rewritten as

$$C(x) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) C_n P_n(x) \qquad 3\text{-}16$$

The eigenvalues of $\tilde{C}$ correspond to the power spectrum of the covariance function $C(x)$.

3.4 Covariance Functions Derived from Covariance of Gravity Anomalies

Consider the quantity $q(\underline{e}_1)$ derived from gravity anomalies on S by the equation,

$$q(\underline{e}_1) = \tilde{M}\Delta g(\underline{e}_1) = \int_S M(\underline{e}_1,\underline{e}_2)\Delta g(\underline{e}_2)\, dS(e_2). \qquad 3\text{-}17$$

Consider further the operator $\tilde{M}$ to be rotation invariant, and Δg to be a rotation invariant random process with covariance $C(\underline{e}_1 \cdot \underline{e}_2)$. Let $Q(\underline{e}_1,\underline{e}_2)$ be the covariance function for $q(\underline{e})$.
Thus $Q(\underline{e}_1,\underline{e}_2) = E[q(\underline{e}_1)q(\underline{e}_2)]$. The spectrum of $\tilde{M}$ is M_k, given by equation 3-15. We wish to show that $q(\underline{e})$ is a rotation invariant random process with power spectrum given by the equation

$$Q_k = |M_k|^2 C_k$$

where C_k is the power spectrum of $C(\underline{e}_1 \cdot \underline{e}_2)$.

Define f and h to be continous functions on S. Then define

$$\xi = \int_S f(\underline{e}) \left(\tilde{M}\Delta g(\underline{e}) \right) dS(\underline{e})$$

$$\eta = \int_S h(\underline{e}) \left(\tilde{M}\Delta g(\underline{e}) \right) dS(\underline{e}).$$

Assuming $\tilde{M}\Delta g(\underline{e})$ is continuous,

$$E[\xi\eta] = E\left[\int_S f(\underline{e}_1)\tilde{M}\Delta g(\underline{e}_1)\,dS(\underline{e}_1) \int_S h(\underline{e}_2)\tilde{M}\Delta g(\underline{e}_2)\,dS(e_2)\right]$$

$$= \int_S f(\underline{e}_1)\left[\int_S Q(\underline{e}_1,\underline{e}_2)h(\underline{e}_2)\,dS(\underline{e}_2)\right]dS(\underline{e}_1)$$

$$= \int_S f(\underline{e}_1)\tilde{Q}h(\underline{e}_1)\,dS(\mathbf{e}_1). \qquad 3\text{-}18$$

But the rotation invariance of $\tilde{M}$ implies

$$M(\underline{e}_1,\underline{e}_2) = M(\underline{e}_1\cdot\underline{e}_2) = M(\underline{e}_2,\underline{e}_1) \quad . \qquad 3\text{-}19$$

Thus

$$E[\xi\eta] = E\left[\int_S f(\underline{e}_1)\left\{\int_S M(\underline{e}_1,\underline{e}_3)\Delta g(\underline{e}_3)\,dS(\underline{e}_3)\right\}dS(\underline{e}_1)\right.$$

$$\left. \times \int_S h(\underline{e}_2)\left\{\int_S M(\underline{e}_2,\underline{e}_4)\Delta g(\underline{e}_4)\,dS(\underline{e}_4)\right\}dS(\underline{e}_2)\right]$$

$$= \int_S f(\underline{e}_1)\left[\int_S\left(\int_S M(\underline{e}_1,\underline{e}_3)\left\{\int_S C(\underline{e}_3,\underline{e}_4)M(\underline{e}_4,\underline{e}_2)\,dS(e_4)\right\}\right.\right.$$

$$\left.\left. \times\, dS(\underline{e}_3)\right)\, h(\underline{e}_2)\,dS(e_2)\right] dS(\underline{e}_1)$$

$$= \int_S f(\underline{e}_1)\left[\int_S L*C*L\; h(\underline{e}_2)\,dS(\underline{e}_2)\right]dS(\underline{e}_1) \qquad 3\text{-}20$$

where A*B represents the convalution of A with B,

$$A*B = \int_S A(\underline{e}_1,\underline{e}_2)\; B(\underline{e}_2,\underline{e}_1)\,dS(e_2) \quad . \qquad 3\text{-}21$$

Comparing terms in 3-18 and 3-20,

$$Q = M*C*M$$

$$\tilde{Q} = \tilde{M}^2\tilde{C} \qquad 3\text{-}22$$

and since C and M are functions of $\underline{e}_1 \cdot \underline{e}_2$, $Q(\underline{e}_1, \underline{e}_2)$ is a function of $\underline{e}_1 \cdot \underline{e}_2$, and by the Funk-Hecke formula (4), $Q(x)$ is the covariance of a rotation invariant random process. Since convolution in the space domain corresponds to a multiplication for the spectral representation,

$$Q_k = M_k C_k M_k = M_k^2 \, C_k = |M_k|^2 \, C_k \; . \qquad 3\text{-}23$$

Since $M_k^2 \geq 0$ and (proven in APPENDIX D) $C_k \geqslant 0$, then from equation 3-23.

$Q_k \geqslant 0$ and Q_k is the spectral representation of the covariance of a rotation invariant integral transform of a rotation invariant random process, $\Delta g(e)$, on S. Thus one result of treating gravity anomalies as a rotation invariant random process on S is that by using equation 3-23, it is extremely easy to obtain the power spectrum of a random process derived as a rotation invariant linear integral transform of gravity anomalies on S.

If $R(\underline{e}_1, \underline{e}_2) = E[\tilde{L}\Delta g(\underline{e}_1) \; \tilde{M}\Delta g(\underline{e}_2)]$ where $\tilde{L}$ and $\tilde{M}$ are rotation invariant operators, then the cross-spectrum R_k is given by

$R_k = M_k \; L_k \; C_k$ and $R(\underline{e}_1, \underline{e}_2) = R(\underline{e}_1 \cdot \underline{e}_2)$, but R_k is not necessarily non-negative.

Chapter 4

COVARIANCE OF GRAVITY ANOMALIES ABOVE THE SURFACE OF A SPHERE

4.1 Introduction

In this chapter the theory developed in chapter 3 will be used to calculate the covariance function of gravity anomalies above the surface of a sphere, given the covariance function for gravity anomalies on the surface of the sphere.

4.2 Covariance Function for Gravity Anomalies on S

Gravity anomalies on S will be treated as a rotation invariant random process on S. Up to now it has been assumed that the covariance function is a known function with discrete non-negative spectrum. For now we will assume that the covariance function takes the form of a tent[1] function. The tent function was chosen to simplify the calculation of the spectrum of the covariance function; there is no guarantee that the spectrum derived will be non-negative, and the actual calculations shows that some terms in the spectrum are indeed negative. Since the tent function is not intended to represent an actual covariance function, this is unimportant. A covariance function obtained from actual measurements of gravity anomalies is later approxi-approximated by a piecewise linear function[2] whose spectrum is found to be non-negative.

If $C(x)$ is the covariance function for gravity anomalies on S,

$x = \cos \quad (x \in [-1,1])$, and

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1. See figure 1, Appendix A, which shows a general line segment on $[-1,1]$ which is a tent function for $a_1 = 1$, $b_2 = 0$. Figure 2 shows the tent function, a specialization of the line segment of figure 1.
2. See figure 3, Appendix A.

if $C(x)$ is a line segment as shown in figure 1, APPENDIX A, then

$$C(x) = b(x-a) \qquad \text{if} \quad a_2 \leqslant x \leqslant a_1 \tag{4-1}$$

$$= 0 \qquad \text{if} \quad x > a_1 \text{ or } x < a_2$$

where

$$b = \frac{b_1 - b_2}{a_1 - a_2} \quad \text{and} \quad a_1 \neq a_2$$

$$a = \frac{a_1 b_2 - b_1 a_2}{b_2 - b_1} \quad \text{and} \quad b_1 \neq b_2$$

and for $C(x)$ a tent function,

$$a_1 = 1$$

$$b_2 = 0. \tag{4-2}$$

Using equation 3-16, we can express $C(x)$ in terms of its power spectrum, C_k.

$$C(x) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1) \; C_k \; P_k(x) \tag{3-16}$$

where

$$C_k = 2\pi \int_{-1}^{1} C(x) \; P_k(x)\, dx. \tag{3-15}$$

We now find an analytic expression for C_k by utilizing the assumed simple form of $C(x)$ (the line segment shown in Figure 1, APP.A). The expression follows from two well known relationships[2] for Legendre polynomials,

$$P_k(x) = \frac{P'_{k+1}(x) - P'_{k-1}(x)}{2k + 1} \quad , \tag{4-3}$$

where

$$P'_k(x) = \frac{d}{dx} P_k(x) \quad ,$$

---

1. See figure 3, APPENDIX A.

2. See page 91, reference (6)

and

$$xP_k(x) = \frac{kP_{k-1}(x) + (k+1)P_{k+1}(x)}{2k+1} \quad . \qquad 4\text{-}4$$

From 3-15 and 4-1,

$$\int_{-1}^{1} C(x)P_k(x)\,dx = b\int_{a_2}^{a_1} (x-a)P_k(x)\,dx$$

$$= b\left\{ -a\int_{a_2}^{a_1} P_k(x)\,dx + \int_{a_2}^{a_1} x\,P_k(x)\,dx \right\} \quad . \qquad 4\text{-}5$$

Now, using equation 4-3,

$$\int_{a_2}^{a_1} P_k(x)\,dx = \int_{a_2}^{a_1} \frac{P'_{k+1}(x) - P'_{k-1}(x)}{2k+1}\,dx$$

$$= \frac{1}{2k+1}\left[P_{k+1}(a_1) - P_{k+1}(a_2) - P_{k-1}(a_1) + P_{k-1}(a_2) \right] . \qquad 4\text{-}6$$

Now from relation 4-4, and using 4-3,

$$\int_{a_2}^{a_1} x\,P_k(x)\,dx = \int_{a_2}^{a_1} \frac{kP_{k-1}(x) + (k+1)P_{k+1}(x)}{2k+1}\,dx$$

$$= \frac{1}{(2k+1)} \int_{a_2}^{a_1} \left\{ k\left[\frac{P'_k(x) - P'_{k-2}(x)}{2k-1}\right] + (k+1)\left[\frac{P'_{k+2}(x) - P'_k(x)}{2k+3}\right] \right\} dx$$

$$= \frac{1}{(2k+1)}\left\{ \left(\frac{k}{2k-1}\right)\left[P_k(a_1) - P_{k-2}(a_1) - P_k(a_2) + P_{k-2}(a_2)\right] \right.$$

$$\left. + \left(\frac{k+1}{2k+3}\right)\left[P_{k+2}(a_1) - P_k(a_1) - P_{k+2}(a_2) + P_k(a_2)\right] \right\} . \qquad 4\text{-}7$$

Combination of these results yields

$$\int_{-1}^{1} C(x) P_k(x)\, dx$$

$$= \frac{b}{(2k+1)}\left[\frac{(2k+1)}{(2k-1)(2k+3)}\left\{P_k(a_1)-P_k(a_2)\right\} + \frac{k}{2k-1}\left\{P_{k-2}(a_2)-P_{k-2}(a_1)\right\}\right.$$

$$+ \frac{(k+1)}{(2k+3)}\left\{P_{k+2}(a_1)-P_{k+2}(a_2)\right\} + a\left\{P_{k+1}(a_2)-P_{k+1}(a_1)\right.$$

$$\left.\left. + P_{k-1}(a_1)-P_{k-1}(a_2)\right\}\right] \triangleq \frac{f(a_1,a_2,b_1,b_2,k)}{2k+1} \quad . \qquad \text{4-8}$$

Thus from equation 3-15,

$$C_k = \frac{2\pi\; f(a_1,a_2,b_1,b_2,k)}{2k+1} \quad . \qquad \text{4-9}$$

4.3 Covariance of Gravity Anomalies at an Altitude

Let $Q(\underline{e}_1 \cdot \underline{e}_2) = Q(x)$ be the covariance of gravity anomalies at a height $(r-R)$ above S. Since gravity anomalies at radius r are expressed as a rotation invariant operator $\tilde{M}$ (equation 2-5) operating on gravity anomalies on S, the theory developed in Chaper 3 applies. The spectrum of $Q(x)$ is obtained from equation 3-23. Comparing equations 2-7 and 3-16, the spectrum of the operator $\tilde{M}$ is found to be

$$M_k = \left(\frac{R}{r}\right)^{2(k+1)} \qquad \text{4-10}$$

and the spectrum of the covariance function is given by 4-8 and 4-9,

$$Q_k = \left(\frac{R}{r}\right)^{2(k+1)} C_k \qquad \text{4-11}$$

and so from equation 3-16 and 4-9,

$$Q(x) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \left[\left(\frac{R}{r} \right)^{2(n+1)} C_n \right] P_n(x) . \qquad 4\text{-}12$$

Let $\gamma = \left(\frac{R}{r} \right)^2$. 4-13

Substituting the expression for C_k given by 4-8 and 4-9 into the expression for $Q(x)$ (equation 4-12), and simplifying, we obtain $Q(x)$ for our simplified model of $C(x)$,

$$Q(x) = \frac{(a_1-a_2)(b_1+b_2)\gamma}{4} + \frac{(a_1-a_2)\left[(2a_1+a_2)b_1 + (2a_2+a_1)b_2\right]\gamma^2}{4} x$$

$$+ \frac{1}{2} \sum_{k=2}^{\infty} \gamma^{k+1} f(a_1,a_2,b_1,b_2,k) P_k(x) \qquad 4\text{-}14$$

where the first term in the above equation (corresponding to the n=0 term in the summation) is obtained from equation 3-15,

$$C_0 = 2\pi b \int_{a_2}^{a_1} (x-a)\, dx$$

$$= \pi (a_1-a_2)(b_1+b_2)$$

and equation 4-11,

$$Q_0 = \gamma C_0 .$$

Then from equation 4-12,

$$Q(x) = \frac{1}{4\pi} Q_0 \; P_0(x) + \text{higher order terms in } \gamma$$

$$= \frac{1}{4\pi} \gamma\pi(a_1-a_2)(b_1+b_2)1 + \text{h.o.t.}$$

$$= \frac{(a_1-a_2)(b_1+b_2) \; \gamma}{4} + \text{h.o.t.}$$

and the second term in (4-14) is obtained from equation 3-15 where $n=1$:

$$C_1 = 2b\pi\int_{a_2}^{a_1} (x-a)x \; dx$$

$$= \frac{\pi}{3} (a_1-a_2)\Big[(2a_1+a_2)b_1 + (2a_2+a_1)b_2\Big] .$$

Thus, from equation 4-11,

$$Q_1 = \gamma^2 \; C_0$$

and equation 4-12 yields

$$Q(x) = \frac{(a_1-a_2)(b_1+b_2)\gamma}{4} + \frac{3}{4\pi} Q_1 P_1(x) + \text{h.o.t.}$$

$$= \frac{(a_1-a_2)(b_1+b_2)\gamma}{4} + \frac{(a_1-a_2)\Big[(2a_1+a_2)b_1 + (2a_2+a_1)b_2\Big]\gamma^2}{4} \; x$$

$$+ \text{h.o.t.} \quad .$$

4.4 An Upper Bound for Truncation Error

A digital computer was used to obtain Q(x), the covariance function for our simpler model of gravity anomalies at altitude, by performing the calculations and summation indicated by equation 4-14. Since only a finite number of terms can be retained from the infinite sum of equation 4-14, an upper bound on the truncation error incurred when only N terms are retained will now be presented. Then, given some error criterion, the number of terms which must be retained to satisfy the criterion can be determined. For $x \varepsilon [-1,1]$ the truncation error(ε) which results from approximating Q(x) by the first N terms of the summation in equation 4-14 will be bounded above by the following expression which is derived in APPENDIX D;

$$\varepsilon < \left[\frac{(\gamma+12+24|a|)}{12(1-\gamma)} + \frac{\gamma}{3(2^{N-1})(2-\gamma)} \right] |b| \gamma^{N+1} \quad . \qquad 4\text{-}15$$

4.5 Computations Performed

For the first set of computations, the covariance function of gravity anomalies on S is assumed to be the tent function shown in Figure 2, APPENDIX A. The constant b_1 is assumed to be unity, which is equivalent to normalizing the variance $\left(C(0)\right)$ of gravity anomalies on S. A computer program was written in FORTRAN IV to calculate the covariance function for gravity anomalies at a distance r from the center of a sphere of radius R, on which the covariance function is defined by the tent function. For simplicity the error criterion was chosen to keep the absolute errors due to truncation smaller than some given constant ε. The program used is PROGRAM 1, APPENDIX B. It accepts a fixed constant ε, and using the error bound given by equation 4-15, calculates N, the number of terms which must be retained in the summation expression for Q(x), the covariance at radius r.

The program calculates Q(x) for R=1.0 and various values of r and a_2, with $\varepsilon = .01$.

A second set of computations was carried out using data for the actual covariance function which was approximated by the piecewise linear function shown in FIGURE 3, APPENDIX A. The eight line segments were chosen to fit the data points shown by crosses in Figure 3,

APPENDIX A, obtained from page 57 of reference (3), and represent the covariance function for gravity anomalies over the surface of the earth calculated from accurate measurements of gravity anomalies over limited regions of the land masses (see reference (3)). The program used to carry out the computation of Q(x) in this case is PROGRAM 1 in APPENDIX B. The next section presents the results from both sets of computations.

4.6 Results and Conclusions Regarding Upward Continuation

For the first set of computations R, the radius of the sphere, was fixed and set equal to 1.0, corresponding to one earth radius, approximately 6380 km. Then r, the radius at which Q(x) (the covariance function for gravity anomalies) is determined, was varied (from 1.0001 to 1.1) to find Q(x) for various altitudes above the earth's surface (638 meters to 638 km above the surface of the earth). For each different altitude (value of r) the parameter a_2 which defines the covariance function C(x) (the tent function shown in Figure 2, APPENDIX A) was varied from 0.6 to -1.0. The graphs displayed in Figures one to fifteen in APPENDIX C were created by program 2, APPENDIX B from data generated by PROGRAM 1, APPENDIX B. Each graph shows the covariance function for gravity anomalies at an altitude for a given covariance function on the earth's surface (C(x) specified by a_2 and shown as a tent function) and a fixed altitude (specified by r). As the altitude goes to zero (as r approaches R) the curves for Q(x) approach the tent function which represents C(x). This should be the case. The upward continuation formula (Poisson's Integral) yields a function which approaches a prescribed function on S (the surface of a sphere, or the earth's surface in this case) as r approaches R, or the altitude goes to zero. In our case the prescribed function corresponds to gravity anomalies on S, with covariance function C(x). The covariance of gravity anomalies at an altitude given by the upward continuation formula $\left(Q(x)\right)$ should approach $C(x)$ as the altitude goes to zero (r approaches R). Thus this result seems justified. Graphs one to fifteen show that as the altitude increases from 0 to 638 km the covariance function $Q(x)$ becomes attenuated. Statistically this corresponds to a greater certainty in our knowledge of gravity anomalies. This behaviour is as expected, since as the altitude becomes very large(r increases) the sphere can be treated as a point mass, and since we have assumed that the mass of the sphere is known exactly, the gravity anomalies will be zero with probability one, or variance zero. Thus as the altitude (or as r) increases, it is expected that $Q(x)$ will become attenuated. The first 23 terms in the spectral representation for $C(x)$ are shown in Figure 16 for various values for a_2. Note that all of the C_k's are not non-negative as the theory demands, but since the tent functions representing $C(x)$ were never intended to resemble the actual covariance function for gravity anomalies on the surface of the earth this is not surprising.

The second set of computations differs from the first only in that $C(x)$, the covariance function for gravity anomalies on the earth's surface, is the piecewise linear function approximating the covariance function obtained from gravimetry by Kaula (3) (see Figure 3, APPENDIX A). Figures 17 to 21 in APPENDIX C show $Q(x)$ plotted against x as the altitude goes from 638 km. to 0 km. (as r goes from 1.1 to 1.00 where R=1.0 is the radius of the sphere). These graphs were created from by Program 3, APP. B from data generated by Program 1, APP. B. As the altitude increases $Q(x)$ becomes attenuated. This is expected for the same reasons as were given in the preceeding paragraph for single line segments (tent functions). As the altitude goes to zero $Q(x)$ approaches the piecewise linear function $C(x)$ as the theory predicted. Figure 22 of APPENDIX C shows the spectrum of $C(x)$, the covariance of gravity anomalies on S (the surface of the earth), obtained from actual measurements of gravity anomalies (3). While C_1 and C_5 are negative, they are small with respect to other non-negative coefficients, and so the spectrum $\left\{C_k\right\}$ is essentially non-negative as the theory demands. These computations have demonstrated how the theory previously derived can be applied, given a covariance function for gravity anomalies on S $\left(C(x)\right)$ and a rotation invariant operator (Poisson's Integral). The covariance function for gravity anomalies at any altitude which is obtained could be used to give statistical information about gravity anomalies along a satellite's orbit and hence help predict its motion.

Chapter 5

COVARIANCE FUNCTION FOR ANOMALIOUS POTENTIAL ON AND ABOVE THE SURFACE OF A SPHERE

5.1 Introduction

In this chapter the theory developed in Chapter 3 is used to calculate the covariance function of the anomalous potential above or on the surface of a sphere. The information available is the covariance function for the rotation invariant random process on S, gravity anomalies. A linear rotation invariant operator on gravity anomalies (Stokes' Formula) yields anomalous potential at an altitude, just as the upward continuation formula expresses gravity anomalies at an altitude. Comparing equations 2-11 and 3-16, the spectrum of $\tilde{G}$ (the Stokes' operator from section 2.3) is

$$G_k = \left(\frac{R}{r}\right)^{n+1} \left(\frac{1}{R(n-1)}\right) . \qquad 5\text{-}1$$

5.2 Covariance of Anomalous Potential at an Altitude

Let R(x) be the covariance function of the anomalous potential at an altitude of (r-R) earth radii above S. The spectrum of R(x) is obtained from equation 3-23 where M_k is replaced by G_k from equation 5-1. The covariance function for gravity anomalies on S, C(x), will be the line segment defined in section 4.2, and thus C_k is defined by 4-8 and 4-9, and

$$R_k = \left(\frac{R}{r}\right)^{2(k+1)} \frac{1}{R^2(k-1)^2} . \qquad 5\text{-}2$$

Equations 3-16 and 4-9 then give an expression for R(x),

$$R(x) = \frac{1}{4\pi} \sum_{n=0}^{\infty}{}^{1} (2n+1) \left[\left(\frac{R}{r}\right)^{2(n+1)} \frac{C_n}{R^2(n-1)^2}\right] P_n(x) . \qquad 5\text{-}3$$

If the term in square brackets in equation 5-3 is replaced by

$$\frac{C_n}{R^2(n-1)^2} \quad ,$$

the new summation has the property that each term dominates the corresponding term in 5-3. If only N terms are used in evaluating the new sum, the truncation error incurred will be greater than or equal to truncation error incurred by approximating R(x) by N terms of the infinite summation 5-3. Comparing the expression for Q(x) (4-12) and R(x) (5-3), and looking at equation 4-14, we see that R(x) can be expressed as

$$R(x) = \frac{(a_1-a_2)(b_1+b_2)\gamma}{4R^2} + \frac{1}{2}\sum_{k=2}^{\infty} \frac{\gamma^{k+1}}{R^2(k-1)^2} f(a_1,a_2,b_1,b_2,k)P_k(x). \qquad 5\text{-}4$$

5.3 An Upper Bound for Truncation Error

An upper bound for the truncation error (ε) incurred by approximating R(x), expressed as an infinite series by equation 5-4, by a finite sum is derived in APPENDIX D, and presented below,

$$\varepsilon < \frac{\gamma\ |b|\,(13+24\ |a|)}{3\ R^2(2)^{N+1}} \quad . \qquad 5\text{-}5$$

5.4 Computations Performed

As in Section 4.6, the first set of computations assumes a covariance function of gravity anomalies on S which takes the form of the tent function shown in FIGURE 2, APPENDIX A. The constant b_1 is assumed to be unity as before. Program 1, APP. B calculates the covariance function for anomalous potential for various altitudes above the surface of the sphere (the earth) and for various values of the parameter a_2 used to specify C(x), the covariance of gravity anomalies on the surface.

For the second set of computations C(x) is taken to be the piecewise linear function shown in Figure 3, APPENDIX A. This function represents the covariance function for gravity anomalies on the earth's surface obtained from gravimetry as described in section 4.6. The covariance function for anomalous potential R(x) is calculated for various altitudes by PROGRAM 1, APP. B.

5.5 Results

For the first set of computations R, the radius of the sphere, was set equal to 1.0 corresponding to one earth radius (approximately 6380 km) and then R(x) was calculated for various altitudes from 0 to 638 km and for various tent functions defining C(x) (or various values of a_2). The graphs displayed in Figures one to fiteen in APP.C were created by Program 2, APP. B from data generated by Program 1, APPENDIX B. Each graph shows the covariance function for anomalous potential, R(x), as a broken line, and C(x), the covariance function for gravity anomalies on the surface, as a tent function. The functions R(x) are normalized so that on the surface of the earth the variance $\left(R(0)\right)$ is unity. As in Chapter 4, as the altitude increases R(x) becomes attenuated, but the functions themselves seem quite unreasonable. This result is quite easily explained. As was previously mentioned the tent functions representing C(x) are not intended to represent actual covariance functions for gravity anomalies on the surface. Graph 16, APP. C, shows that the spectrum of the tent functions has large negative members as well as a large C_1 term $\left(\text{where } C_1 \text{ is the coefficient of } P_1(\cos\Theta)\right)$. Equation (5-3) shows that R(x) is a function of every element of the spectrum C_k except for C_1. Since R(x) is independent of C_1, this means that every covariance function for gravity anomalies derived from any given covariance function by changing C_1 in its spectrum will give rise to the same covariance for anomalous potential, R(x). This observation confirms the physically motivated assumption that an actual covariance function for gravity anomalies must have the member C_1 of its spectrum equal to zero.

For the second set of computations C(x) is the covariance function of Figure 3, APP. A derived from actual measurements of gravity (3). Figure 22, APP. C, shows the spectrum of C(x) and here the C_1 member is so small with respect to the other members of the spectrum that it may be neglected. Graph 17 to 21 in APPENDIX C show R(x) as a dashed curve for various altitudes from zero to 638 km. As the altitude increase R(x) becomes attenuated. The explanation for this is the same as that proposed in section 4.6 where the same attenuation was observed for the covariance function of gravity anomalies.

5.6 Verifying Statistical Assumptions

This section verifies the validity of the assumption that gravity anomalies can be considred as a rotation invariant process. If anomalous potential is considered to be a stationary random process which is ergodic as well, then the coefficients in the associated Legendre function expansion for the potential will gives the covariance function for anomalous potential. Reference 3 gives the following formula for the calculation of R(x), the covariance function for anomalous potential on S:

$$R(x) = \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} (\delta\overline{C}_{nm}^2 + \delta\overline{S}_{nm}^2) P_n(x) \qquad 5\text{-}6$$

where $\delta\overline{C}_{nm}$ and $\delta\overline{S}_{nm}$ are the normalized harmonic coefficients of the anomalous potential[1]. From Chapter 2, refernce (13), if the arbitrary mean radius of the earth used in the expression for the anomalous

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1. See page 21, equation (2-15) of reference (13).

potential (equation 2.15 reference 13) is set equal to the semi-major axis of the earth ellipsoid (equation 2.14 reference 13), then

$$\delta\overline{C}_{nm} = \overline{C}_{nm} - \overline{C}^{(\mu)}_{nm} \qquad 5\text{-}7$$

$$\delta\overline{S}_{nm} = \overline{S}_{nm} \qquad 5\text{-}8$$

where $\overline{C}_{nm}$ and $\overline{S}_{nm}$ are the normalized spherical harmonic coefficients used in expressing the potential as a sum of normalized associated Legendre functions, and $\overline{C}^{(\mu)}_{nm}$ may be calculated (equation 2.13, (13)).

Data obtained from observing satellite motions and from gravimetry were used by Köhnlein(11) to determine values for $\overline{C}_{nm}$ and $\overline{S}_{nm}$. Using equation 2.13 in reference(13), values for $\overline{C}^{(\mu)}_{nm}$ were obtained, where $\overline{C}^{(\mu)}_{nm}$ consists only of even zonal harmonics $\overline{C}^{(\mu)}_{2n,0}$. Since $\overline{C}^{(\mu)}_{6,0}$ is very small in comparison to $\overline{C}_{6,0}$, $\overline{C}^{(\mu)}_{k,0}$ was considered zero for $k \geq 6$. Thus using 5-7 and 5-8 it is possible to obtain values for $\delta\overline{C}_{nm}$ and $\delta\overline{S}_{nm}$, and so, from equation 5-6, R(x), the covariance function for anomalous potential on S. In section 5-6 anomalous potential on S was calculated from C(x), the covariance of gravity anomalies on S, where gravity anomalies were considered to be a rotation invariant random process on S. Graph 23, APP. C shows R(x) obtained from Köhnlein's model of potential[1] (satellite motions and gravimetry), shown by the solid line, and R(x) obtained from Kaula's (3) data (the covariance function of gravity anomalies on S obtained from gravimetry) shown by the dashed line. The very close agreement between these covariance functions (on the scales for which the graphs agree) derived from independent sources (gravity anomalies on the surface of the earth and potential derived from satellite observations) confirms the validity of treating gravity anomalies as a rotation invariant random process over the surface of a sphere.

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1. This calculation was performed by PROGRAM 4, APP. B.

Chapter 6

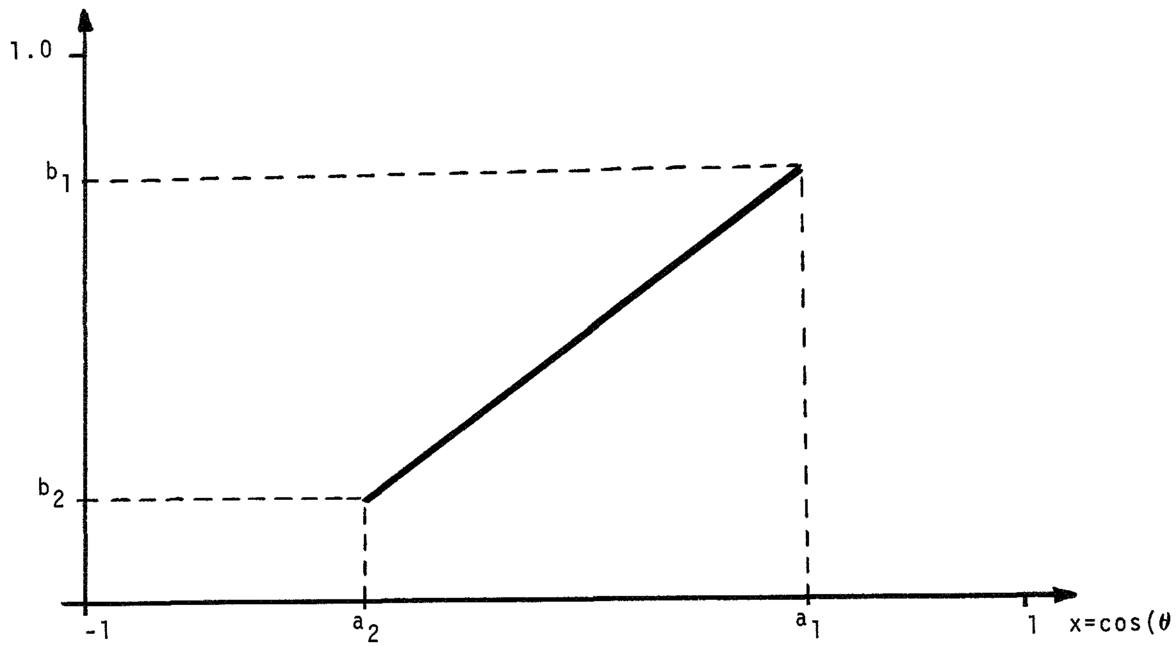
CONCLUSION

One conclusion which may be drawn is that if gravity anomalies are modelled as a rotation invariant random process on the surface of the earth (or a sphere) then it is easy to calculate the covariance function for the anomalous potential and gravity anomalies at any altitude. Further, the covariance function for gravity anomalies and anomalous potential at any altitude will be independent of position on the earth, and so will also be rotation invariant random processes. While it has not been directly proven that gravity anomalies form a rotation invariant random process, it has been shown that this assumption is a consistent one when results derived from it are compared with similar results from independent (satellite) data. To be more precise, what has been shown is that if anomalous potential is an ergodic rotation invariant random process then assuming gravity anomalies to be a rotation invariant random process does not give rise to any contradictions. Thus there are situations in which gravity anomalies may be treated as a rotation invariant random process on the surface of a sphere. Finally if gravity anomalies are modelled as a rotation invariant random process then the covariance function must not have a C_1 term in its spectrum.

APPENDIX A

GRAPHS OF COVARIANCE FUNCTIONS
OF GRAVITY ANOMALIES

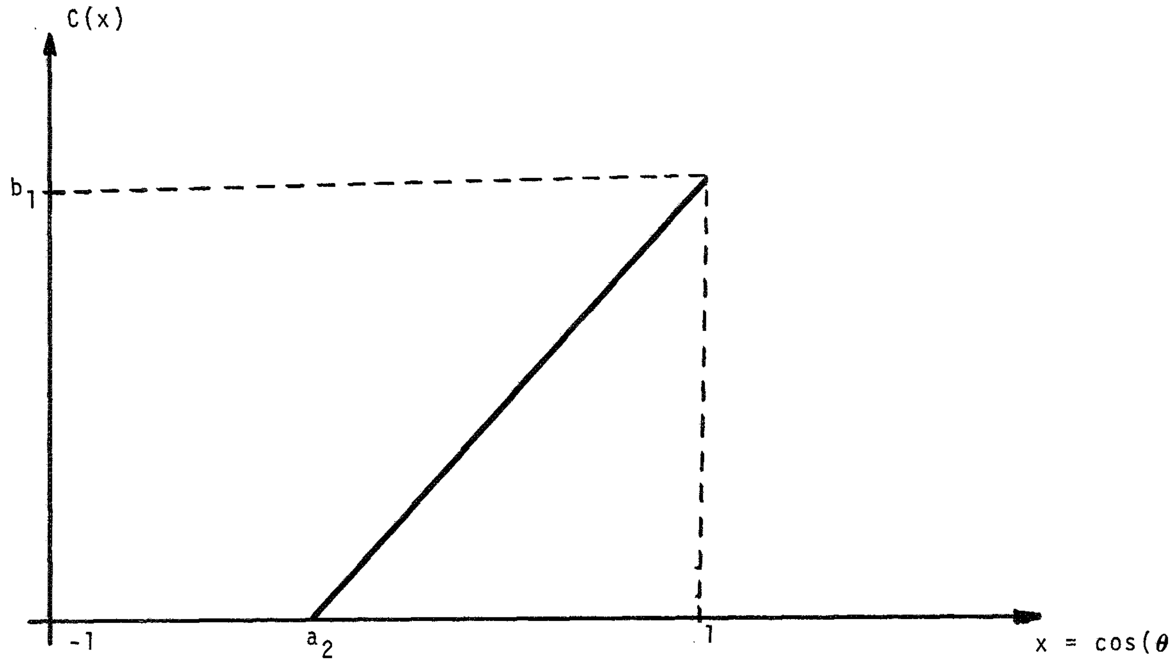
FIGURE 1.



$$C(x) = b[x-a]$$

$$\text{where } b = \frac{b_1 - b_2}{a_1 - a_2}$$

$$a = \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1}$$



$$C(x) = b(x-a_2)$$

$$\text{where } b = \frac{1}{1-a_2}$$

FIGURE 2

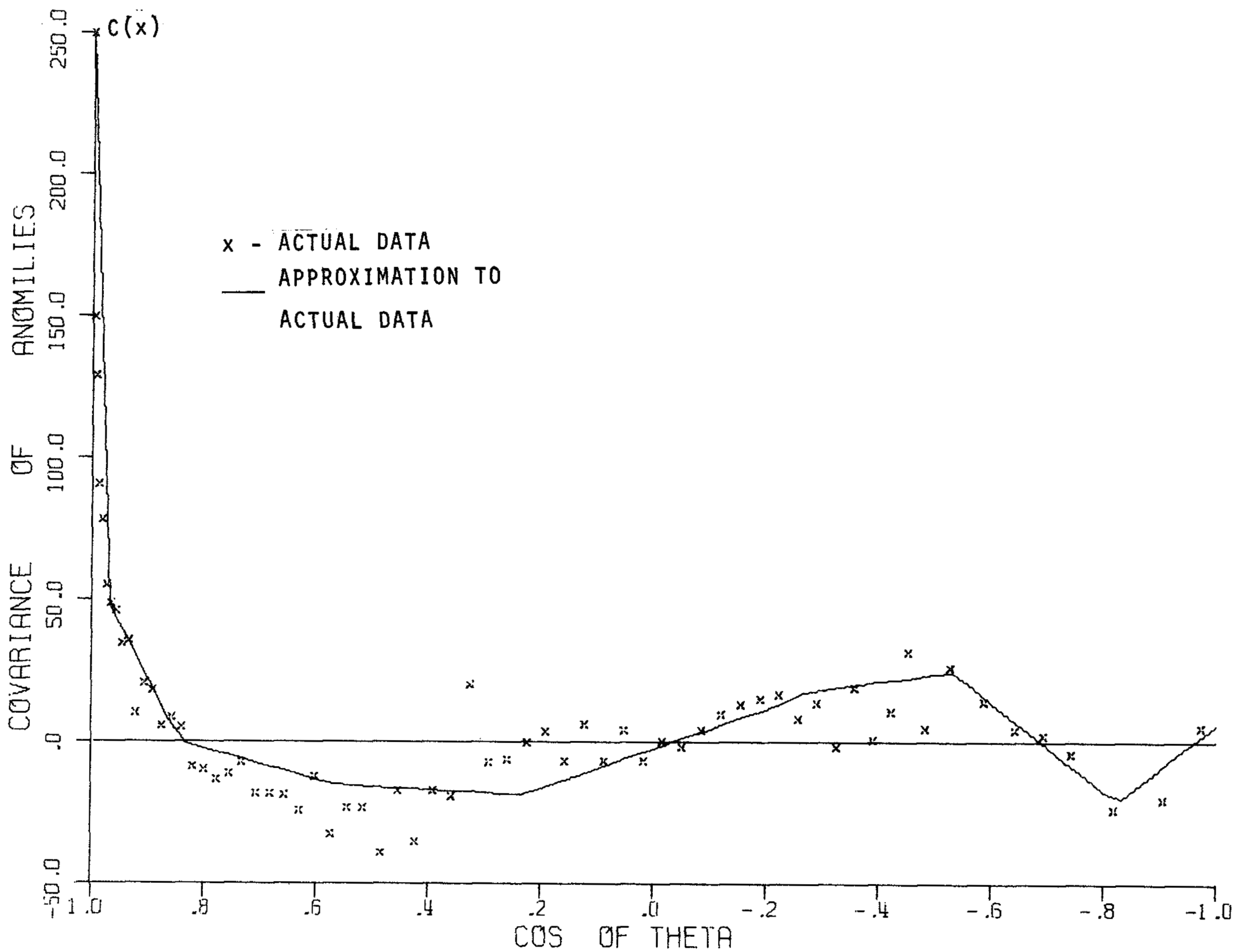

C(x)
x - ACTUAL DATA
APPROXIMATION TO ACTUAL DATA
COVARIANCE OF ANØMILIES
250.0
200.0
150.0
100.0
50.0
.0
-50.0
1.0
.8
.6
.4
.2
.0
-.2
-.4
-.6
-.8
-1.0
COS OF THETA

FIGURE 3.

APPENDIX B

COMPUTER PROGRAMS

PROGRAM 1 - 1

```
      DIMENSION C(202,10),B(208),RX(202,10),PL(4,70),PA(202,2)
      DIMENSION N(70),X(202),NP(70),A(70),BB(70)
      DIMENSION T(70,2),B1(202),RX1(202,10)
      ER=.01
      KD=10
      IR=1
  908 KLH=0
      PRINT 199
  199 FORMAT(1X,17HNL=N,NCH=0,QUIT=9)
      READ 201,ICH
  201 FORMAT(I2)
      IF(ICH.EQ.0)GO TO 876
      IF(ICH.EQ.9)GO TO 999
      IF(ICH.LT.5)GO TO 661
      IF(IR.GT.1)GO TO 327
      READ(60,204)((PL(I,J),I=1,4),NP(J),N(J),J=1,ICH)
      IR=IR+1
      GO TO 327
  661 CONTINUE
      PRINT 200
  200 FORMAT(1X,26HA1,A2,B1,B2 4F7.4 NP,N 2I3 )
      READ 204,((PL(I,J),I=1,4),NP(J),N(J),J=1,ICH)
  204 FORMAT(4F7.4,2I3)
      ISAVE=ICH
  327 KLH=KLH+1
      IF(KLH.GT.ICH)GO TO 328
      NSV=N(KLH)
      K1=N(KLH)+1
      K3=K1+1
      DO 5 J=1,2
      PA(1,J)=PL(J,KLH)
      PA(2,J)=0.5*(3.0*PL(J,KLH)**2-1.0)
      DO 5 I=3,K3
      PA(I,J)=PA(I-1,J)*PL(J,KLH)*FLOAT(2*I-1)
    5 PA(I,J)=(PA(I,J)-PA(I-2,J)*FLOAT(I-1))/FLOAT(I)
      K13=N(KLH)+3
      K12=K13-1
      B1(K12)=0.0
      B1(K13)=0.0
      B(K12)=0.0
      B(K13)=0.0
      K2=NP(KLH)+1
  876 CONTINUE
      IF(ICH.NE.0)GO TO 729
      KLH=1
      N(KLH)=NSV
      ICH=ISAVE
  729 CONTINUE
      IF(KLH.NE.1)GO TO 329
```

PROGRAM 1 - 2

```
      PRINT 202
  202 FORMAT(1X,14HR0,R1,R2 3F6.4)
      READ 205,R0,R1,R2
  205 FORMAT(3F6.4)
      GAMMA=R0**2/(R1*R2)
  329 A(KLH)=(PL(1,KLH)*PL(4,KLH)-PL(3,KLH)*PL(2,KLH))/
     1 (PL(4,KLH)-PL(3,KLH))
      BB(KLH)=(PL(3,KLH)-PL(4,KLH))/(PL(1,KLH)-PL(2,KLH))
      NKLH=N(KLH)
      DO 888 I=6,NKLH
      IF(A(KLH).GE.0.0)GO TO 777
      AAA=-A(KLH)
      GO TO 778
  777 AAA=A(KLH)
  778 IF(BB(KLH).GE.0.0)GO TO 790
      BBB=-BB(KLH)
      GO TO 791
  790 BBB=BB(KLH)
  791 CONTINUE
      GGG=0.0
      IF(I.GT.100)GO TO 533
      GGG=GAMMA/(3.0*2.0**(I-1)*(2.0-GAMMA))
  533 ECK=((GAMMA+12.0+24.0*AAA)/(12.0*(1.0-GAMMA))
     1 +GGG)*GAMMA**(I+1)*BBB
      IF(ECK.GE.ER)GO TO 888
      N(KLH)=I
      GO TO 889
  888 CONTINUE
  889 PRINT 887,N(KLH)
  887 FORMAT(1X,I5)
      K1=N(KLH)+1
      DO 11 J=1,K2
      X(J)=FLOAT((NP(KLH)/2)+1-J)/FLOAT(NP(KLH)/2)
      DO 10 II=1,K1
      I=N(KLH)+1-II
      I1=I+1
      IF(I.NE.1)GO TO 6
      A9=3.1415927*(PL(1,KLH)-PL(2,KLH))/3.0
      C(I1,KLH)=A9*((2.*PL(1,KLH)+PL(2,KLH))*PL(3,KLH)+(2.0*PL(2,KLH)
     1+PL(1,KLH))*PL(4,KLH))
      BC0=C(I1,KLH)*GAMMA**2*3.0/(4.0*3.14159278)
      AC0=0.0
      GO TO 7
    6 IF(I.NE.0)GO TO 8
      C(I1,KLH)=(PL(1,KLH)-PL(2,KLH))*(PL(3,KLH)+PL(4,KLH))*3.14159278
      BC0=C(I1,KLH)*GAMMA/(4.0*3.14159278)
      AC0=BC0/R0**2
      GO TO 7
    8 IF(I.NE.2)GO TO 12
      ZZ=0.0
      GO TO 9
   12 ZZ=PA(I-2,2)-PA(I-2,1)
    9 C(I1,KLH)=(2.0*3.14159278*BB(KLH)/FLOAT(2*I+1))*(FLOAT(2*I+1)*
```

PROGRAM 1 - 3

```
  1 (PA(I,1)-PA(I,2))/FLOAT((2*I-1)*(2*I+3)) + ZZ
  2 *FLOAT(I)/FLOAT(2*I-1) + (PA(I+2,1)-PA(I+2,2))*FLOAT(I+1)/
  3 FLOAT(2*I+3) + A(KLH)*(PA(I+1,2)-PA(I+1,1)+PA(I-1,1)-PA(I-1,2)))
    BCO=C(I1,KLH)*GAMMA**(I+1)*FLOAT(2*I+1)/(4.0*3.14159278)
    ACO=BCO/(FLOAT(I-1)**2*R0**2)
  7 B(I+1)=(FLOAT(2*I+1)*X(J)/FLOAT(I+1))*B(I+2)
  1 - (FLOAT(I+1)/FLOAT(I+2))*B(I+3) + ACO
    B1(I+1)=(FLOAT(2*I+1)*X(J)/FLOAT(I+1))*B1(I+2)
  1 - (FLOAT(I+1)/FLOAT(I+2))*B1(I+3) + BCO
 10 CONTINUE
    RX1(J,KLH)=B1(1)
 11 RX(J,KLH)=B(1)
    GO TO 327
328 DO 400 I=1,K2
    RX1(I,KD)=0.0
    RX(I,KD)=0.0
    C(I,KD)=0.0
    DO 400 J=1,ICH
    RX1(I,KD)=RX1(I,KD)+RX1(I,J)
    RX(I,KD)=RX(I,KD)+RX(I,J)
400 C(I,KD)=C(I,KD)+C(I,J)
401 FORMAT(1X,8F9.5)
    Z5=R1-R0
    IF(Z5.GE.0.00001)GO TO 404
    WRITE(68,401)(C(I,KD),I=1,24)
404 K5=1
    DO 500 I=1,K2
502 CONTINUE
    IF(X(I).GE.PL(2,K5))GO TO 501
    K5=K5+1
    IF(K5.LE.ICH)GO TO 502
    Y=0.0
    GO TO 503
501 Y=BB(K5)*(X(I)-A(K5))
503 CONTINUE
    WRITE(68,410)I,X(I),Y,RX1(I,KD),RX(I,KD)
410 FORMAT(I3,4F10.5)
504 FORMAT(1X,I3,12F9.5)
500 CONTINUE
    GO TO 908
999 CONTINUE
    CALL EXIT
    END
```

PROGRAM 2 - 1

```
      DIMENSION D(71),XAXIS(16),YAXIS(32),XP(100),CO(71)
      DIMENSION DP(97),AP(97),DA(70)
      DATA XAXIS/'COS ',' OF ','THET','A    '/
      DATA YAXIS/'COVA','RIAN','CE  ','    D','F   ','  AN','OMIL',
     1'IES '/
      CALL NEWPLT('M6844','8158','WHITE ','BLACK')
      KC=0
    4 CONTINUE
      CALL AXIS1(0.0,0.0,YAXIS,32,04.0,90.0,-0.25,0.25,2,0,0.8)
      CALL AXIS1(0.0,0.0,XAXIS,16,05.0,0.0,1.0,-.2,1,0,0.5)
      READ(5,50)R1,R2,R0,A,N,EPSLN,NP
   50 FORMAT(3X,3F10.4,3X,F4.2,I5,F10.4,I3)
      KC=KC+1
      IF(R1.GT.100.0)GO TO 999
      READ(5,51)(XP(J),CO(J),D(J),DA(J),J=1,61)
   51 FORMAT(3X,4F10.5)
      IF(KC.EQ.1)GO TO 101
      IF(KC.NE.5)GO TO 100
      KC=0
      GO TO 100
  101 Z11=DA(1)
  100 DO 1 I=1,61
      CO(I)=CO(I)*3.2
      D(I)=D(I)*3.2
    1 DA(I)=DA(I)*3.2/Z11
      Z=D(1)+0.8
      CALL PLOT1(0.0,Z,3)
      DO 2 I=2,61
      XINC=2.5*(1.0-XP(I))
      Z=D(I)+0.8
      CALL PLOT1(XINC,Z,2)
    2 CONTINUE
      Z=CO(1)+0.8
      CALL PLOT1(0.0,Z,3)
      DO 12 I=2,61
      XINC=2.5*(1.0-XP(I))
      Z=CO(I)+0.8
      CALL PLOT1(XINC,Z,2)
   12 CONTINUE
      Z=DA(1)+0.8
      CALL PLOT1(0.0,Z,3)
      DO 96 I=2,61,2
      XI1=2.5*(1.0-XP(I))
      XI2=2.5*(1.0-XP(I+1))
      Z=DA(I)+0.8
      CALL PLOT1(XI1,Z,2)
      Z=DA(I+1)+0.8
      CALL PLOT1(XI2,Z,3)
   96 CONTINUE
      CALL PLOT1(14.0,0.0,-3)
```

PROGRAM 2 - 2

```
      GO TO 4
  999 CONTINUE
      CALL PLOT1(14.0,0.0,-3)
      CALL AXIS1(0.0,0.0,XAXIS,16,08.0,0.0,1.0,-.2,1,0,0.8)
      CALL AXIS1(0.0,0.0,YAXIS,32,6.0,90.0,-50.00,50.00,1,0,1.0)
      CALL PLOT1(0.0,6.0,3)
      READ(5,51)(XP(J),CU(J),D(J),DA(J),J=1,61)
      Z11=DA(1)
      DO 106 J=2,61,2
      Y=4.0*(1.0-XP(J))
      Z=DA(J)*5.0/Z11 + 1.0
      CALL PLOT1(Y,Z,2)
      Y=4.0*(1.0-XP(J+1))
      Z=DA(J+1)*5.0/Z11 + 1.0
      CALL PLOT1(Y,Z,3)
  106 CONTINUE
      CALL PLOT1(0.0,6.0,3)
      DO 956 J=1,91
      READ(5,200)DP(J),AP(J)
      WRITE(6,200)DP(J),AP(J)
  200 FORMAT(3X,2F10.5)
  956 CONTINUE
      Z12=AP(1)
      DO 201 J=2,91
      X=4.0*(1.0-DP(J)/10.0)
      Y=AP(J)*5.0/Z12 + 1.0
      CALL PLOT1(X,Y,2)
  201 CONTINUE
      CALL PLOT1(20.0,0.0,-3)
      CALL ENDPLT
      CALL EXIT
      END
```

PROGRAM 3 - 1

```
      DIMENSICN D(71),XAXIS(16),YAXIS(32),XP(100),CO(71)
      DIMENSICN DP(67),AP(67),DA(70)
      DATA XAXIS/'COS ',' OF ','THET','A   '/
      DATA YAXIS/'COVA','RIAN','CE  ','   O','F   ','   AN','OMIL',
     1'IES '/
      DATA DP/243.0, 145.4,125.3,88.11, 75.96, 53.57,47.40, 45.18,
     1 33.8C, 34.51, 9.94, 20.01, 17.61, 5.53, 8.42, 4.95, -8.28,
     2 -9.69, -12.7, -10.8, -6.87, -17.7, -17.6, -18.0, -23.4, -11.8,
     3 -31.7, -22.5, -22.0, -38.1, -16.6, -34.3, -16.9, -18.5, 19.7,
     4 -7.04, -6.02, -0.18, 3.41, -6.38, 5.84,  -6.46, 4.37, -6.68, .18,
     5 -1.79, 4.03, 9.55, 12.79, 14.59, 16.08, 7.86, 13.32, -1.87,
     6 18.53, .69, 10.27, 30.82, 4.55, 25.53, 13.83, 4.14, 2.24, -4.36,
     7 -23.1, -19.7, 5.13/
      DATA AP/0.0,5.,7.,9.,11.,13.,15.,17.,19.,21.,23.,25.,27.,29.,31.,
     1 33.,35.,37.,39.,41.,43.,45.,47.,49.,51.,53.,55.,57.,59.,61.,
     2 63.,65.,67.,69.,71.,73.,75.,77.,79.,81.,83.,85.,87.,89.,91.,
     3 93.,95.,97.,99.,101.,103.,105.,107.,109.,111.,113.,115.,117.,
     4 119.,122.,126.,130.,134.,138.,145.,155.,167./
      CALL NEWPLT('M6844','8158','WHITE ','BLACK')
    4 CONTINUE
      CALL AXIS1(0.0,0.0,YAXIS,32,04.0,90.0,-0.25,0.25,2,0,0.8)
      CALL AXIS1(0.0,0.0,XAXIS,16,05.0,0.0,1.0,-.2,1,0,0.5)
      READ(5,50)R1,R2,RC,A,N,EPSLN,NP
   50 FORMAT(3X,3F10.4,3X,F4.2,I5,F10.4,I3)
      IF(R1.GT.100.0)GO TO 999
      READ(5,51)(XP(J),CO(J),D(J),DA(J),J=1,61)
   51 FORMAT(3X,4F10.5)
      DO 1 I=1,61
      CO(I)=CO(I)*3.2
      D(I)=D(I)*3.2
    1 DA(I)=DA(I)*32.0
      Z=D(1)+0.8
      CALL PLOT1(0.0,Z,3)
      DO 2 I=2,61
      XINC=2.5*(1.0-XP(I))
      Z=D(I)+0.8
      CALL PLOT1(XINC,Z,2)
    2 CONTINUE
      Z=DA(1)+0.8
      CALL PLOT1(0.0,Z,3)
      DO 96 I=2,61,2
      XI1=2.5*(1.0-XP(I))
      XI2=2.5*(1.0-XP(I+1))
      Z=DA(I)+0.8
      CALL PLOT1(XI1,Z,2)
      Z=DA(I+1)+0.8
      CALL PLOT1(XI2,Z,3)
   96 CONTINUE
      CALL PLOT1(0.0,0.8,3)
      CALL PLOT1(5.0,0.8,2)
```

PROGRAM 3 - 2

```
      CALL PLOT1(14.0,0.0,-3)
      GO TO 4
  999 CONTINUE
      CALL PLOT1(14.0,0.0,-3)
      CALL AXIS1(0.0,0.0,XAXIS,16,08.0,0.0,1.0,-.2,1,0,0.8)
      CALL AXIS1(0.0,0.0,YAXIS,32,6.0,90.0,-50.00,50.00,1,0,1.0)
      CALL PLOT1(0.0,6.0,3)
      DO 106 J=2,61
      Y=4.0*(1.0-XP(J))
      Z=CO(J)*5.0/3.2 + 1.0
      CALL PLOT1(Y,Z,2)
  106 CONTINUE
      CALL PLOT1(0.0,5.86,3)
      DO 111 I=1,67
      YY=AP(I)*3.14159/180.0
      X=4.0*(1.0-COS(YY))
      Y=(DP(I)*5.0/243.0)+1.0
      CALL SYMBL5(X,Y,0.05,-5,0.0,-1)
  111 CONTINUE
      CALL PLOT1(0.0,1.0,3)
      CALL PLOT1(8.0,1.0,2)
      CALL PLOT1(20.0,0.0,-3)
      CALL ENDPLT
      CALL EXIT
      END
```

PROGRAM 4 - 1

```
      DIMENSION C(300),PX(16),CA(18,18),CB(18,18),CN(18),ANG(200)
2     READ(5 ,1)I,J,A,B,K,L,Q,D
1     FORMAT(2(2I3,2E17.8))
      IF(I.LT.0)GO TO 3
      CA(I+1,J+1)=A
      CB(I+1,J+1)=B
      CA(K+1,L+1)=Q
      CB(K+1,L+1)=D
      CB(I+1,1)=0.0
      GO TO 2
3     DO 4 I=3,15,4
      J=I+1
      K=J+1
      L=K+1
      READ( 5,5)CA(I,1),CA(J,1),CA(K,1),CA(L,1)
5     FORMAT(2(6X,2F17.8))
4     CONTINUE
      CA(3,2)=0.0
      CB(3,2)=0.0
      DO 90 I=3,16
      DO 90 J=1,I
      WRITE(6,91)I,J,CA(I,J),CB(I,J)
91    FORMAT(1X,2I5,2F17.10)
90    CONTINUE
      DO 6 I=3,16
      SUM=0.0
      DO 7 J=1,I
7     SUM=SUM+CA(I,J)**2+CB(I,J)**2
      WRITE(6,17)I,SUM,SUM
17    FORMAT(1X,I5,2F17.10)
6     CN(I)=SUM
      DO 10 I=1,181,2
      RAD=FLOAT(I-1)*3.14159265/180.0
      ANG(I)=COS(RAD)
      Y=ANG(I)
      PX(2)=Y
      PX(3)=.5*(3.0*Y**2-1.0)
      DO 11 L=4,16
   11 PX(L)=(PX(L-1)*Y*FLOAT(2*L-3)-PX(L-2)*FLOAT(L-2))/FLOAT(L-1)
      SUM=0.0
      DO 12 K=3,16
12    SUM=SUM+CN(K)*PX(K)
      WRITE(6,17)I,PX(16)
      C(I)=SUM
10    CONTINUE
      DO 15 I=1,181,2
      Z=ANG(I)
      WRITE( 6,16)Z,C(I)
      R1=C(I)*100000000000.0
      R2=Z*10.0
```

PROGRAM 4 - 2

```
      WRITE(7,21)I,R2,R1
   21 FORMAT(I3,2F10.5)
16    FORMAT(5X,2E17.10)
15    CONTINUE
      CALL EXIT
      END
```

APPENDIX C

COVARIANCE OF ANOMALOUS POTENTIAL AND GRAVITY ANOMALIES AT AN ALTITUDE vs COS(θ)

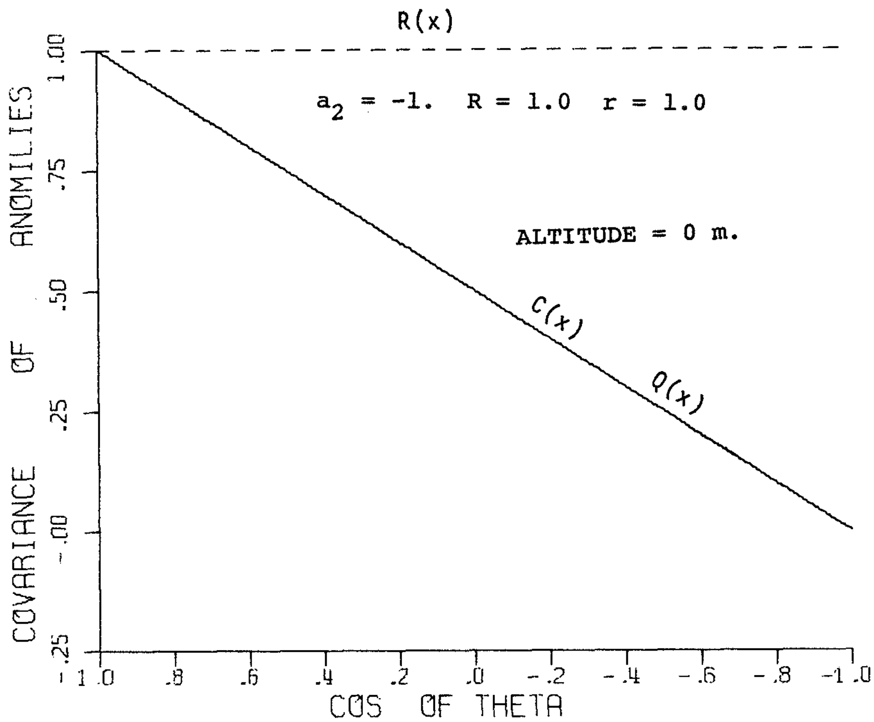


Figure 1

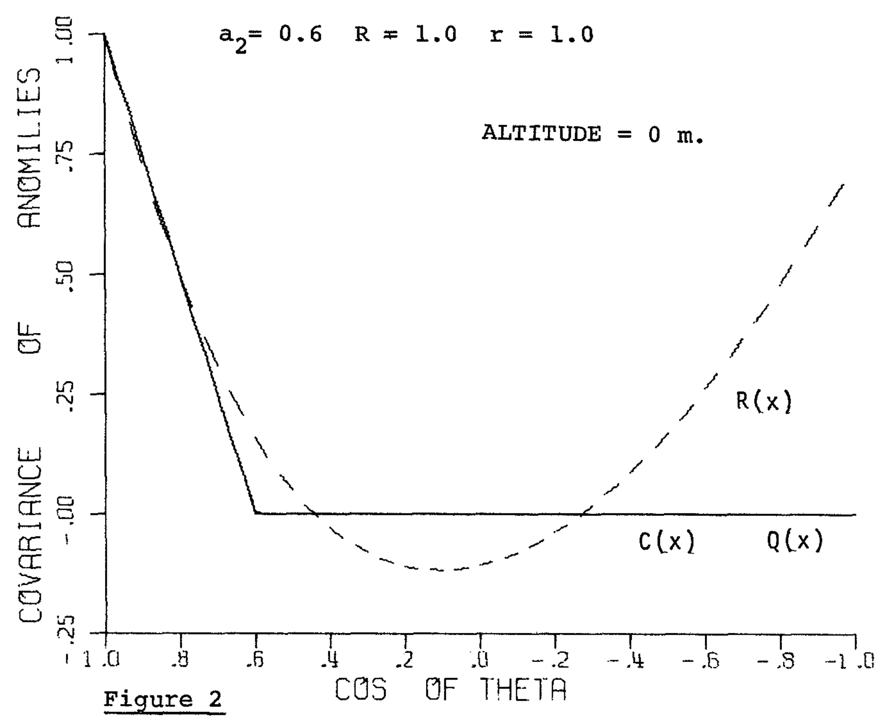


Figure 2

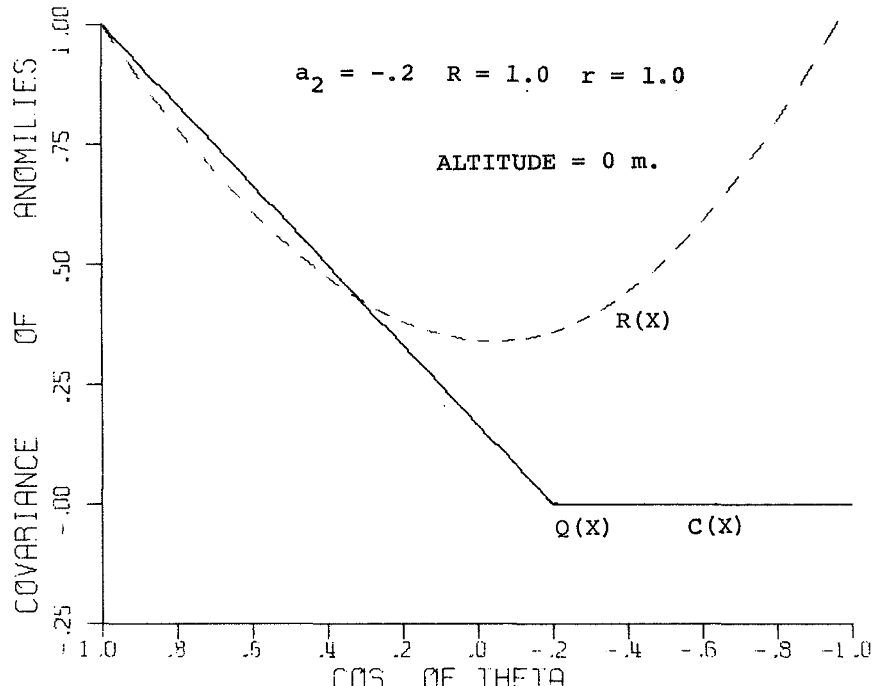


Figure 3

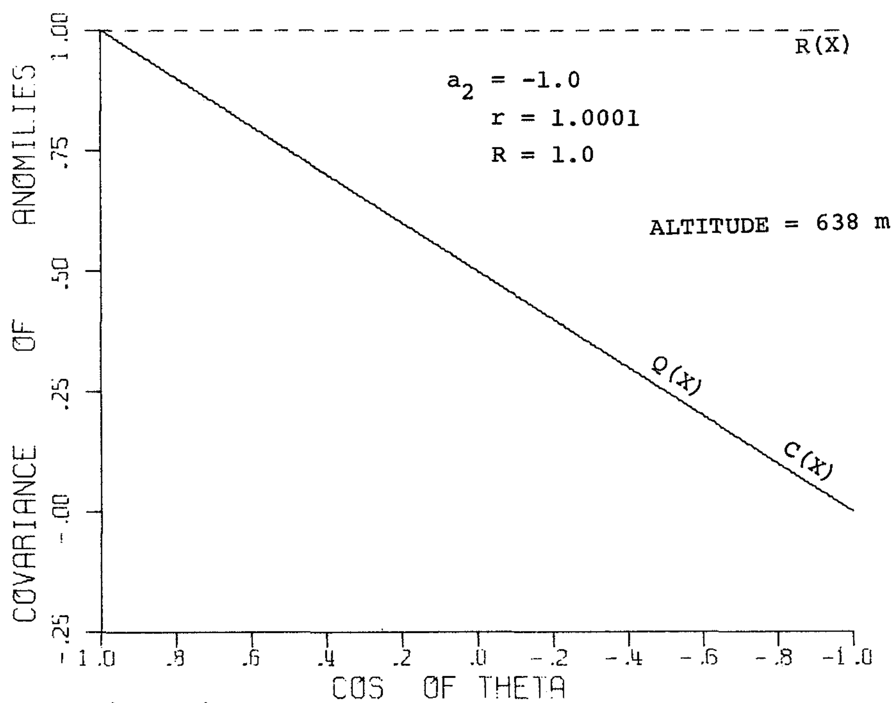


Figure 4

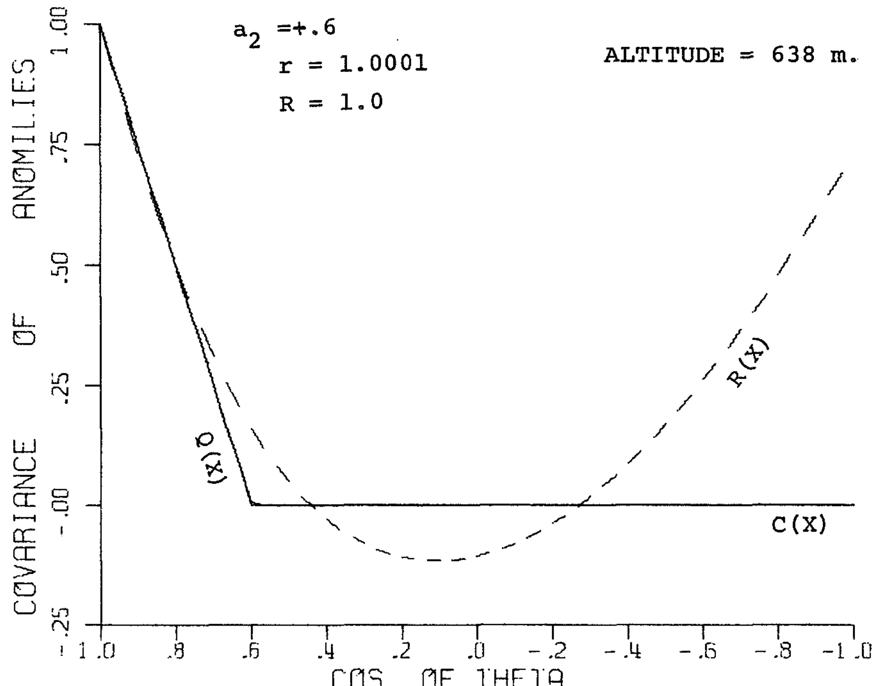


Figure 5

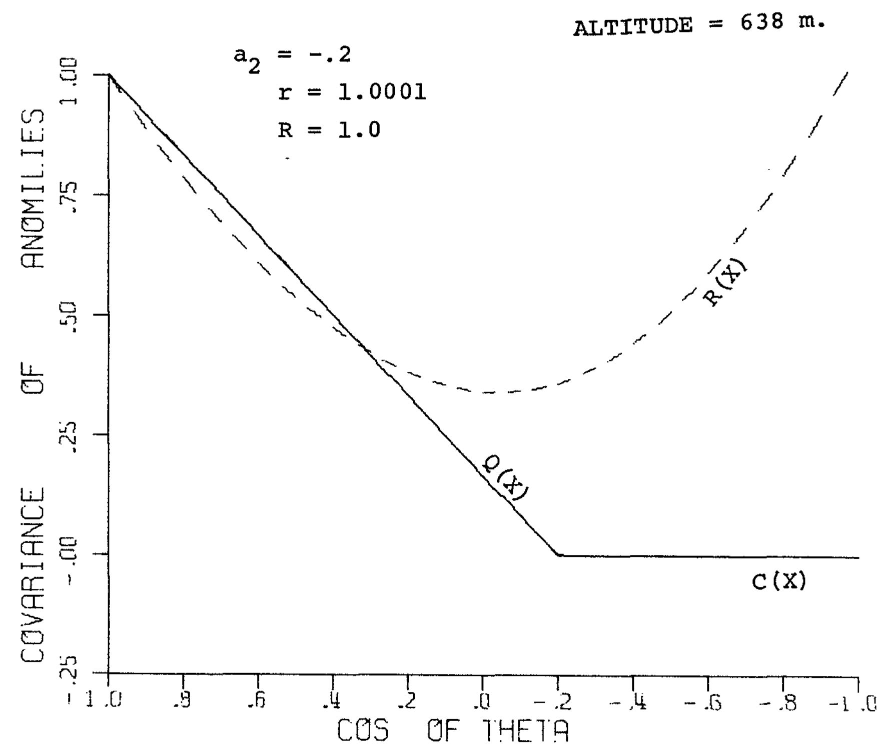


Figure 6

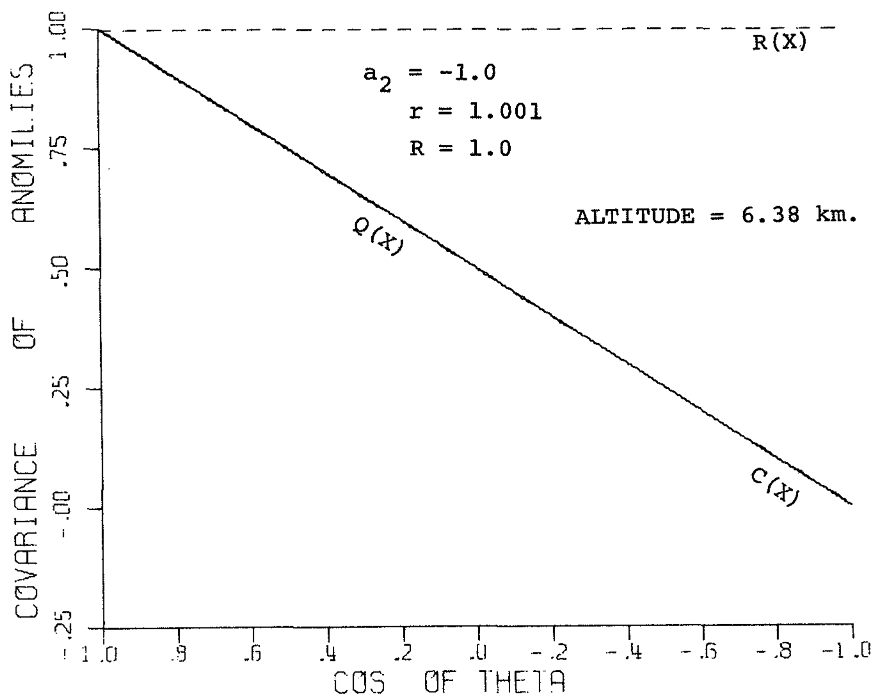


Figure 7

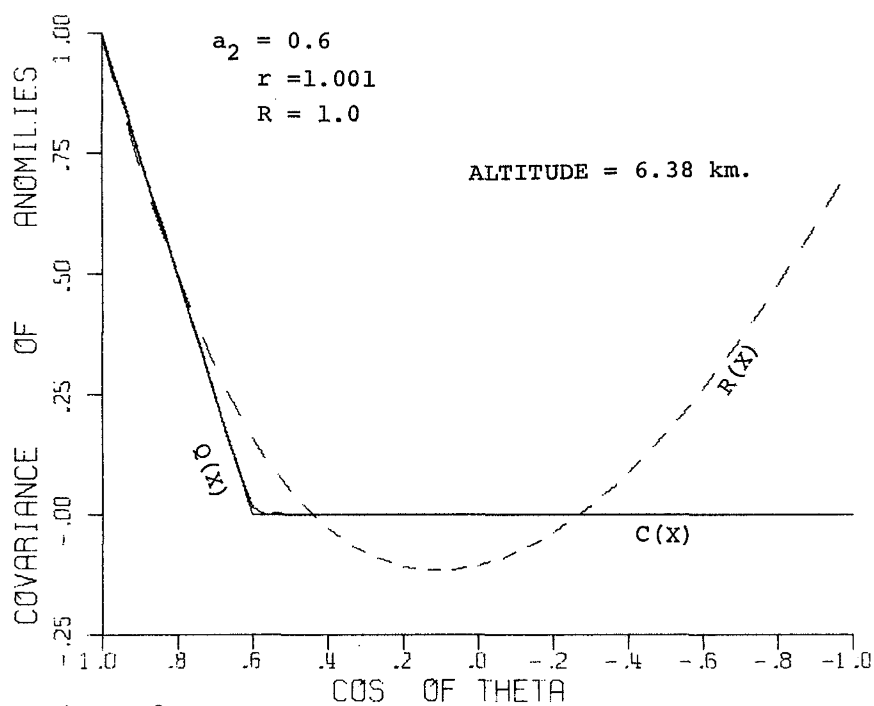


Figure 8

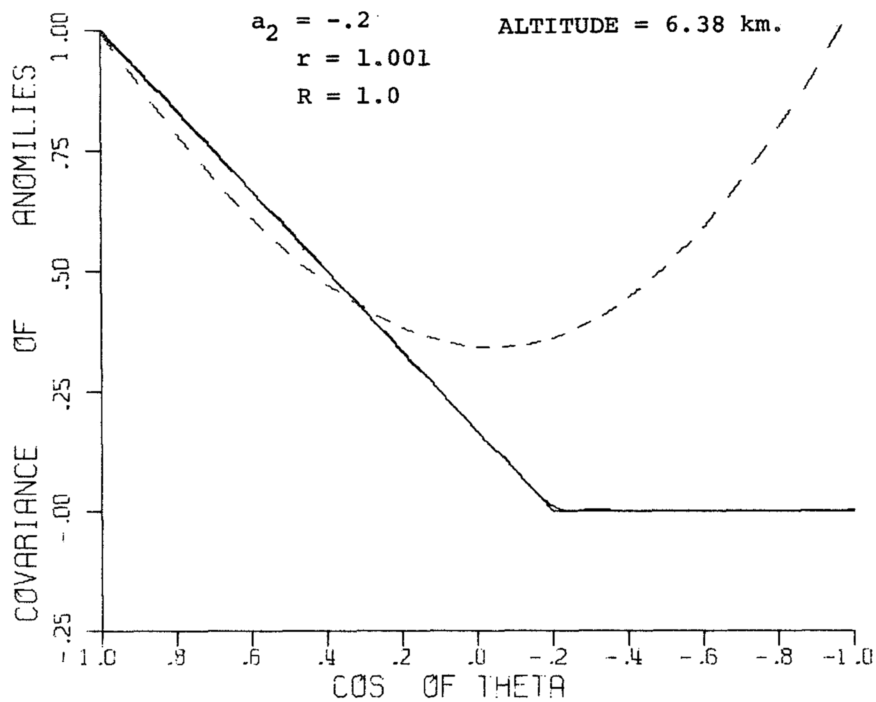


Figure 9

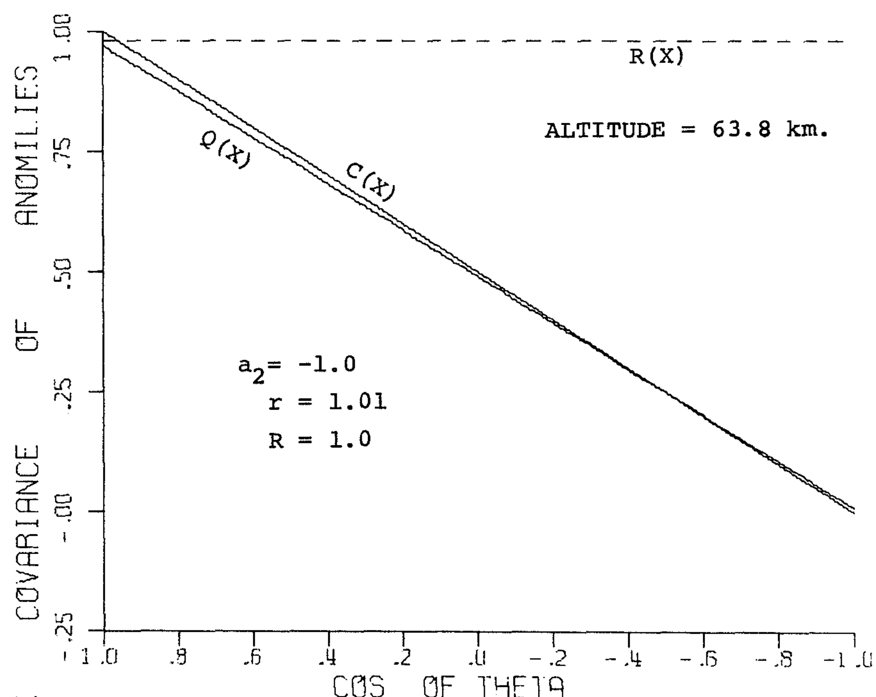


Figure 10

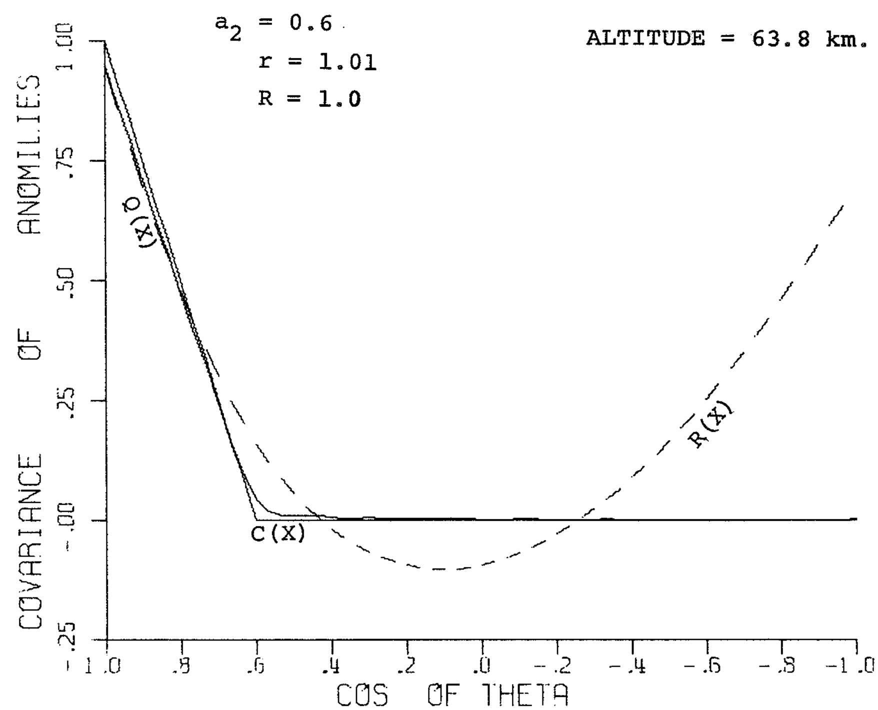


Figure 11

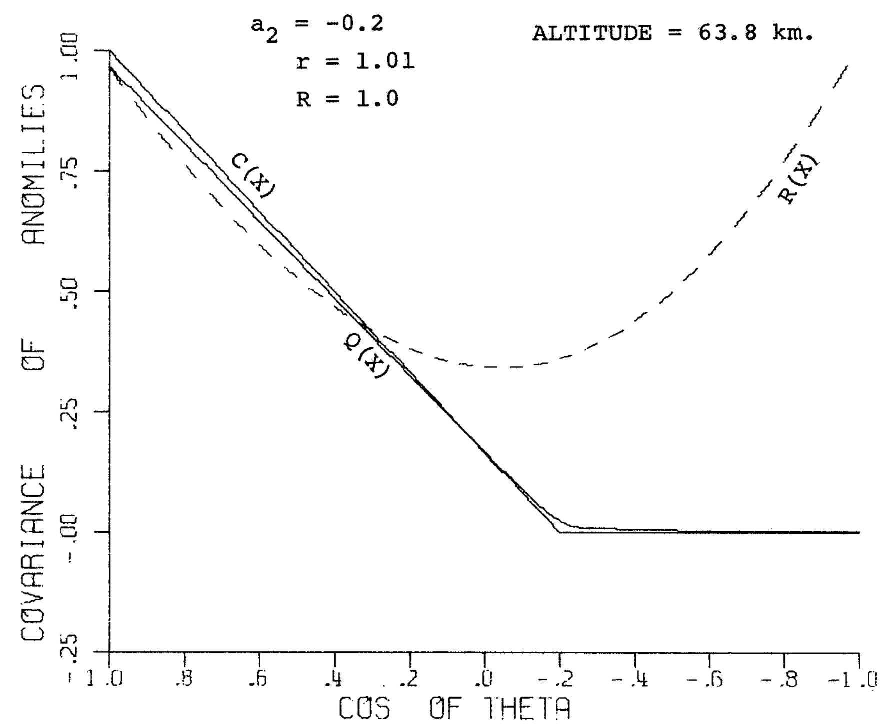


Figure 12

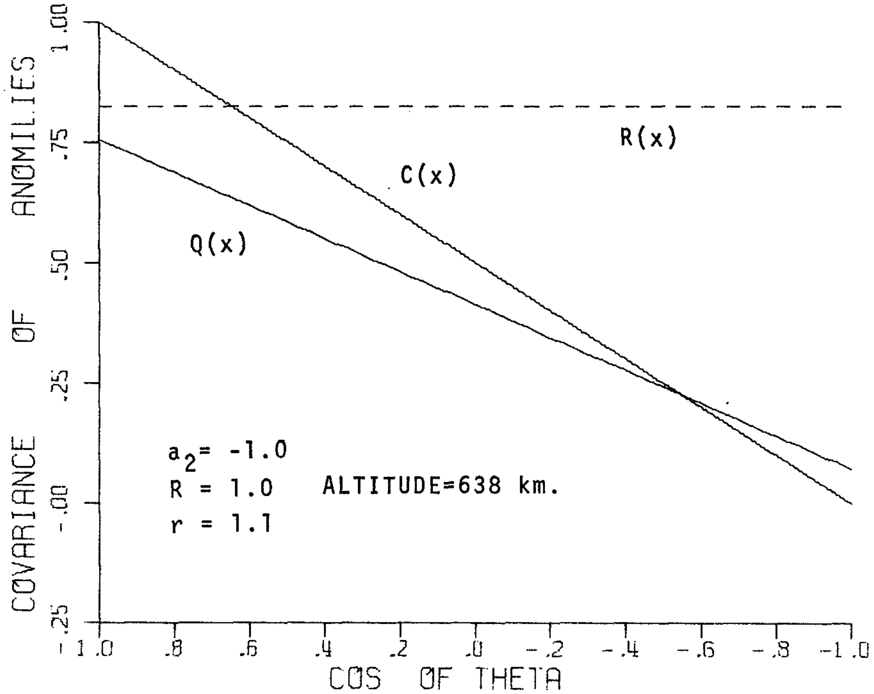


Figure 13

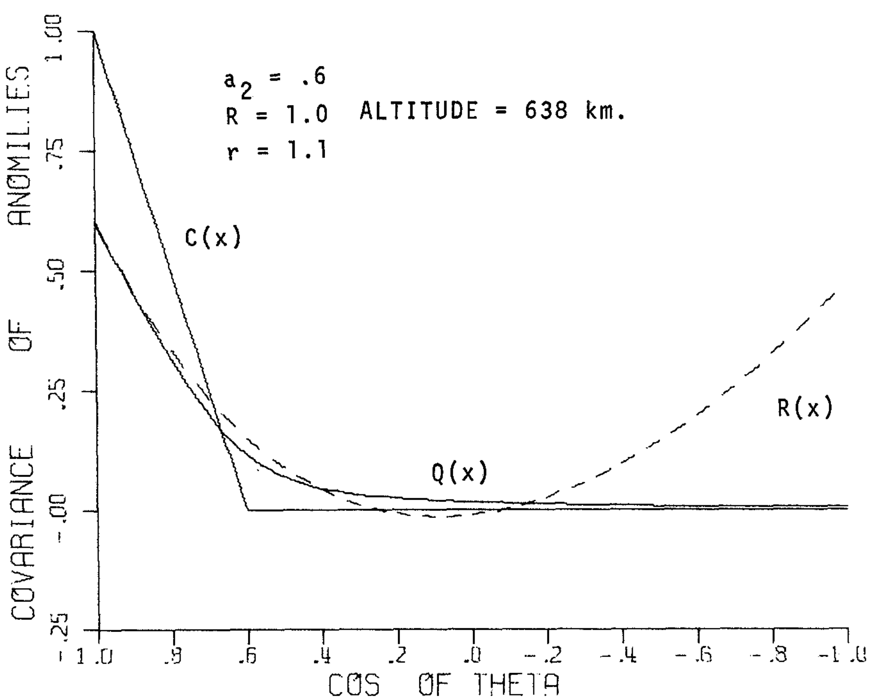
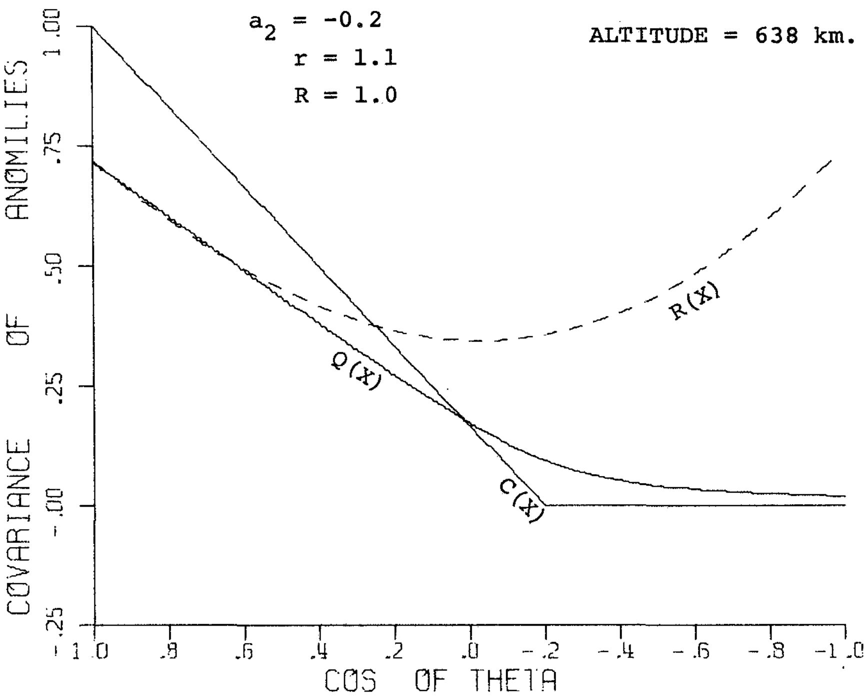


Figure 14



a_2 = -0.2
r = 1.1
R = 1.0
ALTITUDE = 638 km.
R(X)
Q(X)
C(X)
COVARIANCE OF ANOMILIES
1.00
.75
.50
.25
-.00
-.25
1.0
.8
.6
.4
.2
.0
-.2
-.4
-.6
-.8
-1.0
COS OF THETA

Figure 15

SPECTRUM OF C(X) - C_k

k	a_2= 0.6	a_2= -.2	a_2= -1.0
0	1.25664	3.76991	6.28319
1	1.08909	2.26195	2.09440
2	0.80425	0.60319	0.0
3	0.48255	-.12064	0.0
4	0.20374	-.07338	0.0
5	0.01930	0.05308	0.0
6	-.06048	0.01255	0.0
7	-.06099	-.02722	0.0
8	-.02466	0.00189	0.0
9	0.01041	0.01426	0.0
10	0.02427	-.00562	0.0
11	0.01712	-.00696	0.0
12	0.00127	0.00590	0.0
13	-.01014	0.00267	0.0
14	-.01114	-.00496	0.0
15	-.00425	-.00020	0.0
16	0.00366	0.00365	0.0
17	0.00694	-.00112	0.0
18	0.00454	-.00234	0.0
19	-.00048	0.00168	0.0
20	-.00403	0.00121	0.0
21	-.00393	-.00175	0.0
22	-.00103	-.00033	0.0
23	0.00203	0.00152	0.0

Figure 16

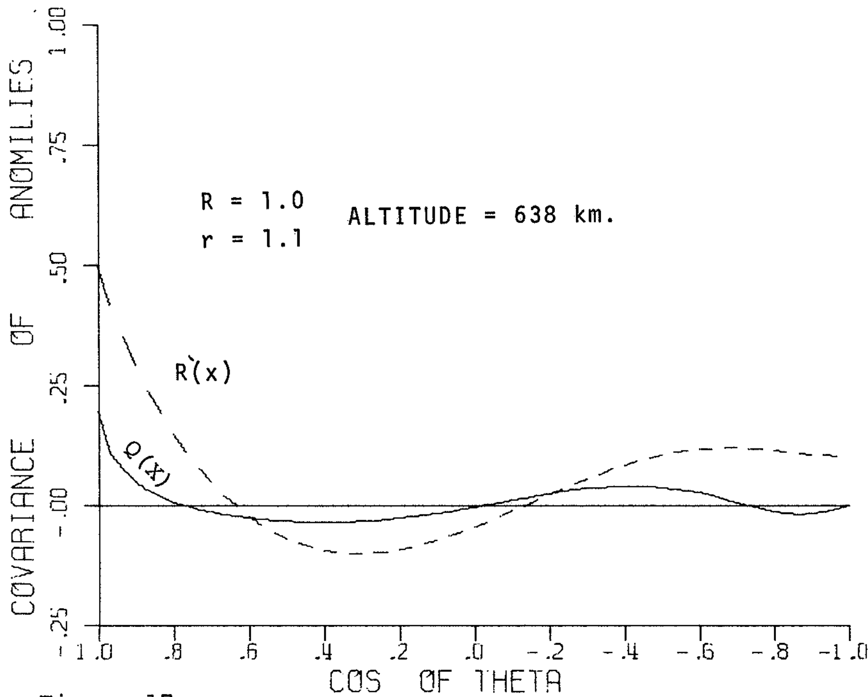


Figure 17

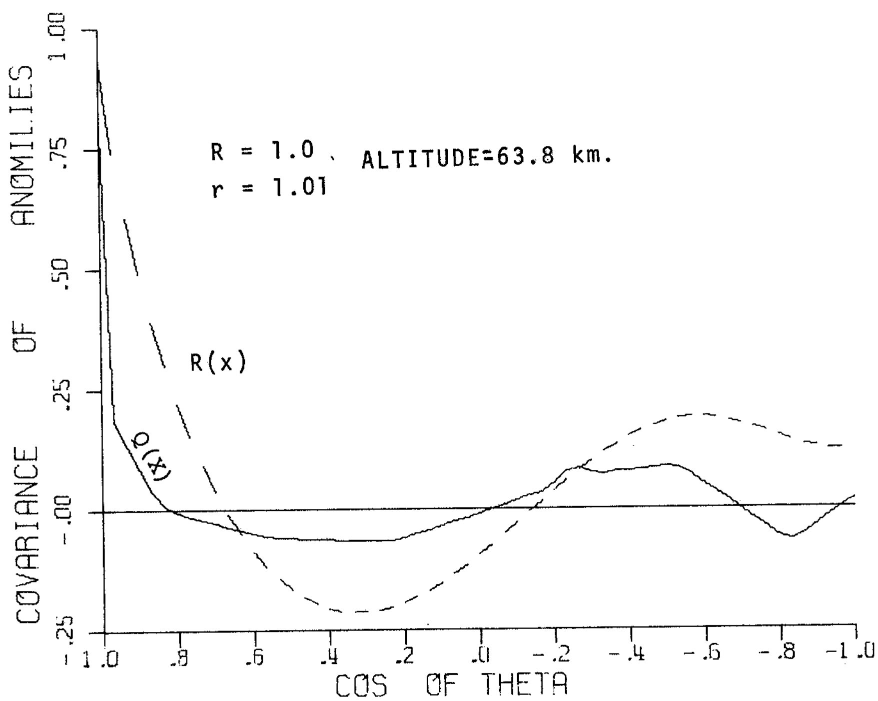


Figure 18

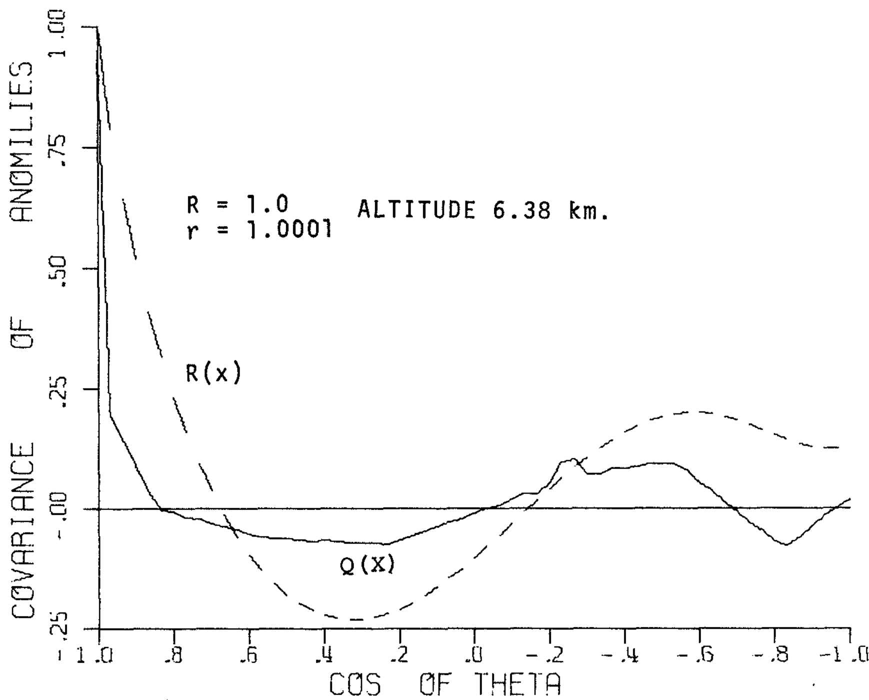


Figure 19

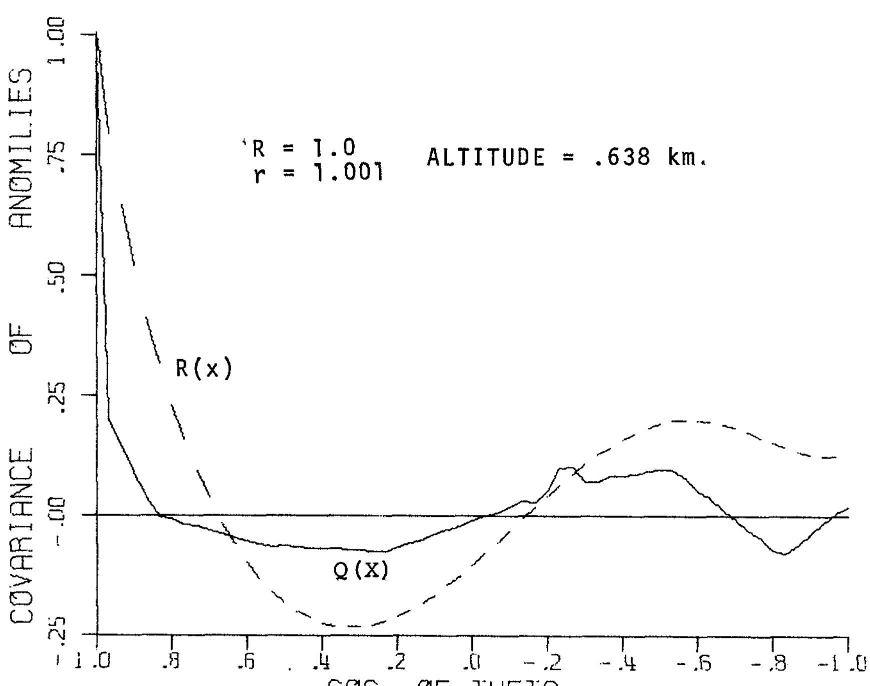


Figure 20

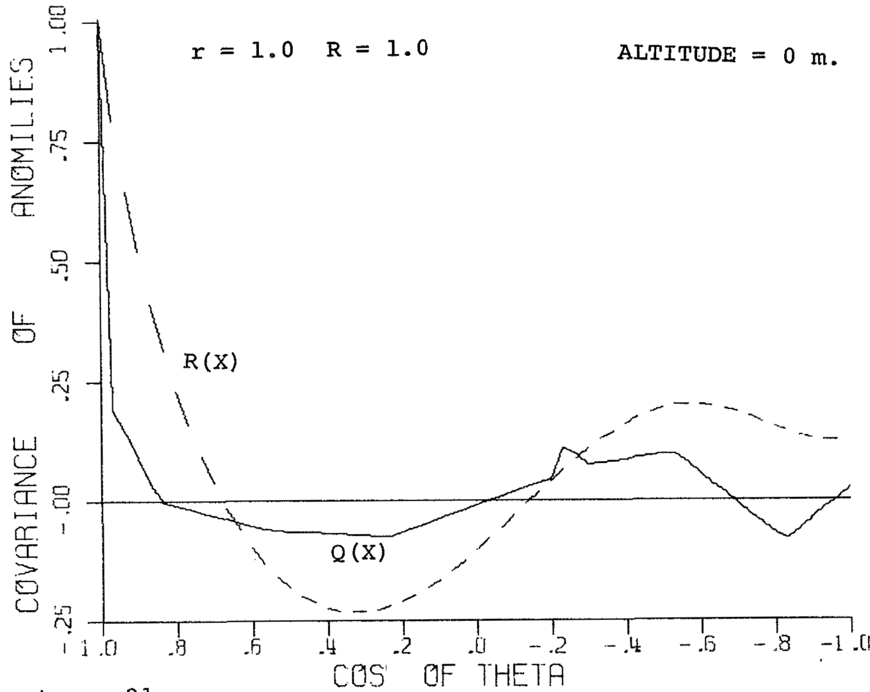

r = 1.0 R = 1.0
ALTITUDE = 0 m.
COVARIANCE OF ANOMILIES
1.00
.75
.50
.25
-.00
-.25
R(X)
Q(X)
1.0
.8
.6
.4
.2
.0
-.2
-.4
-.6
-.8
-1.0
COS OF THETA

Figure 21

SPECTRUM OF C(X) - C_k

k	C_k
0	0.07649
1	-.00004
2	0.09130
3	0.30600
4	0.10003
5	-.02792
6	0.10371
7	0.02808
8	0.02478
9	0.02882
10	0.00798
11	0.02519
12	0.01336
13	0.01473
14	0.02033
15	0.01457
16	0.01755
17	0.02045
18	0.01671
19	0.00680
20	0.01179
21	0.01396
22	0.00334
23	0.00634

Figure 22

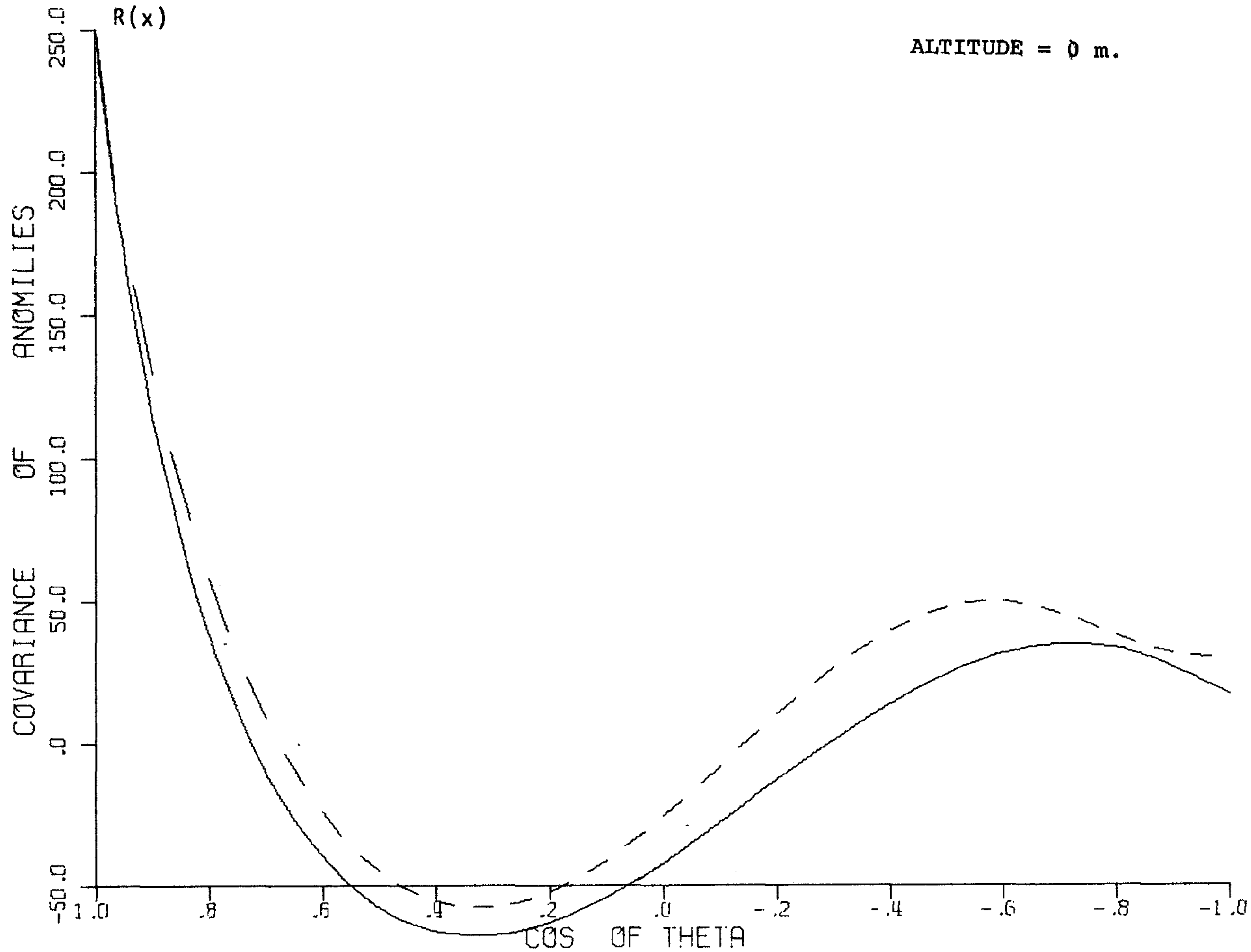

R(x)
ALTITUDE = 0 m.
COVARIANCE OF ANOMILIES
250.0
200.0
150.0
100.0
50.0
.0
-50.0
1.0
.8
.6
.4
.2
.0
-.2
-.4
-.6
-.8
-1.0
COS OF THETA

Figure 23

APPENDIX D

PROVING THAT THE DISCRETE POWER SPECTRUM IS NON-NEGATIVE AND DERIVING UPPER BOUNDS FOR TRUNCATION ERRORS

DISCRETE POWER SPECTRUM

The covariance function C(x) for a rotation invariant random process $x(\underline{e})$ defined on the surface of a sphere has a discrete spectrum:

$$C_k = 2\pi \int_{-1}^{1} C(x)\ P_k(x)\,dx \ . \qquad (3\text{-}15)$$

Now we wish to show that this spectrum is non-negative, or that $C_k \geq 0$. Let f_m be a spherical harmonic function of degree m. Then if $x(\underline{e})$ is a rotation invariant random process, from equations 3-9 and 3-8

$$E\left[\left\{\int_S f_k(\underline{e})\,x(\underline{e})\,ds(\underline{e})\right\}^2\right]$$

$$= \int_S \left[\int_S E[x(\underline{e}_1)\,x(\underline{e}_2)]\ f_k(\underline{e}_2)\,dS(\underline{e}_2)\right] f_k(\underline{e}_1)\,dS(\underline{e}_1)$$

$$= \int_S \tilde{C}\ f_k(e_1)\ f_k(\underline{e}_1)\,dS(\underline{e}_1)$$

$$\geq 0 \ . \qquad \text{D-1}$$

From equations (3-14), (3-16), D-1, and page 126, reference(6);

$$\int_S \left[\tilde{C}\ f_k(\underline{e}_1)\right]\ f_k(\underline{e}_1)\,dS(\underline{e}_1)$$

$$= \int_S C_k\ f_k(\underline{e}_1)\ f_k(e_1)\,dS(e_1)$$

$$= C_k \int_S f_k(\underline{e}_1)\ f_k(\underline{e}_1)\,dS(\underline{e}_1)$$

$$= \frac{4\pi}{2n+1}\, C_k \ . \qquad \text{D-2}$$

But from equation 3-18,

$$C_k \frac{4\pi}{2n+1} \geq 0, \quad \frac{4\pi}{2n+1} > 0, \text{ and thus } C_k \geq 0 \text{ for all } k.$$

UPPER BOUND FOR TRUNCATION ERROR

1. Using Poisson's Integral

This section develops an upper bound for the truncation error incurred when N terms in the infinite sum given by 4-14 are retained. Using the notation of section 4.4, define S_n so that

$$S_n = \frac{(a_1-a_2)(b_1+b_2)\gamma}{4} + \frac{(a_1-a_2)\left[(2a_1+a_2)b_1 + (2a_2+a_1)b_2\right]\gamma^2}{4} x$$

$$+ \frac{1}{2} \sum_{k=2}^{\infty} \gamma^{k+1} f(a_1,a_2,b_1,b_2,k)\, P_k(x)$$

$$= \sum_{k=0}^{n} \nu_k \,. \qquad \text{D-3}$$

Then

$$Q(x) = \lim_{n\to\infty} S_n \,.$$

First we wish to show that the series $\left\{S_n\right\}$ converges to $Q(x)$ absolutely. If this can be shown, then the arrangement of terms in the summation defining $Q(x)$ can be changed without affecting the convergence of the series.

If $\left\{S_n\right\}$ converges absolutely, then the sequence $\overline{S}_n = \sum_{k=0}^{n} |\nu_k|$ converges. We shall show that this convergence is uniform with respect to to x as well.

$$Q(x) \leq \sum_{n=0}^{\infty} |\nu_n|$$

$$\leq \frac{|a_1-a_2|\,|b_1+b_2|\gamma}{4} + \frac{\gamma^2 |a_1-a_2| \left(|2a_1+a_2|\cdot|b_1| + |2a_2+a_1|\cdot|b_1|\right)|x|}{4}$$

$$+ \frac{|b|}{2} \sum_{k=2}^{\infty} \gamma^{k+1} \left[\frac{(2k+1)}{(2k-1)(2k+3)} \,|P_k(a_1) - P_k(a_2)| \right.$$

$$+ \frac{k}{2k-1} \left|P_{k-2}(a_2)-P_{k-2}(a_1)\right| + \frac{(k+1)}{2k+3}\left|P_{k+2}(a_1)-P_{k+2}(a_2)\right|$$

$$+ |a| . \left|P_{k+1}(a_2)-P_{k+1}(a_1)+P_{k-1}(a_1)-P_{k-1}(a_2)\right| . |P_k(x)|$$

by the trangle inequality.

Since $-1\leq a_1\leq 1$; $-1\leq a_2\leq 1$; $-1\leq b_1\leq 1$; $-1\leq b_2\leq 1$; $|x|\leq 1$;

$0<\gamma<1$; $|P_k(x)|\leq 1$; $a_1 \neq a_2$; $b_1 \neq b_2$;

It is easily seen that:

$$Q(x) < \gamma + 3\gamma^2 + \frac{|b|}{2} \sum_{k=2}^{\infty} \gamma^{k+1}\left[\frac{1}{2} + \frac{4}{3} + \frac{1}{2} + 4|a|\right].1$$

$$< 4 + \frac{|b|}{2} (3+4|a|) \sum_{k=2}^{\infty} \gamma^{k+1}$$

$$< 4 + |b|\left(\frac{3+4|a|}{2}\right) \sum_{k=0}^{\infty} \gamma^{k} .$$

But $|\alpha|<1$ so $\sum_{k=0}^{\infty} \gamma^k$ is a geometric seriws which dominates

$\left\{\overline{S}_n(x)\right\}$ and converges, which implies S_n is absolutely convergent, and so the order of summation in equation 4-14 can be changed without affecting the convergence.

By changing the order of summation in equation 4-14, letting C_{22} represent the sum of the first two terms in the series, and applying the triangle inequality;

$$Q(x) < C_{22} + \sum_{n=2}^{\infty} \gamma^{n+1} \frac{|b|}{2} \frac{(2k+1)}{(2k-1)(2k+3)} |P_k(a_1) - P_k(a_2)| . |P_k(x)|$$

$$+ \sum_{n=2}^{\infty} \gamma^{n+1} \frac{|b| \; k}{2(2k-1)} |P_{k-2}(a_2) - P_{k-2}(a_1)| . |P_k(x)|$$

$$+ \sum_{n=2}^{\infty} \gamma^{n+1} \frac{|b| (k+1)}{2(2k+3)} |P_{k+2}(a_1) - P_{k+2}(a_2)| . |P_k(x)|$$

$$+ \sum_{n=2}^{\infty} \gamma^{n+1} \frac{|b| . |a|}{2} |P_{k+1}(a_2) - P_{k+1}(a_1) + P_{k-1}(a_1)$$

$$- P_{k-1}(a_2)| . |P_k(x)|$$

$$< C_{22} + \frac{\gamma|b|}{2} \sum_{n=2}^{\infty} \gamma^n \left[\frac{2}{2n} + \frac{2}{2} + \frac{2}{2} + 4|a| \right]$$

$$= C_{22} + \gamma|b|(1+2|a|) \sum_{n=2}^{\infty} \gamma^n + \frac{\gamma|b|}{2} \sum_{n=2}^{\infty} \frac{\gamma^n}{n} \quad . \qquad \text{D-4}$$

An upper bound for the truncation error which results from approximating the infinite sum (defining $R(x)$) by the first N terms of the summation will be denoted by $E(x)$.

$$E(x) = \gamma|b|(1+2|a|) \sum_{n=N+1}^{\infty} \gamma^n + \frac{\gamma|b|}{2} \sum_{n=N+1}^{\infty} \frac{\gamma^n}{n} \quad . \qquad \text{D-5}$$

Using the following relationship[1]

$$\int_0^{\gamma} \sum_{n=2}^{N} \xi^{n-1} d\xi = \sum_{n=2}^{N} \frac{\gamma^n}{n} \qquad \text{D-6}$$

---

1. Page 176, reference (8)

and noticing that, since $\gamma<1$, $\xi<1$, and

$$\sum_{n=N+1}^{\infty} \frac{\gamma^n}{n} = \sum_{n=2}^{\infty} \frac{\gamma^n}{n} - \sum_{n=2}^{N} \frac{\gamma^n}{n}$$

then it follows that

$$\sum_{n=N+1}^{\infty} \frac{\gamma^n}{n} = \int_0^{\gamma} \left(\sum_{n=2}^{\infty} \xi^{n-1} - \sum_{n=2}^{N} \xi^{n-1} \right) d\xi$$

$$= \int_0^{\gamma} \left[\frac{\xi}{1-\xi} - \left(\frac{\xi}{1-\xi} - \frac{\xi^N}{1-\xi} \right) \right] d\xi$$

$$= \int_0^{\gamma} \frac{\xi^N}{1-\xi}\, d\xi$$

and so equation D-5 becomes

$$E(x) = \gamma|b| \left\{ \frac{(1+2|a|)\gamma^N}{1-\gamma} + \frac{1}{2} \int_0^{\gamma} \frac{\xi^N}{1-\xi}\, d\xi \right\} . \qquad \text{D-7}$$

Since this bound is very conservative, a very rough approximation to the integral in the above expression should not affect the usefulness of the upper bound $E(x)$. So using Simpson's rule[1]

$$\int_0^{\gamma} \frac{\xi^N}{1-\xi}\, d\xi \approx \frac{\gamma}{6} \left[\frac{4\left(\frac{\gamma}{2}\right)^N}{1-\frac{\gamma}{2}} + \frac{\gamma^N}{1-\gamma} \right] . \qquad \text{D-8}$$

---

1. See page 117, reference (5)

Thus for $x \varepsilon [-1,1]$ the truncation error (ε) which results from approximating $Q(x)$ by the first N terms of the summation in equation 4-14 will be bounded above by $E(x)$, and so by subsituting D-8 into D-7;

$$\varepsilon < \left[\frac{(\gamma+12+24|a|)}{12(1-\gamma)} + \frac{\gamma}{3(2^{N-1})(2-\gamma)}\right]\gamma^{N+1}|b| \ . \qquad \text{D-9}$$

II. Using Stokes' Formula

This section develops an upper bound for the truncation error which results when only N terms in the infinite summation, given by equation 5-4, are retained. As was mentioned in section 5.3 replacing

$$\frac{\gamma^{n+1}}{R^2(n-1)^2} \quad \text{by} \quad \frac{1}{R^2(n-1)^2}$$

gives a new series which dominates the old one.

An upper bound for the truncation error incurred when using only N terms in this new summation will thus be an upper bound for the truncation error using the old summation (5-4). Thus from 5-4;

$$R(x) \leq \frac{(a_1-a_2)(b_1+b_2)\gamma}{4R^2} + \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{R^2(k-1)^2} f(a_1,a_2,b_1,b_2,k)P_k(x)$$

and from D-4,

$$R(x) < \frac{(a_1-a_2)(b_1+b_2)\gamma}{4R^2} + \gamma|b|(1+2|b|)\sum_{n=2}^{\infty}\frac{1}{R^2(n-1)^2}$$

$$+ \frac{\gamma|b|}{2}\sum_{n=2}^{\infty}\frac{1}{R^2\ n(n-1)^2} \ .$$

If we let $(n-1)^2 = n^2$ (this will have little effect on the validity of this very conservative upper bound) then

$$R(x) < \frac{\gamma|b|}{R^2}\left[\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^3} + (1+2|a|)\sum_{n=2}^{\infty}\frac{1}{n^2}\right] . \qquad \text{D-10}$$

It is well known in analysis that

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \leq \sum_{i=1}^{\infty}\left[2^{(1-p)}\right]^i < \infty \quad \text{if } p>1 \text{ , and } \frac{1}{n^p} \leq \left[2^{(1-p)}\right]^{n-1}$$

so

$$\sum_{n=2}^{\infty}\frac{1}{n^3} \leq \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$$

and

$$\sum_{n=N+1}^{\infty}\frac{1}{n^3} \leq \sum_{n=N}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{3(2)^N} . \qquad \text{D-11}$$

Similarly

$$\sum_{n=N+1}^{\infty}\frac{1}{n^2} \leq \frac{1}{(2)^{N-1}} . \qquad \text{D-12}$$

If $E(x)$ is the upper bound on the truncation error incurred by using only N terms in the expression for $R(x)$ given by 5-4, then from D-10, D-11, and D-12,

$$E(x) < \frac{\gamma|b|(13+24|a|)}{3R^2(2)^{N+1}} . \qquad \text{D-13}$$

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