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FINAL REPORT

AN ANALYSIS AND EVALUATION OF GUIDANCE MODES

By
C. G. Pfeiffer
1 May 1970

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Prepared under Contract No. NAS 12-593
by
TRW Systems Group
Redondo Beach, California
for
Electronics Research Center
National Aeronautics and Space Administration

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LIST OF SYMBOLS

<u>Symbol</u>	<u>Meaning</u>
a	Thrust acceleration vector or arbitrary constant
$ a $	Thrust acceleration magnitude
α	Final value of Lagrange multiplier vector
α_i	Eigenevalues.
A, B, C, D	Guidance parameters
b	Massless closest approach distance or arbitrary constant
β	Steering angle bias or mass flow rate
c, c_i	Arbitrary constant
c_1	Specific angular momentum or arbitrary constant
c_3	vis-viva energy or arbitrary constant
d	Arbitrary constant
$\delta r, \delta v$	Deviations from nominal r and v
E	Eccentric anomaly
e	Eccentricity
f	Vector function of equations of motion
F	Partial derivative $\left[\frac{\partial f}{\partial x} \right]$
g	Gravitational acceleration
G	Partial derivative $\left[\frac{\partial f}{\partial u} \right]$
γ	Flight path angle
H	Hamiltonian
h	Angular momentum

LIST OF SYMBOLS (Cont.)

<u>Symbol</u>	<u>Meaning</u>
I	Identify matrix
i	Subscript
j	Subscript
k	Steering angle feedback vector $\left(\frac{\partial u}{\partial v}\right)$ or subscript
L	Orthogonal matrix
λ	Time-varying Lagrange multiplier vector
m	Vehicle mass
M	End condition matrix $\left[\frac{\partial q}{\partial x}\right]$
n	Number of components of a vector
μ	Gravitational constant
N	Normality matrix or number of end conditions
v	Constant Lagrange multiplier vector
o	Subscript denoting time zero
p	Semi-latus rectum, guidance parameter, or performance index
ψ	Arbitrary function
q	End condition vector
r	Radius vector
r	Radial distance
r_i	Components of r
ρ	Radius of cruvature of reachable set of points
s	Subscript denoting nominal values
t	Time
t_g	Time-to-go to burnout
θ	Downrange angle or true anomaly

LIST OF SYMBOLS (Cont.)

<u>Symbol</u>	<u>Meaning</u>
U	State transition matrix
u	Steering angle
u_c, u_r	Circumferential and radial components of u
v	Velocity vector
$ v $	Velocity magnitude
v_g	Velocity-to-be gained
v_i	Components of v
v_r	Required velocity
v_∞	Asymptotic velocity
w	Weighting function
ω	Angular rate
x	State vector = (v,r)
y	Augmented state vector
z	Arbitrary vector function

Notation

In general, lower case Roman letters are column vectors and upper case Roman letters are matrices, where the superscript T denotes the transpose. When there is no possibility for confusion, scalars will also be represented by Roman letters. The independent variable is the time t , and (t_0, T) are, respectively, the beginning and end of a guidance phase.

SUMMARY

This Report describes and analyzes the general characteristics of powered flight guidance algorithms (modes) which have been developed for space missions. The modes are classified according to their characteristics, measures of performance are defined, and each subclass is assigned a qualitative performance rating. Selected guidance modes which have been described in the literature are classified and described. The targeting problem, which is the task of describing desired end conditions for a guidance phase, is discussed.

ACKNOWLEDGEMENTS

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1.0 INTRODUCTION AND SUMMARY

1.1 Modes for Advanced Guidance

A guidance mode is an algorithm for calculating a steering program and thrust termination time for a rocket vehicle so as to cause mission objectives to be met. Many modes have been proposed and studied, and some have been used in real time applications. Looking forward to advanced near-earth and deep space missions, where there will be a need for versatile, reliable and cost-effective guidance systems, the primary purpose of the study reported here has been to survey the state of the art in guidance mode development in order to classify modes according to their essential features. Supporting material, which is an integral part of the results obtained, will be found in the Interim Report (Reference 1). This work was supported by NASA Electronics Research Center under contract NAS 12-593. The technical monitor was W.E. Miner.

The task undertaken here was a challenging one, for it is true that there are many ingeniously devised guidance schemes which might all perform reasonably well for most space missions (number of proposed guidance modes $\cong$ number of people who have studied the guidance problem). The essential difficulty follows from the fact that guidance systems tend to be special purpose devices which need to be, and usually can be, tailored to each specific mission. A general purpose guidance system is as rare as a general purpose internal combustion engine. Nevertheless, it is possible to sort out characteristics of various approaches to the guidance problem and offer opinions about which offer the most promise. The progress achieved toward this end is probably best summarized in the classification and evaluation results of Sections 8 and 9.* The conclusions are qualitative and controversial, but in any case it is hoped that the basic approach taken here will be useful for planning guidance mode development for advanced missions.

*The classification method is a much iterated version of an approach first suggested by W.E. Miner.

1.2 Some Historical Background

Algorithms for real time guidance of large rocket vehicles based upon linear perturbation methods had been developed by the early 1950's. A nominal trajectory which satisfied mission objectives was pre-calculated, predicted target errors were identified as linear combinations of position and velocity deviations, the target errors were calculated in real time from the difference between measurement data and corresponding stored values, and steering and shut-off was designed to null these errors. The simplest form of this approach was open-loop autopilot steering to follow the nominal steering history, followed by thrust termination when the integral of thrust acceleration reached the nominal value. This method is still surprisingly effective for some present day applications (Reference 2).

A new direction was taken in the mid 1950's, when the practical need to generate guidance equation coefficients for a wide range of missions made it apparent that simple guidance laws must be developed which require a minimum of pre-flight computation and storage. This led to the development of "explicit hit" equations for specifying burnout conditions (Page, et al), based upon a two-body (conic) approximation of the coasting arc. The invention of the required velocity concept, coupled with the velocity-to-be-gained cross product steering law (Laning, Battin, et al), provided a simple, accurate method for engine shut-off and steering near thrust termination. Assuming that the required velocity could be calculated at any instant from the conic formulae, an "explicit" guidance law was now available near burnout. This led to a separation of the guidance problem into two parts: (1) targeting, consisting of the pre-flight task of specifying the conic conditions to be achieved at burnout, and numerically determining the nominal values of the conic parameters, and (2) real-time guidance, consisting of the task of applying steering and thrust termination laws which will achieve the end

conditions in a near optimum fashion. Efforts toward solving the latter problem have led to the development of parameterized functional forms of the steering laws, which have been called "explicit" or "closed form" guidance schemes (Cherry, Smith, Perkins, Teren, et al).

A parallel effort in guidance technology arose with the development of the high-speed digital computer. It had long been recognized that the calculus of variations offered the theoretical basis for constructing optimum trajectories and closed loop guidance laws, and the computer made the practical implementation of this approach feasible. The general theory of an optimum rocket trajectory to specified end conditions was formulated (Breakwell, et al), solutions based upon simplified models were worked out (Miele, et al), extensive numerical results were obtained (Melbourne, Sauer, et al), the steepest descent method of trajectory generation (Kelley, Bryson, Denham) and the "second variation" guidance scheme were developed (Kelley, Bryson, Breakwell, Speyer), and a general form of closed loop optimum guidance was studied and simulated (Hoelker, Miner, et al). These efforts have led to proposed guidance modes which are based upon the calculus of variations solution of the optimum trajectory problem, implemented either by real time iteration (Brown, Johnson, et al) or by a Taylor series expansion about an optimum nominal trajectory.

In this Report we shall follow the historical development by separating the guidance and targeting problems, and defining two major classes of guidance modes: (1) parameterized, where a functional form is imposed on the steering law, containing parameters to be chosen in real time to satisfy targeted end conditions, and (2) optimal, where the steering law is obtained via the calculus of variations. Both classes of guidance modes have many characteristics in common, since the form of a parameterized law is always chosen to yield near-optimum performance. It should be remarked that the word parameterized was chosen rather than explicit, because as will be seen, the latter term implies unnecessary restrictions and is misleading.

1.3 Scope of Effort

The study under contract has been conducted in two phases. Phase I, which is described in the Interim Report (Reference 1), was concerned with analyzing and simulating representative guidance modes. Some topics covered were:

- o separability of guidance, navigation, and error analysis
- o general structure of guidance modes
- o simulation of a hypothetical vehicle for an advanced Jupiter mission
- o simulation of an Atlas-Centaur vehicle for conventional missions
- o development of a "precise" form of parameterized guidance
- o discussion of the targeting problem
- o treatment of state and control variable constraints
- o switching time conditions for staging rocket vehicles
- o impulsive stochastic guidance

Probably one of the more important results of the Phase I study was the recognition that supposedly special purpose algorithms could be readily applied to advanced missions if one were willing to perform real time integration and iteration. For example, simulation results showed that the TRW hybrid guidance mode performed well on conventional missions, but not on a long-thrust-arc advanced mission, where the approximations inherent in the algorithm were not valid. After replacing the approximations with a numerical integration and iteration technique, however, excellent performance was achieved. This result led to further consideration of what has now been classified as parameterized guidance based upon a precise model.

The Phase II effort reported here has been directed toward analyzing and summarizing the general characteristics of guidance modes, recognizing that numerical results for specific applications cannot lead to conclusions which are generally applicable. The conditions for optimality have been discussed in some detail, since these conditions are the starting point for

the development of any guidance mode. To simplify the discussion, a model guidance problem has been constructed, consisting of a single phase (e.g., a single stage rocket vehicle) moving in a vacuum with the goal of achieving targeted end conditions (assumed given by pre-flight calculations) in minimum time. Some of the topics covered are:

- o optimality conditions
- o approximation of optimum steering
- o guidance mode stability
- o guidance near the final time
- o treatment of constraints
- o classification of guidance modes
- o evaluation of guidance mode performance
- o survey of guidance modes
- o discussion of the targeting problem

All of these topics have been covered from the most general point of view possible in an attempt to draw useful conclusions pertinent to future applications.

1.4 Conclusions and Recommendations

Based upon the results of this study, it is possible to come to the following conclusions and recommendations:

- 1) A method for classifying and qualitatively evaluating the performance of guidance modes has been devised, and it is recommended that this conceptual framework be applied in the future development of guidance technology.
- 2) Applying (1), the parameterized form of guidance, where a specified functional form containing free parameters is imposed as a steering law, appears to be the best candidate for future development. This approach, which is presently the basis for most practical guidance laws, is simple,

versatile and requires a minimum of pre-flight preparation. The loss of optimality should be minimal. It is likely that real time numerical integration and iteration will be called for in order to predict end conditions and compute the guidance parameters for advanced missions, but this seems to be within the state of the art. It is therefore recommended that this approach be studied further.

- 3) The form of parameterized guidance which forces the steering angle to be a linear function of time, such as is presently employed in the IGM mode for Apollo missions, appears to be a near-optimal law for a surprisingly wide range of missions. It is therefore recommended that studies be carried out to determine if this simple, easily implemented form of a guidance law is applicable to advanced missions.
- 4) A simple measure of optimality can be used for evaluating guidance mode performance (see Section 3.6).
- 5) A simple measure of stability can be used as a design criteria and for evaluating guidance mode performance (see Sections 5.5 - 5.6).
- 6) The well known velocity-to-be gained cross product steering law can be interpreted as an optimal steering law when time-to-go to burnout is small. This or an equivalent scheme is required near burnout, where the end conditions become uncontrollable (abnormal). This guidance mode is therefore recommended for final steering to thrust termination.
- 7) Explicit development of targeting manifolds (numerical description of desired end conditions), based upon asymptotic matching of conics, appears to be capable of yielding errors which are negligibly small compared to other sources of guidance system error. Some numerical integration is required, but the computing load should be small compared to the numerical integration presently required to improve approximate explicit targeting. It is therefore recommended that an efficient form of the asymptotic matching method be developed,

perhaps exploiting a numerical integration technique analogous to the hybrid conic method (Section 11.7). The goal should be to develop an explicit targeting scheme which could be used in real time.

It is hoped that these conclusions and recommendations will be useful for future development of cost effective guidance techniques and pre-flight design procedures.

2. ANALYSIS OF GUIDANCE MODES

2.1 Introduction

A guidance mode is an algorithm for calculating a steering program and thrust termination time for a rocket vehicle which will cause specified end conditions to be achieved. If there are more guidance parameters to be chosen than there are end conditions (e.g., a continuous steering program has an infinite number of parameters, which are the values of the function at a continuum of times), then some performance index will be minimized or a "near optimum" (optimal) form of the steering function will be arbitrarily specified. For example, the guidance mode may be designed to achieve with minimum propellant expenditure a position and velocity vector at burnout which, when considered as initial conditions for the resulting coasting orbit, will cause the vehicle to pass through a specified point in space at a specified time.

A guidance law may be continuous, as in launch vehicle guidance, where a continuous steering program must be chosen, or impulsive, as in midcourse guidance, where the rocket vehicle is pointed in a fixed direction and thrust is applied for a relatively short time so as to apply essentially an impulse of acceleration. We shall be primarily concerned here with modes for continuous guidance, since impulsive guidance is a relatively straightforward special case. That is, if the desired position and velocity manifold is specified, (see Section 2.2), then the impulsive guidance task is simply to specify the three components of the velocity increment, i.e., the integral of the acceleration impulse which will achieve a point on the manifold.

Many forms of guidance modes have been proposed and analyzed, and some have been used in practice. The main purpose of this study is to analyze, classify, and to the extent possible, devise measures of performance of such guidance modes.

2.2 Guidance and Targeting

Guidance may be thought of as a two point boundary value problem. For example, a midcourse correction is essentially a step change in the components of the velocity vector, achieved by applying thrust in a fixed direction for a relatively short time, and chosen to satisfy three desired end conditions at some specified terminal time. Suppose $x = (r, v)$ is the state vector, r = position vector, v = velocity vector, and $x^+ = x^- + (0, \Delta v)$, where (+,-) refer to, respectively, the state prior to and after the correction Δv . The components $(\Delta v_1, \Delta v_2, \Delta v_3)$, which are integrals of the thrust acceleration components, are to be thought of as guidance parameters. If the equations of motion are $\dot{x} = f(x)$, then Δv is chosen to cause $R = r(T)$ to be the desired end condition. Given the uncorrected state x^- , this problem can be solved by iteration. That is, if $\Delta v(i)$ is the i^{th} guess of Δv , and if $\Delta R(i) = R(i) - R(\text{desired})$ is the error residual associated with $\Delta v(i)$, calculated by numerical integration of the equations of motion, then

$$\Delta v(i + 1) = - \left[\frac{\partial R}{\partial v} \right]^{-1} \Delta R(i) \quad (2.1)$$

where $\Delta v(0) = 0$. This procedure is repeated until the error in calculating $\Delta v(i)$ is acceptably small. Applied mathematicians sometimes call this procedure the "shooting method" for solving two point boundary value problems.

The case of launch vehicle (injection) guidance is not essentially different, for then there may be certain parameters, such as the initial value of the rocket steering angle and the angle rate, which control the injection (burnout) conditions, and which must be found as the solution of a targeting problem. Some performance index may be optimized in the process if there are more free parameters than desired end conditions; a calculus of variations problem arises if there are an infinite number of parameters to be chosen (i.e., if the steering angle program has not been parameterized but is instead an arbitrary function defined by values at an infinite number of times.) In general, then, the guidance problem is the task of choosing the guidance parameters (perhaps optimally) to satisfy desired end conditions.

Solution of the guidance problem by real time numerical integration and iteration is possible for midcourse guidance, because adequate computing time is available prior to executing the correction, and a good first guess of the final trajectory is available. This is not possible for injection guidance, however, where revised guidance parameters must be recalculated by a relatively small onboard computer as often as once per second as revised navigation data is obtained. In this case one usually employs simplified techniques for predicting burnout conditions, and relates these to the desired end conditions R by specifying a description of a manifold in state space which is the locus of desired injection conditions. That is, if the mission objective is to achieve some given position vector at a fixed terminal time T , and (r,v) is the state vector at completion of the guidance phase at time t , where $T > t$, then the goal of the guidance system is to cause

$$\begin{aligned} R_1(r,v) &= 0 \\ R_2(r,v) &= 0 \\ R_3(r,v) &= 0 \end{aligned} \tag{2.2}$$

Implicitly we have $v = v(r)$ from the above relationships, which specifies the manifold. The $v(r)$ is sometimes called the required velocity, and if v is the actual velocity prior to guidance termination, $v - v(r)$ is sometimes called the velocity-to-be-gained.*

The preflight task of generating the functions $R_i(r, v)$ is sometimes called "targeting", as opposed to guidance, and we shall therefore make the following:

Definition: Targeting is the task of specifying functions of position and velocity (and perhaps time), which, if satisfied at burnout of a rocket vehicle, will cause desired mission objectives to be met. Guidance is the task of choosing the steering and thrust termination parameters so as to achieve at burnout a state vector on (or acceptably close) to the manifold described by these functions.

Note that, by definition, impulsive guidance reduces to the simple task of determining the required Δv from the solution of the targeting problem, and calculating the guidance acceleration to be applied as $a = \left(\frac{\Delta v}{\Delta t} \right)$, where Δt is the burning time, given by $(|\Delta v|/|a|)$.

The solution of the targeting problem could be stored in the vehicle in the form of a table of numbers. This approach poses a formidable preflight preparation and storage task if the guidance system must be prepared to fly one of many possible trajectories (e.g., as in interplanetary missions, where many launch opportunities are open). Fortunately, it can be shown analytically and by numerical simulation that for many space missions

*The fact that guidance may not be terminated at the fixed nominal time t does not essentially change the problem, because varying the guided rocket engine shut-off time by a small δt has the effect of adding a delta function of acceleration at t , yielding a step change of velocity of magnitude (acceleration) $\times (\delta t)$.

the injection guidance manifold can be simply specified in terms of the osculating orbital elements. For example, in the case of an interplanetary mission the injection guidance task can be said to be accomplished with adequate accuracy if the osculating hyperbola at burnout achieves the nominal asymptotic velocity (v_∞). Since v_∞ can be calculated at injection as a function of instantaneous position and velocity, it can be said that the required velocity is $v(r) = v_\infty(r)$. Since the nominal trajectories depend upon the launch time (t_L), present practice is to precalculate and store $v_\infty(t_L)$ in the on-board guidance computer. These numbers are generated preflight by an iterative procedure such as described above. In effect, by this approach one transforms to the new state variables $v(r)$, r , and specifies the manifold by the simple equations $v_\infty(r) = \text{constant}$. This transformation will be discussed further in Section 2.3, the targeting problem will be discussed further in Section 11.

2.3 The Model Guidance Problem

The performance of a guidance mode can only be described when one specifies the mission objectives and the characteristics of the vehicle and trajectory to be flown. To more clearly illustrate the ideas to be presented here, we shall restrict ourselves to considering a model guidance problem which is representative of an important class of applications one might reasonably consider for an advanced guidance system. It will be seen, however, that the basic approach to guidance mode analysis can be extended to more general cases.

It will be assumed that (1) the rocket vehicle is moving exo-atmospherically, (2) in the presence of a single gravitating body, where (3) the thrust acceleration magnitude is a specified function of time (e.g., a chemical rocket), and (4) the coast time between rocket stages is a fixed staging time. The latter assumption is equivalent to considering a single powered flight "phase", where the mission has been divided into phases which are targeted to achieve end conditions suitable for the succeeding phase. This approach may be justified by a dynamic programming argument. It will be assumed (5) that the desired end conditions at burnout are functions of

position and velocity only (e.g., combinations of orbital elements of the coast trajectory). Since one must, at least implicitly, define a performance index to be optimized in order to define the steering law (or the functional form of the steering law), it will be assumed (6) that the guidance mode will be designed to achieve the desired burnout conditions in minimum, or near minimum, time. This is a reasonable operational goal, and is equivalent to minimizing propellant expenditure for the case of a chemically propelled rocket with fixed coast durations between stages.

Taking an inertial coordinate system centered in the gravitating body, the equations of motion are

$$\left. \begin{aligned} \dot{v} &= g(r) + a(u,t) \\ \dot{r} &= v \end{aligned} \right\} = f(v,r,u,t) \quad (2.3)$$

where v is the velocity vector, r is the position vector, g is the gravitational acceleration vector given by

$$g = - \frac{\mu r}{|r|^3} \quad (2.4)$$

and μ is the gravitational constant. The $a(t)$ is the thrust acceleration vector given by (see Figure 2.1).

$$a(t) = |a(t)| \begin{bmatrix} \cos u_1(t) \cos u_2(t) \\ \sin u_1(t) \\ \cos u_1(t) \sin u_2(t) \end{bmatrix} \quad (2.5)$$

where $|a(t)|$ is the pre-specified time varying magnitude of the thrust acceleration vector, and $u_1(t)$, $u_2(t)$ are steering angles.

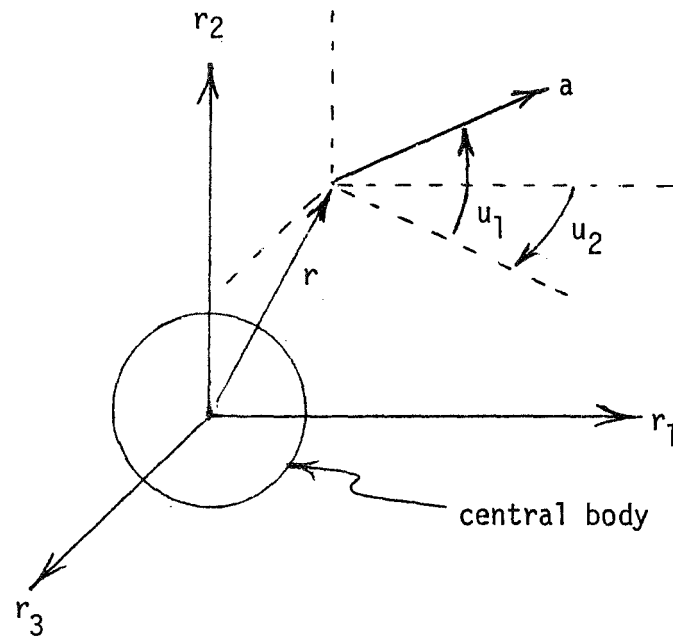


Figure 2.1: The Coordinate System

Given an initial state $r_0 = r(t_0)$, $v_0 = v(t_0)$ at some time t_0 during the thrusting phase, the problem is to choose the control functions $u_1(t)$, $u_2(t)$, and the burnout time T so as to achieve desired end conditions in minimum time (minimum T). The end conditions will be specified by certain functions $q_i(r,v) = 0$ for $i = 1, 2, \dots, N$, where $N \leq 6$.

2.4 Necessary Conditions for Optimality

The necessary conditions for optimality offer both a test which can be applied to a guided trajectory and an aid in analytically constructing such a trajectory. In Part 3 the optimality conditions will be developed in a form convenient for guidance mode analysis; we shall only summarize the results here.

The minimum time problem is most easily treated by introducing the state variable $s = t$, so that the equations of motion become

$$\frac{d}{dt} \begin{pmatrix} v \\ r \\ s \end{pmatrix} = \begin{bmatrix} a+g \\ v \\ 1 \end{bmatrix} = f(v,r,u,t) \quad (2.6)$$

Introduce the Lagrange multiplier vector

$$\lambda^T(t) = (\lambda_v^T(t), \lambda_r^T(t), \lambda_s(t)) \quad (2.7)$$

and define the Hamiltonian

$$H = \lambda^T f = \lambda_v^T(a+g) + \lambda_r^T v + \lambda_s \quad (2.8)$$

In this formulation, the Lagrange multipliers are determined by their end conditions and

$$\dot{\lambda}_v^T = -\lambda^T \left(\frac{\partial H}{\partial v} \right) = -\lambda_r^T \quad (2.9)$$

$$\dot{\lambda}_r^T = -\lambda^T \left(\frac{\partial H}{\partial r} \right) = -\lambda_v^T \left(\frac{\partial g}{\partial r} \right) \quad (2.10)$$

$$\dot{\lambda}_s = \lambda^T \left(\frac{\partial H}{\partial s} \right) = 0 \quad (2.11)$$

The trajectory itself is determined by its initial condition and the equations of motion

$$[\dot{v}, \dot{r}, \dot{s}]^T = f^T = \left(\frac{\partial H}{\partial \lambda} \right) \quad (2.12)$$

The control functions $u_1(t)$ and $u_2(t)$ are determined by the optimality conditions

$$\frac{\partial H}{\partial u_i} = \lambda_v^T \left(\frac{\partial a}{\partial u_i} \right) = 0 \quad , \quad i = 1, 2 \quad (2.13)$$

and the final time T is determined by

$$H(T) = 0 \quad (2.14)$$

It is easily shown from (2.13) that the thrust vector must be collinear with λ_v , which is sometimes called the primer vector.

It can be seen from the above that the problem of calculating the time T and the control functions $u_1(t)$, $u_2(t)$ is equivalent to finding $\lambda(t)$. Let the end conditions specified by the functions $q(r, v) = 0$ for $i = 1, 2, \dots, N$. Since s is to be minimized subject to these constraints, at the final time T define the performance index

$$p = -s + \sum_i v_i q_i(r, v) \quad (2.15)$$

where the v_i are constant multipliers. Then the time-varying Lagrange multiplier $\lambda(t)$ must satisfy at the final time T

$$\lambda_v^T(T) = \left(\frac{\partial p}{\partial v} \right) = \sum_i v_i \left(\frac{\partial q_i}{\partial v} \right) \quad (2.16)$$

$$\lambda_r^T(T) = \left(\frac{\partial p}{\partial r} \right) = \sum_i v_i \left(\frac{\partial q_i}{\partial r} \right) \quad (2.17)$$

$$\lambda_s(T) = \left(\frac{\partial p}{\partial s} \right) = -1 = \lambda_s(t) \quad (2.18)$$

These are sometimes called the transversality conditions. Normally one must find $\lambda(t)$ by an iterative procedure, which involves an initial guess of the value $\lambda(t_0)$.

If the functions $q_i(r, v)$ are constants of the motion on a coast arc, then at burnout (time $T + 0$) we have

$$\begin{aligned} 0 &= \frac{d}{dt} \sum_i v_i q_i = \sum_i v_i \left[\left(\frac{\partial q_i}{\partial v} \right) \dot{v} + \left(\frac{\partial q_i}{\partial r} \right) v \right] \\ &= \lambda_v g + \lambda_r v \end{aligned} \quad (2.19)$$

Thus the shut-off condition $H(T) = 0$ reduces to

$$0 = \lambda_v a - 1 \quad (2.20)$$

Since $a = |a| \frac{\lambda_v}{|\lambda_v|}$, it follows that shut-off occurs when

$$|a| = |\lambda_v|^{-1} \quad (2.21)$$

Note that one might pose a different optimization problem by choosing to maximize the final mass (m) of a single stage chemical rocket, where

$$\dot{m} = -\beta = \text{const} \quad (2.22)$$

$$|a| = \frac{V_e \beta}{m_0 - \beta t} \quad (2.23)$$

$$V_e = \text{effective exhaust velocity} = \text{const.} \quad (2.24)$$

In this case m is the additional state variable to be optimized (instead of s), and the performance index becomes

$$p^* = -(m - m_0) + \sum_i v_i^* q_i(r, v) \quad (2.25)$$

where the superscript star refers to the maximum mass problem. The Hamiltonian is (since $\dot{m} = -\beta$)

$$H^* = \lambda_v^*(a + g) + \lambda_r^* v - \lambda_m^* \beta \quad (2.26)$$

and, as in (2.20), at $t = T$ we have

$$H^*(T) = 0 = \lambda_v^* a - \lambda_m^* \beta \quad (2.27)$$

Since $\lambda_m^*(T) = 1$,

$$|a| = \beta |\lambda_v^*|^{-1} \quad (2.28)$$

But we have $(m - m_0) = \beta t$, so that, given a solution to the minimum time problem, a solution of the maximum mass problem can be found by setting $p^* = \beta p$, from which it follows that $\lambda_v^* = \beta \lambda_v$ and $\lambda_r^* = \beta \lambda_r$.

Then the steering angles for the two problems, which are determined by the ratio of $\lambda_v(t)$ components, and the shut-off criterion, are identical, and we have the intuitively obvious result that minimizing burnout time is equivalent to maximizing burnout mass of a chemical rocket.

2.5 Optimal and Parameterized Guidance

Since the guidance problem must be solved many times during flight as revised estimates of position and velocity are obtained by the navigation system, the solution of the optimization problem described in Section 2.4 can be a formidable computational task. According, the guidance mode designer chooses a sub-optimal steering and shut-off law which may be classified as:

Optimal Guidance: where the steering law is obtained by approximating the solution of the optimization equations (see section 2.4) in some way.

or

Parameterized Guidance: where a functional form of the steering law is chosen which is supposed to approximate the true optimum law, and parameters in this form are chosen in real time to satisfy the end conditions.

For example, optimal guidance might be achieved by calculating and pre-storing a truncated Taylor series expansion describing the steering law as a function of position, velocity, and time, or by a real time iteration procedure, where approximations are made of the equations of motion and/or the Lagrange multiplier equations. Parameterized guidance might be achieved by assuming that the steering angle is the linear function of time $u_i(t) = a_i + b_i t$, $i = 1, 2$, where a_i and b_i are chosen in real time to satisfy the end conditions. Approximations of the equations of motion and/or the equations to determine a_i and b_i may also be made. Clearly the types of approximations made to solve both the optimal and parameterized guidance problems (and to analytically justify any functional form chosen for a parameterized steering law) are the key characteristics of a guidance mode. Various approximations will be discussed further in Section 4.0.

Further subclasses of the optimal and parameterized guidance will be defined in Section 8.0. It will be seen that the central theme of this report is to analyze guidance modes by developing a classification method, classifying typical guidance modes, defining measures of performance, and to the extent possible, making judgements of the relative performance of the various sub-classes.

2.6 Measures of Guidance Mode Performance

The measures of guidance mode performance which will be considered in this report are:

1. Optimality - given that there is a performance index to be minimized, say propellant expenditure, how does the obtained value of the performance index compare to the theoretical minimum?
2. Stability - does the guidance mode call for high frequency attitude changes, or can small input errors result in large non-standard maneuvers of the vehicle?
3. Accuracy - given that approximations are introduced into the derivation and mechanization of the guidance equations, what are the resulting errors in the desired terminal conditions? (There are also navigation, computation, and mechanization errors, but, insofar as these are separable from guidance mode errors, i.e., assuming superposition of effects, they need not be considered in the design of a guidance mode.)
4. Constraint Compatibility - is the guidance mode capable of generating commands and a trajectory which satisfy all imposed control and state variable constraints? If so, how difficult is the task of incorporating the constraints into the analytical formulation?

5. Preflight preparation - what is the cost in time and money of preflight preparation of the guidance equations. In particular, how long does it take to prepare the guidance system to accomplish a given mission? (The "quick reaction" problem.)
6. Verification requirements - what is the cost in time and money of preflight verification of guidance system performance? (Another aspect of the "quick reaction" problem.)
7. Flexibility - what are the types of missions which the guidance mode can perform, and how well can it adapt to changes in the mission, such as variations of launch azimuth? (Another aspect of the "quick reaction" problem.)
8. Region of applicability - what is the range of perturbations which can be adequately treated by the guidance mode?
9. Computer factors - what are the real time on-board and/or earth-based computer requirements, in particular, how much storage space is required, what is the length of the computing cycle for each iteration of the guidance equations, and how complex must the computer be?
10. Growth potential - what is potential applicability of the guidance mode to future missions?

These measures of performance will be discussed further in the sequel, and employed in a qualitative way to judge classes of guidance modes.

2.7 Linearized Analysis

The quantitative evaluation of guidance mode performance will call upon well known techniques for linearizing the equations of motion about some nominal (reference) trajectory, thus considering only the first order (first variation) effects of the steering and shut-off laws. In particular, the optimality of a guidance mode can be evaluated if the optimum trajectory is chosen as the nominal. This perturbation approach, which is the basis of most guidance system performance analysis, will be exploited throughout this report.

Suppose we are given a nominal steering program $[u_{1s}(t), u_{2s}(t)]$ and a nominal shut-off time T , and wish to study the effect of varying the steering angles and shut-off time by a small amount. Define the state and control deviations from nominal as

$$\begin{aligned}\delta r(t) &= r(t) - r_s(t) \\ \delta v(t) &= v(t) - v_s(t) \\ \delta u_i(t) &= u_i(t) - u_{is}(t) \quad i = 1, 2 \\ \delta T &= T - T_s\end{aligned}$$

where the subscript s refers to the nominal values, and $\delta(\cdot)$ refers to a deviation at a fixed time t . Assuming that the δ quantities are small, the equations of motion may be linearized by representing them by a truncated Taylor series expansion:

$$\delta \dot{v} = \dot{v} - \dot{v}_s = \left(\frac{\partial g}{\partial r} \right) \delta r + \left(\frac{\partial a}{\partial u} \right) \delta u + O(\delta^2) \quad (2.29)$$

$$\delta \dot{r} = \delta v \quad (2.30)$$

where the neglected higher order terms are denoted by $O(\delta^2)$. For simplicity, u_2 has been set equal to zero (planar motion) and only $\delta u = \delta u_1$ has been treated. Then

$$\left(\frac{\partial g}{\partial r}\right) = \frac{-\mu}{|r|^3} I + \frac{3\mu}{|r|^5} \begin{bmatrix} (r_1^2) & (r_1 r_2) & (r_1 r_3) \\ (r_1 r_2) & (r_2^2) & (r_2 r_3) \\ (r_1 r_3) & (r_2 r_3) & (r_3^2) \end{bmatrix} \quad (2.31)$$

$$\left(\frac{\partial a}{\partial u}\right) = |a| \begin{bmatrix} -\sin u_1 \\ \cos u_1 \\ 0 \end{bmatrix} \quad (2.32)$$

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \delta v \\ \delta r \end{pmatrix} &= \begin{bmatrix} 0 & \frac{\partial g}{\partial r} \\ I & 0 \end{bmatrix} \begin{pmatrix} \delta v \\ \delta r \end{pmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ 0 \end{bmatrix} \delta u \\ &\equiv F \begin{pmatrix} \delta v \\ \delta r \end{pmatrix} + G \delta u \end{aligned} \quad (2.33)$$

where I is the identity, matrix, and $F(t)$ and $G(t)$ are as indicated, (calculated on the nominal trajectory).

Introduce the state transition matrix $U(T,t)$, defined by

$$\frac{d}{dt} U(T,t) = -U(t,t) F(t) \quad (2.34)$$

and $U(T,T) = I$. It follows that

$$\frac{d}{dt} \left[U \begin{pmatrix} \delta v \\ \delta r \end{pmatrix} \right] = U G \delta u \quad (2.35)$$

and hence

$$\begin{bmatrix} \delta v(T) \\ \delta r(T) \end{bmatrix} = U(T,t) \begin{bmatrix} \delta v(t) \\ \delta r(t) \end{bmatrix} + \int_t^T U(T,s) G(s) \delta u(s) ds \quad (2.36)$$

Equation (2.36) describes the linear (first) variation of v and r at the fixed time T . Considering now the effect of varying the final time T :

$$\begin{pmatrix} dv \\ dr \end{pmatrix} = \begin{bmatrix} v(T+dt) - v_s(T) \\ r(T+dt) - r_s(t) \end{bmatrix} = \begin{bmatrix} \delta v \\ \delta r \end{bmatrix}_T + \begin{bmatrix} a+g \\ v \end{bmatrix}_T dT \quad (2.37)$$

where $\delta v(t)$ and $\delta r(T)$ are given by (2.36). Equations (2.36)-(2.37) are the fundamental equations of linear guidance analysis. Since only a first order expansion of the equations of motion were employed, they describe the first variation of the final state vector.

Since the end condition functions $q_i(r,v)$, $i = 1,2,\dots,N$, are of primary interest here, (2.36)-(2.37) can be used to describe their first variations. Then

$$dq_i = \left(\frac{\partial q_i}{\partial r} \right) dr + \left(\frac{\partial q_i}{\partial v} \right) dv \quad i = 1,2,\dots,N \quad (2.38)$$

where the partial derivatives are evaluated at the final time T , and

$$\begin{aligned} \begin{pmatrix} dq_1 \\ \vdots \\ dq_N \end{pmatrix} &= \begin{bmatrix} \left(\frac{\partial q}{\partial v} \right) & \left(\frac{\partial q}{\partial r} \right) \end{bmatrix} \begin{pmatrix} dv \\ dr \end{pmatrix} \\ &= M(t,t) \begin{bmatrix} \delta v(t) \\ \delta r(t) \end{bmatrix} + \int_t^T \eta(t) \delta u(t) dt + K dt \end{aligned} \quad (2.39)$$

where

$$M(T,t) = \begin{bmatrix} \left(\frac{\partial q}{\partial v} \right) & \left(\frac{\partial q}{\partial r} \right) \end{bmatrix} [U(T,t)] \quad (2.40)$$

$$\eta(t) = [M(T,t) \begin{bmatrix} \frac{\partial a(t)}{\partial u(t)} \end{bmatrix}] \quad (2.41)$$

$$K = \begin{bmatrix} \left(\frac{\partial q}{\partial v} \right) & \left(\frac{\partial q}{\partial r} \right) \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} \quad (2.42)$$

In (2.39) we have used the fact that the functions $q_i(r,v)$ are constant on the coast arc, so that

$$\dot{q}_i = 0 = \left(\frac{\partial q_i}{\partial r} \right) v + \left(\frac{\partial q_i}{\partial v} \right) g \quad (2.43)$$

and the $[a,v]$ terms drop out of the expression for the first variation with respect to dT . Equation (2.39), which describes the first order behavior of the end conditions, is the result we shall exploit in the sequel.

3. OPTIMALITY OF GUIDANCE MODES

3.1 Introduction

Necessary conditions for an optimum trajectory (see Section 2.4) provide the rationale for both optimal and parameterized guidance. Furthermore, a measure of the optimality of a guidance mode can be developed from these results which is one of the measures of performance described in Section 2.6. In this section, the optimality conditions will be developed in a form convenient for the analysis of both optimal and parameterized guidance modes, and an approximate optimality measure will be devised. It will be seen that the optimality conditions are obtained by applying and extending the first variation (first order) analysis described in Section 2.8.

3.2 The First Necessary Conditions for Optimality

Suppose that a nominal trajectory is given which meets the desired end conditions $q_i(r,v) = 0$, $i = 1,2,\dots,N$, and one wishes to test whether or not the time T is a minimum. This is accomplished by varying the controls $u(t)$ and T to see if the same end conditions can be achieved in a smaller time.

Holding the initial conditions at time $t = 0$ fixed, and employing equation (2.39), it follows that a necessary condition for optimality is that the matrix N must be singular, where

$$N = \int_0^T \eta(t) \eta^T(t) dt \quad (3.1)$$

The proof is by contradiction. If N were not singular, we could choose

$$\delta u(t) = -\eta^T(t) N^{-1} K\epsilon \quad (3.2)$$

where ε is some arbitrary, small positive number, to obtain

$$\begin{bmatrix} dq_1 \\ \vdots \\ dq_N \end{bmatrix} = N N^{-1} K \varepsilon + K dT = K(\varepsilon + dT) \quad (3.3)$$

Thus $dT = -\varepsilon$ yields $dq_i = 0$ for $i = 1, 2, \dots, N$, which is a contradiction of the minimum time hypothesis. Since N is therefore singular, there must be a constant vector $\dot{v}^T = (v_1 \dots v_N)$ such that

$$0 = v^T N v = \int_0^T (v^T \eta(t))^2 dt \quad (3.4)$$

which implies the

First necessary condition for optimality:

There is a constant vector v such that

$$0 = v^T \eta(t) = 0 = \lambda^T(t) \left[\frac{\partial f}{\partial u}(x(t), u(t), t) \right] \quad (3.5)$$

for all $0 \leq t \leq T$, where

$$\lambda^T(t) = \lambda^T(T) [U(T, t)] = [\lambda_v^T(t) : \lambda_r^T(t)] \quad (3.6)$$

$$\lambda^T(T) = v^T \left[\left(\frac{\partial q}{\partial v} \right) : \left(\frac{\partial q}{\partial r} \right) \right] \quad (3.7)$$

and f describes the equations of motion (2.3). Note that (3.7) defines the end conditions (2.9)(2.10), and by the definition of the state transition matrix $U(T, t)$ we have

$$\dot{\lambda}^T(t) = -\lambda^T(t) \left[\frac{\partial f}{\partial x}(t) \right] \quad (3.8)$$

From the form of the equations of motion we again note that (3.5) implies that the thrust vector is parallel to $\lambda_v(t)$.

The $\lambda_v(t)$ and $\lambda_r(t)$ are to be interpreted as partial derivatives of a quantity p with respect to $v(t)$ and $r(t)$, respectively, where $p = \sum_i v_i q_i(r(T), v(T)) - T$. As in Section 2, one can define the hamiltonian

$$H = \lambda_v^T (a+g) + \lambda_r^T v - 1 \quad (3.9)$$

which has the interpretation

$$H = \left[\frac{\partial p}{\partial v(t)} \right] \dot{v} + \left[\frac{\partial p}{\partial r(t)} \right] v + \left(\frac{\partial p}{\partial T} \right) \quad (3.10)$$

At the final time we then have $H(T) = \frac{dp}{dT}$, which must be zero if the control parameter T is chosen optimally. Since the v_i may be scaled by an arbitrary common factor, the shut-off condition

$$H(T) = 0 = \lambda_v^T a - 1 = |\lambda_v| |a| - 1 \quad (3.11)$$

determines the scaling factor corresponding to the desired end conditions.

3.3 An Approximate Form of the Second Variation

Thus far only the first variation has been considered in deriving necessary conditions for optimality. Further necessary conditions follow from analysis of the second variation of the state variables, which is obtained by including the second order terms of the Taylor series expansion of the equations of motion. A complete analysis of the second variation is given in Reference (3), leading to a double integral expression for the second variation of the end conditions with respect to $u(t)$, but, since this is rather cumbersome for the purpose of guidance mode analysis, an approximate form will be developed here.

Express the i^{th} component of the acceleration equation (2.3) as

$$\begin{aligned} v_i = v_{si} + \left(\frac{\partial g_i}{\partial r} \right) \delta r + \left(\frac{\partial a_i}{\partial u} \right) \delta u \\ + \frac{1}{2} \delta r^T \left(\frac{\partial^2 g_i}{\partial r_j \partial r_k} \right) \delta r + \frac{1}{2} \left(\frac{\partial^2 a_i}{\partial u^2} \right) \delta u^2 + O(\delta^3) \end{aligned} \quad (3.12)$$

$$i = 1, 2, 3$$

where, for simplicity, only a scalar $\delta u = \delta u_1$ will be considered (planar motion), and the gravity second derivative term denotes a 3x3 matrix. The neglected third order terms are denoted by $O(\delta^3)$. Differentiating the first derivative expressions as given by (2.31) and (2.32), we have terms of the type

$$\left(\frac{\partial^2 g_1}{\partial r_1^2} \right) = \frac{3 \mu r_1}{|r|^5} \left[3 - 5 \left(\frac{r_1}{|r|} \right)^2 \right] \quad (3.13)$$

$$\left(\frac{\partial^2 g_1}{\partial r_1 \partial r_2} \right) = \frac{3 \mu r_2}{|r|^5} \left[1 - 5 \left(\frac{r_2}{|r|} \right)^2 \right] \quad (3.14)$$

and

$$\left(\frac{\partial^2 a_i}{\partial u^2} \right) = \begin{cases} -|a| \cos u_1 & \text{for } i = 1 \\ -|a| \sin u_1 & \text{for } i = 2 \\ 0 & \text{for } i = 3 \end{cases} \quad (3.15)$$

The second variation approximation follows from the assumption that the δr^2 terms are negligible compared to the δu^2 terms. If $|r| = r_1$, $r_2 = r_3 = 0$, then the δr^2 terms are of the order

$$\left[6 \frac{\mu r_1}{|r|^3} \left(\frac{\delta r_1}{|r|} \right)^2 \right] = 6 \left(\frac{\delta r}{|r|} \right)^2 |g| \quad (3.16)$$

where $g = \frac{\mu}{|r|^2}$. If $|a| \geq |g|$ and $\cos u \approx 1$, then

$$\left(\frac{\delta r^2 \text{ terms}}{\delta u^2 \text{ terms}} \right) \approx \frac{6 \left(\frac{\delta r}{|r|} \right)^2}{\delta u^2} = \text{ratio} \quad (3.17)$$

This ratio can be estimated by a crude simple analysis. Suppose that an initial error $\delta r(0)$ is to be corrected in time t , and assume that the thrust vector is nearly perpendicular to r during that period. Assume a constant steering angle variation, chosen from

$$0 = \delta r(T) = \delta r(0) + \frac{1}{2} \delta \ddot{r} T^2 = \delta r(0) + \frac{1}{2} \delta u |a| T^2 \quad (3.18)$$

so that

$$\delta u = \frac{-2 \delta r(0)}{|a| T^2} \quad (3.19)$$

Then

$$\text{ratio} = 6 \left(\frac{|a| T^2}{2 |r|} \right)^2 \quad (3.20)$$

and if $\frac{|a| T^2}{2}$, which is the downrange distance increment due to the thrust acceleration acting during T , is $\epsilon |r|$, then $\text{ratio} = 6 \epsilon^2$. A typical values of ϵ might be at most 0.1, so that $\text{ratio} \leq 0.1$. In summary, it may be asserted that the δr^2 terms are small compared to the δu^2 terms if the steering law tends to keep the altitude variations from the nominal trajectory equal to small values.

The simplified extension of equation (2.33) now becomes

$$\frac{d}{dt} \begin{pmatrix} \delta v \\ \delta r \end{pmatrix} = \begin{bmatrix} 0 & \left(\frac{\partial g}{\partial r}\right) \\ I & 0 \end{bmatrix} \begin{pmatrix} \delta v \\ \delta r \end{pmatrix} + \begin{bmatrix} \frac{\partial a}{\partial u} \\ 0 \end{bmatrix} \delta u + \frac{1}{2} \begin{bmatrix} \frac{\partial^2 a}{\partial u^2} \\ 0 \end{bmatrix} \delta u \quad (3.21)$$

where

$$\left(\frac{\partial^2 a}{\partial u^2}\right) = -|a| \begin{bmatrix} \cos u_1 \\ \sin u_1 \\ 0 \end{bmatrix} = -a(t) \quad (3.22)$$

Again introducing the state transition matrix, and proceeding as in Section 2.7, we have

$$\begin{bmatrix} dq_1 \\ \vdots \\ dq_N \end{bmatrix} = M(T, t) \begin{bmatrix} \delta v(t) \\ \delta r(t) \end{bmatrix} + \int_t^T n(t), \delta u(s) ds - \frac{1}{2} \int_t^T M(T, s) a(s) \delta u^2(s) ds + K dT \quad (3.23)$$

3.4 The Second Necessary Condition for Optimality

From this result one can deduce further necessary condition for optimality. From the first order results of Section 2.7, it can be seen that, except for the linear combination $\sum v_i dq_i$, the end condition functions dq_i can be set equal to arbitrary values by applying the control $\delta u(t)$. Thus, choose a continuous $\delta u(t)$ for $0 \leq t \leq T$, plus a doublet impulse function

$$\hat{\delta u}(t) = \begin{cases} c & \text{for } t_1 - \epsilon \leq t \leq t_1 \\ -c & \text{for } t_1 \leq t \leq t_1 + \epsilon \end{cases} \quad (3.24)$$

where ϵ is a small value. This impulse has negligible effect on the end conditions dq_i as compared to the first order effect of $\delta u(t)$, except for the linear combination

$$\begin{aligned}
\sum v_i dq_i &= v^T \left[\left(\frac{\partial q}{\partial v} \right) : \left(\frac{\partial q}{\partial r} \right) \right] \begin{bmatrix} dv(T) \\ dr(T) \end{bmatrix} \\
&= -\lambda_V^T(t_1) a(t_1) c^2_\epsilon + \lambda_V^T(T) a(T) dt
\end{aligned} \tag{3.25}$$

Thus dT can be made equal to a negative number unless

$$-\lambda_V^T(t_1) a(t_1) = H_{uu}(t_1) \leq 0 \quad \text{for all } 0 \leq t_1 \leq T \tag{3.26}$$

where we have introduced the hamiltonian expression (3.9) . This is the classical Legendre-Clebsch condition. It is clearly true in this case, because the thrust vector is proportional to $\lambda_V(t)$, and hence

$$H_{uu}(t) = -|a(t)| |\lambda_V(t)| \tag{3.27}$$

3.5 A Geometrical Interpretation of Optimality

The analysis of the first and second variation of a trajectory has a geometrical interpretation which aids the study of optimality. Define the normalized (by $-H_{uu}$) control variation

$$\hat{\delta u}(t) = [\delta u(t)] [|\lambda_V(t)| |a(t)|]^{1/2} \tag{3.28}$$

Thus the first variation control terms of equation (3.23) become

$$\int_0^T n_i(t) \delta u(t) dt = \int_0^T n_i^*(t) \hat{\delta u}(t) dt \quad i = 1, \dots, N \tag{3.29}$$

where

$$n_i^*(t) = [|\lambda_V(t)| |a(t)|]^{-1/2} n_i(t) \tag{3.30}$$

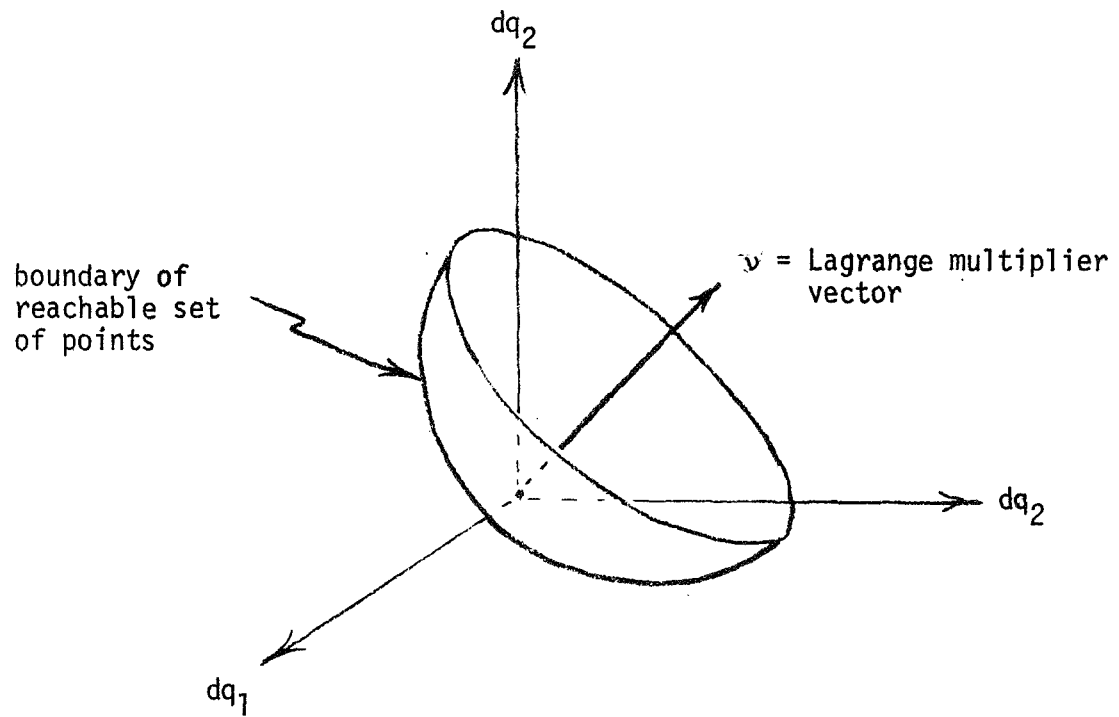


Figure 3.1: Geometrical Interpretation of Optimality

Consider the N dimensional space with coordinates $dq_1 \dots dq_N$, as shown in figure 3-1. Make the orthogonal coordinate transformation

$$\hat{dq} = L dq$$

where the matrix L is chosen to diagonalize the matrix

$$N^* = \int_0^T \begin{bmatrix} (\eta_1^*)^2 & (\eta_1^* \eta_2^*) & \dots \\ (\eta_1^* \eta_2^*) & (\eta_2^*)^2 & \dots \\ \dots & \dots & (\eta_N^*)^2 \end{bmatrix} dt \quad (3.31)$$

That is,

$$LN^*L^T = \begin{bmatrix} 0 & & & \\ & \rho_2 & & 0 \\ & & \ddots & \\ 0 & & & \rho_N \end{bmatrix} \quad (3.32)$$

The first eigenvalue is zero since the matrix N of equation (3.1), and hence N^* , is singular on an optimal trajectory. Clearly the first row of L must be a unit vector proportional to the vector $v^T = [v_1, v_2, \dots, v_N]$. Assuming the second variation terms are negligible compared to the first variation terms when they appear as a sum, equation (3.23) yields

$$d\hat{q}_1 = -\frac{1}{2} \int_0^T [\delta \hat{u}(t)]^2 dt \quad (3.33)$$

$$d\hat{q}_i = \int_0^T \hat{n}_i(t) \delta \hat{u}(t) dt \quad i = 2, \dots, N \quad (3.34)$$

where T is held fixed and

$$\hat{n}_i(t) = \sum_{j=1}^N L_{ij} \eta_j^*(t). \quad (3.35)$$

The $\hat{q}_i$ for $i = 2, \dots, N$ are called the "first order controllable" end conditions. Note that, by this construction, the $\hat{n}_i(s)$ are orthogonal; that is,

$$\int_0^T \hat{n}_i(s) \hat{n}_j(s) ds = \begin{cases} 0 & \text{for } i \neq j \\ \rho_i & \text{for } i = j \end{cases} \quad (3.36)$$

The boundary of the set of points reachable for arbitrary $\delta u(t)$ and fixed T can be determined by seeking the minimum value of $d\hat{q}_1$, subject to $d\hat{q}_i =$ specified for $i = 2, \dots, N$. That is, solve the accessory minimum problem

$$\text{minimize } dp = d\hat{q}_1 + \sum_{i=2}^N \alpha_i d\hat{q}_i \quad (3.37)$$

where the α_i are Lagrange multipliers. Then

$$0 = -\delta\hat{u}(t) + \sum_{i=2}^N \alpha_i \hat{n}_i(t) \quad (3.38)$$

Using the orthogonality relation (3.36), we multiply the above equation by $n_j(t)$ and integrate to obtain

$$\int_0^T \hat{n}_i(t) \delta\hat{u}(t) dt = d\hat{q}_i = \alpha_i \rho_i \quad i = 2, 3, \dots, N \quad (3.39)$$

Then the Lagrange multipliers are

$$\alpha_i = \left(\frac{d\hat{q}_i}{\rho_i} \right) \quad (3.40)$$

and, again applying (3.36), we have

$$\begin{aligned} d\hat{q}_1 &= \frac{1}{2} \int_0^T \left[\sum_{i=2}^N \alpha_i n_i(t) \right]^2 dt \\ &= \frac{1}{2} \sum_{i=2}^N \left(\frac{d\hat{q}_i^2}{\rho_i} \right) \end{aligned} \quad (3.41)$$

This paraboloid describes the boundary of the readable set of points, and the ρ_i are the radii of curvature at the origin (see figure 3.1).

3.6 A Measure of Optimality

Consider now a near-optimum trajectory which satisfies the end conditions. Expand the equations of motion about the true optimum trajectory, and interpret $\delta\hat{u}(t)$ to be [near optimum $u(t)$] - [optimum $u(t)$]. Since the end conditions are met on the near optimum trajectory, equations (3.33)-(3.34) become

$$0 = dq_1 = -\frac{1}{2} \int_0^T \delta\hat{u}^2 dt + K_1 dT \quad (3.42)$$

$$0 = dq_i = \int_0^T \hat{\eta}_i \delta\hat{u} dt + K_i dT \quad i = 2, \dots, N \quad (3.43)$$

Applying (3.36) it follows that the $\delta\hat{u}(t)$ which satisfies (3.43) is

$$\delta\hat{u}(t) = -dT \sum_{i=2}^N K_i \frac{\hat{\eta}_i(t)}{\rho_i} \quad (3.44)$$

where, from (3.42),

$$dT = \frac{1}{2K_1} \int_0^T \delta\hat{u}^2 dt \quad (3.45)$$

But $K_1 = \lambda_V^T(T) a(T)$, and replacing $\delta\hat{u}(t)$ with $\delta u(t)$ according to (3.28), we have

$$dT = [2|\lambda_V(T)||a(T)|]^{-1} \int_0^T [|\lambda_V(t)||a(t)|] \delta u^2(t) dt \quad (3.46)$$

Equation (3.46) is the desired measure of optimality, for it describes the additional time required to achieve the desired end conditions when using the non-optimum control. Another form can be obtained by defining the weighting factor

$$w(t) = \left[\frac{\lambda_V^T(t) a(t)}{\lambda_V^T(T) a(T)} \right] \quad (3.47)$$

so that

$$dT = \frac{1}{2} \int_0^T w(t) \delta u^2(t) dt \quad (3.48)$$

3.7 The Approximate Second Variation Guidance Mode

The equations developed in section 3.5 to describe approximate second order behavior of the end conditions with respect to the control can be used to develop an optimal, closed loop steering law. The result is a special form of what is called the second variation guidance mode, which will be discussed in section 10. Since the approach is based upon an expansion about a nominal trajectory, the method is representative of an optimal guidance mode which employs stored coefficients.

Equation (3.38) describes a control variation which is optimum in the sense that it yields a point on the boundary of the reachable set of points. Let $d\hat{q}_1, d\hat{q}_2, \dots, d\hat{q}_N$ be the open loop end condition variations due to an observed position and velocity error at some time t_0 . Then, from (3.38), and (3.40), the $\delta\hat{u}(t)$ which nulls the first order controllable components is

$$\delta\hat{u}(t) = - \sum_{i=2}^N \left(\frac{d\hat{q}_i}{\rho_i} \right) \hat{n}_i(t) \quad t_0 \leq t \leq T \quad (3.49)$$

Neglecting the second order effect of $\delta\hat{u}(t)$, the corresponding change of thrusting time is

$$dT = \frac{d\hat{q}_1}{|\lambda_V(T)| |a(T)|} \quad (3.50)$$

These equations can be put into a more convenient form by removing the orthogonal transformation L of (3.32). Suppose the end conditions are described such that v lies along the q_1 axis, so that $\eta_1^*(t) = 0$. Consider the $N-1$ dimensional, first order controllable end condition space

$$d\hat{q}_C = L_C dq_C \quad (3.51)$$

where L_C is composed of the first $N-1$ rows of L , and $dq_C^T = [dq_2, \dots, dq_N]$. The matrix N^* of equation (3.31) takes the form

$$N^* = \begin{bmatrix} 0 & 0 \\ 0 & N_C \end{bmatrix} \quad (3.52)$$

where

$$L_C N_C^* L_C^T = \begin{bmatrix} \rho_2 & & 0 \\ & \ddots & \\ 0 & & \rho_N \end{bmatrix} \quad (3.53)$$

Equation (3.49) can be written

$$\begin{aligned} -\delta \hat{u}(t) &= [\hat{n}_2(t), \dots, \hat{n}_N(t)] \begin{bmatrix} \rho_2 & & 0 \\ & \ddots & \\ 0 & & \rho_N \end{bmatrix}^{-1} d\hat{q}_C \\ &= [n_2^*(t), \dots, n_N^*(t)] L_C^T [L_C N_C^* L_C^T]^{-1} L_C dq_C \\ &= [n_2^*(t), \dots, n_N^*(t)] [N_C^*]^{-1} dq_C \end{aligned} \quad (3.54)$$

Multiplying both sides of (3.54) by $|\lambda_v(t)| |a(t)|^{-1/2}$, and introducing (3.28) and (3.30),

$$\delta u(t) = -|\lambda_v(t) a(t)|^{-1} \left\{ \eta_C^T(t) [N_C^*]^{-1} dq_C \right\} \quad (3.55)$$

Thus the approximate second variation algorithm calls for steering to correct the first order controllable end conditions, where the influence functions $\eta_i(t)$ are weighted as indicated above.

4.0 APPROXIMATIONS OF OPTIMUM STEERING

4.1 Introduction

It can be a formidable task to numerically solve the optimization equations in order to determine the thrust direction, i.e., the direction of the primer vector $\lambda_v(t)$. Accordingly, the guidance mode designer may analyze the properties of the primer vector in order to choose a functional form for the steering angle (u) which is supposed to be nearly optimum. This approach is called parameterized guidance, since certain free parameters in the form $u(p,t)$ are then chosen in real time to satisfy the end conditions. For example, an analysis of the primer vector behavior in a uniform gravity field might lead one to choose the tangent of the thrust angle as a linear function of time, or, using small angle approximations, to choose the angle itself as a linear function of time. In this section, we shall examine properties of the primer vector by using certain simplifying approximations of the equations of motion. This analysis will then be exploited to discuss the linear form of parameterized steering. This particular form was chosen for special attention because it is the basis of the Iterative Guidance Mode, which has been used with much success in real time applications (the Apollo mission) and in trajectory design.

4.2 Motion in a Uniform Gravity Field

Consider the motion of a rocket vehicle in the (r_1, r_2) plane, and assume that the gravitational acceleration is a pre-specified function of time: $g = g(t)$. For example, g could be a constant vector in the $-r_2$ direction. Then $(\frac{\partial g}{\partial r}) = 0$, the F matrix of equation (2.33) is a constant, and it is easily shown that the state transition matrix is given by

$$U(T,t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t_g & 0 & 1 & 0 \\ 0 & t_g & 0 & 1 \end{bmatrix} \quad (4.1)$$

where $t_g = \text{time-to-go} = T-t$.

Since the primer vector (λ_v) is a linear combination of elements of the state transition matrix, it must be a linear function of time. From Equations (3.6) and (4.1) we obtain the general expression for the primer vector in a uniform gravity field:

$$\lambda_v^T = \alpha^T U(T, t) = [\alpha_1 + \alpha_3 t_g, \alpha_2 + \alpha_4 t_g] \quad (4.2)$$

$$\text{where } \alpha^T = v^T \left[\left(\frac{\partial q}{\partial v} \right) : \left(\frac{\partial q}{\partial r} \right) \right] = \lambda^T(T) \quad (4.3)$$

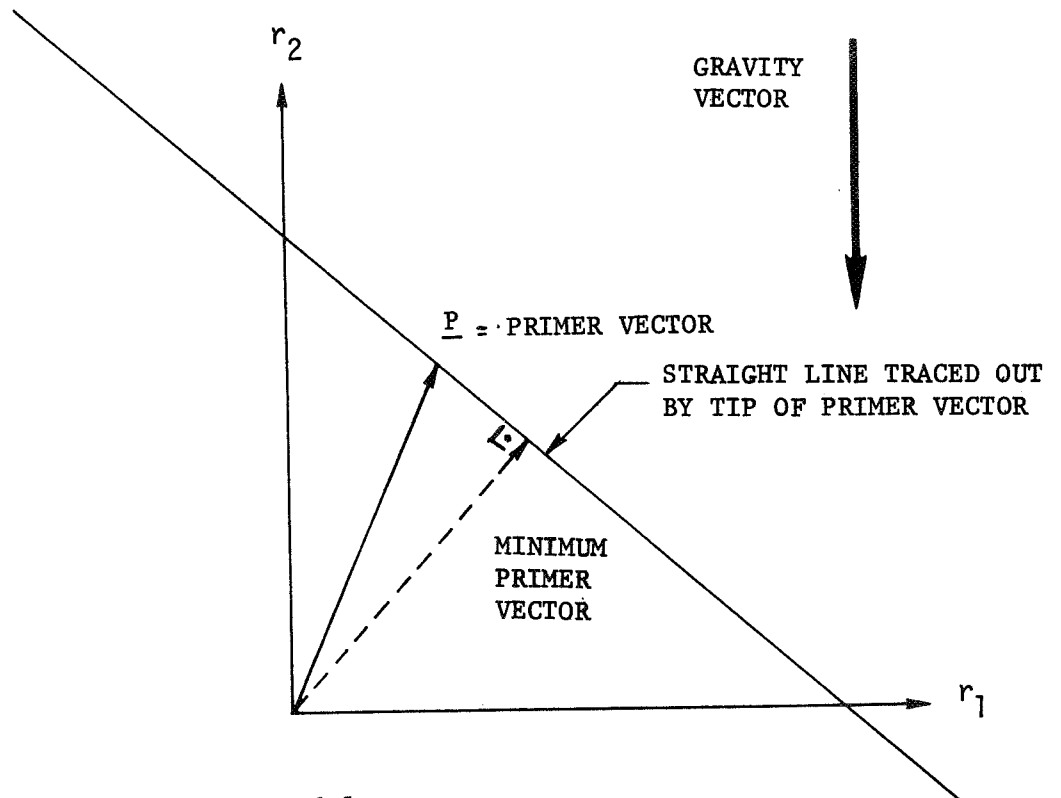


Figure 4.1
PRIMER VECTOR IN A
UNIFORM GRAVITATIONAL FIELD
(TYPICAL)

From Equation (4.2) we note that the tip of the primer vector moves with uniform speed along a straight line. This is illustrated in Figure 4.1 Since the unit steering vector $[\cos u, \sin u]$ must be taken parallel to the primer vector, the functional dependence of the optimum pitch angle with time is given by

$$\tan u(t) = \frac{\lambda_2(t)}{\lambda_1(t)} = \frac{\alpha_2 t_g + \alpha_4}{\alpha_1 t_g + \alpha_3} \quad (4.4)$$

Equation (4.4) is known as the bilinear-tangent law. Letting

$$a = \begin{bmatrix} \alpha_3 \\ \alpha_4 \end{bmatrix} \text{ and } b = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (4.5)$$

Equation (4.2) may be rewritten as

$$\lambda_v(t) = a + t_g b. \quad (4.6)$$

It is easy to show that the primer vector of minimum length is given in terms of the vectors a and b of Equation (4.6) by

$$\lambda_v(\min) = a - \frac{a \cdot b}{|a|^2} b \quad (4.7)$$

Define t^* by

$$t^* = T + \frac{a \cdot b}{|b|^2} \quad (4.8)$$

Then Equation (4.2) can be written as

$$\lambda_v = \lambda_v(\min) + (t^* - t) b \quad (4.9)$$

It follows that

$$\lambda_V^T(\min) b = 0 \quad (4.10)$$

If a coordinate system is now chosen so that r_1 is along $\lambda_V(\min)$, then in that system the primer vector has the following simple form

$$\lambda_V(t) = \begin{bmatrix} c \\ d - \dot{e}t \end{bmatrix} \quad (4.11)$$

where

$$c = |\lambda_V(\min)|, \quad d = t^* |b|, \quad \dot{e} = |\dot{b}|. \quad (4.12)$$

In this coordinate system the primer vector consequently traces out a vertical line segment, and the equation for the pitch angle becomes

$$\tan u = \frac{d}{c} - \dot{e}t \quad (4.13)$$

We have shown in effect that by rotating the coordinate system the bilinear-tangent law (Eq. 4.4) is reduced to the linear-tangent law (Eq. 4.13). It follows that by simply changing the reference direction from which u is measured the bilinear-tangent law is transformed into the linear tangent law, and we can always write

$$\tan (u-u_0) = (\text{constant}) + (\text{constant})(t) \quad (4.14)$$

as the most general case for motion in a uniform gravity field. Thus, as an approximate steering law for the inverse square gravity law, one might choose (4.14), or, simpler still, let the angle itself be a linear function of time. The latter functional form has been used with much success in applications (viz, the IGM), and will be discussed below.

4.3 On the Linearity of the Steering Angle for the Uniform Gravity Model

The behavior of the optimum pitch angle in a uniform gravity field is given by Equation (4.14). This equation states that the tangent of the angle varies uniformly with time in a suitable coordinate system. Since the tangent is a non-linear function, one can immediately conclude that in general, the pitch angle does not vary linearly with time. Furthermore, since a uniform gravity field is but a special case of a central force field (corresponding to the case where the length of the thrusting arc is small compared to the distance from the center of force), we may further conclude that the general steering history of a rocket in an inverse-square force field is non-linear.

Experience shows, however, that in many cases of practical interest the steering angle may be approximated as a linear function of time with negligible loss of performance. This is not an unreasonable conclusion, since the arc tangent of a linear function might be approximated reasonably well over some range of the independent variable. Alternatively, this conclusion can be justified by showing that the angular acceleration is small. From Equation (4.4), it follows that

$$\dot{u} = \frac{\lambda_1 \dot{\lambda}_2 - \lambda_2 \dot{\lambda}_1}{\lambda_1^2 + \lambda_2^2} \quad (4.15)$$

$$\ddot{u} = \frac{-2 (\lambda_1 \ddot{\lambda}_1 + \lambda_2 \ddot{\lambda}_2)}{(\lambda_1^2 + \lambda_2^2)^2} \quad (4.16)$$

But $\ddot{u}$ is zero when $|\lambda_v|$ is a minimum, where $\lambda_1 \dot{\lambda}_1 + \lambda_2 \dot{\lambda}_2 = 0$, and is also zero when $|\lambda_v| \rightarrow \infty$. Thus, an average value of $\dot{u}(t)$ might approximate the steering angle quite well.

The validity of the linear assumption depends upon the segment of the straight line (4.2) traced out by the primer vector in a particular case, and hence is intimately connected with the boundary conditions. Three examples will now be given to illustrate this.

Example 1 - Problem:

The end conditions require the attainment of a certain velocity, but the position is unconstrained. Find the primer vector.

Solution:

By the end condition equations (2.16) and (2.17), it can be seen that the last two components of the vector α must vanish. But then from Equation (4.2), we see that the primer vector is a constant. The pitch angle is constant, and therefore, linear.

Example 2 - Problem:

The end conditions require the attainment of a certain position, but the velocity is unconstrained. Find the primer vector.

Solution:

The end conditions require that the vector a of Equation (4.5) be zero. The primer vector moves along a straight line passing through the origin but always stays to one side of the origin. At the final time the primer vector vanishes. The pitch angle is again constant.

Example 3 - Problem:

The rocket is initially at the origin with zero velocity. At the final time it is to have an altitude $h \geq 0$ and a (downward) velocity $-v$. Choose the coordinate system so that r_2 corresponds to (positive) altitude. The final conditions then are: $r_2(T) = h$, $v_2(T) = -v$, and r_1 and v_1 are unconstrained. Find the primer vector.

Solution:

The tip of the primer vector moves along the λ_2 axis. Initially the tip lies above the origin, but at the final time it is below the origin. Hence, at some critical time the primer vector passes through the origin. At this time "thrust reversal" occurs, and the pitch angle exhibits a jump of π (radians). Consequently, the pitch angle is "highly" non-linear, albeit it is linear everywhere except at the jump.

Note that example 3 is not typical of the cases being considered in this Report, for the minimum time criterion requires shut-off when the magnitude of the primer vector becomes less than a certain value (Equation 2.21). Thus, at least for the case of chemical rockets, the minimum value of $|\lambda_v(t)|$ should be attained at burnout. As a practical matter, we assume that example 3 would be accomplished by a two-burn (two phase) mission, with a coast between. The problem would then be treated as two minimum time problems, where the end conditions of the first phase are related to the initial conditions of the second by solution of the targeting problem.

In summary, it might be said that the linear steering law can be a reasonably good approximation for the constant gravity case if the end conditions are "compatible", which means a set of end conditions which cause the primer vector to attain its minimum value at burnout.

4.4 Motion in a Linearized Gravity Field

The uniform gravity assumption only applies to cases where the thrusting arc length is short in comparison with the distance from the principal center of force. A more sophisticated model is required to adequately treat cases where this assumption breaks down. A model which is probably adequate to handle thrusting arcs up to 90° is the "linearized gravity field" model.

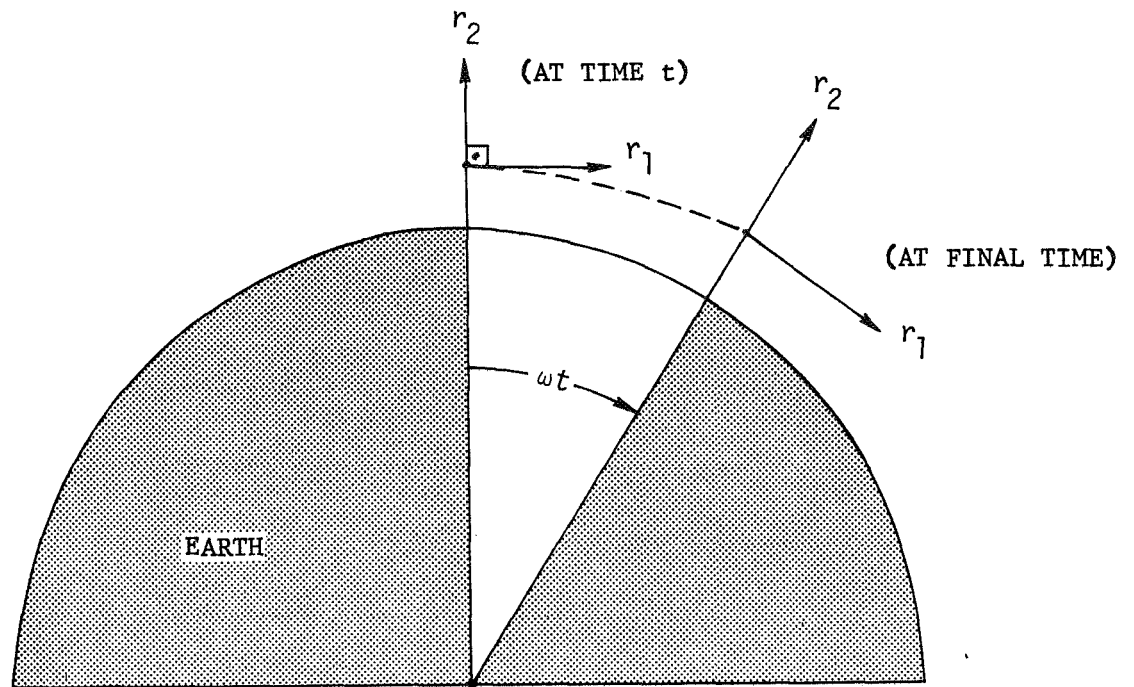


Figure 4.2
UNIFORMLY ROTATING
COORDINATE SYSTEM

Consider a particle moving in a circular orbit about the principal attracting body. The radius and starting location of this hypothetical particle are imagined to be so chosen so as to minimize the maximum distance between the particle and the rocket vehicle during the thrusting arc of the latter. For example, if the rocket is initially in a circular orbit and accelerates to escape conditions, the reference point should probably be chosen in the same circular orbit but somewhat ahead of the initial position of the rocket. Thus, as the rocket accelerates it catches up and ultimately passes the reference point, and the maximum distance between the two is thereby minimized. The origin of a Cartesian coordinate system is taken at the previously defined reference particle. The r_1 -axis is taken in the direction of motion and the r_2 -axis is the direction of the radius vector from the center of force to the particle. The system, therefore, rotates at a uniform rate which is equal to the angular velocity of the origin about the center of force. This system is illustrated in

Figure 4.2. The gravity field is now taken as the gravity vector at the origin plus the first order expansion about the origin. The equations for the gravity field so obtained are

$$g = g_0 + \begin{bmatrix} -\omega^2 r_1 \\ 2\omega^2 r_2 \end{bmatrix} \quad (4.17)$$

$$\omega = \sqrt{\frac{\mu}{r^3}} \quad (4.18)$$

Moreover the constant term (g_0) in the gravity field is equal to the acceleration of the origin, and therefore does not appear in the equation of motion. The function f of Equation (2.3) is given by

$$f = \begin{bmatrix} -2\omega v_2 + a_1 \\ -3\omega^2 r_2 + 2\omega v_1 + a_2 \\ v_1 \\ v_2 \end{bmatrix} \quad (4.19)$$

where a_1 , a_2 are the components of the thrust acceleration.

The F matrix of Equation (2.33) is therefore a matrix of constants, and this equation can be readily integrated to furnish the state transition matrix. The resulting matrix is given below in its entirety, even though only the "velocity part" is required for the primer vector:

$$U(T,t) = \begin{bmatrix} 4 \cos \phi - 3 & -2 \sin \phi & 0 & -6\omega(1 - \cos \phi) \\ - & - & - & - \\ 2 \sin \phi & \cos \phi & 0 & -3\omega \sin \phi \\ - & - & - & - \\ \frac{1}{\omega}(4 \sin \phi - 3\phi) & -\frac{2}{\omega}(1 - \cos \phi) & 1 & 6(\sin \phi - \phi) \\ - & - & - & - \\ \frac{2}{\omega}(1 - \cos \phi) & \frac{\sin \phi}{\omega} & 0 & 4 - 3 \cos \phi \end{bmatrix} \quad (4.20)$$

where

$$\phi = \omega t_g = \text{range-angle-to-go} \quad (4.21)$$

Note that if ϕ is small then $\cos \phi \approx 1$ and $\sin \phi \approx \phi$ and, accounting for the rotation of the coordinate system, Equation (4.20) reduces to Equation (4.1), as it should. This corresponds to stating that if the range-angle-to-go is small a constant gravity assumption is valid. Hence, if the initial range-angle-to-go is less than about 20° , the constant gravity model applies and the present model offers nothing new. If, on the other hand, the initial range to go is greater than about 90° , then the present model will generally prove inadequate as well. The linearized gravity model is therefore of interest in cases where the initial range-to-go is in the range of 20 to 90° .

The general expression for the primer vector is given by

$$\lambda_V = \begin{bmatrix} \frac{\alpha_3}{\omega}(4\sin\phi-3\phi) + \alpha_4 \frac{2}{\omega}(1-\cos\phi) + \alpha_1(4\cos\phi-3) + 2\alpha_2 \sin \phi \\ -\alpha_3 \frac{2}{\omega}(1-\cos\phi) + \frac{\alpha_4}{\omega} \sin\phi - 2\alpha_1 \sin\phi + \alpha_2 \cos \phi \end{bmatrix} \quad (4.22)$$

where the α vector is as defined in Equation (4.3).

On the interval $20 - 90^\circ$ the term $(4 \sin \phi - 3\phi)$ is generally much smaller than the other terms, attaining a maximal value of 0.05. It will consequently be dropped. With this modification, it can be seen that the primer vector traces out an ellipse, for

$$\lambda_1(t) = -2a \cos \phi + 2b \sin \phi + c_1 \quad (4.23)$$

$$\lambda_2(t) = d \cos \phi + a \sin \phi + c_2 \quad (4.24)$$

where

$$\begin{aligned} a &= -2 \alpha_1 + \frac{\alpha_4}{\omega} \\ b &= \alpha_2 \\ c_1 &= 2 \frac{\alpha_4}{\omega} - 3 \alpha_1 \\ d &= \alpha_2 + 2 \frac{\alpha_3}{\omega} \\ c_2 &= -2 \frac{\alpha_3}{\omega} \end{aligned} \quad (4.25)$$

Then

$$\begin{bmatrix} a & b \\ d & -\frac{a}{2} \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 - c_1 \\ \lambda_2 - c_2 \end{bmatrix} = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} \quad (4.26)$$

and

$$\begin{aligned} \left[\frac{a}{2} (\lambda_1 - c_1) + b(\lambda_2 - c_2) \right]^2 + [d(\lambda_1 - c_1) - a(\lambda_2 - c_2)]^2 \\ = \left(\frac{a^2}{2} + bd \right)^2 \end{aligned} \quad (4.27)$$

The rate at which the primer vector moves around the ellipse is given by the rule that the eccentric anomaly increases at the constant rate ω , which is the rate at which ϕ decreases. The situation is illustrated in figure 4.3.

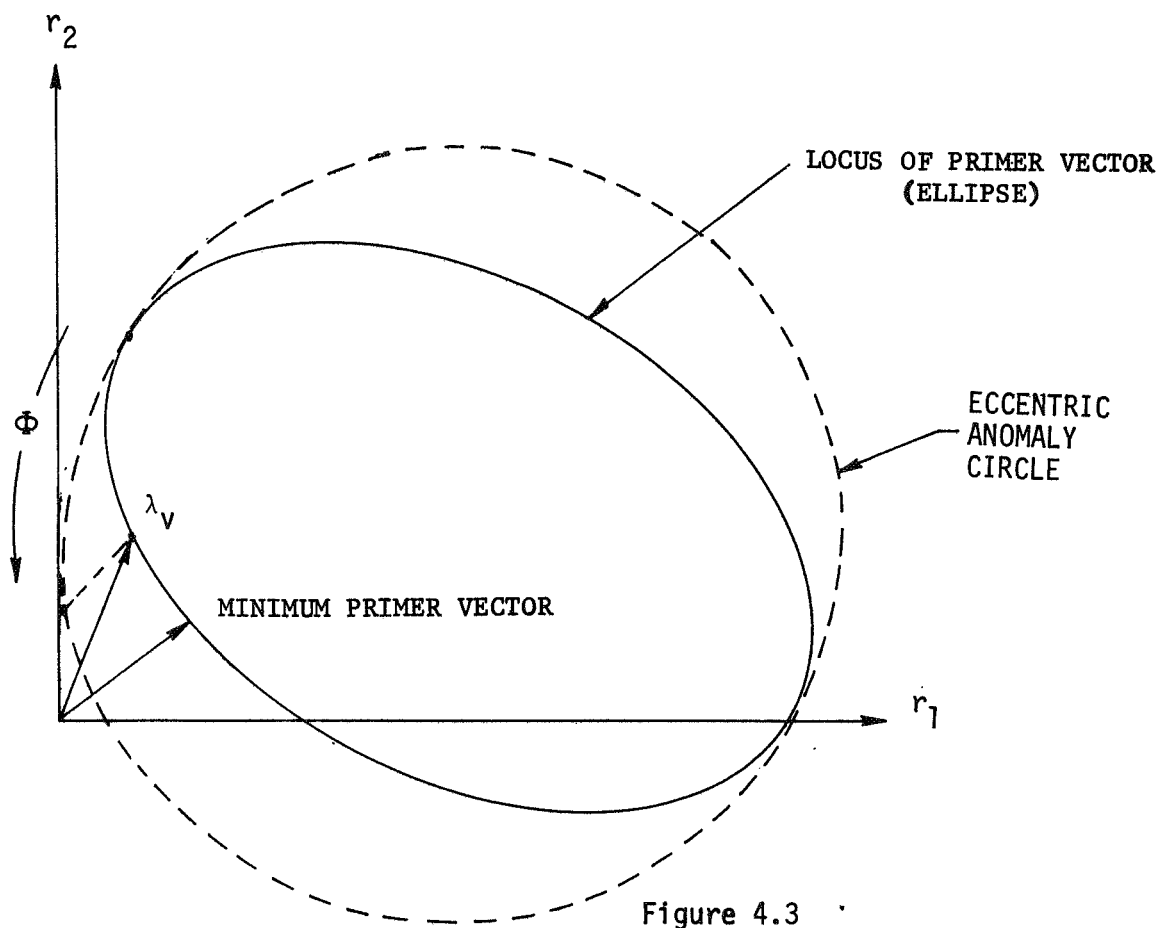


Figure 4.3
TYPICAL LOCUS OF
PRIMER VECTOR

A solution likely to be of interest is obtained when the "downrange" coordinate r_2 is free. The transversality conditions require that α_3 vanish. Then $d = b$, $c_2 = 0$, and

$$\tan u(t) = \frac{\lambda_2(t)}{\lambda_1(t)} = \frac{\sin(\phi + \beta)}{k - 2 \cos(\phi + \beta)} \quad (4.28)$$

where

$$\beta = \tan^{-1} \left(\frac{b}{a} \right) \quad (4.29)$$

$$k = \frac{c_1}{\sqrt{a^2 + b^2}} \quad (4.30)$$

Some examples from this family of solutions are given in Section 4.5.

As in the case of motion in a constant gravity field, a plausible argument for the linear angle approximation can be made. Essentially, it is required that the primer vector trace out a well behaved region of the ellipse determined by the end conditions, where, as before, the minimum time criterion causes the primer vector to achieve its minimum length only at the final time. This analysis will not be pursued here, instead, numerical examples will be presented.

4.5 Numerical Examples

This section presents some actual cases of optimal steering histories. These have been extracted from Reference 4. This reference is of interest since it contains an evaluation of both the linear tangent law and the IGM mode (linear pitch angle) as they apply to various Earth-based mission. The results will not be duplicated here, but the basic conclusion formed in the above reference is that linear steering is superior to linear tangent steering. Furthermore, linear steering was shown to be very good in comparison with the "exact" calculus of variations steering, in terms of injection payload.

The trajectories analyzed in Reference 4 are all based upon a Titan-3D/Centaur launch vehicle. This vehicle has a relatively long burn duration and a quite varied acceleration profile, making it a good candidate for study.

As has been pointed out, Earth-launched missions present quite a severe test for the linearity of the pitch angle because of the requirement to climb out of the atmosphere. This requires that even upper stages (as opposed to the booster stage) expend some of their thrust in climbing. If, in addition, the thrusting arc range angle is large, a further strain is imposed upon the linearity.

The end conditions to which the vehicles were targeted are presented in Table 4.1. The thrust pointing histories as obtained from the calculus

of variations are presented in Figure 4.4. The pitch angle is given in a space-fixed system with the reference being the horizontal at burnout. The basically linear trend in the steering histories is apparent.

The tangents of the pitch angles are presented in Figure 4.5. For short thrusting arc linearity is observed, but marked departure from linearity is found in the longer range cases. However, this effect is somewhat exaggerated since the coordinate system in which the data are presented is not in general the coordinate system which reduces the bilinear tangent law to the linear tangent law. Assuming the uniform gravity field approximation, the theory of Section 4.2 can be used to straighten the tangent curves. Example calculations have been carried out to determine the optimum coordinate system, resulting in reasonably linear behavior.

4.6 Comparison to the Linearized Gravity Model

The constants k and β of Equation (4.28) can be adjusted empirically and the resulting pitch histories compared to actual results obtained using the calculus of variations as presented in the last section. It was found that if k is assigned the approximate value 2.8, and β the approximate value -10° , the resulting family of curves (which depend on the range angle) includes members very similar to cases presented in the last section.

Figure 4.6 shows two such pitch histories. The pitch histories of trajectories 2 and 5 (see Section 4.5) are also presented for comparison. The similarity in the trends is apparent. Since no attempt was made to "fine tune" the constant k and β to obtain the best possible agreement, it would appear that the data might be fitted quite well with Equation (4.28). Thus, Equation (4.28) is a candidate form for parameterized guidance, where k and β are adjusted to meet end conditions, or it can be used to justify simpler forms, such as linear steering.

In summary, it can be said that numerical and analytic results indicate that the simple, easily implemented linear steering law may be a near optimum approximation for a rather wide range of missions. Further studies of the applicability of this form of guidance mode to advanced missions are therefore recommended.

Table 4.1
INJECTION CONDITIONS OF SAMPLE TRAJECTORIES*

Mission	Altitude (NM)	Velocity (FPS)	Path Angle (Deg)	Range Angle Traversed (Deg)
OA0	416.	24,522	0.08	28.60
Mars Probe	102.6	38.933	0.00	46.05
Jupiter Probe	102.6	48,007	1.36	49.50
Trajectory No. 1	150.0	24,000	0.01	19.07
Trajectory No. 2	150.0	28,000	0.0	27.22
Trajectory No. 3	150.0	32,000	0.0	34.39
Trajectory No. 4	150.0	36,000	0.0	40.54
Trajectory No. 5	150.0	40,000	0.0	45.33

* Extracted From Reference 4.

Figure 4.4 SAMPLE OPTIMAL PITCH ANGLES

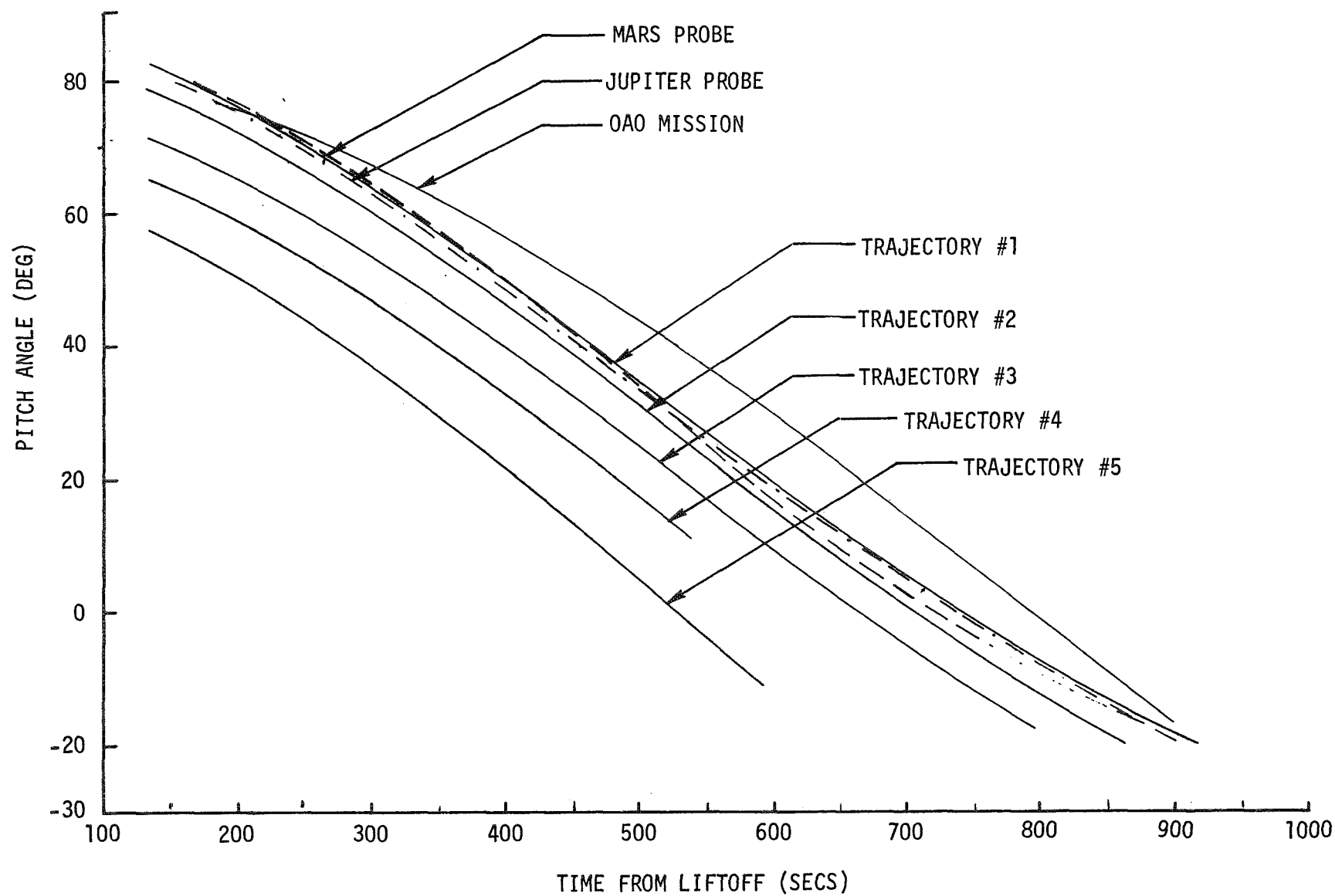


Figure 4.5 TANGENTS OF OPTIMAL PITCH ANGLES

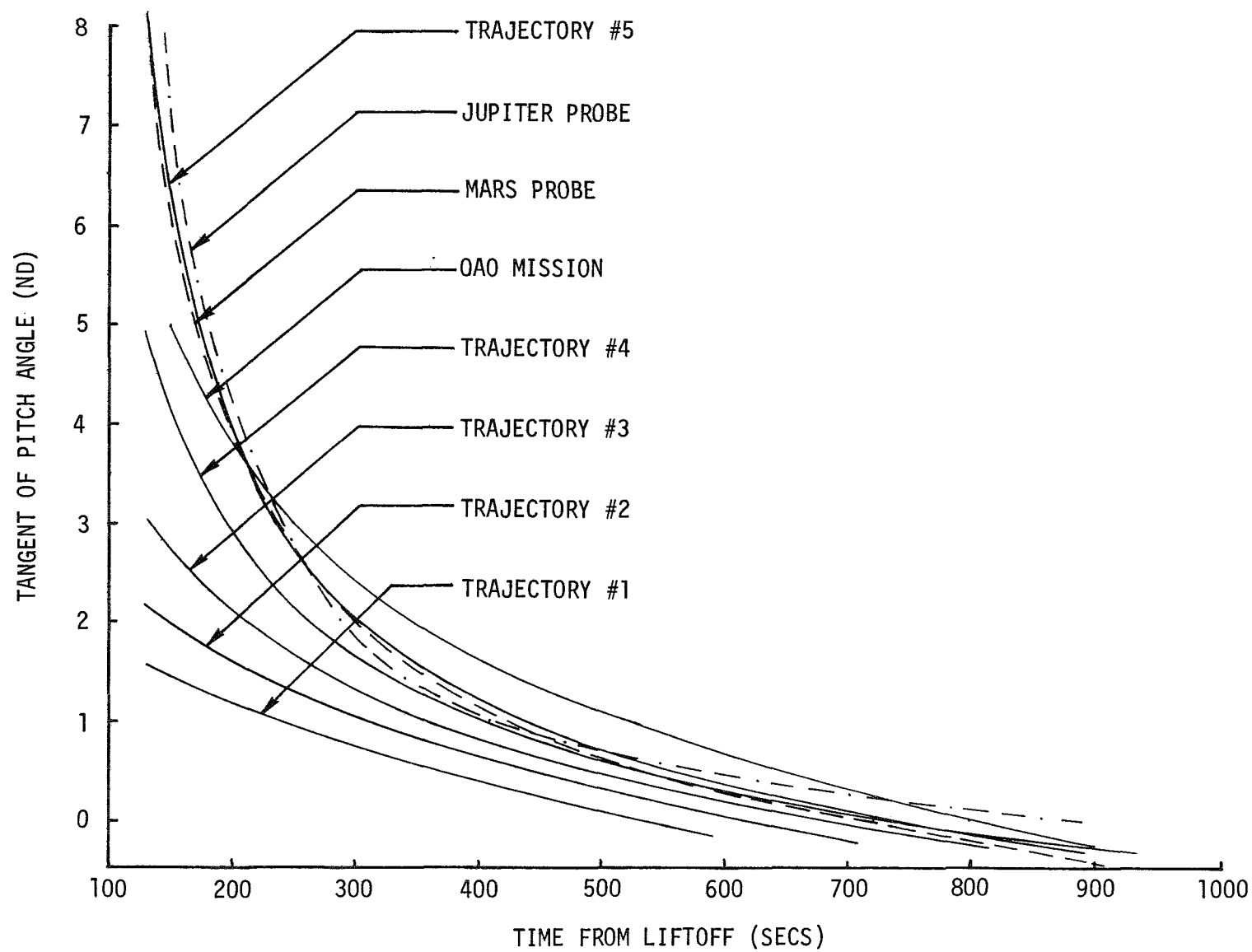
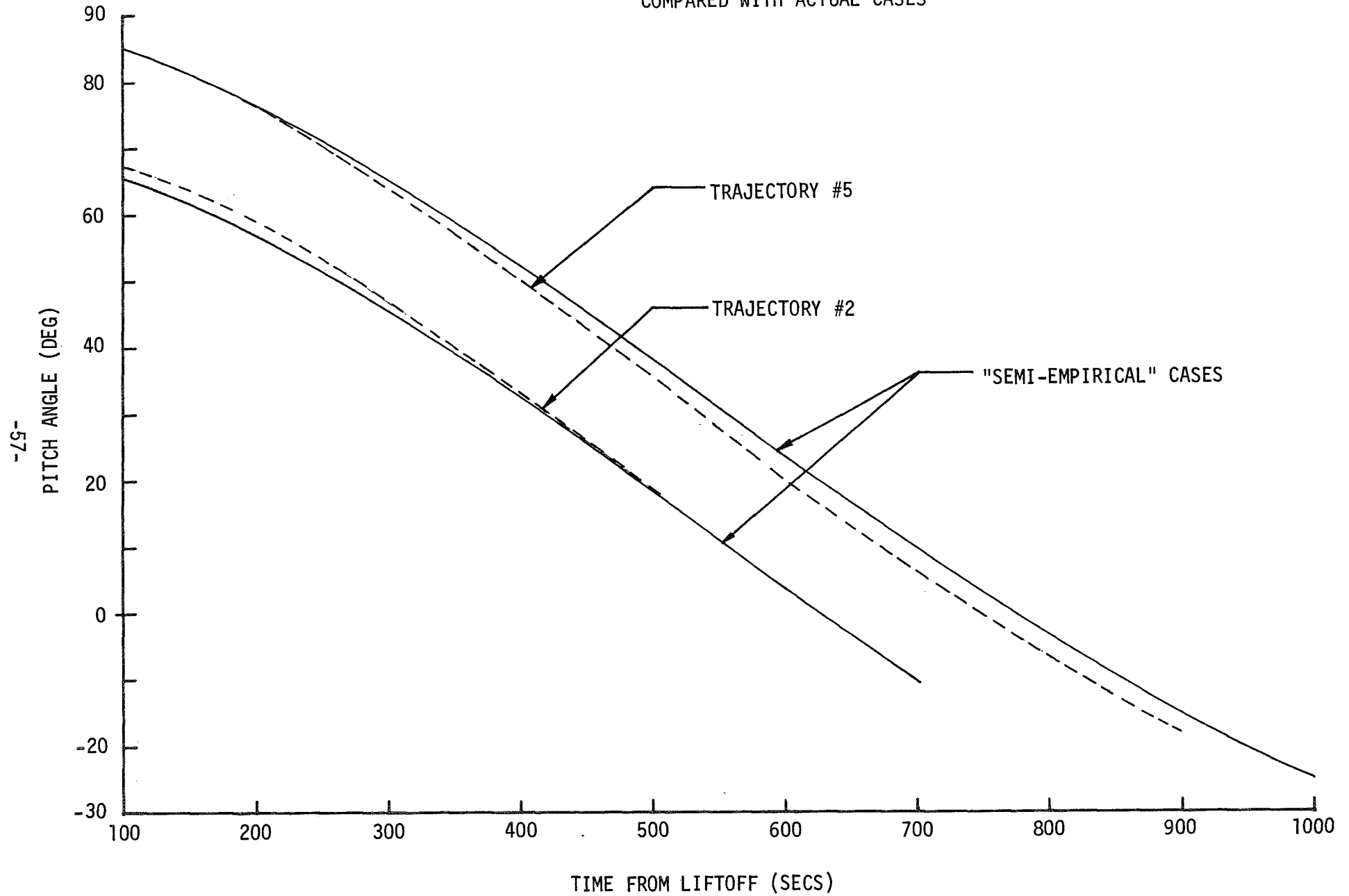


Figure 4.6 SEMI-EMPIRICAL SOLUTIONS
COMPARED WITH ACTUAL CASES



5. GUIDANCE MODE STABILITY

5.1 Introduction

We shall say that a guidance mode is stable if steering applied to a perturbed trajectory always causes the magnitude of all position and velocity deviations to become smaller, where the deviations are measured from some nominal trajectory. This is a rather strong definition of stability, and in general, not easy to test. A useful necessary condition for stability will be derived, however, which will offer a quantitative measure of guidance mode stability.

5.2 Stability of the Linearized System

Let the state vector be represented by $x^T = (v^T, r^T)$, and assume the closed loop guidance law $u(x, t)$, where (u_1, u_2) are the steering of angles generated by the guidance mode. The closed loop equations of motion become

$$\dot{x} = f(x, u(x, t), t) \quad (5.1)$$

Introducing a nominal trajectory $x_s(t)$, and the deviations $\delta x(t) = x(t) - x_s(t)$, the linearized equations become

$$\delta \dot{x} = F(t) \delta x \quad (5.2)$$

where

$$F(t) = \left[\frac{\partial f}{\partial x} \right] + \left[\frac{\partial f}{\partial u} \right] \left[\frac{\partial u}{\partial x} \right] \quad (5.3)$$

and $F(t)$ is evaluated on the nominal trajectory. If $F(t)$ can be approximated by a constant matrix, $F(t) = F$, then it is well known (Reference 5) that the system (5.2) is stable if the eigenvalues of F all have negative real parts.

Another test which also applies to the time varying case, is as follows: Let the closed-loop state transition matrix associated with (5.2) be $U(t, s)$, where $U(t, t) = I$ and

$$\frac{d}{ds} U(t,s) = -U(t,s) F(s) \quad (5.4)$$

Then, for any two times t_1, t_2 ,

$$\delta x_2 = U(t_2, t_1) \delta x_1 \quad (5.5)$$

where $\delta x_i = \delta x(t_i)$. Following the approach of reference 6, let L_2 be an orthogonal transformation such that

$$L_2 [U(t_2, t_1) U^T(t_2, t_1)] L_2^T = D_{21} = \begin{bmatrix} \alpha_1^2(t_2, t_1) & 0 \\ 0 & \ddots \alpha_6^2(t_2, t_1) \end{bmatrix} \quad (5.6)$$

where the eigenvalues α_i^2 are guaranteed positive because $[U(t_2, t_1) U^T(t_2, t_1)]$ is positive definite. Let $U(t_2, t_1) = U_{21}$, and define the matrix

$$L_1 = D_{21}^{-1/2} L_2 U_{21} \quad (5.7)$$

It follows that L_1 is orthogonal, since

$$L_1 L_1^T = D_{21}^{-1/2} D_{21} D_{21}^{-1/2} = I \quad (5.8)$$

and, since $U(t_i, t_i) = I$, $L_1 = L_2$ if $t_1 = t_2$. Then

$$\delta y_i = L_i \delta x_i \quad (5.9)$$

so that

$$\begin{aligned} \delta y_2 &= L_2 U_{21} L_1^T L_1 \delta x_1 \\ &= L_2 U_{21} U_{21}^T L_2 D_{21}^{-1/2} L_1 \delta x_1 \\ &= D_{21}^{1/2} \delta y_1 \end{aligned} \quad (5.10)$$

Thus, each component of y_2 is of the form

$$\delta y_2^{(i)} = \alpha_i(t_2, t_1) \delta y_1^{(i)} \quad i = 1, 2, \dots, 6 \quad (5.11)$$

and the system is stable if $\alpha_i(t_2, t_1) < 1$ for all t_2, t_1 .

Since the trace of a matrix is invariant under an orthogonal rotation, we must have

$$(\alpha_1 \alpha_2 \dots \alpha_6) = |D_{21}^{1/2}| = |U_{21}| < 1 \quad (5.12)$$

But it can be shown that (reference 5, page 10)

$$|U_{21}| = \exp \int_{t_1}^{t_2} \text{trace } F(t) dt \quad (5.13)$$

Since $(t_2 - t_1)$ may be chosen arbitrarily small, we have the following:

Necessary condition for linear stability:

$$\text{trace } F(t) = \sum_{i=1}^6 \left(\frac{\partial f_i}{\partial x_i} \right) + \left(\frac{\partial u}{\partial x_i} \right) < 0$$

This condition insures that $|U_{21}|$ always diminishes for any time (t_1, t_2) .

Note that the stability condition is, in general, necessary but not sufficient, for consider the simple example

$$\begin{aligned} \dot{x}_1 &= 2 x_1 \\ \dot{x}_2 &= -3 x_2 \end{aligned}$$

Then

$$\text{trace } F(t) = 2 - 3 = -1 < 0$$

but x_1 goes to infinity with time.

5.3 A Statistical Interpretation of Stability

The necessary condition for stability can be obtained from a statistical point of view. Suppose that $P_2(\delta x_2)$ is the probability density function (p.d.f.) of δx at time t_2 .

Using the rule for transforming density functions, and employing 5.5, the p.d.f. of δx at time t_2 is obtained from the p.d.f. of δx at time t_1 by

$$\begin{aligned} P_2(\delta x_2) &= P_1(\delta x_1(\delta x_2)) \left| \frac{\partial x_1}{\partial x_2} \right| \\ &= P_1(U_{21}^{-1} \delta x_2) |U_{21}|^{-1} \end{aligned} \quad (5.14)$$

Thus the density P_2 will increase with time if $|U_{21}|$ diminishes with time. Since the mean value of δx_2 is zero (the mean value of x_2 is the nominal), then δx_2 tends to approach zero with probability 1.0. This is the statistical interpretation we seek, and the necessary condition for stability follows as before. Note, however, that $|U_{21}| \rightarrow 0$ does not guarantee that $\delta x \rightarrow 0$, as the simple example of Section 5.2 indicates. In this case, the probability density is spread out along the line $x = (x_1, 0)$.

5.4 The Necessary Condition for Non-Linear System Stability

The necessary condition for stability also applies to the non-linear case. If $x_2(x_1)$ is the solution of the non-linear differential equation (5.1), then

$$\begin{aligned} P_2(x_2) &= P_1(x_1(x_2)) \left| \frac{\partial x_1}{\partial x_2} \right| \\ &= P_1(x_1(x_2)) |U_{21}|^{-1} \end{aligned} \quad (5.15)$$

where the state transition matrix U_{21} is evaluated along the nominal trajectory corresponding to the initial condition x_1 . Again we note that the probability density increases with time if $|U_{21}|$ diminishes with time, and the state x tends toward the nominal trajectory values.

McReynolds obtained another derivation of this result in Reference 7. The Fokker-Planck equation describing the evaluation of the p.d.f. is

$$\frac{\partial P}{\partial t} + \sum \frac{\partial}{\partial x_i} [P f_i(x, u(x, t), t)] = 0 \quad (5.16)$$

The characteristics of this partial differential equation are solutions of equation (5.1). Along these characteristics we have

$$\dot{P} = -P \sum_i \left(\frac{\partial f_i}{\partial x_i} \right) \quad (5.17)$$

for which the solution is

$$P(t_2) = P(t_1) \exp \left[- \int_{t_1}^{t_2} \sum_i \left(\frac{\partial f_i}{\partial x_i} \right) + \left(\frac{\partial f_i}{\partial u} \right) \left(\frac{\partial u}{\partial x_i} \right) \right] \quad (5.18)$$

Imposing the condition that $P(t_2)$ must always increase for all times t_1, t_2 , we have the previously obtained result.

5.5 A Necessary Condition for Guidance Mode Stability

Suppose we write the equations of motion as

$$\dot{x} = \begin{pmatrix} \dot{v} \\ \dot{r} \end{pmatrix} = \begin{bmatrix} g + a(u) \\ v \end{bmatrix} = f[x, u(x, t), t] \quad (5.19)$$

where u is the steering angle which is supposed to be continuously dependent upon the state. Note that the first three components of f only depend upon the first three components of x (i.e., the velocity components) via the feedback law $u(x, t)$, and the last three components of f are independent of the last three components of x (i.e., the position components). Then

$$\sum_{i=1}^6 \left(\frac{\partial f_i}{\partial x_i} \right) = \sum_{i=1}^3 \left(\frac{\partial a}{\partial u_1} \right) \left(\frac{\partial u_1}{\partial v_i} \right) + \left(\frac{\partial a}{\partial u_2} \right) \left(\frac{\partial u_2}{\partial v_i} \right) \quad (5.20)$$

Let the linearized velocity feedback law be represented by

$$k_1^T(t) = \left(\frac{\partial u_1}{\partial v_1}, \frac{\partial u_1}{\partial v_2}, \frac{\partial u_1}{\partial v_3} \right) \quad (5.21)$$

$$k_2^T(t) = \left(\frac{\partial u_2}{\partial v_1}, \frac{\partial u_2}{\partial v_2}, \frac{\partial u_2}{\partial v_3} \right) \quad (5.22)$$

and define unit vectors perpendicular to the acceleration vector

$$e_2 = \begin{bmatrix} -\sin u_1 \cos u_2 \\ \cos u_1 \\ -\sin u_1 \sin u_2 \end{bmatrix} = \frac{\partial a}{\partial u_1} \frac{1}{|a|} \quad (5.23)$$

$$e_3 = \begin{bmatrix} -\sin u_2 \\ 0 \\ \cos u_2 \end{bmatrix} = \frac{\partial a}{\partial u_2} \frac{1}{|a|} \quad (5.24)$$

where the acceleration vector is given by (see figures 2-1 and 5-1)

$$a = |a| \begin{bmatrix} \cos u_1 \cos u_2 \\ \sin u_1 \\ \cos u_1 \sin u_2 \end{bmatrix} |a| e_3 \quad (5.25)$$

Then the necessary condition for stability is

$$\sum_{i=1}^3 \left(\frac{\partial a_i}{\partial u} \right) \left(\frac{\partial u}{\partial v_i} \right) = |a| (l_1 \cdot k_1 + \cos u_2 l_2 \cdot k_2) < 0 \quad (5.26)$$

Seeking a slightly stronger stability criterion which avoids the pitfall illustrated by the simple example of Section 5.2, we shall say that a guidance mode is stable if $(\ell_1 \cdot k_1) < 0$ and $(\ell_2 \cdot k_2) \cos u_2 < 0$.

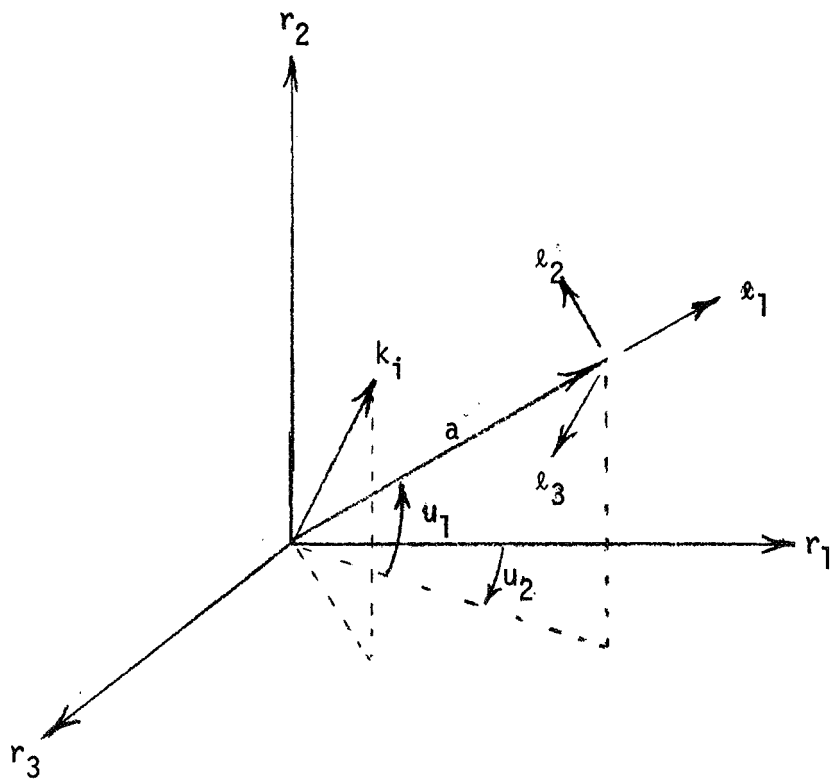


Figure 5.1: The Steering Vectors

5.6 Interpretation of the Stability Condition

Suppose that the rocket vehicle is traveling nearly in a plane and let r_1, r_2 axes be chosen to lie in this plane (nominal $u_2(t) \approx 0$). Let the feedback law be chosen so that $u_1(t)$ corrects only in plane errors, and $u_2(t)$ corrects only out-of-plane errors, that is

$$\frac{\partial u_1}{\partial v_3} = \frac{\partial u_2}{\partial v_1} = \frac{\partial u_2}{\partial v_2} = 0 \quad (5.27)$$

Choose a coordinate system at time t aligned with the acceleration vector, where the unit basis vectors are ℓ_1, ℓ_2, ℓ_3 , (as defined above - see Figure 5.1). In this system:

$$k_1 = \left(\frac{\partial u_1}{\partial v_1} \right) \ell_1 + \left(\frac{\partial u_1}{\partial v_2} \right) \ell_2 \quad (5.28)$$

$$k_2 = \left(\frac{\partial u_2}{\partial v_3} \right) \ell_3 \quad (5.29)$$

where the velocity v_1, v_2, v_3 is measured in the ℓ_1, ℓ_2, ℓ_3 system. The statement $(\ell_1 \cdot k_1) < 0$ implies that k_1 must lie below the ℓ_1 axis. That is, $\left(\frac{\partial u_1}{\partial v_1} \right)$ must be negative. The statement $(\ell_2 \cdot k_2) \cos u_2 < 0$ implies that k_2 be opposite to ℓ_3 if $\cos u_2$ is positive, which we assume by choosing the r_1, r_2 axis as the plane of motion. That is, $\left(\frac{\partial u_1}{\partial v_1} \right)$ must be negative. Let $\delta v(t)$ be the velocity deviation from nominal at the time t , so that the corresponding steering angle correction is

$$\delta u_1(t) = - \left| \frac{\partial u_1}{\partial v_2} \right| \delta v_2 + \left(\frac{\partial u_1}{\partial v_1} \right) \delta v_1 \quad (5.30)$$

$$\delta u_2(t) = - \left| \frac{\partial u_2}{\partial v_3} \right| \delta v_3 \quad (5.31)$$

The corresponding acceleration is

$$\delta a_i = |a| \delta u_i l_i \quad i = 2,3 \quad (5.32)$$

Thus the physical meaning of the stability condition can be stated in the following:

Stability Rule: Any given component of δv in the direction perpendicular to the acceleration vector must call for an oppositely directed steering correction from the guidance mode.

A guidance mode will be said to be stable if this condition applies. It is interesting to note that the stability rule places no restriction on the position feedback.

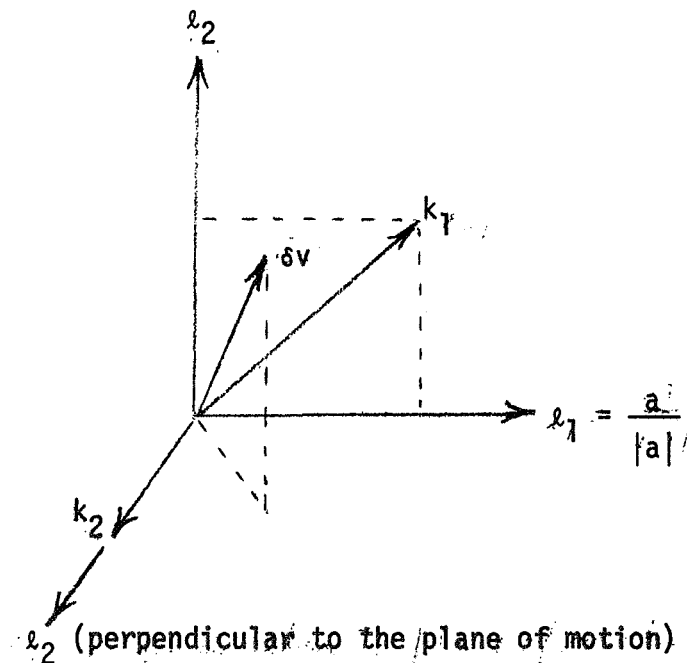


Figure 5.2: Coordinate System for Stability Analysis

5.7 Stability of the Second Variation Algorithm

We shall now show that the approximate form of the second variation algorithm described in Section 2.7 is indeed stable. Given a measurement of $\delta v(t)$ and $\delta r(t)$, let the first order controllable components of the end conditions be

$$dq_C = [M_v : M_r] = \left[\frac{\partial q_C}{\partial v}(T) : \frac{\partial q_C}{\partial r}(T) \right] [U(T, t)] \quad (5.33)$$

Considering only a velocity error, equation (3.35) yields

$$\delta u(t) = -|\lambda_v(t) a(t)|^{-1} \eta_C^T(t) [N_C^*]^{-1} M_v \delta v(t) \quad (5.34)$$

But, from the definition of η and ℓ_2

$$\eta_C(t) = M_v \ell_2 |a| \quad (5.35)$$

Let

$$\delta v(t) = (\delta v \cdot \ell_1) \ell_1 + (\delta v \cdot \ell_2) \ell_2 + (\delta v \cdot \ell_3) \ell_3 \quad (5.36)$$

Then the control component due to a velocity error in the ℓ_2 direction is

$$[\delta u(t)]_{\ell_2} = -|\lambda_v|^{-1} \left\{ \ell_2^T M_v^T [N_C^*]^{-1} M_v \ell_2 \right\} (\delta v \cdot \ell_1) \quad (5.37)$$

It can be seen that a positive ℓ_2 component of $\delta v(t)$ does indeed call for a negative value of $\delta u(t)$, since the kernel matrix is positive definite.

6. GUIDANCE NEAR THE FINAL TIME

6.1 Introduction

The behavior of a guidance mode near thrust termination determines the algorithm's ultimate accuracy, for the result of approximation errors occurring early in flight are corrected by final steering and shut-off. It is intuitively clear that an arbitrary number of end conditions cannot be all set equal to zero if errors occur near burnout, for pointing the thrust vector and varying the burning time by a small amount can only change the velocity components. This degradation of capability can be understood in terms of the concept of controllability, and the required guidance mode behavior at the final time can be understood in terms of the concept of velocity-to-be gained. These ideas will be discussed in this Section.

6.2 Controllability and Abnormality at the Final Time

Equation (2.39) states that the first order change of the end conditions attainable by applying steering and thrust termination are given by

$$dq = \int_{t_0}^T n(t) \delta u(t) dt + K dT \quad (6.1)$$

where dq is an n dimensional vector. A system is said to be first order controllable of order r if r linear combinations of end conditions can be set equal to arbitrary values. A necessary and sufficient condition that all end conditions be controllable is that the n by n matrix

$$N = \int_{t_0}^T n(t) n^T(t) dt \quad (6.2)$$

have full rank (all n eigenvalues non-zero). If so, one can choose an arbitrary ϵ and

$$\delta u(t) = \eta^T(t) N^{-1} \epsilon \quad (6.3)$$

yielding

$$\int_{t_0}^T [\eta(t) \eta^T(t)] N^{-1} \epsilon dt = \epsilon \quad (6.4)$$

But it has already been shown (Section 3.2) that a necessary condition for optimality is that N be singular. Thus, an optimal trajectory with n end conditions is controllable with respect to steering of order at most $n-1$. The additional degree of freedom afforded by the shut-off time variation dT will, in general, make the end conditions completely controllable.

It should be noted that the singular matrix N may have rank less than $n-1$ (more than one zero eigenvalue), in which case the end conditions would not be completely controllable. This corresponds to abnormality (see Reference 3), and is assumed to be not the case for $(T-t_0)$ sufficiently large.

Abnormality does tend to arise of the final time, however. Suppose that $(T-t_0)$ is small so that

$$\eta(t) \cong \eta(T) = \left[\frac{\partial q}{\partial V}(T) \right] \left[\frac{\partial a}{\partial u}(T) \right] \quad (6.5)$$

Then,

$$N = \left[\eta(T) \eta^T(T) \right] (T-t_0) \quad (6.6)$$

and the rank of N is equal to the rank of $n(T)$. If there are two steering angles (pitch and yaw), and the rank of $(\partial q / \partial v)$ is ≥ 2 , then the rank of N is 2. Thus near the final time the trajectory is controllable of order 2 with respect to the steering angle, that is, the trajectory is abnormal at the final time if the dimension of q is greater than $n=3$. Including the degree of freedom afforded by the shut-off time variation, the trajectory is controllable of order 3 at the final time. In other words, the only control available at the final time is the velocity impulse achieved by pointing the thrust vector and adjusting the burning time.

6.3 The Velocity-to-be-Gained

The velocity-to-be gained concept is useful for designing a guidance mode to operate near the final time. Suppose that the components of the gravitational acceleration can be represented as functions of time (not position), so that

$$v(T) = v(t_0) + \int_{t_0}^T g(t)dt + \int_{t_0}^T a(t)dt \quad (6.7)$$

Let the velocity to be gained (v_g) at t_0 be the velocity added by the thrust acceleration between t_0 and T . For an optimal trajectory, where the thrust acceleration is applied in the direction of $\lambda_v(t)$, we have

$$v_g = \int_{t_0}^T a(t)dt = \int_{t_0}^T |a(t)| \frac{\lambda_v(t)}{|\lambda_v(t)|} dt \quad (6.8)$$

If $(T-t_0)$ is small,

$$v_g = \frac{\lambda_v(T)}{|\lambda_v(T)|} |a(T)| (T-t_0) \quad (6.9)$$

Thus v_g can be thought of as a step change in velocity at the final time, applied in the direction of the primer vector $\lambda_v(T)$.

A v_g which satisfies the end conditions always exists if early steering has been properly applied, but only the three velocity components can be changed to correct perturbations occurring very near burnout. Recognizing this, suppose that only three end conditions need be corrected, q_1, q_2, q_3 . Let the required velocity, $v_r(r)$, be the velocity required to achieve the end conditions at a given position vector r . Such a velocity always exists, for one can solve

$$\delta q = \left(\frac{\partial q}{\partial v} \right) \delta v + \left(\frac{\partial q}{\partial r} \right) \delta r = 0 \quad (6.10)$$

to obtain

$$\delta v = - \left(\frac{\partial q}{\partial v} \right)^{-1} \left(\frac{\partial q}{\partial r} \right) \delta r \quad (6.11)$$

In general, a required velocity can be found if there are three (or less) end conditions to achieve and the matrix $(\partial q / \partial v)$ of (6.10) is invertable. Then, from (6.11) the required velocity at the final time is

$$v_r(r) = v_s(T) - \left(\frac{\partial q}{\partial v} \right)^{-1} \left(\frac{\partial q}{\partial r} \right) [r - r_s(T)] \quad (6.12)$$

where s denotes nominal values. The instantaneous velocity to be gained is thus

$$v_g = v_r(r) - v \quad (6.13)$$

As a practical matter one usually defines the desired end conditions in terms of conic elements, in which case the components of $v_r(r)$ can be calculated as explicit functions of r . For example, if q_1 = deficit in desired specific energy = $c_3 - (|v|^2 - 2\mu/r)$, where c_3 is a specified constant, then the required speed is

$$|v| = \sqrt{c_3 + \frac{2\mu}{r}}$$

Explicit expressions for the components of v_g for the cases of interplanetary and circular orbit injection are presented in Reference 8, pages 124-126.

6.4 The Cross Product Steering Law

The impulsive approximation of v_g is not adequate when there is a long time-to-go to burnout. Accordingly, a cross product steering law has been suggested (Reference 8), which has proved itself in applications. This law is near optimum in the following sense. From (6.8) note that

$$\dot{v}_g(t) = \frac{|a(t)|}{|\lambda_v(t)|} \lambda_v(t) \quad (6.14)$$

That is, the rate of change of v_g is in the direction of $\lambda_v(t)$. But from the previous discussion it can be seen that the primer vector $\lambda_v(t)$ is nearly parallel to the instantaneous $v_g(t)$ near the final time. Thus, v_g and $\dot{v}_g$ are nearly parallel, and near-optimum steering law can be defined by

$$v_g \times \dot{v}_g = 0 \quad (6.15)$$

It can be seen that this is indeed the optimum steering at the final time. Shut-off occurs when $|v_g| = 0$, i.e., when all components of v_g are simultaneously zero.

Following Reference 8, suppose $v_r(r)$ is explicitly known, so that

$$\dot{v}_g = \dot{v}_r - \dot{v} = \left(\frac{\partial v_r}{\partial r} \right) v - g - a \quad (6.16)$$

where a is the thrust acceleration. Let

$$\left(\frac{\partial v_r}{\partial r} \right) v - g = b \quad (6.17)$$

Then it is shown in reference 8 that

$$a = b + \left[\sqrt{|a|^2 - |b|^2 + (i_g \cdot b)^2} - (i_g \cdot b) \right] i_g \quad (6.18)$$

where

$$i_g = \frac{v_g}{|v_g|} \quad (6.19)$$

Since b is the rate of change of v_g with $|a| = 0$, then $b(T) = 0$ is a consequence of the fact that $v_g(t) = 0$ after burnout. Hence, a becomes parallel with v_g near the final time, as expected.

6.5 End Conditions for Guidance Modes

The previous discussion has indicated that at most three velocity components can be controlled near the final time, which means that at most three end conditions can be met. Then if the matrix $(\partial q / \partial v)$ is not singular, either the velocity-to-be gained or some equivalent technique can be employed near burnout to satisfy end conditions in a near optimum way. Since as many as six end conditions can be satisfied by steering over a long thrusting arc, it is clear that the guidance mode may have to make a transition in flight to the case where only three end conditions are to be achieved. If so, a transition criterion must be devised. For example,

it may be apriori desired to attain a circular satellite orbit at burnout with 100 n.mi. altitude, but at some time near burnout this goal would be abandoned in favor of a circular orbit at arbitrary altitude.

This consideration is not serious for most guidance work, for three end conditions are usually sufficient. For example, lunar or interplanetary injection guidance requires that three functions of position and velocity be nulled in order to pass through a given point in space at a fixed time (or, approximately, that the three components of the hyperbolic excess velocity be held fixed). This is not the case of rendezvous problems, however, where velocities as well as positions must be fixed, but these are usually accomplished using multiple phase guidance with coast periods between phases. In any case, attention must be given to the behavior of the guidance mode near burnout.

Since the velocity-to-be-gained cross product steering law has proved itself in applications, and because it can be interpreted as an optimal steering law when time-to-go to burnout is small, this guidance mode is recommended for final steering to thrust termination.

7. TREATMENT OF CONSTRAINTS

7.1 Introduction

Another important consideration in the development of a guidance mode is the analytical treatment of constraints on the mission, the trajectory, and the vehicle. Indeed, it is sometimes the case that these requirements affect the form of a guidance algorithm more than any other factor, for constraints are of prime importance and are difficult to treat analytically. Constraints are basically of two types: mission constraints and vehicle constraints. The mission constraints are those which exist independently of any vehicle, while vehicle constraints are those required by the particular launch vehicle.

Examples of mission constraints would be:

- Total mission time. The time from launch to mission completion may be fixed. This is particularly important in reliability analysis, where the probability of failure is a monotonically increasing function of time.
- Tracking. It may be that the trajectory will be required to pass over certain tracking stations at various times during the mission.
- Range Safety. It may be required that the trajectory not pass over certain parts of the earth because of the safety considerations. These constraints can be described in terms of certain impact probabilities, and translate themselves into such considerations as launch azimuth constraints and yaw steering constraints.
- Launch Window Duration. The length of time available to get to launch a vehicle may be restricted by many practical considerations.
- Launch Window Availability. Certain days or months may be restricted for numerous reasons.
- Solar Visibility. Some payloads may require solar power or heating or attitude reference at phases of the mission.

Examples of vehicle dependent constraints which may be specified are as follows:

- The vehicle attitude at various times (e.g., at staging or injection).
- The vehicle attitude rate at various times.
- The velocity correction capability remaining at burnout.
- Maximum dynamic pressure.
- The heating integral, which is the integral of dynamic pressure times velocity.
- Dynamic pressure at first stage separation.
- Angle of attack at first stage separation.
- First stage impact distance.
- Radar look angles, for the case of a vehicle which is radio guided.

There are other constraints of these types which might be specified. It is important to note that the constraints can be rather complex to describe analytically, and the requirement that none be violated at any time during a guidance phase poses a potentially severe restriction on the form of a guidance mode.

7.2 State and Control Variable Constraints

From the point of view of guidance theory, all constraints translate themselves into one of two types:

- State variable constraints

Consisting of certain functional specifications of the trajectory state.

For example, the specification that the dynamic pressure must never exceed a certain number is a state variable constraint, as is the specification that the heating integral shall be less than a certain number.

- Control variable constraints

The specification of vehicle control functions, such as the steering law.

For example, if the steering attitude at injection is specified to be a certain value, or the injection attitude rate is specified to be a certain value, these would be control variable constraints.

The control variable or the state variable constraints could be in equality or inequality constraints. For example, specification of a radar look angle as a precise function of time on the nominal trajectory would be a state variable equality constraint. Specification of a radar look angle as lying between two given bounds would be a state variable inequality constraint. Specification of final injection attitude as a fixed number would be a control variable equality constraint, while specification of the final injection attitude as lying within certain bounds would be a control variable inequality constraint. Control variable constraints can usually be treated without too much difficulty, particularly with a parameterized guidance law.

7.3 Treatment of State Variable Constraints

The analytical treatment of state variable constraints is generally difficult. The reader is referred to References 9 and 10 for a discussion of this problem from the point of view of optimal control theory. In general, the theory states that one may design a trajectory without regard to constraints if it occurs that the resulting trajectory satisfies all constraints. If this is not the case, a different trajectory must be constructed. For example, in the case of a state variable inequality constraint one must develop a trajectory broken into segments, where on some segments an ordinary unconstrained optimization problem is solved, and on other segments the trajectory moves along the boundary of the constraint inequality surface. The task then is to find the break points, or switching times, where the segments join (see Figure 7.1). With the exception of some special cases, the theory is difficult to implement, and empirical rules are often applied for the design of practical guidance modes. One must be aware of this troublesome problem when evaluating the performance of any given guidance mode.

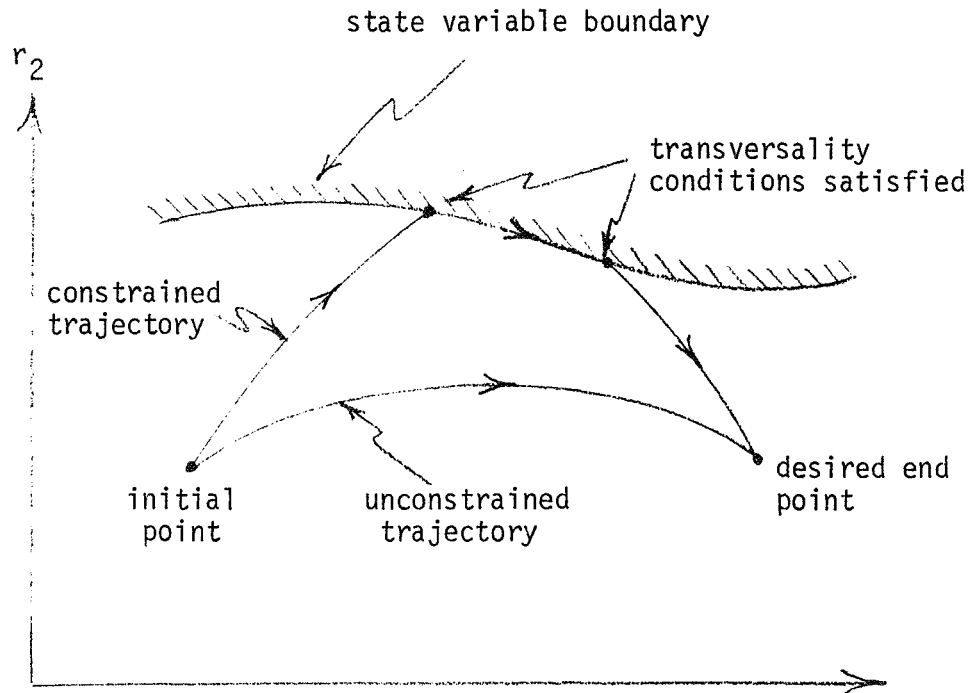


Figure 7.1: Effect of State Variable Constraints

The state variable constraint problem might be efficiently treated by a parameterized guidance law. Suppose that a steering law of specified form $u(p,t)$ is chosen, where p is a parameter vector chosen in real time to meet the end conditions. Substituting $u(p,t)$ into (2.3), the equations of motion take the form

$$\frac{dx}{dt} = f(p,t) \quad (7.1)$$

where $x^T = (v^T, r^T)$ is the state vector. Implicitly we then have $x = x(p,t)$. Suppose there is a state variable inequality constraint of the form

$$0 \leq \psi(x) = \psi(x(p,t)) \quad (7.2)$$

For any given p , the maximum value of ψ is attained when

$$0 = \frac{d\psi}{dt} = \left(\frac{\partial \psi}{\partial x} \right) \left(\frac{dx}{dt} \right) \quad (7.3)$$

Solving for the time (t_m) of maximum ψ from equation (7.3), we have an implicit constraint on p :

$$\psi_{\max} = \psi(x(p, t_m)) \geq 0 \quad (7.4)$$

Thus there is an acceptable region in parameter space, the boundary being determined by setting $q_{\max} = 0$ in equation (7.4). If one can find the intersection of this set of acceptable parameter values with other similarly defined sets corresponding to other state variable constraints, the selection of acceptable p vectors would be possible. If there are more values of p than end conditions, and p can be chosen as an interior point, degrees of freedom are available for optimization.

The constraints determine p , however, if it must be a boundary point in order to satisfy the end conditions. If there are just the right number of values of p to satisfy end conditions, the constraint surfaces provide a means for checking whether or not a given p vector is acceptable.

8. CLASSIFICATION OF GUIDANCE MODES

8.1 Introduction

In this Section we shall develop a method for classifying guidance modes according to the assumptions that were applied in their derivation and implementation. That is, does the mode attempt to solve the calculus of variations problem in real time, or is a supposedly near-optimal form of the steering law imposed? Is the steering law implemented by developing a Taylor series expansion about a nominal trajectory, or is real time iteration employed? If an expansion method is used, is it linear or non-linear? If real time iteration is used, is it based upon a precise or approximate model of the vehicle trajectory and partial derivatives? These questions are the basis of the guidance mode classification, which will be used in the sequel to provide a basis for guidance mode evaluation.

8.2 Optimal Guidance

A guidance mode will be called optimal if the thrust vector is pointed in the direction of the primer vector, $\lambda_v(t)$, where $\lambda_v(t)$ is obtained either by an exact integration of the Lagrange multiplier equations (2.9 and 2.10) or by an approximation of these equations. In general, the primer vector would be approximated, for acceptable performance can usually be obtained with approximations which greatly simplify the computation load.

One method for implementing optimal guidance is to use a Taylor series expansion. Conceptually, a nominal trajectory and a set of perturbed trajectories corresponding to perturbed initial conditions are calculated, along with the associated primer vectors. Then one could store

$$\lambda_v(t, x_i) = \lambda_{vs}(t) + \left[\frac{\partial \lambda_v}{\partial x_i}(t) \right] \delta x_i + \frac{1}{2} \left[\frac{\partial^2 \lambda_v}{\partial x_i^2}(t) \right] \delta x_i^2 + \dots \quad (8.1)$$

where the subscript s refers to the nominal value, and $\delta x_i^T = [\delta v^T(t_i), \delta r^T(t_i)]$ is the perturbed state at $t_i \leq t$, where a closed-loop correction is to be applied as a function of δx_i . More directly, the steering angle itself could be stored:

$$u(t, x_i) = u_s(t) + \left[\frac{\partial u}{\partial x_i}(t) \right] \delta x_i + \frac{1}{2} \left[\frac{\partial^2 u}{\partial x_i^2} \right] \delta x_i^2 + \dots \quad (8.2)$$

A stored expansion method will be called linear if the steering angle is stored as a linear function of the state, that is, if second and higher order terms in (8.2) are dropped. Otherwise, it will be called non-linear. In either case the coefficients must be redefined for each time t_i where a new measurement from the navigation system is introduced to calculate a closed loop connection. The time differences $(t_{i+1} - t_i)$ are known as the guidance cycle time. Typical cycle times are from 1.0 to 10 seconds, which means that many coefficients must be calculated and stored for a given nominal trajectory. Furthermore, many nominal trajectories must be developed if there is a wide range of mission opportunities.

In general, it can be said that the expansion approach has the advantage of being easily to implement in real time, since only a stored table look-up is required to calculate the $u(t)$ for any given $x(t_i)$. The approach has the disadvantage of requiring much storage and preflight preparation, and being rather inflexible in the sense that the expansion is valid only in some region of an a priori nominal trajectory.

The second variation method is useful for determining the linear terms. In with this method one analytically develops the expressions for $\left[\partial u / \partial x_i(t) \right]$

for arbitrary times t, t_f in terms of matrix integrals of first and second partial derivatives, calculated on the nominal trajectory. An approximate form of this method is described in Section 3.6; other forms will be described in Section 9.3 and 9.4. This approach could be extended to treat the higher order terms (which would probably be too unwieldy to be practical), or the higher order terms could be obtained by numerically differencing the first order terms. In any case, it is clear that specially tailored methods are required if the general non linear expansion method is to be made computationally feasible. Recent studies toward that end are reported in Reference 11.

A straightforward approach to optimal guidance would be to solve the optimization equations in real time. Since no closed form solution is available, an iteration technique will be required, and such guidance modes will be said to be based upon real time iteration. The mode will be said to be based upon a precise model if the exact equations of motion and Lagrange multiplier equations are solved. The mode will be said to be based upon an approximate model if either or both of these equations are replaced by an approximate form.

The precise model requires iteration in order to determine the initial conditions of the Lagrange multipliers which will satisfy the desired end conditions (state and transversality), that is, a two-point boundary value must be solved. One might require a steepest descent method (Reference 12) if no good initial guess of the trajectory properties is available, but otherwise a Newton-Raphson procedure would be used. For example, introducing the Lagrange multiplier v of equation (2.11), which implicitly determines the shut-off time via equation (2.13), we have the implicit functional relationship

$$q = q(v) \quad (8.3)$$

where q are the desired end conditions. Then given a reasonable guess of v , we have (approximately)

$$dq^i = \text{desired } q - q(i^{\text{th}} \text{ guess of } v)$$

$$= \left(\frac{\partial q}{\partial v} \right) \delta v^i \quad (8.4)$$

and the calculated i^{th} improvement of v is

$$\delta v^i = \left(\frac{\partial q}{\partial v} \right)^{-1} dq \quad (8.5)$$

This procedure can be iterated to convergence, where $(\partial q / \partial v)^i$ is calculated on the best-guess trajectory corresponding to the i^{th} guess of v . This matrix can be generated, for example, by numerical differencing of trajectories corresponding to different values of v . The v can be initially approximated to adequate accuracy for convergence if one has a reasonable guess of what the steering history will be. This specifies the trajectory and the state transition matrices, and v is inferred from the steering angles. Other forms of the Newton-Raphson method are described in References 13, 14, 15, and 16. These calculations can be rather formidable, but advances in the state of the art in on-board computers (see Section 9.7) indicate that this approach should not be ruled out.

In practice one would usually seek some approximation technique which will yield adequate performance. Several levels can be defined:

- (I) Approximation of the equations of motion - such as the uniform gravity model or the linearized gravity model of Section 4.
- (II) Approximations of the first variation - where the exact trajectory is calculated, but the state transition matrices are approximated as per the uniform or linearized gravity model, or some other approximate model.
- (III) Approximations of the second variation - where the exact trajectory and state transition matrices are calculated, but the second variation terms are approximated by a method such as is described in Section 3.4.

All of these methods can lead to significant simplification of the real time iterations. It should be noted that level I does not yield the true optimum steering, and also leads to an error in the desired end conditions, which must

be corrected by closed loop steering and shut-off at the final time based upon information obtained from the navigation system. Level II suffers no loss of accuracy, but does not yield the true optimum steering. Level III converges to the true optimum trajectory, but the rate of convergence is potentially not as rapid as if the true second variation terms were employed.

8.3 Parameterized Guidance

A guidance mode will be called parameterized if the steering law is explicitly determined by a functional form of the type $u(p,x,t)$, where x is the instantaneous state, and p is a parameter vector chosen in real time to satisfy the end conditions. There may be two types:

- parameterized state - where a functional form is apriori assigned to components of the state vector, and the steering angle is inferred from that form.
- parameterized steering angle - where a functional form is apriori assigned directly to the steering angles.

As an example of parameterized state, suppose the distance from the center of the earth (r) is defined to be

$$\frac{|\ddot{r}|}{|a|} = p_1 + p_2 t = \text{function_of } (u, x, t) \quad (8.6)$$

where p_1 and p_2 are constants and $|a|$ is the magnitude of the thrust acceleration. The steering angle u is then explicitly calculated as a function of p , x , and t , where p is adjusted to meet the end conditions. This is the form imposed by the TRW Hybrid guidance mode, which is extensively analyzed in the Interim Report (Reference 1) and discussed in Section 10.4. An example of parameterized steering is found in the Iterative Guidance Mode (IGM), where

$$u(t) = p_1 + p_2 t \quad (8.7)$$

These apparently different approaches are in fact intimately related, and will not be distinguished in the classification of guidance modes. For example, if a constant gravity vector (g) is assumed, the steering angle for the above described parameterized state is found from

$$\frac{|\ddot{r}| - |g|}{|a|} = \sin u = A + Bt - \frac{|g|}{|a|} \quad (8.8)$$

If rocket vehicle has constant flow rate, so that $|a|^{-1} = (\text{constant}) + (\text{constant})(t)$, it can be seen that this case corresponds to $\sin u = \text{linear function of time}$. More complicated forms follow from a consideration of the general case, but the essential feature of parameterized guidance is that the steering angle can be explicitly found as a function of x , p , and t .

As previously pointed out, the form of the steering law is chosen on the grounds of optimality. That is, numerical or approximate analytical results may indicate that the form (8.6) or (8.7) describes the behavior of the optimum steering law reasonably well for a given class of missions. Based on the discussion of Section 4, and the numerous numerical results of many investigators, it seems that the approach is widely applicable. The parameterized state method has the advantage of placing a direct control on the shape of the trajectory, but has the disadvantage of potentially causing awkward steering maneuvers, such as angle or angle rate discontinuities at staging. The parameterized steering angle method has the advantage of placing a direct control on the form of the steering history, but has the disadvantage of potentially allowing large deviations from a nominal path. Since any guidance mode must be tailored somewhat for specific missions, either disadvantage can be overcome. In particular, application of the stability rule (Section 5.6) should obviate the objection to parameterized steering.

In general, it can be said that parameterized steering has the potential advantage of being a simple, flexible, widely applicable law which requires a minimum of real-time computation and storage. It has the disadvantage of providing potentially poor optimality if an inappropriate functional form is chosen for a given mission.

Given a functional form $u(p, x(t), t)$, the problem remains to find the parameter vector p which will satisfy the end conditions. This can be done by a Newton-Raphson procedure. Substituting this expression into the equations of motion, we have the implicit functional relationship

$$q = q(p, T) \quad (8.9)$$

where q are the desired end conditions and T is the final time. If the number of parameters plus one is the same as the number of end conditions, then with the i^{th} guess of p we have

$$\begin{aligned} dq^i &= q(\text{desired}) - q(i^{th} \text{ guess of } p) \\ &= \left(\frac{\partial q}{\partial p} \right) \delta p^i \end{aligned} \quad (8.10)$$

where $(\partial q / \partial p)$ can be inverted. The calculated i^{th} improvement in p is given by

$$\delta p^i = \left(\frac{\partial q}{\partial p} \right)^{-1} dq^i \quad (8.11)$$

This procedure is iterated to convergence, where $(\partial q / \partial p)$ is calculated on the best-guess trajectory corresponding to the i^{th} guess of p . This matrix can be generated, for example, by numerically differencing trajectories corresponding to different values of p . An optimization problem can be solved if there are more parameters p_j than required to meet the end conditions. If the final time T is to be minimized, define the performance index

$$J = T + \sum_{i=1}^N \alpha_i q_i(p, T) \quad (8.12)$$

where the α_i are Lagrange multipliers. Then the optimality conditions are

$$\frac{\partial J}{\partial p} = 0 = \alpha^T \left(\frac{\partial q}{\partial p} \right) \quad (8.13)$$

$$\frac{\partial J}{\partial T} = 0 = 1 + \alpha^T \left(\frac{\partial q}{\partial T} \right) \quad (8.14)$$

Given an i^{th} guess α^i and p^i , resulting in $(\partial J / \partial p)^i$, $(\partial J / \partial T)^i$, and dq^i all non-zero, the desired α , p , and T can be found by a Newton-Raphson procedure:

$$0 = \begin{bmatrix} \frac{\partial J}{\partial p} \\ \frac{\partial J}{\partial T} \\ dq \end{bmatrix}^i + \begin{bmatrix} \frac{\partial}{\partial p} \left(\alpha^T \frac{\partial q}{\partial p} \right) & \frac{\partial}{\partial T} \left(\alpha^T \frac{\partial q}{\partial p} \right) & \left(\frac{\partial q}{\partial p} \right)^T \\ \vdots & \vdots & \vdots \\ \alpha^T \left(\frac{\partial^2 q}{\partial p \partial T} \right) & \alpha^T \left(\frac{\partial^2 q}{\partial T^2} \right) & \left(\frac{\partial q}{\partial T} \right)^T \\ \vdots & \vdots & \vdots \\ \left(\frac{\partial q}{\partial p} \right) & \left(\frac{\partial q}{\partial T} \right) & 0 \end{bmatrix}^i \begin{bmatrix} \delta p \\ dT \\ \delta \alpha \end{bmatrix}^i \quad (8.15)$$

$$\begin{bmatrix} \delta p \\ dT \\ \delta \alpha \end{bmatrix}^i = - \left[\text{above matrix} \right]^{-1} \begin{bmatrix} \frac{\partial J}{\partial p} \\ \frac{\partial J}{\partial T} \\ dq \end{bmatrix}^i \quad (8.16)$$

Equation (8.16) describes the i^{th} calculated correction to p , T , and α . This procedure was implemented in the analysis of the TRW hybrid guidance mode described in the Interim Report (reference 1), where the partial derivatives were obtained by differencing numerically integrated trajectories.

A parameterized guidance mode which uses the above described Newton-Raphson procedure, or some equivalent technique, will be called a real-time iteration scheme. It will be said to be based upon a precise model if the exact equations of motion are used to calculate the trajectory and partial derivatives, otherwise it will be said to be based upon an approximate model. As in the case of optimal guidance the approximations can be of three levels:

- (I) Approximation of the trajectory
- (II) Approximation of the first partial derivatives
- (III) Approximation of the second partial derivatives

As before, level I leads to a degradation of accuracy and optimality, level II leads to a degradation of optimality, and level III causes a reduced speed of convergence of the Newton Raphson iterations. It is to be noted that most proposed parameterized guidance modes call upon level I approximations, which are chosen so as to yield a "closed form" solution for p if T is assumed known. The thrusting time T is then obtained by iteration. Thus, these schemes have sometimes been called "explicit" guidance modes. This terminology has not been employed here, however, because it is neither possible nor particularly advantageous to eliminate some numerical iteration.

As in the case of optimal guidance, stored expansion techniques can be used for parameterized guidance. Thus one could calculate and store

$$p = p_s + \left(\frac{\partial p}{\partial x_i} \right) \delta x_i + \frac{1}{2} \left(\frac{\partial^2 p}{\partial x_i^2} \right) \delta x_i^2 + \dots \quad (8.17)$$

where the subscript s refers to the value on a nominal trajectory, the partial

derivatives are evaluated on the nominal trajectory, and δx_i is the measure error in the state at some time t_i . The value of p is used in $u(p, x, t)$ for $t \geq t_i$ to steer the vehicle, where enough values of t_i are chosen to yield adequate closed-loop performance. The expansion will be called linear if only the linear terms in equation (8.17) are retained, otherwise it will be called non-linear. The comments of section 8.2 on stored expansion methods also apply here.

8.4 Summary of Guidance Mode Classification

The proposed classification of guidance modes is summarized in Figure 8. The performance of various sub-classes will be qualitatively assessed in Section 9, and examples of the various subclasses will be described in Section 10.

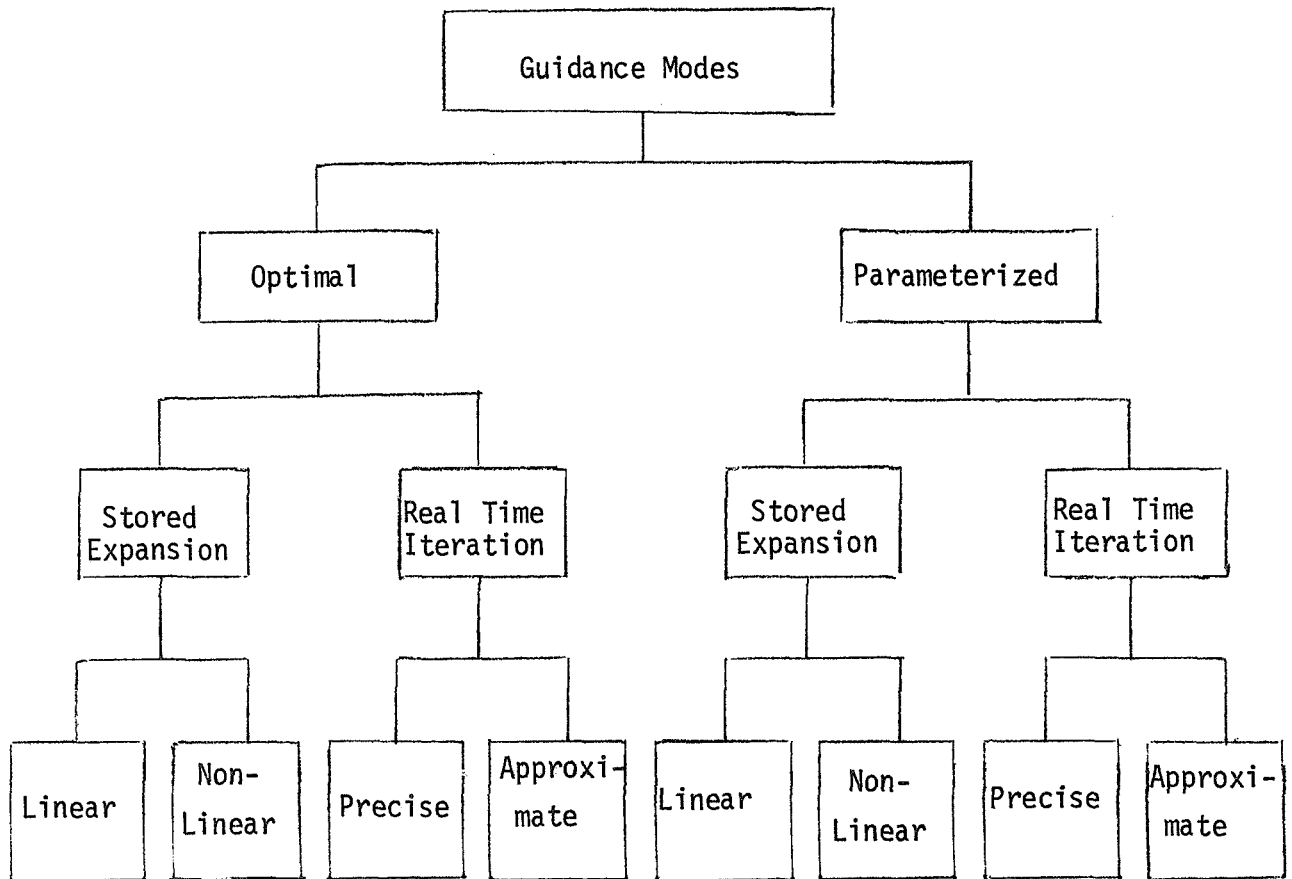


Figure 8.1: Classification of Guidance Modes

9. EVALUATION OF GUIDANCE MODE PERFORMANCE

9.1 Introduction

It is in general the case that guidance mode performance must be evaluated on a qualitative basis. It is true that many algorithms may be capable of satisfying a given mission objective with a given space vehicle. It is also true that all algorithms will require some special tailoring for every specific application. Thus, the mission planner is confronted with a choice of guidance modes which seem to have few features in common, and yet all can be made to work if enough competent engineering design is applied. The choice of a guidance mode depends upon factors which are hard to quantify, which depend upon the type mission to be performed, the state of the art in such areas as on-board computers, and individual assessment of the approach used to derive a given guidance mode.

Attempting to address these questions on a rational basis, we shall discuss the performance measures defined in Section 2.6, and then qualitatively evaluate each of the sub-classes described in Section 8 according to these measures. The author's judgement will intrude in this process; readers with differing views are invited to apply their own assessments.

9.2 Qualitative Evaluation of Performance

The measures of performance introduced in Section 2.6 are optimality, stability, accuracy, constraint compatibility, pre-flight preparation, verification requirements, flexibility, region of applicability, computer factors, and growth potential. Each of these will be discussed, and each guidance mode class or sub-class (as appropriate) will be rated as "poor", "adequate", or "good", according to the author's judgement of how well it may perform with respect to a given performance measure. Essentially, poor means that a significant drawback exists, adequate means that a potential drawback exists but is either marginally acceptable or probably can be overcome, and good means that no difficulties can be foreseen.

The evaluation is supposed to apply to rather general class of advanced missions. Since the performance of a specific guidance mode is highly mission dependent, the author has taken the liberty of assuming that a given mode sub-class will or will not have the capability to perform well if competent engineering design is applied. That is, advances in the state of the art in guidance mode capability must be postulated, and evaluation of a given approach must depend upon applying one's experience and judgement to assess the probability of being able to overcome inherent difficulties. To be fair, it is necessary to postulate that the best performing member of a given subclass is being evaluated. Thus, if any guidance mode in a subclass can be rated good, the sub-class will be rated good. A rating of poor will be assigned only if all members of the sub-class perform poorly.

Following the evaluation of individual performance measures, an overall assessment of each sub-class will be made by averaging each of the 10 performance measures for each subclass, with poor = -1, adequate = 0, good = +1. An average value of 0 or less will be rated poor, an average value of 0.1 to 0.5 will be rated adequate, and an average value of 0.6 to 1.0 will be rated good.

9.3 Optimality

The optimality criterion developed in Section 3.8 shows that the loss of optimality of a guidance mode can be expressed in terms of a certain integral. The numerical result depends upon the particular application, and no general conclusions can be drawn by examining the mathematical basis for the guidance mode. It can be said, however, that a mode which is based upon the optimization equations, or which has a near-optimum functional form of the steering law, can be said to have good optimality. Assuming that parameterized guidance modes will be designed with optimality in mind, and that they will be stable, it is fair to rate all guidance modes as having good optimality.

9.4 Stability

The stability rule developed in Section 5.6 states that an instantaneous velocity error must call for instantaneous corrective steering. This behavior tends to drive the vehicle toward a nominal trajectory and hence tends to cause small variations in the steering angle during the guidance phase of flight. Thus, a guidance mode which has this property shall be said to be stable, otherwise it will be considered unstable. Since it was shown in Section 5.7 that the second variation steering algorithm is stable, it can be said that a guidance mode which is based upon the optimization equations, or which has a near-optimum functional form of the steering law, can be said to have good stability. Thus all guidance modes will be assumed to have good stability.

9.5 Accuracy

The accuracy of a guidance system depends upon many factors, such as the mathematical approximations introduced in the guidance mode, the numerical errors in the computer, the errors in mechanizing the guidance commands, and the measurement errors in the navigation sub-system. From the point of view of guidance mode selection, however, only the approximation errors are of concern.

Since the ultimate accuracy is determined by the behavior near thrust termination, where errors incurred early in flight are corrected by the closed loop final correction (see Section 6), a fair comparison must assume that an accurate final steering and shut-off technique equivalent to the velocity to be gained method will always be employed near burnout. Thus, accuracy is determined by how well a given guidance mode will deliver the vehicle to the burnout region. If a stored expansion method is used, based upon a Taylor series expansion about a nominal trajectory, arbitrary accuracy can be achieved by carrying enough terms in the expansion. If real time iteration method is used, the accuracy is determined by the form of the approximations used to predict end conditions. In general, it can be said that modes employing real-time iteration tend to be more accurate because of the inherent ability to

treat large variations from the nominal trajectory, and stored expansion methods tend to be less accurate in this sense. Thus, all stored expansion methods will be rated adequate, real time iteration schemes using approximate models will be rated adequate, and real time iteration schemes using precise models will be rated good.

9.6 Constraint Compatibility

The treatment of constraints was discussed in Section 7. Because of the difficulty of handling state variable inequality constraints with the optimal control formulation, it was concluded that parameterized guidance modes are superior from the point of view of constraint compatibility. As a practical matter, the treatment of constraints is one of the most important guidance mode performance measures, since the violation of a constraint implies a severe degradation of mission reliability. Thus all optimal guidance modes will be rated poor, and all parameterized guidance modes will be rated good.

9.7 Preflight Preparation

Preflight preparation is required to generate preflight guidance parameters. For example, a mode employing stored coefficients based upon a Taylor series expansion about a nominal trajectory would require the generation of a number of nominal trajectories, and the associated partial derivatives, for a number of feasible launch opportunities. A mode employing a real time iteration technique might require the generation of constants to describe an equivalent approximate gravity vector for a set of nominal trajectories. In either case, numbers to describe the manifold of desired end conditions, as determined by the solution of the targeting problem, would be required. In general, it can be said that stored expansion methods require considerably more pre-flight preparation than real-time iteration schemes, and hence will be rated "poor". Real time iteration schemes will be rated "good", since there are usually designed to minimize pre-flight preparation.

9.8 Verification Requirements

Preflight guidance verification is required in order to check that the system will perform adequately in the presence of perturbations due to the vehicle, environment, and navigation system, and to insure that all pre-flight input has been properly generated and stored. Selected perturbed cases are chosen for these tests, the required number being more or less proportional to the complexity of the guidance mode. Optimal modes have the disadvantage of requiring starting values for the iterations to find the primer vector, and stored expansion methods have the disadvantage of a limited range of applicability. Both of these factors can only be evaluated by extensive simulation. Thus modes using stored expansions will be rated poor. Optimal modes using real time iteration will be rated adequate, assuming that reliable techniques for starting the iterations can be devised. Parameterized modes using real time iteration will be rated good.

9.9 Flexibility

A guidance mode is said to be flexible if it can be used for a wide range of missions, and a wide range of different trajectories within a mission type (e.g., variations of the launch window). Stored expansion schemes are inherently inflexible, but, since theoretically any mission can be treated if enough terms in the Taylor series are retained, non-linear expansions will be rated adequate. Linear expansion will be rated poor. All real time iteration schemes will be rated good, since they are designed for flexibility.

9.10 Region of Applicability

The region of applicability is the range of vehicle, environment, and navigation errors which can be corrected by the guidance mode. Linear stored expansion modes will be rated poor, since the large thrust acceleration dispersions can cause the vehicle to move beyond the linear range. Non-linear stored expansion modes will be rated adequate, since any perturbation can theoretically be treated by keeping enough terms in the Taylor series. Since

real time iteration schemes are inherently able to treat almost any perturbation, approximate schemes will be rated adequate and precise schemes will be rated good .

9.11 Computer Factors

Computer complexity can be measured in terms of the number of instructions per second which need to be executed in order to calculate a guidance command, and by the storage requirement. The importance of these factors depends upon the capability of the guidance computer. Reference 17 presents a survey of the state of the art in on-board computers, where the author indicates that a fictitious aerospace computer of the not-too-distant future might have an add time of 1μ sec, a multiply time of 8μ sec, a divide time of 10μ sec, and a memory of as many as 131,000 words. These numbers are comparable to those of a present day "third generation" computer, for example the CDC 6600 machine. Thus, it appears that many trajectory computations presently carried out on the ground can be performed on board, and almost any guidance mode might be feasible from the point of view of computer factors. The main considerations are therefore cost and reliability. Thus, parameterized modes using real time iteration will be rated as good , since they are designed to have low computation loads. Optimal modes using real time iteration with a precise model will be rated as poor because of the large number of computations required to find the exact primer vector. Optimal modes using real time iteration with an approximate model will be rated adequate, because the approximations are introduced to reduce computation. Stored expansion modes will be rated as poor because of the large (and costly) storage requirement.

9.12 Growth Potential

Growth potential reflects the applicability of a guidance mode to future missions. For example, what is the potential applicability to re-entry guidance of a satellite shuttle vehicle of a mode designed for interplanetary injection from earth parking orbit? Such a question is of course highly

dependent on the form of the mode. Since stored expansion schemes are theoretically capable of treating any mission, but may require significant mission dependent modifications, it is fair to rate them adequate . Since an optimum solution always exists for any physically realizable mission, optimal schemes using real time iteration will be rated good . Similarly, parameterized modes using real time iteration with a precise model will also be rated good . Parameterized modes using real time with an approximate model will be rated only adequate , because significant modifications of the approximations may be required.

9.13 Summary of Guidance Mode Evaluation

The factors assigned to each of the guidance mode sub-classes are summarized in Table 9.1, along with the overall rating described in Section 9.2. It can be seen that all stored expansion methods turn out to be poor, all parameterized modes using real time iteration turn out to be good, and all optimal modes using real time iteration turn out to be adequate. Thus the conclusion is that the real time iteration method should be emphasized for future guidance mode development, and the parameterized form is preferred.

Table 9.1 Summary of Guidance Mode Evaluation

			Optimality	Stability	Accuracy	Constraint Compatibility	Pre-flight Preparation	Verification Requirements	Flexibility	Region of Applicability	Computer Factors	Growth Potential	Average Value
Optimal Steering Law	Stored Expansion	Linear	G	G	A	P	P	P	P	P	P	A	P
		Non-Linear	G	G	A	P	P	P	A	A	P	A	P
	Real Time Iteration	Approximate Model	G	G	A	P	G	A	G	A	A	G	A
		Precise Model	G	G	G	P	G	A	G	G	P	G	A
Parameterized Steering Law	Stored Expansion	Linear	G	G	A	G	P	P	P	P	P	A	P
		Non-Linear	G	G	A	G	P	P	A	A	P	A	P
	Real Time Iteration	Approximate Model	G	G	A	G	G	G	G	A	G	A	G
		Precise Model	G	G	G	G	G	G	G	G	G	G	G

P = poor, A = adequate, G = good

10.0 A SURVEY OF GUIDANCE MODES

10.1 Introduction

In this section we shall summarize the characteristics of selected guidance modes which have been described in the literature, and classify each according to the method of Section 8. The examples chosen are supposed to be representative, but by no means constitute a complete summary of all published work or of all guidance modes which have been applied in practice. Indeed, it is the case that there is relatively little published work describing guidance schemes which have been successfully used. Perhaps this is because such modes are usually rather simple and special purpose as compared to the theoretical methods, and hence thought not to be of little general interest. In any case, examples are offered here which illustrate and supposedly typify the state of the art in guidance mode development.

10.2 Optimal Guidance with Real Time Iteration

These guidance modes are characterized by thrust vector steering which follows the primer vector $\lambda_v(t)$, where $\lambda_v(t)$ is obtained by real time iteration. The equations to be solved can be based upon either an approximate or precise mathematical model.

- a) A Steepest-Ascent Method for Solving Optimum Programming Problems by A.E. Bryson and W.F. Denham*

The steepest descent method can be used to develop the optimum steering program and shut-off time when a good apriori guess of trajectory characteristics is not available. As applied to the determination of optimal flight paths for rocket-propelled vehicles, the optimum programming problem can be stated as follows:

Determine the thrust direction control program, $u(t)$, in the interval $t_0 \leq t \leq T$, so as to maximize

$$P = P(x(T), T) \quad (10.1)$$

*Reference 18

subject to the constraints

$$q = q(x(T), T) = 0 \quad (10.2)$$

$$\dot{x}(t) = f(x(t), u(t), t) \quad (10.3)$$

where

$$t_0 \text{ and } x(t_0) \text{ given}$$

and T determined by $\Omega = \Omega(x(T), T) = 0$, where the $u(t)$ is a 2×1 matrix of control variable programs (pitch and yaw steering), $x(t)$ is an $n \times 1$ matrix of state variable programs (including position and velocity component functions), q is a $p \times 1$ matrix of terminal constraint functions, f is an $n \times 1$ matrix of known functions (the equations of motion) of $x(t)$, and $\Omega = 0$ is the stopping condition that determines cutoff time, T .

This technique starts with an assumed nominal control variable program $u^*(t)$ and uses it with the given initial conditions in the differential equations of motion to calculate, numerically, the state variable program $x^*(t)$ until $\Omega = 0$. In general, this nominal "path" will not satisfy the terminal conditions $q = 0$, or yield the maximum value of payload. Small perturbations about the nominal control variable program, $\delta u(t)$, are considered, where

$$\delta u(t) = u(t) - u^*(t) \quad (10.4)$$

These perturbations will cause perturbations in the state variable program, $\delta x(t)$, where

$$\delta x(t) = x(t) - x^*(t) \quad (10.5)$$

Substituting these relations into the equations of motion and including only first order terms in the perturbations, the resulting linear differential equation for δx is

$$(\delta \dot{x}) = \left(\frac{\partial f}{\partial x} \right)^* \cdot \delta x + \left(\frac{\partial f}{\partial u} \right)^* \cdot \delta u, \quad (10.6)$$

where the $()^*$ indicates evaluation along the nominal path. To determine the changes in P , q , and Ω (dP , dq , $d\Omega$) corresponding to δu , the linear differential equations adjoint to (10.6) are introduced.

For steepest-ascent, it is desired to find the $\delta u(t)$ program that maximizes performance increase over the nominal for a specified value of the integral

$$(dS)^2 = \int_{t_0}^T \delta u^T(t) \cdot W(t) \cdot \delta u(t), dt \quad (10.7)$$

given values of dq and $d\Omega = 0$, where $()^T$ indicates a matrix transpose. The values of dq are chosen to bring the nominal solution closer to the desired terminal constraints. Chose of dS , the "length" of the step in the control variable program, is made to insure that the perturbations $\delta u(t)$ will be small enough for the linearization introduced in (10.6) to be valid. $W(t)$ is an arbitrary positive semi-definite 2×2 symmetric matrix of weighting functions chosen to improve convergence of the steepest-ascent procedure. The $\delta u(t)$ program yielding a maximum value of dP is obtained by first writing dP as a function of $\delta x(t_0)$, dq , dS , and δu (by properly combining the equations adjoint to (10.6) with (10.7)) and then setting its first variation equal to zero, regarding $\delta x(t_0)$, dq , and dS as constants.

A new nominal path is obtained by using $u_{\text{new}} = u_{\text{old}} + \delta u$, and the above procedure is repeated with u_{new} . This iteration continues until the terminal constraints $q = 0$ are satisfied and the gradient, $\frac{dP}{dS}$, approaches zero.

This guidance mode would be classified as either approximate model or precise model, depending upon whether or not approximations are introduced in equation (10.6).

b) Real-Time Optimal Guidance by K.R. Brown and G.W. Johnson*

A Newton-Raphson procedure can be used to develop the optimum steering program and shut-off time if one has apriori trajectory information good enough to get the process started.

The equations of motion for a chemically propelled rocket vehicle in a central force field may be approximated, for guidance purposes, by

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3} + a(t) \cdot \frac{\mathbf{u}}{|\mathbf{u}|}, \quad (10.8)$$

where $\mathbf{r}$ and $\dot{\mathbf{r}}$ are the position and velocity vectors in an inertial Cartesian coordinate system with origin at the center of the attracting mass, μ is the gravitational constant, $a(t)$ is the time history of thrust acceleration magnitude, and $\frac{\mathbf{u}}{|\mathbf{u}|}$ is the unit vector in the thrust direction. The unnormalized function $u(t)$ is considered to be a control vector whose magnitude is arbitrary.

Fuel optimal trajectories for prescribed thrust acceleration magnitude $a(t)$ are defined by (10.8) together with a differential equation for optimal control,

$$\ddot{\mathbf{u}} = -\frac{\mu \mathbf{u}}{r^3} + \frac{3\mu (\mathbf{r}^T \mathbf{u})}{r^5} \mathbf{r} \quad (10.9)$$

A full set of boundary-value conditions are supplied by the six components of initial state,

$$\mathbf{r}(t_0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}_0 \text{ at } t_0 \quad (10.10)$$

*Reference 16

the k mission constraints on final state

$$q_i(r_T, \dot{r}_T) = 0, \quad i = 1, \dots, k \leq 6 \text{ at } T \quad (10.11)$$

and the 6-k transversality conditions on final state, control, and control rate,

$$\dot{u}_f^T \alpha_j(r_T, \dot{r}_T) - u_f^T \beta_j(r_T, \dot{r}_T) = 0, \quad j = k+1, \dots, 6 \quad (10.12)$$

where $\alpha_j(r_T, \dot{r}_T)$ and $\beta_j(r_T, \dot{r}_T)$ are any 6-k independent solutions of

$$\frac{\partial q_i(r, \dot{r})}{\partial r} \alpha_j + \frac{\partial q_i(r, \dot{r})}{\partial \dot{r}} \beta_j = 0, \quad i = 1, \dots, k. \quad (10.13)$$

Equations (10.8) through (10.13) state the optimal trajectory problem as a well-posed boundary-value problem in order differential equations. A numerical integration algorithm can be used for the solution of (10.8) and (10.9) as an initial-value problem, in order to reduce the problem of constructing optimal control histories to the problem of choosing the appropriate initial values for control and control rate, u_0 , $\dot{u}_0$, and the injection time, T , such that the required conditions on final state, control, and control rate, given by (10.11) and (10.12) are met. This is a finite dimensional search problem.

Let the unknown initial conditions u_0 and $\dot{u}_0$ and the unknown final time T be combined into a seven-vector y of independent variables, and let the six constrained functions of r_T , $\dot{r}_T$, u_T , and $\dot{u}_T$ together with the (arbitrarily constrained) magnitude of u_0 be considered as a seven-vector z of dependent variables. Then the boundary-value relationship may be represented by $z = f(y)$, and the boundary-value

problem is to find a value of y that is a vector zero of $z^* - f(y)$, where z^* is the desired value of z .

An iteration of Newton's method of zeroing $z^* - f(y)$ consists of solving the system

$$\left(\frac{\partial f(y_i)}{\partial y_i} \right) (y_{i+1} - y_i) = z^* - f(y_i) \quad (10.14)$$

for y_{i+1} , where $\left(\frac{\partial f(y_i)}{\partial y_i} \right)$ is the Jacobian matrix of partial derivatives of components of f with respect to components of y , evaluated at y_i .

At the beginning of each guidance cycle, u_0 and $\dot{u}_0$ together with r_0 and $\dot{r}_0$ (as determined by the navigation system) are used to initialize a numerical solution of (10.8) and (10.9) together with the associated adjoint differential equations defining the dynamics of the partial derivatives of $u(t)$, $\dot{u}(t)$, $r(t)$, and $\dot{r}(t)$ with respect to u_0 and $\dot{u}_0$. The solution is continued to T , at which point an added evaluation of (10.8) and (10.9) furnishes also the derivatives of r_f , $\dot{r}_f$, u_f and $\dot{u}_f$ with respect to T . Thus, r_T , $\dot{r}_T$, u_T , and $\dot{u}_T$ are computed simultaneously with the computation of their partials with respect to u_0 , $\dot{u}_0$, and t_f . Then f and its partials are computed, and (10.14) can be solved for y_{i+1} .

These computations were initiated, say, at t_i and have used up most of a guidance cycle. Therefore, the revised control policy $y_{i+1} = (u_0, \dot{u}_0, t_{f_{i+1}})$, which was based on the state at t_i , is already out of date. So the remaining time in the guidance cycle is used to extrapolate the new u_0 and $\dot{u}_0$ forward in time (according to (10.9)) to t_{i+1} where they become the new current control policy for the new guidance cycle. In practice even the linear extrapolation $u(t_{i+1}) = u(t_i) + \dot{u}(t_i) \Delta t$ is accurate over intervals much longer than anticipated guidance cycles.

The guidance mode would be classified as either approximate model or precise model, depending upon whether or not approximations are introduced in Equation (10.14).

c) Rocket Vehicle Guidance by Charles W. Sarture*

This paper describes guidance modes based upon the "velocity to be gained" cross product steering law, which, according to the discussion of Section 6.4, can be considered an optimal mode employing real time iteration. This concept is based on the fact that at each point of the powered-flight trajectory of a rocket vehicle, a "required velocity" vector

$$v_R = v_R(r;t) \quad (10.15)$$

may be defined as that velocity vector which, together with the current position vector, would yield an instantaneous free-flight trajectory satisfying the mission constraints. Both explicit and perturbation guidance equations can be used to compute the required velocity for a rocket in flight. For the case of a missile traveling in a vacuum about a spherical earth, with explicit expressions for the desired end conditions available, the components of v_R can be determined as explicit functions of the instantaneous position and velocity. Alternatively, v_R can be determined numerically from a pre-calculated and stored table, obtained as the solution of the targeting problem.

Once the vehicle is out of the atmosphere, the guidance steering system can begin to perform its primary function, that of reducing

$$v_G = v_R - v \quad (10.16)$$

*Reference 19. Other authors (e.g., Battin of Reference 8) have previously developed this concept.

to zero, where v_G is the so-called "velocity-to-be-gained", v_R , is the required velocity, and v is the actual velocity. One of the best known general steering methods for accomplishing this is "cross-product steering", where the vehicle is caused to have an attitude rate proportional to $v_G \times v_G^*$.

It is first shown that, for all current high-acceleration rocket vehicles, causing the thrust vector to be pointed along the v_G vector guarantees that v_G will be driven monotonically to zero. That is, the vector components of v_G are driven to zero as its magnitude approaches zero. Then letting i_u be the unit vector along v_G , it follows that

$$\omega = i_u \times \dot{i}_u = \frac{1}{v_G} (v_G \times \dot{v}_G), \text{ where} \quad (10.17)$$

ω is the required turning rate of the v_G vector. If the thrust vector is to be kept aligned with the v_G vector, the missile turning rate must be equal to ω . In practice, the commanded vehicle turning rate is of the form

$$\omega_c = K(v_G \times \dot{v}_G) \quad (10.18)$$

where K is a constant which may or may not be a function of v_G . The pitch and yaw commanded rates will be equal to

$$\omega_p = -K (v_{Gx} \dot{v}_{Gz} - v_{Gz} \dot{v}_{Gx}) \quad (10.19)$$

$$\omega_y = K (v_{Gx} \dot{v}_{Gy} - v_{Gy} \dot{v}_{Gx}) \quad (10.20)$$

*This method is based largely on work done by J.M. Bachar, F. Baskin, and D.W. Whitcombe at Space Technology Laboratories, Inc.

where x, y , and z represent the roll, pitch, and yaw components of a vector, respectively. The cross-product steering scheme causes the vehicle to fly a nearly constant attitude trajectory, which is desirable from a propellant expenditure standpoint. In a practical system where steering errors occur due to both static and dynamic effects, the thrust termination signal is usually given when $v_G \leq K_{co}$, where K_{co} is some allowable tolerance on cutoff velocity.

The guidance mode will be classified as either approximate model or precise model, depending upon whether or not approximations are introduced to calculate v_R .

10.3 Optimal Guidance with Stored Coefficients

These guidance modes are characterized by thrust vector steering which follows the primer vector $\lambda_v(t)$, where the steering angle as a function of the measured state is obtained via a Taylor series expansion about a nominal trajectory. The expansions can be either linear or non-linear in the state variables.

- a) Optimization and Control of Nonlinear Systems Using the Second Variation by J.V. Breakwell, J.L. Speyer, and A.E. Bryson*

As applied to thrust direction control for placing a vehicle in orbit with maximum payload and satisfying various mission constraints, this scheme is based on a linear perturbation from a nominal optimum trajectory and, as such, involves the second variation of the calculus of variations. Small changes in the initial and/or final conditions from nominal values are considered, which, it is assumed, require only small changes in the control program to preserve optimality. Using the second variation, this results in a linear feedback control scheme.

*Reference 20

The vector differential equation of motion describing the system is

$$\dot{x} = f(x, u, t), \quad (10.21)$$

where x is a column vector of the six state variables (components of position and velocity, f is a column vector of six known functions of the vehicle state, u is a column vector of the two control variables (pitch and yaw attitude angles), t is time, and $(\dot{})$ denotes a time derivative. A set of initial conditions, $x(t_0)$, is specified, and it is assumed that a set of nominal control variable programs $u^*(t)$ has been determined that maximizes the delivered payload, and yields the specified values of certain other terminal quantities $q(x(T), t)$, where q is a column vector of n known functions ($n \leq 6$), and T is the cutoff time.

Small deviations from this nominal optimum path which might be caused by disturbances, and/or small changes in the terminal values, dq , are considered. The deviations of the state variables may be written

$$\delta x(t) = x(t) - x^*(t), \quad (10.22)$$

where $()^*$ indicates values along the nominal path and unstarred quantities indicate values observed along the actual path. The task is to determine small deviations from the nominal optimum thrust direction control program

$$\delta u(t) = u(t) - u^*(t) \quad (10.23)$$

so that the revised terminal conditions are met and payload is still maximized when the initial values of the state variables differ from the nominal values by $\delta x(t_0)$.

Adjoining the constraints f and q to the performance index P ($\equiv$ delivered payload) by Lagrange multipliers v and $\lambda(t)$, it follows that the augmented index, J , may be written

$$J = (P + v^T q)_T - \int_{t_0}^T \lambda^T (\dot{x} - f(x, u, t)) dt \quad (10.24)$$

where $()^T$ indicates a matrix transpose. Variations of J up to second order due to a change in the control program $\delta u(t)$ and changes in the initial conditions $\delta x(t_0)$ and final conditions $\delta q(T)$ are considered. Interpreting the second order terms for δJ as a quadratic performance index to be extremized by a choice of $\delta u(t)$ for a neighboring optimum path, the variation of δJ , i.e., the second variation of J , $\delta^2 J$, due to a change in the perturbation control program $\delta^2 u$ for a fixed value of δx_{t_0} and δq_T is computed. An interesting feature of this scheme is that the weighting matrix in the quadratic performance index is determined as second partial derivatives of the variational Hamiltonian and the payload, evaluated on the nominal path. Making use of the necessary vanishing of $\delta^2 J$ for arbitrary $\delta^2 u$ for an extremum, it can be shown that a matrix $\Lambda(t_k, t)$ can be pre-calculated so that the control scheme

$$\delta u(t) = -\Lambda(t_k, t) \begin{pmatrix} \delta x(t_k) \\ dq_T(t_k) \end{pmatrix} \quad (10.25)$$

yields the desired "neighboring optimum" control program for $t_k \leq t \leq t_{k+1}$. Here t_k denotes the k^{th} sampling time when the error vector, δx , is determined.

The guidance mode will be classified as either approximate model or precise model, depending upon whether or not approximations are introduced in equation (10.21).

b) Guidance Theory and Extremal Fields by Henry J. Kelley*

This paper describes another form of the second variation method. Essentially, one represents a field of extremals, i.e., a family of optimal trajectories, in the neighborhood of a predetermined extremal serving as a "nominal" trajectory. It turns out that the theoretical development closely parallels that of Jacobi's accessory minimum problem for the second variation.

The equations of motion and the equations governing change in a vehicle's mass comprise a system of differential equations, which may be written in generalized first-order form as

$$\dot{x}_i = f_i(x_1, \dots, x_n, u_1, \dots, u_m, t), \quad i = 1, \dots, n \quad (10.26)$$

where the x_i are "state" variables and the u_j "control" variables.

The optimality criterion of flight performance is usually stated in terms of a minimum of some function $P = P(x_{1T}, \dots, x_{nT}, T)$ of the state variable final values and the final time. The minimization is subject to (10.26) and certain specified initial and terminal conditions on the x_i as subsidiary constraints. The function P with adjoined differential constraints

$$J = P(x_{1T}, \dots, x_{nT}, T) + \int_{t_0}^T \sum_i \lambda_i (-\dot{x}_i + f_i) dt \quad (10.27)$$

is extremized by the control functions $u_j(t)$ of the nominal trajectory, subject to the prescribed boundary conditions. A solution to this minimum problem is desired for slightly different initial conditions $x_{i_0} + \delta x_{i_0}$, $i = 1, \dots, n$, and it is desired to exploit prior

*Reference 21

knowledge of the optimized nominal trajectory to whatever extent possible. Any point along the nominal may be regarded as an initial point, and attention is turned to continuous generation of control increments δu_j as a function of the "current" errors δx_i .

An expansion of (10.27) about the nominal is considered

$$J = J_0 + J_1 + \frac{1}{2} J_2 + \dots, \quad (10.28)$$

in which J_0 corresponds to the nominal path, J_1 the first-order effect of initial value variations and control variations, J_2 the second-order effect, etc. The problem of minimizing J is abandoned in favor of minimizing the second-order approximation to J given by the first three terms of (10.28) subject to the following differential subsidiary conditions

$$\dot{\delta x}_i = \sum_j \left(\frac{\partial f_i}{\partial x_j} \right) \delta x_j + \sum_k \left(\frac{\partial f_i}{\partial y_k} \right) \delta u_k, \quad i = 1, \dots, n, \quad (10.29)$$

where in this linearized version of the system the partial derivatives of the f_i are evaluated along the nominal path and hence are functions of time only. The partial derivatives and Lagrange multiplier functions appearing in the development of (10.28) are those belonging to the nominal (optimal) trajectory.

Adjoining the constraints (10.29) by means of new multipliers, $\delta \lambda_i$, the following Euler-Lagrange equations are obtained:

$$\delta \lambda_i + \sum_j \delta \lambda_j \left(\frac{\partial f_j}{\partial x_i} \right) + \sum_j \frac{\partial^2 H}{\partial x_i \partial x_j} \delta x_j + \sum_k \frac{\partial^2 H}{\partial x_i \partial y_k} \delta u_k = 0 \quad (10.30)$$

where $H = \sum_i \lambda_i f_i$ is the generalized Hamiltonian function. The Weierstrass condition for optimal control variations, δu_k^* , is that the δu_k^* minimize a certain function, h , or

$$\min_{\delta u_k} h = h(\delta u_k^*) \quad (10.31)$$

where

$$\begin{aligned} h = & \frac{1}{2} \left(\sum_i \sum_j \frac{\partial^2 H}{\partial x_i \partial x_j} \delta x_i \delta x_j + 2 \sum_i \sum_k \frac{\partial^2 H}{\partial x_i \partial u_k} \delta x_i \delta u_k \right. \\ & \left. + \sum_k \sum_l \frac{\partial^2 H}{\partial u_k \partial u_l} \delta u_k \delta u_l \right) + \sum_k \frac{\partial H}{\partial u_k} \delta u_k + \sum_i \delta \lambda_i \cdot \\ & \left[\left(\sum_j \frac{\partial f_i}{\partial x_j} \delta x_j + \sum_k \frac{\partial f_i}{\partial u_k} \delta u_k \right) \right] \end{aligned} \quad (10.32)$$

If applied as a guidance mode, it would be classified as either approximate model or precise model, depending upon whether or not approximations are introduced in Equations (10.29), (10.30), and (10.32).

c) An Optimal Linear Feedback Guidance Scheme
by S.E. Dreyfus and J.R. Elliott*

Another development of the linear expansion method is obtained from the dynamic programming point of view. The optimal trajectory problem considered here has the following discrete-time statement:

Find that sequence of controls

$$\{u(t)\}, \quad t = t_0, t_0 + \Delta, t_0 + 2\Delta, \dots$$

such that the dynamical system governed by the difference equations

*Reference 22

$$x_i(t+\Delta) = x_i(t) + f_i(x_1(t), \dots, x_{n-1}(t), t, u(t)) \Delta \quad (10.33)$$

$$x_i(t_0) = x_{i_0}, \quad i = 1, \dots, n-1 \quad (10.34)$$

evolves in such a way that the function

$$P(x_1, (t), \dots, x_{n-1}(t), t)$$

is minimized at some unspecified time T determined by the satisfaction of the m terminal conditions

$$q_j(x_1(T), \dots, x_{n-1}(T), T) = 0, \quad j = 1, \dots, m < n.$$

That is, a control function $u(t)$ is sought that minimizes an objective function P at a time determined by the simultaneous satisfaction of m terminal conditions. For simplicity of notation, the following conventions are made:

$$x_n = t, \quad f_n = 1 \quad (10.35)$$

The problem is stated discretely since it must be solved digitally and used mechanically in a discrete way.

The "optimal return function", $S(x_1, \dots, x_n)$, is defined as the value of P at the terminal time determined by $q_j = 0$, where the vehicle starts with state $x_1, \dots, x_n$ and an optimal trajectory is followed. The optimal control, u^* , is defined by the equation

$$\sum_j \left(\frac{\partial S}{\partial x_j} \right)_{t+\Delta} \left(\frac{\partial f_j}{\partial u} \right)_t = 0 \quad (10.36)$$

The $\left(\frac{\partial S}{\partial x}\right)$ is the Lagrange multiplier $\lambda(t)$. The fact that the value of S is invariant on the optimal trajectory from any initial state, regardless of the particular point, is described by the equation

$$(S)_t = (S)_{t+\Delta} \quad (10.37)$$

By taking partial derivatives of Equation (10.36) with respect to the x_j and keeping in mind that the optimal control decision, u^* , may change due to perturbations in the state, the $\left(\frac{\partial u^*}{\partial x_j}\right)_t$ can be found in terms of the first and second partial derivatives of S evaluated on the nominal trajectory at time $t+\Delta$.

To determine a recursion relation for the $\left(\frac{\partial S}{\partial x_j}\right)$ on the nominal trajectory partial derivatives of Equation (10.37) are taken with respect to the x_j . This yields

$$\left(\frac{\partial S}{\partial x_j}\right)_t = \left(\frac{\partial S}{\partial x_j}\right)_{t+\Delta} + \sum_i \left(\frac{\partial S}{\partial x_i}\right)_{t+\Delta} \cdot \left(\frac{\partial f_i}{\partial x_j}\right)_t, \quad j = 1, \dots, n \quad (10.38)$$

Recursion relations for the second partial derivatives of S are similarly obtained by taking partial derivatives of Equation (10.38).

Given the first and second partial derivatives of S at time $t+\Delta$ along the nominal trajectory, the $\left(\frac{\partial u^*}{\partial x_j}\right)$ can be computed at time t . The recursion relations developed above then yield the first and second partial derivatives of S at time t , thus allowing the continuation of the backward recursion. Once $\left(\frac{\partial u^*}{\partial x_j}\right)$, the change in optimal control dictated by a change in state, is known as a function of time along a nominal (optimal) trajectory, the linear rule

$$\delta u^*(t) = \sum_{j=1}^n \frac{\partial u^*}{\partial x_j}(t) \delta x_j(t) \quad (10.39)$$

yields the change in u^* required to correct for deviations from nominal in the state variables, the δx_j . This is the linear feedback guidance scheme. The near terminal values of the first and second partial derivatives of S that allow the initiation of the recursion relations developed earlier can be obtained by considering appropriate partial derivatives of Taylor series expansions about time t shortly before T of the terminal constraints q_j and the objective function P .

This guidance mode would be classified as either approximate model or precise model, depending upon whether or not approximations are introduced in Equations (10.33) and (10.38).

d) Path Adaptive Mode for Guidance Space Flight Vehicles
by W.E. Miner, D.H. Schmieder, and N.J. Braud*

This approach addresses the most general form of the stored expansion method. Consider the two-dimensional vacuum flight of a rocket under constant thrust and mass flow, in an inverse square gravitational field. Assuming that the initial conditions are known and that certain mission criteria are to be achieved with, say, a minimum time of flight, the calculus of variations can be used to determine the steering function, u , which accomplishes this.

The task undertaken here is to derive this steering function for the vacuum flight of a single rocket stage, with thrust termination occurring at fuel depletion. In this empirical approach, the so-called "two point boundary value problem" described above is solved numerically

*Reference 23

for assumed nominal initial conditions. This procedure is then repeated for a large variety of values that lie within some uncertainty region in phase space about the nominal ignition conditions. Thus, a family of optimum (minimum flight time) trajectories originating near the nominal initial conditions and satisfying the mission criteria is constructed. Moreover, each point on each trajectory of the family can be considered to be another set of initial conditions and the corresponding two point boundary value problem solved for u . Then, from each trajectory of the family points are taken every Δt seconds, providing numerous values of the steering function

$$u = u(r_1, r_2, v_1, v_2, F/m, \dot{m}/m) \quad (10.40)$$

where F is the thrust magnitude, and m is the vehicle mass.

The curve-fitting procedure chosen to functionally represent u according to this tabulation is the method of least squares. The polynomial approximating function used in the least squares procedure may be written in the form

$$u = \sum_{a_{hkpqrs}} r_1^h r_2^k v_1^p v_2^q (F/m)^r (\dot{m}/m)^s \quad (10.41)$$

where the summation is taken over a certain set of non-negative integers h, k, p, q, r, s .

The choice of the set of non-negative integers over which the indices range in the summation determines the specific polynomial form being used. For any such choice, it is desired to find the "best" set of coefficients. "Best" in the least squares approach means that the differences between the polynomial and tabular values of u are minimized, in the sense that the sum of the squares of these differences over all tabulated points is a minimum, i.e.,

$$\Delta u = \sum_b (u_b - \sum_{hkpqrs} a_{hkpqrs} r_{1b}^h r_{2b}^k v_{1b}^p v_{2b}^q (F/m)_b^r (\dot{m}/m)_b^s)^2 \quad (10.42)$$

is a minimum. The index b used here ranges over all of the tabulated points, and u_b is the tabulated value of u at the point $[r_{1b}, r_{2b}, v_{1b}, v_{2b}, (F/m)_b, (\dot{m}/m)_b]$. It is assumed, of course, that the number of tabulated points exceeds the number of coefficients in the polynomial.

A necessary condition for Δu to be a minimum with respect to the a_{hkpqrs} is that

$$\frac{\partial(\Delta u)}{\partial a_{hkpqrs}} = 0 \quad (10.43)$$

for all a_{hkpqrs} . These linear equations are called the normal equations and may be solved simultaneously for the coefficients a_{hkpqrs} .

The ultimate criterion of goodness of this curve-fit must be a weighing of the number and complexity of terms in the polynomial against the degree of optimization and accuracy of meeting mission conditions achieved by the polynomial.

The guidance mode would be classified as either approximate model or precise model, depending upon whether or not approximations are introduced to calculate the trajectories defining the series (10.41).

10.4 Parameterized Guidance Modes Using Real Time Iteration

These guidance modes are characterized by a parameterized form of the thrust steering law, where the parameters are chosen to satisfy desired end conditions by a real time iteration method. The equations to be solved can be based upon either an approximate or precise mathematical model.

a) General Formulation of the Iterative Guidance Mode
by I.E. Smith*

This paper discusses the iterative guidance mode (IGM) and its application to the three-dimensional upper stage vacuum flight of a rocket vehicle. It is an inertial or closed-loop mode in that the only inputs required after liftoff are available from the onboard navigation system. That is, this iterative scheme computes steering commands as a function of the current state of the vehicle - position, velocity, longitudinal acceleration, and gravitational acceleration - and the desired cutoff conditions. The guidance commands are updated each computation cycle, using the updated state of the vehicle. The terminal conditions for the IGM are defined by five quantities: altitude, velocity magnitude, flight path angle, orbital inclination (relative to the equator), and the descending node of the orbit plane (relative to the launch meridian). The terminal velocity vector is constrained to lie in the plane defined by the inclination and descending node. The radius and path angle will also lie in that plane.

Assuming a flat earth with a constant gravitational field, no atmospheric drag, constant thrust and flow rate, the Euler-Lagrange equations produce the following optimal steering law for the attitude angle of the thrust vector

$$\tan u = \frac{C_1 + C_2 t}{C_3 + C_4 t} \quad (10.44)$$

*Reference 24

where u is the angle between the thrust direction and the terminal horizontal. If downrange distance is a free variable (i.e., unconstrained), then the transversality condition associated with that state component requires that C_4 be zero. This condition produces the linear tangent law of the form

$$\tan u = \left(\frac{C_1}{C_3} \right) + \left(\frac{C_2}{C_3} \right) t \equiv A + Bt. \quad (10.45)$$

The analytical equations which make up the IGM algorithm are an approximate closed-form solution of the equations of motion for a vehicle in vacuum flight, using a steering law of the form

$$u = A + Bt. \quad (10.46)$$

The linear optimality of this scheme comes from the use of this first order approximation of the linear tangent law described above.

The equations of motion are integrated by expressing the pitch and yaw steering laws as

$$u_p = \hat{u}_p - K_1 + K_2 t, \quad u_y = \hat{u}_y - K_3 + K_4 t. \quad (10.47)$$

The pitch and yaw control angles (u_p and u_y , respectively) are measured in the guidance coordinate system, a target-centered inertial system. Geometrically, $\hat{u}_p$ represents the attitude angle that would optimally satisfy only the velocity vector constraint at cutoff. The constant angle K_1 is added to enforce the altitude requirement. To avoid incompatibility between the two requirements, a pitch rate, K_2 , is introduced, thus completing the steering law form of (10.47). The parameters K_3 and K_4 play a similar role in the formulation of the yaw steering policy.

Utilizing (10.47), the two-point boundary - value problem is solved analytically as a function of instantaneous state of the vehicle, desired end conditions, vehicle characteristics, and time-to-go (t_g). The closed-form solution requires the postulation of a constant effective gravity vector for a given guidance cycle (the average of the current and terminal gravity vectors). Since the gravity force field is a well-behaved function and the guidance scheme is a null-seeking system, the change in gravity magnitude and direction can be thus approximated with reasonable accuracy.

Since the solutions for the guidance parameters include the time-to-go, an estimate of this variable must be computed. This is accomplished by determining velocity deficiency, ΔV , or the difference between the current desired velocity and the present sensed velocity. The time-to-go can then be computed from the rocket equation

$$t_g = \tau \left(1 - \exp \left(- \frac{\Delta V}{V_e} \right) \right), \quad (10.48)$$

where τ is the specific stage time and V_e is the effective engine exhaust velocity. Since the equation for ΔV is a function of t_g , an iterative scheme is required. The IGM avoids this iteration by making an initial estimate of time-to-go, t_g^1 , and adding a correction term, Δt_g , which is a function of the updated velocity-to-be-gained. Thus, the burn time is calculated as

$$t_g = t_g^1 + \Delta t_g$$

After the first time through the guidance scheme, the computed t_g minus the computer cycle time is used as the approximation for the next computation.

The current downrange transfer angle, ϕ , is required for establishing the guidance coordinate system. This angle is computed using the instantaneous target range and a characteristic range that would be obtained in time t_g by a linear acceleration horizontally directed above a flat earth. The accuracy of this computation is improved by including an approximation for gravity loss which also includes a mission dependent constant. This constant adjusts the effect of gravity loss on the range angle depending on the steepness of the trajectory and the length of flight time.

It can be seen that the iterative guidance mode is to be classified as approximate model, because of the constant gravity assumption.

b) A General, Explicit, Optimizing Guidance Law for Rocket-Propelled Spaceflight by George W. Cherry*

Assuming a single-stage rocket vehicle with constant propellant massflow and constant effective exhaust velocity, the equation for vehicle thrust acceleration can be written

$$|a| = V_e / (\tau - t) \quad (10.49)$$

where V_e is the effective exhaust velocity, τ is the ratio of initial vehicle mass to massflow, and t is the time from epoch, $t = 0$. It is desired to compute a thrust angle[†] regime, $u(t)$, which would result in the acquisition of pre-specified terminal state values with minimum fuel consumption, starting from an initial exoatmospheric vehicle state.

*Reference 25

[†]The thrust angle $u(t)$ is measured with respect to the local horizontal.

The guidance program presented in this paper for fixed-thrust rocket engines has two major segments of computations, or subroutines. The first, called "radial Coordinate Control", computes a thrust angle regime which would result in the satisfaction of input radius and radial rate constraints at a specified time of thrust termination, T . Since the terminal value of horizontal speed is a computationally convenient function of T or time-to-go ($T-t_0$), it is reasonable to select time-to-go so that the predicted horizontal speed is equal to the desired value. The subroutine called "Horizontal Speed Control" adjusts time-to-go so that the predicted and desired values are equal. The computation loop through these two blocks can be described as follows: time-to-go is guessed; $u(t)$ corresponding to this time-to-go is computed in the "radial Coordinate Control" subroutine; using $u(t)$ and time-to-go, the "Horizontal Speed Control" subroutine computes the resulting final horizontal speed; time-to-go is adjusted so that final horizontal speed is equal to the desired value.

In the "Radial Coordinate Control" segment, the following form of the radial acceleration is imposed:

$$r(t) = C_1 p_1(t) + C_2 p_2(t), \quad (10.50)$$

where $p_1(t)$ and $p_2(t)$ are linearly independent, pre-specified functions of time, and C_1 and C_2 are constants chosen to satisfy the boundary conditions. Let $f_{11}(t)$ be the integral over time (from t_0 to t) of $p_1(t)$, and $f_{12}(t)$ the integral of $p_2(t)$. Now let $f_{21}(t)$ be the integral from t_0 to t of $f_{11}(t)$, and $f_{22}(t)$ the integral of $f_{12}(t)$. Then the solution for C_1 and C_2 is expressed in matrix notation as

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} \dot{r}_D - \dot{r}_0 \\ r_D - r_0 - \dot{r}_0(T - t_0) \end{pmatrix} \quad (10.51)$$

where the E matrix, $E = (e_{ij})$, is the inverse of the matrix $F = (f_{ij}(T))$, subscript "o" denotes present values (of range and range rate), and the subscript "D" denotes desired terminal values. The solution thrust angle program is then obtained from the following equation which can be solved for $u(t)$

$$a \cdot \sin u + G_{\text{eff}} = C_1 P_1 + C_2 P_2, \quad (10.52)$$

where G_{eff} represents the combined effect of gravitational and centrifugal acceleration. Subject to the restrictions mentioned above, the choice of $P_1(t)$ and $p_2(t)$ is rather arbitrary. Their form is often selected as polynomial of low degree, with optimization (fuel consumption minimization) achieved by proper choice of polynomial coefficients.

As explained earlier, time-to-go (or equivalently T), needed for evaluation of the integrals $f_{ij}(T)$, is chosen so that the terminal horizontal speed is equal to the desired value. The following equations results

$$t_{\text{go}} = \tau_o \left\{ 1 - \exp \left[- \left(v_{\theta D} - v_{\theta o} + \Delta v_{\theta L} \right) / v_e \right] \right\} \quad (10.53)$$

where t_{go} is the time-to-go ($T - t_o$), v_{θ} is the horizontal component of velocity, and $\Delta v_{\theta L}$ is the horizontal velocity "loss" defined by

$$\Delta v_{\theta L} = \int_{t_o}^T \left\{ a - \left[(1 - \cos u) a_T + \frac{\dot{r}}{r} v_{\theta} \right] \right\} dt \quad (10.54)$$

The "Horizontal Speed Control" subroutine then proceeds as follows. Suppose that a first guess of the value of $\Delta v_{\theta L}$ is made. This estimate is used to compute the corresponding, but probably incorrect, time-to-go. Using this time-to-go, the "Radial Coordinate Control" subroutine

is called on to compute the thrust angle regime which would satisfy the radial boundary conditions by the predicted time of thrust termination. Because the thrust angle regime, remaining time of rocket burning, and thrust acceleration model are all available, it is possible to predict the terminal horizontal speed which would result. The computed deficiency in terminal horizontal speed is then added to the first guess of the loss in horizontal speed to obtain a better estimate of this loss. This iterative procedure is repeated until the computed deficiency in horizontal speed is sufficiently close to zero.

Only one major algorithm is needed to implement this procedure; namely, the block required for predicting the terminal horizontal speed which would result from a given T_{go} and $u(t)$. The prediction of $V_{\theta}(T)$ is accomplished by predicting the terminal value of the nearly linear function

$$H(t) = \exp \left[- \left(V_{\theta}(t) - V_{\theta 0} \right) / V_e \right] \quad (10.55)$$

The prediction of $H(T)$ is obtained by simply expanding $H(t)$ in a truncated Taylor series about $t = t_0$ and then evaluating this expansion at T .

This guidance model is classified as approximate model because of approximation introduced to calculate the horizontal motion (10.55).

c) Explicit Tangent-Steering Guidance for Multi-Stage Boosters
by Frank M. Perkins*

The explicit Tangent-steering Guidance Equations constitute a differential correction procedure using a fixed mathematical form for the pitch and yaw steering angles versus time. These equations are intended as an active guidance scheme initiated after pre-programmed booster steering through the atmosphere and terminated at final vehicle injection. Although there is no known exact algebraic expression for

*Reference 26

the optimum (minimum propellant consumption) pitch program, a simple expression suitable for guidance equations is to diminish the tangent of the angle between the thrust vector and the horizontal at burnout linearly with time. The derivation of the linear tangent program shows that it applies to yaw steering as well as to pitch. It has been shown that if the total integrated effect of gravitation in causing position and velocity changes during guided flight were path-invariant, this linear tangent steering policy would indeed be optimum.

The basic principle of operation may be described by the following equations:

$$R_R = R_B - R_O - R_g \quad (10.56)$$

$$\Delta R_T = R_R - R_T \quad (10.57)$$

$$\Delta(\text{G.P.}) = \left(\frac{\partial R_T}{\partial(\text{G.P.})} \right)^{-1} \Delta R_T \quad (10.58)$$

Equation (10.56) states that the changes in vehicle position and velocity that are required to be contributed by thrust over the remainder of powered flight, R_R , are obtained by subtracting from the mission-required burnout values, R_B , the present values, R_O , from sensed information and an estimate of the changes caused by gravitation, R_g . Rough estimates of position and velocity changes that will be caused by gravitation are obtained by assuming that the average magnitude and direction of gravitation are fixed fractions of the way between the known present and final values.

Position and velocity components contributed by thrust, R_T , and the matrix of partial derivatives of these components with respect to the guidance parameters,

$$\frac{\partial R_T}{\partial(\text{G.P.})} ,$$

are obtained from mathematically exact closed-form expressions* as functions of the five guidance parameters (initial and final thrust attitudes and powered flight time).

The incremental corrective changes needed in the position and velocity components contributed by thrust from now to the end of flight, ΔR_T , are calculated as in Equation (10.57). Incremental corrections to the guidance parameters in the pitch and yaw planes needed to remove the burnout position and velocity errors computed in the previous step, $\Delta(G.P.)$, are obtained as in Equation (10.58). These corrections are then added to the current guidance parameters which came from the last computation cycle. The corrected parameters thus obtained are then updated for the passage of time during the computing cycle and are then used as principal input to the next cycle. A new present position and velocity are obtained from sensed data and the entire procedure repeated again. When the time to burnout is found to be less than or equal to the computation cycle time, a thrust termination command is sent.

The matrix solution referred to may involve the inverse of a 5x5 matrix, using all possible partial derivatives of the five terminal position and velocity components with respect to the five guidance parameters. The only end component not specifically controlled is the downrange distance.

This guidance mode is classified as approximate mode because of the assumption of the path-invariance of gravity.

*These exact closed-form expression result from the parameterized steering law: $\tan u = A + Bt$.

d) Explicit Guidance Equations for Multistage Boost Trajectories
by Fred Teren*

Explicit guidance equations are developed for steering multi-stage launch vehicles to injection conditions that satisfy given mission requirements, e.g., lunar impact criteria. The calculus of variations is used to derive the required boundary conditions on the thrust direction at staging points. The pitch and yaw steering laws are based on an approximate closed-form solution of the equations of motion, solved "continuously" during flight. These laws closely approximate the form prescribed by the calculus of variations for maximum payload.

The guidance law used for the radial component of thrust follows from the imposed functional form

$$a \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = A + Bt + \frac{1}{|a|} \left(\frac{\mu}{|\mathbf{r}|^2} - \omega^2 |\mathbf{r}| \right) \quad (10.59)$$

where a is the thrust acceleration, and $\mathbf{r}$ is the position vector. It follows that

$$|\ddot{\mathbf{r}}| = |a| (A + Bt) \quad (10.60)$$

This form was first introduced by MacPherson, whose results were later expanded by Cherry. Determination of the guidance constants A and B for a given computation cycle is accomplished by specification of the present and desired terminal radius and radial velocity and by estimation of the time to cutoff. Assuming a given angular momentum, $h(T)$, at cutoff, and approximating the velocity required, Δv , to yield the necessary change in angular momentum, Δh , the cutoff time, T , is given by

$$T = \tau [1 - \exp(-\Delta v/v_e)]. \quad (10.61)$$

*Reference 27

to approximate the calculus of variations solution, corrections are provided for A and B which cause thrust direction and rate to be continuous across staging.

The appropriate yaw steering law is derived by using an approximate calculus of variations solution to minimize the payload loss in removing yaw velocity errors:

$$a \cdot h = K \left(\frac{\theta \cdot R_m}{\theta \cdot v} \right) \quad (10.62)$$

$$K = \frac{(h \cdot R_m)_0}{\int_0^t a_T \left(\frac{\theta \cdot R_m}{\theta \cdot v} \right)^2 dt} \quad (10.63)$$

where h is the angular momentum vector and θ is a unit vector perpendicular to the radius vector.

It is assumed that the booster stage is flown through the atmosphere with a pre-specified (open-loop) steering program due to aerodynamic heating and loads. Active, closed-loop guidance is initiated at sustainer ignition and terminated at the time corresponding to the specified injection energy, determined by a quadratic curve-fit of the last three calculated time and energy values prior to cutoff.

The following simplifying assumptions are used in the derivation of guidance laws:

- (1) Spherical Earth, with an inverse-square gravitational force field.*
- (2) No perturbing bodies.*
- (3) No atmospheric effects on upper stages (stages after the booster).
- (4) Constant thrust and specific impulse operation over each of the actively-guided powered flight phases.

This guidance mode is classified as approximate model because of the approximations introduced to calculate the horizontal motion (10.61).

e) Centaur Explicit Guidance Equation Study
by TRW Staff[†]

A prime objective of the Centaur Guidance Study was to evaluate the performance of several explicit guidance techniques using an Atlas-Centaur vehicle over a mission spectrum indicative of future planning for this vehicle. After detailed analysis of features of the guidance sets considered in this study, a "Hybrid" set of guidance equations was developed that incorporated some of the best features of each equation set with additional work by TRW. The functional form imposed is the same as described in (d) above.

A set of pitch steering equations for advanced Centaur guidance was selected. This scheme imposes the following form on the radial acceleration:

$$\ddot{r} = |a(t)| (B + Dt) \quad (10.64)$$

where the r-axis lies along the local vertical direction in the (rotating) guidance plane, (t) is the thrust acceleration of the

* A "Pseudotarget" concept is introduced to compensate for Earth Oblateness and other perturbing forces.

[†] Reference 28

rocket engines, and B and D are guidance parameters (updated each guidance cycle) characteristic of a given rocket stage. The guidance parameters are obtained by integrating the equations of motion forward in time, assuming a pitch steering law of the form (10.64). The calculus of variations shows that the direction cosine, C_{az} , between the thrust vector and the local vertical must be continuous throughout each stage for minimum fuel consumption. Moreover, it can be shown that $\dot{C}_{az}$ must be continuous in the same region. In practice, it is convenient to work with the discontinuities which are defined by

$$\Delta B_i = B_{i+1} - B_i - D_i(T_i + T_{ci}), \Delta D_i = D_{i+1} - D_i \quad (10.65)$$

where $t = 0$ is the present time (during i^{th} stage flight), T_i is the i^{th} stage cutoff time, and T_{ci} is the coast time between stages i and $i+1$. The continuity requirement on C_{az} is then used to determine ΔB_i , while ΔD_i is determined by the continuity requirement on $\dot{C}_{az}$.

In the yaw steering scheme, the current unit normal vector, j , is computed from the expression

$$j = \frac{r \times v}{|r \times v|} \quad (10.66)$$

where r is the vehicle position vector and v the velocity vector. This algorithm attempts to drive r , v , and r_T (an input target vector) into a common plane while minimizing the in-plane velocity loss. Assuming that the square of the direction cosine between the thrust vector and the local normal, C_{ay}^2 , is small compared to unity, the velocity loss due to yaw steering can be approximated by

$$\Delta v_L = \frac{1}{2} \int_0^{t_g} a C_{ay}^2 dt \quad (10.67)$$

where t_g is the time-to-go until final thrust cutoff. The problem of driving the out-of-plane component of the target vector, $r_{Ty} = r_T \cdot j$, to zero while minimizing ΔV_L is then solved by using the calculus of variations and the method of undetermined Lagrange multipliers.

As in other guidance schemes, an estimate of velocity-to-be-gained to injection is needed. This may be viewed as

$$\Delta V = v_R - v \quad (10.68)$$

where ΔV is the vector velocity-to-be-gained, v_R is the instantaneous velocity vector required to yield the predicted desired terminal conditions, and v is the current velocity vector from navigation. Simple conic formulas involving the target vector and predicted desired terminal conditions are employed to yield the required velocity vector, v_R .

An estimate of time-to-go to injection, t_g is also required for use in the steering laws. Essentially, the magnitude, ΔV , of the velocity-to-be-gained vector serves as the independent variable specifying t_g by the rocket equation

$$t_g = \tau \left[1 - \exp \left(- \frac{\Delta V}{V_e} \right) \right] \quad (10.64)$$

where V_e is the effective engine exhaust velocity and τ^* is the specific stage time. When time-to-go becomes less than a given threshold, t_g is counted down to zero and the cutoff discrete generated.

* $\tau = V_e/a(0)$, the ratio of the effective exhaust velocity to the initial thrust acceleration.

This guidance mode is classified as approximate model because of the approximation introduced to calculate the horizontal motion. As demonstrated in the Interim Report (Reference 6), however, the precise model can be implemented by real time iteration for this (or any other) parameterized steering law.

10.5 Parameterized Guidance Modes Using Stored Coefficients

These guidance modes are characterized by a parameterized form of the steering law, where the parameters are determined as a function of measured state by a Taylor series expansion about a nominal trajectory. The expansions can be either linear or non-linear in the state variables.

Few examples of this class of guidance modes are to be found in the literature, even though early guidance modes for military vehicles were all of this form. A good recent-day example is the guidance law applied to the Agena vehicle for Ranger and Mariner lunar and interplanetary missions. In this case, a horizon scanner supplied data to keep the vehicle attitude at a nominal value, and thrust was terminated when the integrated acceleration reached a pre-set value. The required constants were obtained from pre-flight trajectory simulations, with the provision to alter the thrust termination constant based upon first stage (Atlas) burnout conditions. This simple law worked demonstrably well.

10.6 A Synopsis of Parameterized Guidance Modes

Since the results of Section 9 indicated that parameterized guidance laws are of particular interest, Table 10.1 is presented in an attempt to clarify the relative differences between the individual guidance modes described above. In effect, special emphasis is placed on this class since current real-time operational constraints (and on-board computer limitations) dictate a relatively straightforward implementation. All of these modes are either being flown today or are at least under operational consideration.

Among the assumptions common to all of these guidance modes are vacuum rocket flight with constant thrust and specific impulse over each stage, real time iteration, and an approximate model of system dynamics. As previously noted, any of these modes could employ a precise model if real time integrations of the true equations of motion were carried out in the iterations.

Table 10.1: Synopsis of Parameterized Guidance Modes

	Perkins (Linear Tangent)	Teren	Smith (IGM)	TRW Hybrid	Cherry (E-Guidance) Also includes engine throttling capability
Steering Law					
Pitch	$\tan u_p = A + Bt$	$\ddot{r} = a (A + Bt)$	$u_p = \tilde{u}_p - K_1 + K_2 t$	$\ddot{r} = a (B + Dt)$	$r = C_1 P_1(t) + C_2 P_2(t)$
Yaw	$\tan u_y = C + Dt$	approximate COV optimum	$u_y = \tilde{u}_y - K_3 + K_4 t$	approximate COV optimum	$^* \ddot{z} = C_3 P_3(t) + C_4 P_4(t)$
Basic Steering Policy	Diminish tangents of the thrust angles (as measured with respect to the burn-out horizontal) linearly with time.	Starts specified boundary conditions with minimum payload loss in removing yaw (out-of-plane) velocity errors.	Steer to acquire pre-specified terminal state values with minimum propellant consumption	Drive r , v , and r_T (an input target vector) into a common plane while minimizing in-plane velocity loss.	Steer to acquire pre-specified terminal state values with minimum propellant consumption
Guidance Coordinate System	"Canted" pitch-yaw guidance plane	Coordinate system rotating with vehicle	Target centered inertial system	Coordinate system rotating with vehicle	Coordinate system rotating with vehicle
Guidance Parameters	Initial and final thrust attitude and powered flight time	Present and desired terminal radius and radial velocity and time to cutoff	K_1, K_2, K_3, K_4 and velocity-to-be-gained	Desired radius, radial velocity, velocity magnitude, time-to-go, and down-range transfer angle	Present and desired radius and radial rate and time-to-go
Guidance Parameter Update	$\Delta G.P. \cdot \left(\frac{\partial R_T}{\partial (G.P.)} \right)^{-1} \Delta R_T$	Estimation of current time to cutoff	Estimation of time-to-go and down-range transfer angle	Estimation of time-to-go	Estimation of time-to-go
Provision for Staging	Required discontinuities in guidance constants A, B, C, D computed	Required discontinuities in guidance constants A, B computed	No special corrections required for the K_i	Required discontinuities in guidance constants B, D computed	No provision for multiple stages
Gravitational Model for position and velocity losses over powered flight	$g_{av} = k_1 g_0 + k_2 g_T$ $k_1 + k_2 = 1$	$^* g = \frac{\mu r}{ r ^3}$	$g = \frac{1}{2} (g_0 + g_T)$	$g = - \frac{\mu r}{ r ^3}$	$g = \frac{\mu r}{ r ^3}$
Unconstrained terminal state variables	Downrange distance	Orientation of injection orbit plane	Downrange distance	Orientation of injection orbit plane	Downrange distance
Cutoff Criterion	Time to burnout $\leq$ computation cycle time	Energy-time polynomial	Time-to-go $\leq$ minor cycle navigation updates, cutoff on velocity-to-be-gained	When time-to-go $\leq$ present tolerance, then counted down to zero	Time-to-go $\leq$ present tolerance, cutoff on final horizontal speed

* A "pseudotarget" concept is introduced to compensate for Earth oblateness and other perturbing bodies.

* z is the component of acceleration normal to the plane of motion.

11. THE TARGETING PROBLEM

11.1 Introduction

The targeting problem was defined in Section 2.2 to be the task of specifying functions of position and velocity (and perhaps time) which, if satisfied at burnout of a rocket vehicle, will cause desired end conditions to be met. Guidance, on the other hand, was defined to be the task of choosing the steering and thrust termination parameters so as to achieve at burnout a state vector on (or acceptably close) to the manifold described by these functions. Thus, targeting is essentially the pre-flight task of generating a manifold of desired end conditions, which is closely related to the mission planning task, and guidance is the real time task of causing the vehicle to achieve the targeted end conditions.

The discussion thus far has treated only the guidance problem, assuming that the solution of the targeting problem is known. Furthermore, only a single "phase" of guidance has been considered, which would correspond to a single stage vehicle or multiple stages with fixed coast periods. The general case of multiple phases, with variable coast times between them, is assumed to be treated by solving a targeting problem for each phase, where the burnout conditions of each anticipates the guidance to be subsequently applied. Implicit in this approach is the assumption that it makes sense to segment the mission into guidance phases with independent, explicitly given objectives, for otherwise one is faced with the obnoxious requirement to numerically generate the desired end conditions by the equivalent of a dynamic programming technique. A discussion of guidance phases, with typical examples for a lunar mission, is presented in the Interim Report (Reference 1), and will not be repeated here.

In this Section the phasing concept will be assumed, and methods for generating the targeted end conditions will be discussed. The main emphasis will be upon explicit targeting based upon conic formulae.

11.2 Numerical Generation of the Targeting Manifold

Consider a single guidance phase (e.g., a single stage guided rocket) where certain desired mission objectives are expressed in terms of

$$q_i(r(T), v(T)) = 0 \quad i = 1, \dots, N \quad (11.1)$$

where T is the end of the phase (e.g., burnout of the last stage of the booster vehicle). For example, it may be desired to pass through a given point in space at some fixed time $t_F > T$, so that the q_i implicitly define the functional relationships between $r(T)$ and $v(T)$ which will achieve this. One could employ a Newton-Raphson procedure to numerically determine the required velocity $v_r(r, T)$ to satisfy Equation (11.1) and these values could be stored in the form of a Taylor series expansion about some nominal values. The resulting hypersurface $v_r(r, T)$ is said to be the targeting manifold in position and velocity space at time T .

To illustrate this approach, assume that there are three end conditions to be met (see discussion of Section 6.5) and that the matrix $(\partial q / \partial v)$ is invertible. If burnout occurs at or before the nominal time T we must have

$$q(T) = 0 = \left(\frac{\partial q}{\partial v} \right) \delta v(T) + \left(\frac{\partial q}{\partial r} \right) \delta r(T) + \dots \quad (11.2)$$

where $\delta() = () -$ value on the nominal trajectory, and the partial derivatives are calculated on the nominal trajectory at time T . Considering only the linear terms, the required velocity to satisfy Equation (11.2), expressed as a function of the position r at time T , is

$$v_r(r) = v_s(T) - Q [r(T) - r_s(T)] \quad (11.3)$$

where the subscript s denotes a value on the nominal trajectory, and

$$Q = \left(\frac{\partial q}{\partial v} \right)_s^{-1} \left(\frac{\partial q}{\partial r} \right)_s \quad (11.4)$$

The required velocity for burnout at a time $t = T - dT$ can be calculated from

$$v(T) \cong v(t) + g dT \quad (11.5)$$

$$r(T) \cong r(t) + v(t)dT \quad (11.6)$$

where g is the acceleration due to gravity. Then

$$v_r(r, T) = v_r(r, t) + g dT = v_s(T) - Q \left[r(t) + v_r(r, t)dT - r_s(T) \right] \quad (11.7)$$

and

$$v_r(r, t) = \left[I + Q dT \right]^{-1} \left\{ v_s(T) - g dT - Q [r(t) - r_s(T)] \right\} \quad (11.8)$$

But for dT small we have

$$(I + Q dT)^{-1} \cong I - Q dT \quad (11.9)$$

so that

$$v_r(r, t) = v_s(t) - Q [r(t) - r_s(t)] \quad (11.10)$$

where we have substituted the following expression for the nominal velocity on the coasting orbit:

$$v_s(t) = v_s(T) - g dT \quad (11.11)$$

It can be shown that Equation (11.11) also applies for burnout after T . Thus the linear theory indicates that near the final time the mission objectives can be expressed in terms of a required velocity, which depends upon position and time. The targeting manifold then consists of the hypersurface $v = v_r(r, t)$ in position and velocity space at the time t . This manifold can be generated numerically by calculating a nominal trajectory to define $v_s(t)$ and $r_s(t)$, and determining the matrix Q at the nominal burnout time.

The case of targeting multiple guidance phases (e.g., more than one guided rocket stage with variable coast periods between) is more complicated, because the desired burnout conditions of one phase must anticipate the guidance of subsequent phases. In general, the determination of optimum steering and shut-off parameters for a given phase can only be accomplished by also simultaneously optimizing the parameters for all subsequent phases, where the end conditions are defined at final thrust termination. This is potentially a very formidable problem. From the targeting point of view, one would approach it from the dynamic programming point of view. Let $x^{(2)}$ be the state vector at the end of the second phase, and $x^{(1)}$ be the state vector at the end of the first phase. Let $p^{(i)}$ be guidance parameters describing the start/stop times and steering history for the i^{th} phase, where it is assumed that there are enough parameters to set $x^{(1)}$ and $x^{(2)}$ equal to arbitrary values (complete controllability). Let $J(p^{(2)}, x^{(1)})$ be a performance index describing the performance achieved after completion of the second phase for a given $x^{(1)}$, assuming the end conditions are met. The $p^{(2)}$ are chosen to obtain

$$J(\text{OPT}) = \underset{\{p^{(2)}\}}{\text{minimum}} \left[J(p^{(2)}, x^{(1)}) \right] \quad (11.12)$$

It follows that there is an implicit relationship between $x^{(1)}$ and the optimum $p^{(2)}$, and one can write

$$J(\text{OPT}) = \phi(x^{(1)}) \quad (11.13)$$

where ϕ is some function. Thus the performance index for the first phase is implied by Equation (11.13), that is, $p^{(1)}$ is to be chosen to achieve

$$q_i(x^{(1)}) = \left[\frac{\partial \phi}{\partial x_i}(x^{(1)}) \right] = 0 \quad i = 1, 2, \dots, 6 \quad (11.14)$$

These are the targeting conditions for the first phase. These functions could conceivably be numerically generated in the neighborhood of a nominal trajectory, but such a scheme would be difficult to implement in practice.

11.3 Explicit Targeting

A much simpler and more versatile approach to single and multi-phase targeting follows from defining guidance phases such that each has an independent objective which can be explicitly described in terms of osculating conic elements at burnout. For example, phase 1 might have the objective of lifting the rocket vehicle through the atmosphere to achieve a coast ellipse at burnout with maximum energy for a specified apogee distance; the second phase might have the objective of attaining a circular parking orbit of given altitude and inclination, and the third phase might have the objective of attaining a coast hyperbola with specified asymptotic conditions. Then $v_r(r)$ can be explicitly calculated as a non-linear function of r (where the time t does not appear). In this case the manifold is said to be obtained by an explicit targeting technique. The targeting problem reduces to analytically specifying combinations of orbital elements which will satisfy the mission objectives and numerically determining their nominal values.

Development of explicit expressions to describe the targeting manifold depend upon employing constants of the motion obtained from the two body approximation. Suppose that there is a primary conic about some central body which characterizes the mission. For example, this would be an earth centered ellipse for near-earth missions, or an earth centered ellipse to a massless moon (or hyperbola for flight times less than 50 hours) for a lunar transfer mission, or an ellipse about the sun, with massless earth and target planet, for an interplanetary mission. At final burnout the rocket vehicle is on an osculating conic, the elements of which can be directly related to those of the primary conic. For example, the osculating conic is the primary conic for lunar or near earth missions, or is a hyperbola for interplanetary flight. In the latter case the near-earth asymptotic conditions serve as initial conditions for the primary ellipse about the sun, and the end conditions of the primary ellipse serve as asymptotic conditions for the target-centered hyperbola. Suppose that the sequence of two body approximations obtained by somehow patching conics (earth ellipse-moon hyperbola, or earth hyperbola-sun ellipse-target planet hyperbola) is an adequate targeting approximation, in the sense that the conic equations can be used to explicitly relate variations in desired target conditions to variations in osculating orbital elements at burnout with accuracy acceptable for guidance purposes (small error compared to other guidance system error sources). Then one can calculate and store nominal values of appropriate combinations of orbital elements at burnout which will achieve the mission objectives, and the guidance task is to set the variations of these quantities equal to zero. This approach can be thought of as based upon a crude Enke integration of the post-burnout equations of motion. Let the state of the primary conic be described by six conic elements, that is, transform the position and velocity coordinates (r,v) to conic element coordinates $c_1, \dots, c_6$ which are constants of the motion for the two-body problem. Because of perturbations to the two body motion, arising from such things as earth oblateness and "n-body" effects, at some future time we have

$$c(t_F) = c(T) + \int_T^{t_F} \dot{c}(t) dt \quad (11.15)$$

where $\dot{c}(t)$ represents the effect of the perturbations. Given nominal values of $c_1 \dots c_6$ which satisfy mission objectives, the explicit targeting method assumes that the perturbation terms are invariant for all trajectories close to the nominal.

Examples of explicit targeting equations for relating burnout conditions to mission objectives will be presented in Section 11.4. Techniques for patching conics in the presence of non-two body perturbations will be discussed in Section 11.5 - 11.7, and the asymptotic matching of conics will be described in Section 11.8. It will be seen that the asymptotic matching method provides the analytic justification for explicit targeting when more than one conic arc must be treated.

Targeting models for explicitly calculating desired end conditions for deep space missions can be ordered according to increasing levels of precision and computational complexity:

- a. flat earth, vacuum - where gravitational acceleration is represented by a constant vector, and the trajectory after guidance termination is a parabola.
- b. simple conic - where the trajectory after guidance termination is described by the conic formulae with no corrections applied.
- c. simple patched conic - where the trajectory after guidance termination is described by a sequence of conic segments patched at the spheres of gravitational influence of the bodies which perturb the spacecraft.
- d. asymptotically patched conic - where the trajectory after guidance termination is described by a sequence of conic segments as in (c) above, but the patching is accomplished by matching the "asymptotic" values of position and velocity of the individual (hyperbolic) segments.

- e. hybrid patched conic - where a patching technique such as described in (c) or (d) is employed, and corrections to this model are numerically determined from a curve fit of the perturbing acceleration.
- f. asymptotically matched conic - where the corrections to (d) above due to the primary central bodies (e.g., earth-sun-target planet, or earth-moon) are analytically determined, and other perturbation effects are calculated numerically.

There are, of course, other types of approximations which could be employed. For example, an approximate explicit description of motion of a high speed vehicle in a dense atmosphere might be obtained by assuming that the gravitational acceleration is negligible compared to the drag acceleration. In this Report, however, we shall be primarily concerned with the conic approximations. The approximation methods (c) - (f) will be discussed in Sections (11.5) - (11.8).

11.4 Examples of Explicit Targets

Assuming that conic relationships are adequate for describing the post-injection trajectory, one must be able to express the mission objectives in terms of the orbital elements. The conic formulae are then used to define explicit relationships between final speed, velocity path angle, altitude, and downrange angle which must be satisfied in order to meet mission requirements, where the requirements are specified in terms of certain conic parameters such as energy, angular momentum, orientation of the line of apsides, or orientation of the hyperbolic asymptote. In this section, we will give examples of conic formulae which can be used as approximations for explicitly relating end conditions at termination of the injection guidance phase (end of booster guidance) to desired conditions at mission completion. For this discussion it will be assumed that the asymptotically patched conic method (d above) yields adequate accuracy, recognizing that corrections to the conic elements can be applied if necessary by the methods outlined in (e) and (f) above.

The examples given here are for guidance in the plane of motion only, and are primarily applicable to injection guidance, where future guidance corrections are nominally zero. All equations follow from the conic relationship between radial distance (r) and true anomaly (θ), given by

$$r = \frac{p}{1 + e \cos \theta} \quad (11.16)$$

where

$$\begin{aligned} p &= \text{semi-latus rectum} = c_1^2 / \mu = \text{constant} \\ \mu &= \text{gravitational constant} \\ c_1 &= \text{specific angular momentum} = r v \cos \gamma = \text{constant} \\ v &= \text{speed} \\ \gamma &= \text{flight path angle} \\ e &= \text{eccentricity} = \left[1 + \frac{c_1^2 c_3}{\mu^2} \right]^{1/2} = \text{constant} \\ c_3 &= 2 \times (\text{specific energy}) = \text{vis} - \text{viva} = v^2 - \frac{2\mu}{r} = \text{constant} \end{aligned}$$

a. Energy-Angular Momentum Guidance - This form of end conditions specification arises when controlling the apogee ($\theta = \pi$) and perigee distances of a near earth ellipse. Via Equation (11.16), these quantities are completely specified by a , p and e , and hence specific energy, (c_3), and specific angular momentum, (c_1). Let r_I be the injection altitude. Then the required speed (v_r) and velocity path angle (γ_r) at injection are found from

$$c_3 = \text{specified} = v_r^2 - \frac{2\mu}{r_I} \quad (11.17)$$

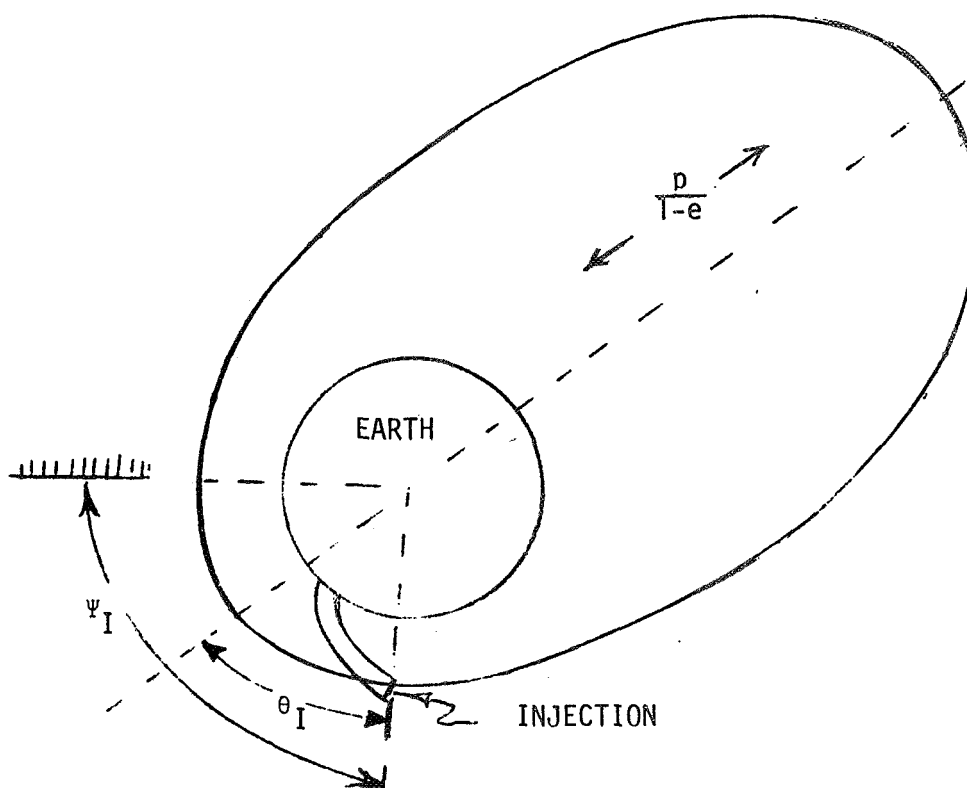
$$c_1 = \text{specified} = r_I v_r \cos \gamma_r \quad (11.18)$$

From these expressions the $v_r(r_I)$ and $\gamma_r(r_I)$ required to obtain the specified c_1 and c_3 can be calculated. Equation (11.18) can be put into different forms better suited for control of low path angles. For example,

$$\sin \gamma_r = \left[1 - \left(\frac{c_1}{r_I v_r} \right)^2 \right]^{1/2} \quad (11.19)$$

is an equivalent form.

FIGURE 11.1: APOGEE/APSIDES GUIDANCE



b. Apogee/Apsides Guidance - This explicit form arises when it is required to control the apogee of a near earth ellipse as well as the direction of the line of apsides. Similar formulas would apply for control of perigee and line of apsides. If ψ_I is the down range angle at injection, relative to some arbitrary inertial reference, and θ is the true anomaly at injection, then the control of line of apsides is achieved by setting $(\psi - \theta)$ equal to a fixed value (see Figure 11.1). The true anomaly at injection is found from

$$\cos \theta = \left(\frac{p - r_I}{r_I e} \right) \quad (11.20)$$

It follows that

$$\sin \theta = \left(\frac{p \tan \gamma}{r_I e} \right) \quad (11.21)$$

Then

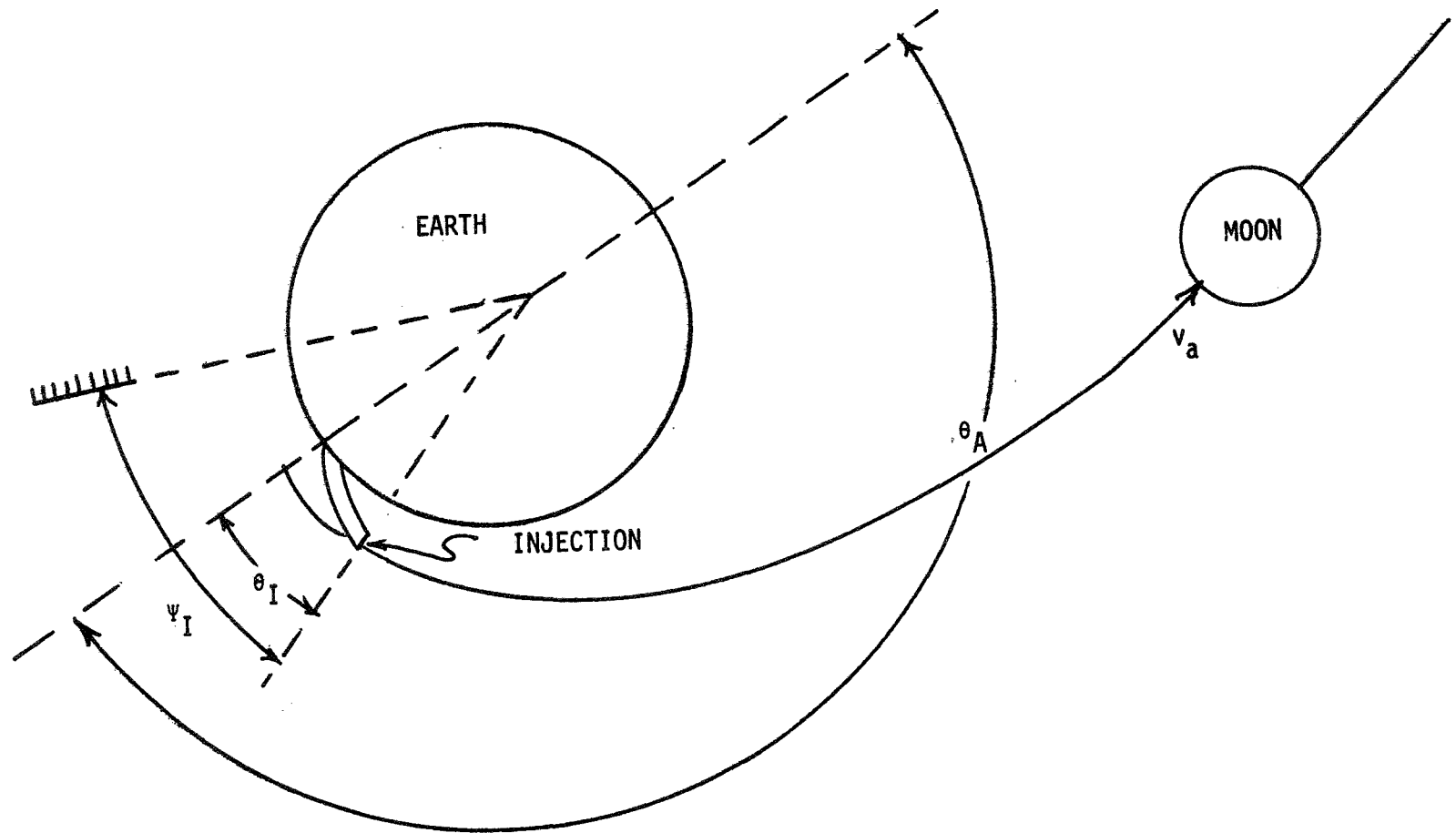
$$\begin{aligned} \cos (\psi - \theta) &= \text{specified} \\ &= \cos \psi \cos \theta + \sin \psi \sin \theta \end{aligned} \quad (11.22)$$

$$\text{apogee distance} = \text{specified} = \left(\frac{p}{1 - e} \right) \quad (11.23)$$

Given ψ_I and r_I , the simultaneous solution of these equations for $v_r(r_I, \psi_I)$ and $\gamma_r(r_I, \psi_I)$ yields the required velocity.

c. Lunar Mission Guidance - Suppose the mission objectives is to fly past the moon at a given distance. It will be shown in Section 11.6 that it is legitimate to pretend that the moon has no mass and construct an equivalent "massless" miss to cause the appropriate fly-by distance to be achieved. Then it follows that conditions at injection can be specified by constructing a conic passing through a massless point which moves according to the lunar ephemeris. Such an approach requires an application of Lambert's Theorem to find the time-of-flight to the moon's distance as a function of the injection

FIGURE 11.2: LUNAR IMPACT GUIDANCE



conditions r , v , and γ . The specification of this time, as well as the angular orientation of the earth-centered conic, will satisfy mission requirements, and thus define the required $v_r(r_I, \psi_I)$ and $\gamma_r(r_I, \psi_I)$. A simpler (but approximate) approach is to introduce the notion of a "pseudo-asymptote", which is defined by the direction and magnitude of the earth-centered conic velocity at the point of closest approach to the massless moon. Controlling the magnitude of this vector at the (specified) lunar distance fixes energy and (approximately) the time of flight. Controlling the direction of this vector fixes (approximately) the orientation of the earth-centered conic. This law has the practical feature of being similar to the "energy-asymptote" specification used for interplanetary missions.

Assuming that the distance to the moon is so large that the velocity and position vectors are approximately parallel, the "pseudo-asymptote" is then determined by (see Figure 11.2).

$$c_3 = \text{specified} = v_r^2 - \frac{2\mu}{r_I} \quad (11.24)$$

$$\begin{aligned} \cos \beta &= \text{specified} = \cos (\psi - \theta_I + \theta_A) \\ &= \cos \psi_I (\cos \theta_A \cos \theta_I + \sin \theta_A \sin \theta_I) \\ &\quad + \sin \psi_I (\sin \theta_A \cos \theta_I - \cos \theta_I \sin \theta_A) \end{aligned} \quad (11.25)$$

where ψ_I is defined as before; θ_I is the injection true anomaly, where $\cos \theta_I$ and $\sin \theta_I$ are determined from Equations (11.16) and (11.24), and θ_A is the true anomaly at the moon's distance, given by

$$\theta_A = \cos^{-1} \left(\frac{p}{r_m} - \frac{1}{e} \right) \quad (11.26)$$

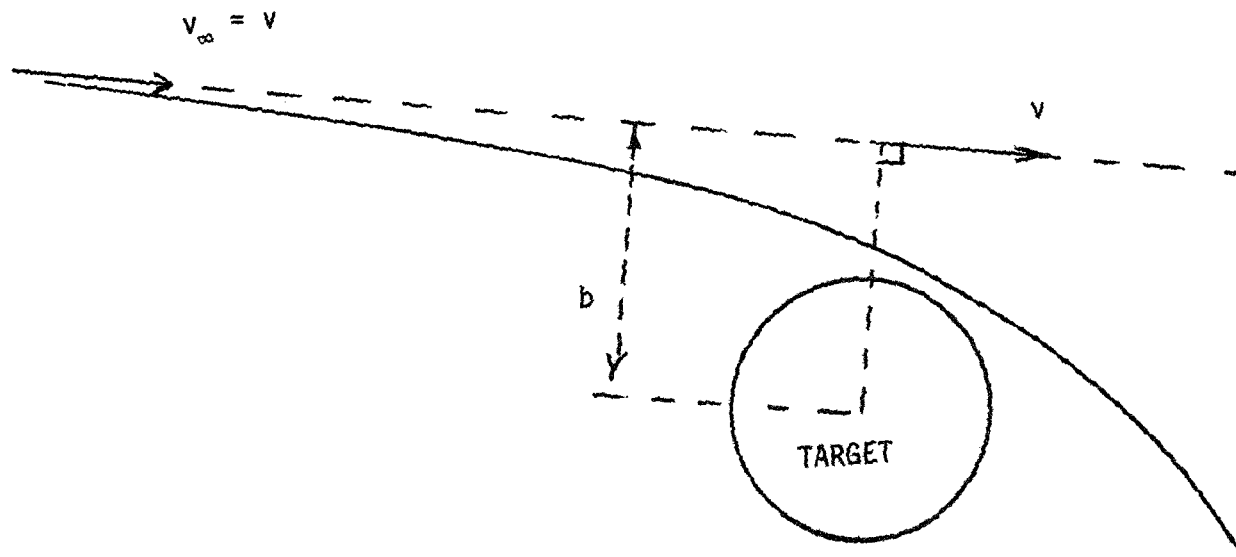


FIGURE 11.3: ASYMPTOTIC PATCHING

where r_m is the moon's distance. Convenient expressions for $\sin \theta_A$ and $\cos \theta_A$ can be obtained from Equation (11.25), and thus the required $v_r(r_I, \psi_I)$ and $\gamma_r(r_I, \psi_I)$ can be calculated.

d. Interplanetary Guidance - A more complicated situation arises in the case of interplanetary missions, where there are three conics to be considered i.e., the near earth escape hyperbola, the heliocentric ellipse, and the approach hyperbola near the target. The basic approach is to pretend that the earth and target planet have no mass and design a heliocentric ellipse which will pass between the massless earth and the massless planet with an acceptable earth escape hyperbola, characterized by the energy and direction of the asymptote. The conic injection conditions are then given by (see Figure 11.3)

$$c_3 = |\bar{v}_\infty|^2 = \text{specified} = v_r^2 - \frac{2\mu}{r_I} \quad (11.27)$$

$$\begin{aligned} \cos \beta &= \text{specified} = \cos (\psi_I - \theta_I + \theta_A) \\ &= \cos \psi_I \cos \theta_A \cos \theta_I + \sin \theta_A \sin \theta_I \\ &\quad + \sin \psi_I \sin \theta_I \cos \theta_A - \cos \theta_I \sin \theta_A \end{aligned} \quad (11.28)$$

where ψ_I and θ_I are defined as before, and θ_A is the asymptotic true anomaly, defined by

$$\cos \theta_A = -\frac{1}{e} \quad (11.29)$$

Note that the angle of the hyperbolic asymptote is obtained in much the same fashion as for the lunar pseudo-asymptote described in (c). Indeed, success with the energy asymptote concept for interplanetary guidance led to the notion of a pseudo-asymptote for lunar guidance.

11.5 The Simple Patched Conic Approximation

In this approach the initial conditions are related to desired final conditions by defining a sequence of conic trajectories, patched together at "spheres of influence" of primary central bodies (e.g., earth-sun-target planet, or earth-moon). These spheres are roughly defined to be at the point on a trajectory where the gravitational attractions of the central bodies are equal. For example, an interplanetary mission would consist of a hyperbola relative to the earth until the vehicle reaches a distance, of, say, 10^6 miles from the earth. The next segment would be an ellipse about the sun until the vehicle reaches a distance of, say, 10^6 miles from the target planet. The third phase would be a hyperbola relative to the target planet. Thus the effect of each central body is included in a piecewise manner on each conic segment, but the advantage of the closed form two-body solution is preserved by ignoring non-two body effects on each segment. The position and velocity relative to a given central body at the point of exit from its sphere of influence, together with the positions and velocities of central bodies, are used to calculate the position and velocity of the vehicle relative to the subsequent central body for initializing the next conic segment.

This well known method has been successfully used for approximate trajectory calculations, but is not accurate enough for final pre-flight targeting.

11.6 The Asymptotically Patched Conic Method

This approach is analogous to the simple patched conic method, but is simpler in the sense that a sphere of influence need not be defined, and more accurate in the sense that non-two body effects due to adjacent central bodies are implicitly treated in each conic segment.

Consider an interplanetary trajectory composed of a geocentric segment, a heliocentric segment, and a target-planet centered segment. The segments near the earth and target planet are hyperbolas, which from the point of view

of an observer on the sun, appear to be characterized by the asymptotic conditions. That is, the hyperbola near the earth is characterized by its departing asymptotic velocity (v_∞), which can be treated as a velocity impulse added at a massless earth, and the time of perigee passage. The hyperbola near the target planet is characterized by its arriving asymptotic velocity and the time of closest approach. Thus the asymptotic patching method treats the earth and target planet as massless points, where the departure and arrival conditions, respectively, are characterized by asymptotic conditions determined from the conic formulae. The lunar mission case is analogous, except that the primary ellipse (or hyperbola) is earth centered, and the arrival conditions to the massless moon are determined by the moon-centered hyperbola.

Referring to Figure 11.3, let the massless closest approach distance be denoted by b and the massless velocity relative to the target at closest approach by v . Recognizing that the asymptotic speed must be $v_\infty = v$, so that

$$v_\infty^2 = v^2 = c_3 \quad (11.30)$$

and that the angular momentum is given by

$$c_1 = bv_\infty \quad (11.31)$$

We note that the energy and angular momentum of the target centered hyperbola are specified by b and v . The orientation of the hyperbola is specified by the inertial direction of $\bar{v}_\infty$.

Thus, the relationship between the conic elements at any point on the central body hyperbola (e.g., at guidance termination near the earth) and the equivalent massless earth injection conditions (v_∞ and b) can be established. It remains to be shown how the time-of-flight to massless closest approach relates to time of flight to actual closest approach. From the conic formula we find that a time from an arbitrary position r to periapsis is

$$t(r) = \left(\frac{a}{v_{\infty}} \right) \left[e \sinh E - E \right] \quad (11.32)$$

where

$$\cosh E = \left(1 + \frac{r}{a} \right) \left(\frac{1}{e} \right) \quad (11.33)$$

and a is the semi-major axis given by $a = (\mu/c_3)$. But for large r ($r \rightarrow \infty$), we have (asymptotically)

$$\cosh E \approx \left(\frac{r}{ae} \right) \approx \left(\frac{\exp E}{2} \right) \approx \sinh E \quad (11.34)$$

$$E \approx \ln \left(\frac{2r}{a} \right) - \ln e \quad (11.35)$$

Then

$$t(r) = \left(\frac{r}{v_{\infty}} \right) - \left(\frac{a}{v_{\infty}} \right) \ln \left(\frac{2r}{a} \right) + \left(\frac{a}{v_{\infty}} \right) \ln e \quad (11.36)$$

The term (r/v_{∞}) is the time to massless closest approach; the term $(a/v_{\infty}) \ln (2r/a)$ is the decrease in that time due to the gravitational acceleration of the target body, and the term $(a/v_{\infty}) \ln e$ is the increase in that time due to the bending of the conic trajectory. Similar results can be obtained for the asymptotic patching of the near-earth hyperbola when launching to a planet, or for the near-moon hyperbola when moving from moon to earth.

It should be noted that the asymptotic distance corrections and the time of flight correction $(a/v_{\infty}) \ln e$ are commonly used in guidance work (Reference 29). The time of flight correction $(a/v_{\infty}) \ln (2r/a)$ has not been applied, however, because it cannot be accurately calculated by setting r = earth-target distance. Instead, a numerical integration is usually performed to calculate actual time of flight to closest approach (t_{CA}), and "linearized time-of-flight" (t_L) is used for guidance purposes, defined by

$$t_L = t_{CA} - \left(\frac{a}{v_{\infty}} \right) \ln e \quad (11.37)$$

This quantity, called the "local" correction by Breakwell (Reference 30), is linear in the sense that the nonlinear dependence of flight time upon the impact parameter b has been eliminated. As an example of the magnitude of this correction, note that for a lunar mission with $v = 1.25$ km/sec and a closest approach distance of 100 km, we have $a = 6,270$ km, $p = 9,190$ km, $e = 2.46$, $\frac{a}{v_\infty} = 5,360$ sec, and

$$\frac{a}{v_\infty} \ln e = 4,830 \text{ sec} = 1.34 \text{ hr} \quad (11.38)$$

The other flight time correction term, called the gross correction by Breakwell (Reference 30), is much larger. If we set r = distance to earth = 384,000 km, then

$$\left(\frac{a}{v_\infty}\right) \ln \left(\frac{2r}{a}\right) = 25,800 \text{ sec} = 7.15 \text{ hrs.} \quad (11.39)$$

Since this correction is adequate only if the motion of the spacecraft relative to the moon is approximately a hyperbola for the entire mission, errors of more than 10% (an hour, say) are to be expected. Thus, one can see the motivation for numerically calculating this term.

The asymptote matching method yields very good results if the gross correction term is numerically computed. There nevertheless do remain some errors in the approximation, for the effect of other bodies during the hyperbolic phase has not been precisely treated. The asymptotic matching method for treating these effects will be discussed in Section 11.8.

11.7 The Hybrid Conic Method

A "hybrid conic" technique has been developed by TRW Systems Group to numerically improve conic approximations of earth-moon and moon-earth trajectories.* For the earth-moon case one takes the earth centered conic

*This work was carried out under contract to the Manned Spacecraft Center, Houston, under Contract No. NAS 9-4810, Phase II and results are reported in Reference 31.

as the zero-order approximation and numerically estimates the effect moon's perturbation, as well as any other perturbations which need be treated. A similar approach, but with primary conics about different central bodies, as the zero order approximation, is used for moon-earth or interplanetary trajectories. In all cases the non-two-body perturbation is numerically approximated in such a way as to make a closed form integration possible.

Let the two-body conic with respect to the central body be denoted by the zero-order solution, $r^{(0)}(t)$, so that the exact solution of the equation of motion relative to the central body is

$$r(t) = r^{(0)}(t) + \delta r(t) \quad (11.40)$$

The $\delta r(t)$ is the correction term on which iterations are performed. The geocentric conic (or the heliocentric conic, as the case may be) is a solution of

$$\ddot{r}^{(0)} = -\mu_0 r/|r|^3 \quad (11.41)$$

The function $\delta r(t)$ is then the position error between the exact and conic trajectories. The differential equation for this error is obtained by subtracting Equation (11.41) from the time acceleration equation; the result is an Encke equation:

$$\delta \ddot{r} = \mu_0 \left(\frac{r^{(0)}}{|r^{(0)}|^3} - \frac{r}{|r|^3} \right) - \sum_{j=1}^n \mu_j \left(\frac{r - r_j}{|r - r_j|^3} + \frac{r_j}{|r_j|^3} \right) \quad (11.42)$$

where the subscripts denote the non-two-body effects. The first term in the error equation (11.42) is referred to as the indirect term, the second term as the direct term.

While in the method of asymptotic expansions (see Section 11.8) an approximate solution to the exact equation of motion is developed, the HPCT proceeds to approximate the perturbation (that is, the right hand side of Equation (11.42)) by functions that can be twice integrated in closed form. For lunar trajectories and interplanetary trajectories where a singularity exists near pericenter passage (with $r = r_j$), the approximating functions are chosen as the inverse powers $(t - t_p)^{-n}$ for $n = 0, 1, 2$; where t_p denotes the best estimate of time of periapsis passage (initially the patched conic value):

$$\ddot{\delta r}(t) = a + \frac{b}{t_p - t} + \frac{c}{(t_p - t)^2} \quad (11.43)$$

The constant coefficients (vectors) a , b , and c are calculated from three values of Equation (11.42), respectively: at the initial time t_0 , at time of arrival at the sphere of influence (t_2) and at an appropriately chosen intermediate time (t_1). Then one forms the following 3 by 3 matrix equation:

$$\begin{bmatrix} \delta \ddot{r}^T(t_0) \\ \delta \ddot{r}^T(t_1) \\ \delta \ddot{r}^T(t_2) \end{bmatrix} = \begin{bmatrix} 1 & (t_p - t_0)^{-1} & (t_p - t_0)^{-2} \\ 1 & (t_p - t_1)^{-1} & (t_p - t_1)^{-2} \\ 1 & (t_p - t_2)^{-1} & (t_p - t_2)^{-2} \end{bmatrix} \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix} \quad (11.44)$$

and finds a , b , c by an matrix inversion. Equation (11.43) is integrated to give $\dot{\delta r}$ and δr which is then used in Equation (11.40) to find a corrected value for r ; this is used in Equations (11.42) and (11.44) to find a new set of values for a , b , and c . The corrections at every iteration are given by the simple expressions

$$\dot{\delta r}(t) = a(t - t_0) - b \ln \frac{t_p - t}{t_p - t_0} + c \frac{t - t_0}{(t_p - t)(t_p - t_0)} \quad (11.45)$$

$$\delta r(t) = \frac{1}{2} a(t - t_0)^2 + b \left[(t_p - t) \ln \left(\frac{t_p - t}{t_p - t_0} \right) + t - t_0 \right] - c \ln \left[\frac{t_p - t}{t_p - t_0} + \frac{t - t_0}{t_p - t_0} \right] \quad (11.46)$$

The evaluation of the perturbation is only carried to some point within the lunar sphere of influence (because of the singular term), and the closest approach conditions are determined by calculating a patched lunar-centered hyperbola from that point to closest approach. In general, more than two iterations are required in order to converge to stationary end conditions, with the perturbation term being evaluated each time. The calculations to find a , b , and c are very rapid, however, because the times at which the ephemeris of the disturbing body is read does not change in the iteration and the inversion of the three by three matrix is trivial. Some numerical results for earth-moon transfer are presented in Figures 11.4 and 11.5.

It should be noted that Equation (11.42) could be represented as a second order differential equation in $\delta r(t)$ if one approximated the indirect term by a first order expansion in $\delta r(t)$. In this case, the solutions $\dot{\delta r}(t)$ and $\delta r(t)$ would be obtained as integrals of a state transition matrix operating on the direct term. This method would require the calculation of the state transition matrix, however, which might destroy the rapid computation feature of the hybrid conic method.

FIGURE 11.4: HYBRID AND CONIC POSITION ERRORS

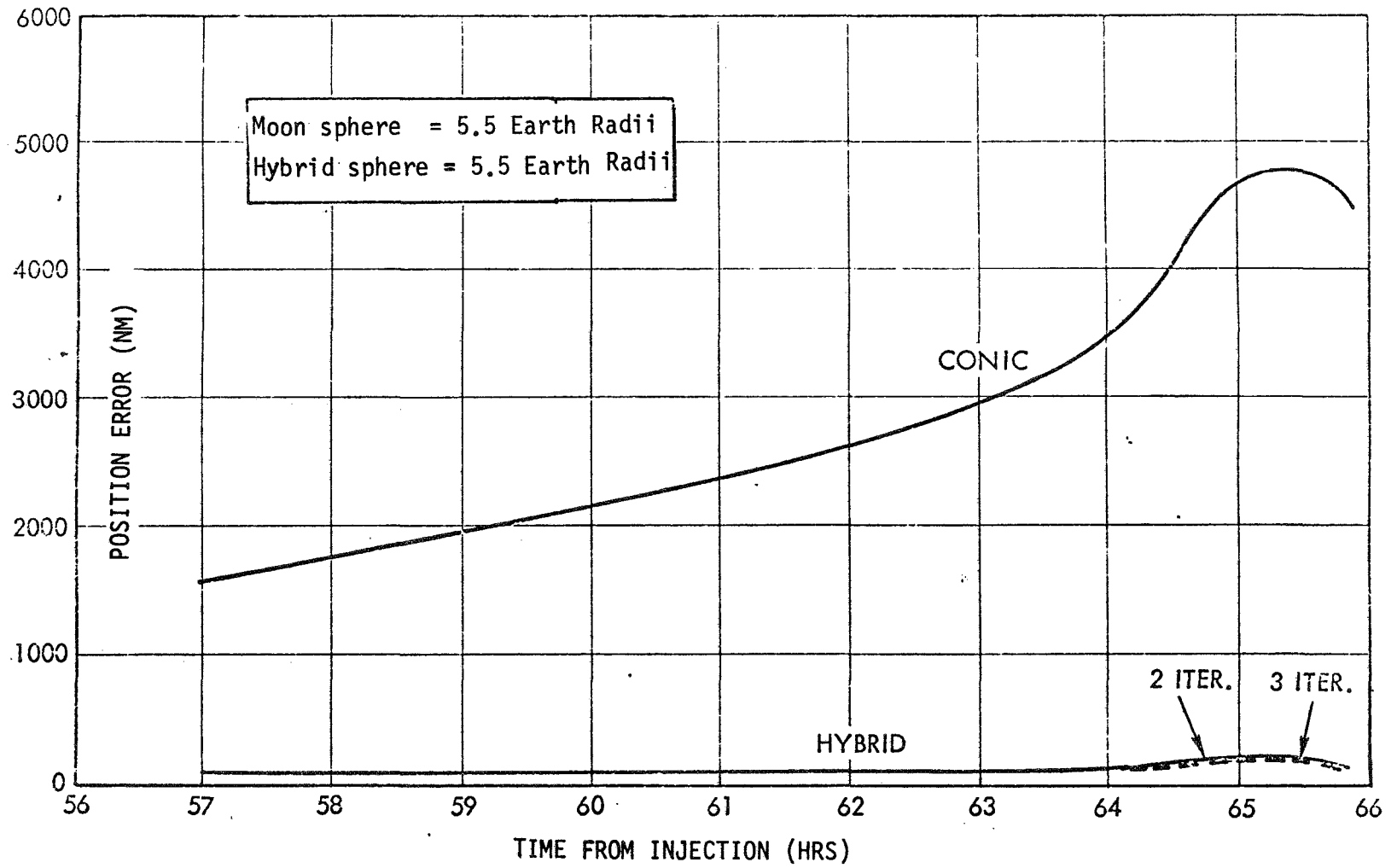
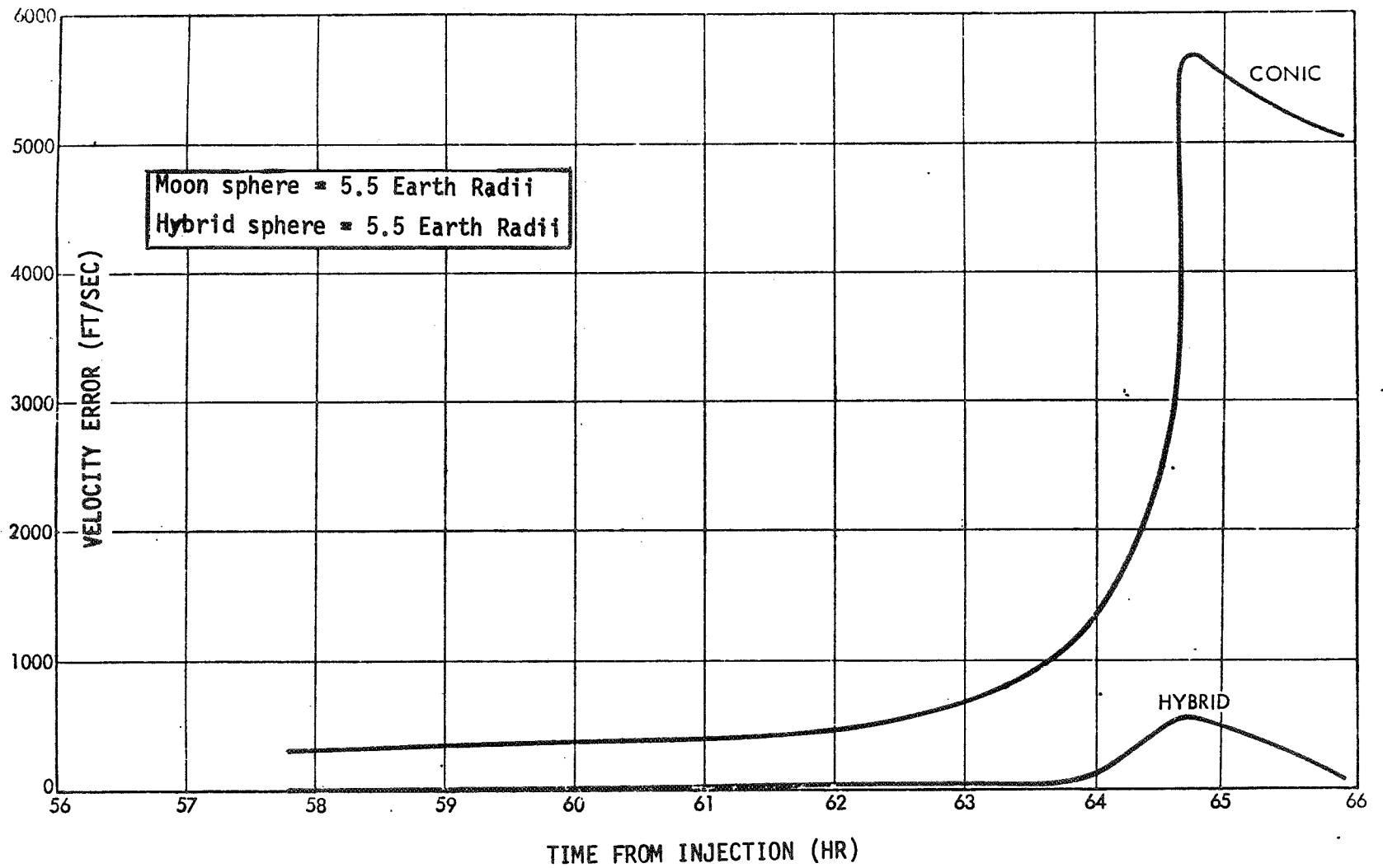


FIGURE 11.5: TRANSLUNAR HYBRID AND CONIC VELOCITY ERRORS



11.8 The Asymptotic Matching Method

The Asymptotic patching method discussed in Section 11.6 assumes that an interplanetary trajectory can be approximated as an ellipse about the sun, with the masses of the earth, target planet, and other bodies set equal to zero. The elements of the near-earth and near-target-planet hyperbolas are then calculated by assuming, respectively, that the massless initial and final conditions describe the asymptotic positions and velocities. Lunar trajectories are similarly treated, except that the primary conic is earth centered, and only the near-moon hyperbola must be approximated by its asymptotic conditions. This approach has worked well for preliminary trajectory design and targeting, but approximation errors are present which must be corrected by a final, preflight numerical integration. In particular, the gross time-of-flight bias is a source of concern.

The asymptotic matching method, developed by Breakwell and Perko (Reference 30), and applied by Carlson (Reference 32), is a semi-analytic method for improving the accuracy of these asymptotic conditions. The primary conic about the central body (the sun for an interplanetary flight, the earth for earth-moon flight) is considered the "zero order" solution, and the effect of neglected gravitational bodies is calculated in the form of perturbation integrals. This constitutes the "outer expansion". Next, the asymptotically patched, zero-order target centered hyperbola is treated as the reference orbit, and the effect of gravitational accelerations due to non-target masses are calculated in the form of perturbation integrals. This constitutes the "inner-expansion". Both expansions are carried out up to terms of order ϵ^2 (position errors of order $\epsilon^2 \times$ central body-target distance), where

$$\epsilon = \frac{\text{target mass}}{\text{central body mass}}$$

The spacecraft position and velocity as determined by either expansion must agree "on the asymptote" of the target hyperbola, which is considered to be the region where

$$\frac{\text{spacecraft-target distance}}{\text{central body-target distance}} = \text{order of } (\epsilon^{1/2})$$

It follows that like coefficients of the inner and outer expansions can be equated in order to determine the elements of the target centered hyperbola. A similar procedure is carried out to treat the effects of the mass of the earth for interplanetary travel. The procedure will be outlined here by considering an earth-moon transfer trajectory (i.e., a case where only the perturbation of the target planet is of interest).

Consider an earth-centered, earth-to-moon transfer conic (an ellipse for flight times greater than 50 hours), where the acceleration is given by

$$\ddot{r} = \mu_0 \frac{r}{|r|^3} + p(r, t) \quad (11.47)$$

The μ_0 is the gravitational constant of the earth, and p is the perturbation due to the moon, given by

$$p(r, t) = -\mu_1 \left[\frac{r - r_1}{|r - r_1|^3} + \frac{r_1}{|r_1|^3} \right] \quad (11.48)$$

where the subscript 1 denotes the moon. Let $r^{(0)}$ denote the zero-order solution obtained by setting $p = 0$, that is, the earth centered conic for a zero mass moon. Let t_0 be the initial time, corresponding to injection near the earth, t_1 be the final time, corresponding to the time of closest approach to the massless target, and v_1 be the zero-order velocity relative to the moon

at time t_1 . Since the local time correction for a non-zero impact parameter vector (b) can be easily calculated (see Section 11.6), assume that the zero order solution passes through the center of the moon. Furthermore, for simplicity, let the initial conditions (at t_0) be held fixed. Then the first order correction is

$$\begin{bmatrix} \delta \dot{r}(t) \\ \delta r(t) \end{bmatrix}^{(1)} = \int_{t_0}^t \begin{bmatrix} U_{VV}(t,s) \\ U_{rV}(t,s) \end{bmatrix} p^0(s) ds \quad (11.49)$$

where

$$p^0(s) = p(r^0(s), s) \quad (11.50)$$

is the gravitational acceleration due to the moon, evaluated on the zero-order solution. The state transition matrix elements, also evaluated on the zero-order solution, are

$$U_{VV}(t,s) = \left[\frac{\partial v(t)}{\partial v(s)} \right] \quad (11.51)$$

$$U_{rV}(t,s) = \left[\frac{\partial r(t)}{\partial v(s)} \right] \quad (11.52)$$

In general, the n^{th} order solution could be expressed as

$$r^{(n)}(t) = r^{(0)}(t) + \delta r^{(1)}(t) + \dots \delta r^{(n)}(t) \quad (11.53)$$

where $\delta r^{(1)}(t)$ is as given above, and $\delta r^{(2)}(t)$ is similarly determined, but with $p^{(0)}(s)$ replaced by

$$p^{(1)}(s) = \left[\frac{\partial p}{\partial r} (r^{(0)}(s), s) \right] \delta r^{(1)}(s) \quad (11.54)$$

The series (11.53) is the outer expansion, which can be very accurate if enough terms are kept. Only the first order correction, and the "singular" portion of the second order correction are required for asymptotic matching up to terms of order ϵ^2 , however.

The computation of $\delta r^{(i)}(t)$ is most conveniently carried out in a non-rotating frame centered in the moon, where the r_1 -axis is aligned with the zero-order velocity vector of the spacecraft relative to the moon. Performing a Taylor series expansions of $[r^{(o)}(t) - r^{(moon)}(t)]$ and the state transition matrices, it can be shown that the r_1 -component of $\delta \ddot{r}^{(1)}(t)$ is composed of a singular term (going to infinity as $t \rightarrow t_1$) plus bounded terms, and that the other components of $\delta \ddot{r}^{(1)}$ are composed of bounded terms. Subtracting out the singular term and evaluating it analytically, one obtains expressions for $\delta r^{(1)}(t)$ and $\delta \dot{r}^{(1)}(t)$ which contain analytic expressions plus bounded integral terms. These are then employed to determine the second order correction, which again contains a singular term in the r_1 acceleration component. Thus an expansion of $r(t) - r^{(o)}(t)$ is obtained to order

$$\epsilon^2 = \left(\frac{\mu_1}{\mu_0} \right)^2$$

which holds for distances

$$\frac{r_1(t)}{|r^{(o)}(t_1)|} \approx \text{of order } \epsilon^{1/2}$$

This defines the asymptotic region, where matching will take place.

Suppose now that the zero-order hyperbola relative to the moon is treated as the zero-order approximation, and the gravitational acceleration of the earth is treated as the perturbation. Expressions analogous to the above would be obtained to describe this inner expansion relative to the moon. As in the case of the outer expansion, it is possible to develop expressions for position and velocity relative to the moon which have errors of order ϵ^2 , and which hold in the asymptotic region.

Ignoring the terms of order ϵ^2 , the inner and outer expansions may be compared in the asymptotic region to equate the coefficients of terms of like form. Then we find the elements of the "matched" target centered hyperbola to be

Hyperbolic excess velocity:

$$v_{\infty} = v_1 \left[1 - \frac{\mu_1}{|v_1|^3 (t_1 - t_0)} \right] + \Delta v_b \quad (11.55)$$

Time of periapsis passage:

$$t_p = t_1 - \frac{\mu_1}{|v_{\infty}|^3} \left\{ \ln \left[\frac{2 \mu_1 (t_1 - t_0)}{|v_{\infty}|^3 e} \right] - 2 \right\} - \frac{v_{\infty}^T \Delta r_b}{|v_{\infty}|} \quad (11.56)$$

Impact parameter vector:

$$b = \left[I - \frac{v_{\infty} v_{\infty}^T}{|v_{\infty}|^2} \right] \Delta r_b \quad (11.57)$$

where

$$e = \left[1 + \frac{|v_{\infty}|^2 |b|^2}{\mu_1^2} \right]^{1/2} \quad (11.58)$$

$$\Delta r_b = \int_{t_0}^{t_1} \left[U_{rv}(t_1, t) p^0(t) - \frac{\mu_1 v_1}{|v_1|^3 (t_1 - t)} \right] dt \quad (11.59)$$

$$\Delta v_b = \int_{t_0}^{t_1} \left[U_{vv}(t_1, t) p^0(t) - \frac{\mu_1 v_1}{|v_1|^3 (t_1 - t)^2} \right] dt \quad (11.60)$$

and $p^0(t)$ is the zero-order perturbation given by equation (11.50). Note that Δr_b and Δv_b are numerically calculated as bounded integrals. That is, the $p^0(t)$ has a second order singularity as $t \rightarrow t_1$, but this has been subtracted out in the above integral expressions (Recall that $U_{rv} \rightarrow I(t_1 - t)$ as $t_1 \rightarrow t$). Note that the bounded integral terms represent the perturbation correction obtained by the hybrid conic method of Section 11.7, but with the singular term and local time correction subtracted out.

A similar analysis can be carried out for the effect of the earth on interplanetary trajectories, and for the effect of other perturbing bodies, and the first order effect of variations in initial conditions can be linearly superimposed. Thus these calculations result in the following simplified trajectory prediction technique:

- 1) Determine the zero-order trajectory as the central body conic which passes through the center of the massless target.
- 2) Calculate the state transition matrices on the zero-order trajectory.
- 3) Numerically evaluate the bounded integral terms to account for the effects of target mass, earth mass*, and other perturbing bodies.
- 4) Determine the elements of the target centered hyperbola by the above formulae.

It is to be noted that the bounded integral terms are invariant with respect to first order initial condition variations. Thus, the procedure can be used for trajectory design and targeting by applying the local time correction to adjust any desired terminal conditions to equivalent zero-miss

* For the case of interplanetary flight, where the sun is the central body.

conditions, and using the state transition matrices evaluated on the zero-order solution to adjust the initial conditions to achieve the desired target hyperbola. This fixes the zero-order reference trajectory. The guidance system objective, then, is to control the zero order reference trajectory, that is, to attain elements of the near-earth conic which result in certain nominal target hyperbolic elements (e.g., the nominal b). Since the asymptotic matching method results in errors of order ϵ^2 , which are small compared to other sources of guidance system error, it can be thought of as the analytic justification for explicit targeting.

11.9 Conclusions and Recommendations

Explicit targeting for space missions, based upon the assumption that the post injection trajectory can be adequately represented for guidance purposes by a central body conic with asymptotically patched near-earth and target-centered hyperbolas, has proved to be a useful technique. The asymptotic matching concept offers analytical justification of this approach, for it is shown that, aside from the calculable local time correction, non-two body perturbations to the zero-order conic are indeed invariant to order ϵ^2 . This implies a position error of less than 100 miles for a lunar trajectory, which is acceptably small compared to, say, navigation errors at injection.

It still remains to numerically calculate the perturbation terms in order to design reference trajectories and determine the targeting manifold. At present this is accomplished with precise numerical integration. In the future one might employ some rapid calculation technique, such as the hybrid conic method, or an approximate form of the asymptotic matching method. These two approaches are closely related, for in both cases the non-two body perturbations to the primary conic are integrated and comparable accuracy should be expected. The hybrid conic method has the advantage of being very rapid, but cannot be carried all the way to closest approach to the target, and seems to require more than one iteration. The asymptotic matching method has the advantage of requiring only one iteration, but requires an evaluation of the state transition matrix and hence a more elaborate numerical integration.

The hybrid conic approximation could be carried to closest approach if the singular perturbation term were removed, but this should not affect the required number of iterations. Calling upon the results of Breakwell and Perko, and assuming that the hybrid conic approximation yields errors of order ϵ^2 , one might conclude that one iteration should be sufficient if the perturbation were to be evaluated on the zero order trajectory which passed through the center of a massless target.* That is, in the presently applied hybrid conic method the time of periapsis passage for the first iteration is chosen as the patched conic closest approach time, which, because of the local time correction and the portion of the gross time correction included in the target sphere of influence, could be several hours different from the time of passage through a massless target. This increment in t_p in Equation (11.43) could easily account for the need for extra iterations. If so, one would expect that a single iteration of the hybrid patched conic method, after correction for the analytic terms in Equations (11.55) to (11.58), would produce the bounded integral terms of Equations (11.59) and (11.60). On the other hand, it must then also be true that the bounded integral terms in the asymptotic matching method can be rapidly evaluated to sufficient accuracy with some simple numerical approximation. In any case, the evidence indicates that a rapid, single iteration numerical method can be developed for calculating the perturbations to the zero order solution.

Calling upon the best features of both methods, the following approach is recommended as a candidate for trajectory design and targeting:

- 1) Generate the zero order solution describing the central body conic, with asymptotic patching of earth and target.
- 2) Use matched asymptotic equations for computing the elements of the target-centered hyperbola.

* This is intuitively reasonable, for we are perturbing the zero order solution and t_p should be evaluated on that solution.

- 3) Compute the bounded integrals by numerically fitting the perturbation, as in the hybrid conic method.

Further study is called for to see if such a method can be developed which is sufficiently rapid and accurate. If so, it might be possible to solve the targeting problem in real time with a relatively small computer, thereby greatly reducing the requirement for preflight preparation.

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