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BAYESIAN ANALYSIS FOR AN EXPONENTIAL SURVEILLANCE MODEL*

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Bayesian estimation and design are considered for a dichotomous response surveillance model for defectives. The probability of a defective item after storage time t is assumed to be given by $F(t) = 1-\exp(-\lambda t)$, $0 < t, \lambda < \infty$. Surveillance is carried out by removing k lots from storage and observing the size (n_i), storage time (t_i) and number of defectives (r_i) for each lot. The r_i are assumed to be independently and binomially distributed with respective expectations $n_i F(t_i)$. A prior gamma distribution is assumed to be available for λ. The posterior distribution of λ is derived as well as its moments, including the mean which is the Bayes estimate assuming quadratic loss. A formula for the corresponding Bayes risk is derived as are some recursive relationships to aid in computation. A design procedure is given for selecting surveillance times and sample sizes successively one lot at a time. Tables of optimum surveillance times and Bayes risks at these times as functions of the prior parameters and sample size are provided to help with this selection procedure.

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1. INTRODUCTION

Bayesian estimation and design are considered for a dichotomous response surveillance model for defectives. The probability of a defective item, $F(t)$, is assumed to be related to the time t after storage at which surveillance testing is performed by the cumulative distribution function

$$F(t) = 1 - \exp(-\lambda t),\ 0 < t,\ \lambda < \infty. \tag{1.1}$$

This distribution function is appropriate if an item is defective whenever it has one or more defects and when the number of such defects is a Poisson random variable with mean λt. The function (1.1) also arises when the failure time of each item is exponentially distributed with mean $(1/\lambda)$. Surveillance is carried out by testing k lots of size n_i at respective times t_i and finding r_i defectives in the i-th lot. The r_i's are assumed to be independently and binomially distributed with means $n_i F(t_i)$, $i = 1,2,\ldots,k$.

It is assumed that prior information on λ can be summarized by specifying a gamma distribution with probability density function

$$g(\lambda|\alpha,\beta) = \Gamma^{-1}(\alpha+1)\beta^{\alpha+1}e^{-\lambda\beta}\lambda^{\alpha},\ \lambda>0,\ \alpha>-1, \beta>0 \tag{1.2}$$

which has mean and variance

$$E(\lambda|\alpha,\beta) = (\alpha+1)/\beta\ ; \tag{1.3}$$

$$V(\lambda|\alpha,\beta) = (\alpha+1)/\beta^2\ , \tag{1.4}$$

respectively. This distribution is in the extended natural conjugate family for this problem and is flexible. Moreover, considerable past information is usually available in surveillance situations for uses in specifying prior parameters. A quadratic loss function is also assumed for λ in the determination of its Bayes estimate denoted by $\hat{\lambda}$.

The surveillance setting for testing for defectives is described by Hillier [6]. The assumption of an exponential distribution function as specified by (1.1) is common in reliability life testing problems (see Epstein [5]). It also arises frequently in epidemiology and biological assay through its relationship with the Poisson distribution (see Cornell and Speckman [3]). Bayesian estimation for this problem when the probability of a defective is fixed is considered, for instance, by Bracken [2]. His work is not appropriate for our surveillance situation because his prior on λ would involve the design parameter t.

In Section 2 derivations are presented of formulas for the posterior probability density function of λ, its mean which is our Bayes estimate $\hat{\lambda}$, other posterior moments of λ, the joint marginal probability function of the r_i's and the Bayes risk. Recursion relationships to assist in using these formulas are given in Section 3. Then the selection of the time at which to carry out a surveillance test and of the corresponding lot size are considered in Section 4. The results presented there can be applied successively to select the complete set of k lots inspected during a surveillance program. Tables of optimum surveillance times are included in Section 4 followed by tables of corresponding Bayes risks for use in selecting lot sizes.

2. BAYES RESULTS FOR k LOTS

Let $\underline{r} = (r_1,\ldots,r_k)$ be the number of defective items in lots with respective sizes $\underline{n} = (n_1,\ldots,n_k)$ corresponding to inspection times $\underline{t} = (t_1,\ldots,t_k)$. From the assumptions stated in Section 1, the joint probability function of the elements of $\underline{r}$ given λ is

$$\ell(\underline{r}|\lambda) = \{\exp[-\lambda\sum_i t_i(n_i-r_i)]\}\prod_i \binom{n_i}{r_i} \prod_i (1-e^{-\lambda t_i})^{r_i}, \tag{2.1}$$

where the subscript i ranges from 1 through k unless otherwise specified. Binomial expansion of the last product term in (2.1) leads to

$$\ell(\underline{r}|\lambda) = \prod_i \binom{n_i}{r_i} D(\lambda,\underline{r};\ 0,0),$$

where

$$D(\lambda,\underline{r};\gamma,\beta) = \sum_{j_1=0}^{r_1}\cdots\sum_{j_k=0}^{r_k} (-1)^{j_1+\cdots+j_k} \prod_i \binom{r_i}{j_i}\lambda^{\gamma}$$

$$\exp\{-\lambda[\sum_i t_i(n_i-r_i) + \sum_i t_i j_i + \beta]\}\ .$$

We also define

$$S(\underline{r};\ \gamma,\ \beta) = \sum_{j_1=0}^{r_1}\cdots\sum_{j_k=0}^{r_k} (-1)^{j_1+\cdots+j_k)} \prod_i\binom{r_i}{j_i} \tag{2.2}$$

$$[\sum_i t_i(n_i-r_i) + \sum_i t_i j_i + \beta]^{-\gamma}$$

and note that

$$\int_0^{\infty} D(\lambda,\underline{r};\gamma,\beta)\ d\lambda = \Gamma(\gamma+1)\ S(\underline{r};\gamma+1,\beta). \tag{2.3}$$

Using (2.3) along with the prior probability density function on λ given by (1.2), we can write the corresponding posterior probability density function for λ as

$$h(\lambda|\underline{r}, \alpha, \beta) = \Gamma^{-1}(\alpha+1)\, D(\lambda, \underline{r}; \alpha, \beta)/S(\underline{r}; \alpha+1, \beta). \tag{2.4}$$

Similarly, use of equation (2.3) enables the marginal probability function of $\underline{r}$ to be written as

$$f(\underline{r}|\alpha, \beta) = \prod_i \binom{n_i}{r_i} \beta^{\alpha+1} S(\underline{r}; \alpha+1, \beta) \tag{2.5}$$

and, with $\gamma = \alpha + h$, the ω-th moment of the posterior distribution of λ about the origin to be written as

$$\mu'_\omega(\lambda|\underline{r}, \alpha, \beta) = \prod_{j=1}^{\omega} (\alpha+j)\, S(\underline{r}; \alpha+\omega+1, \beta)/S(\underline{r}; \alpha+1, \beta). \tag{2.6}$$

The Bayes estimate $\hat{\lambda}$ is given by setting $\omega = 1$ in (2.6) and the posterior variance of λ can be determined by using equation (2.6) in the calculation of $\mu'_2 - (\mu'_1)^2$.

The Bayes risk is defined as

$$R(\alpha, \beta) = E_{\lambda,\underline{r}}(\hat{\lambda}-\lambda)^2$$

where $E_{\lambda,\underline{r}}$ denotes expectation with respect to the joint distribution of λ and $\underline{r}$. Similarly defining $E_{\underline{r}}$ and $E_{\lambda|\underline{r}}$, we have

$$R(\alpha,\beta) = E_{\underline{r}}\, E_{\lambda|\underline{r}}(\hat{\lambda}-\lambda)^2 = \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} f(\underline{r}|\alpha,\beta) E_{\lambda|\underline{r}}(\hat{\lambda}-\lambda)^2 \tag{2.7}$$

where $f(\underline{r}|\alpha, \beta)$ is given by (2.5). Now $E_{\lambda|\underline{r}}(\hat{\lambda}-\lambda)^2 = E_{\lambda|\underline{r}}(\lambda^2)-\hat{\lambda}^2$ can be evaluated using (2.6). Substituting the resultant expression into (2.7) and noting that

$$\sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} f(\underline{r}|\alpha,\beta) = 1$$

implies from (2.5) with α replaced by $\alpha + 2$ that

$$\beta^{-(\alpha+3)} = \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \prod_i \binom{n_i}{r_i} S(\underline{r};\ \alpha+3,\beta),$$

we find that

$$R(\alpha,\beta) = (\alpha+2)(\alpha+1)\beta^{-2}-(\alpha+1)^2\beta^{\alpha+1} \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \prod_i \binom{n_i}{r_i} \tag{2.8}$$

$$S^2(\underline{r};\ \alpha+2,\beta)/S(\underline{r};\ \alpha+1,\beta).$$

Although the expressions presented in this section for Bayesian analysis of surveillance data are complex, computations using them are well within the capacities of modern computing equipment. Moreover, when α is chosen to be a positive integer the $S(\underline{r};\ \alpha,\ \beta)$ functions which occur in all of these expressions satisfy recursive relationships which can be utilized in these computations and which are derived in the next section.

3. RECURSIVE RELATIONSHIPS

The expressions derived in section 2 all require the evaluation of $S(\underline{r};\gamma,\beta)$ functions as given by equation (2.2). When $k = 1$, we can suppress the subscript i in (2.2) and write

$$S(r;\gamma,\beta) = t^{-\gamma} \sum_{j=0}^{r} (-1)^j \binom{r}{j} (j+k)^{-\gamma} \tag{3.1}$$

where $K = n - r + \beta/t$. For $k = 2,3,\ldots$ we can similarly write

$$S(\underline{r};\gamma,\beta) = t_k^{-\gamma} \sum_{j_1=0}^{r_1} \cdots \sum_{j_{k-1}=0}^{r_{k-1}} (-1)^{j_1+\ldots+j_{k-1}} \prod_{i=1}^{k-1} \binom{r_i}{j_i} \sum_{j_k=0}^{r_k} (j_k+K_k)^{-\gamma} \tag{3.2}$$

with $K_k = \sum_{i=1}^{k} t_i(n_i-r_i)/t_k + \sum_{i=1}^{k-1} t_i j_i/t_k + \beta/t_k$. Comparison of equations (3.1) and (3.2) reveals that for any positive integer k, the evaluation of $S(\underline{r};\gamma,\beta)$ involves the evaluation of one or more sums of the form

$$T^{(\gamma)}(K) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} (j+K)^{-\gamma} \tag{3.3}$$

for suitably selected r and K. We develop recursive relationships for evaluating $T^{(\gamma)}(K)$ in this section for integer values of γ.

First consider the Beta integral

$$\int_0^1 x^K(1-x)^r dx = \Gamma(K)\Gamma(r+1)/\Gamma(K+r+1) = r!/\prod_{j=0}^{r}(j+K),\ k > 0,$$

which can also be integrated in the form

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} \int_0^1 x^{K-1+j} dx = \sum_{j=0}^{r} (-1)^j \binom{r}{j}/(j+K) = T^{(1)}(K). \tag{3.4}$$

Thus

$$T^{(1)}(K) = r!/\prod_{j=0}^{r}(j+K). \tag{3.5}$$

Differentiating $T^{(1)}(K)$ as given by the middle expression in (3.4), we obtain

$$\frac{\partial T^{(1)}(K)}{\partial K} = (-1) \sum_{j=0}^{r} (-1)^j \binom{r}{j} / (1+K)^2 = -T^{(2)}(K). \qquad (3.6)$$

Evaluating (3.6) using (3.5) yields

$$T^{(2)}(K) = n! \; [\sum_{j=0}^{r} 1/(j+K)] \prod_{j=0}^{r} (j+K)^{-1}. \qquad (3.7)$$

We can extend (3.6) to any positive integer γ to give the recursion formula

$$T^{(\gamma)}(K) = -(\gamma-1)^{-1} \; \partial T^{(\gamma-1)}(K)/\partial K \qquad (3.8)$$

which could in turn be used to develop an equation for $T^{(\gamma)}(K)$ in the way (3.7) was derived for $\gamma = 2$. In particular, for $\gamma = 3$ it can be shown that

$$T^{(3)}(K) = (r!/2!) \; \{[\sum_{j=0}^{r} \frac{1}{(j+K)^2}] + [\sum_{j=0}^{r} \frac{1}{(j+K)}]^2\} \prod_{j=0}^{r} (j+K)^{-1}.$$

Now define the polygamma function

$$\psi^{(\gamma)}(K) = \frac{d^\gamma}{dK^\gamma} \ln \Gamma(K) \qquad (3.9)$$

where

$$\Gamma(K) = \int_0^\infty e^{-x} x^{K-1} dx = \Gamma(r+1+K)/[K(1+K)\dots(r+K)]. \qquad (3.10)$$

Taking logarithms and derivations with respect to K of both sides of (3.10), we find that

$$\frac{d \ln \Gamma(K)}{dK} = \psi^{(1)}(K) = \psi^{(1)}(K+r+1) - \frac{1}{K} - \frac{1}{1+K} - \cdots - \frac{1}{r+K}$$

which shows that

$$\sum_{j=0}^{r} 1/(j+K) = \psi^{(1)}(r+1+K) - \psi^{(1)}(K). \tag{3.11}$$

Differentiation of both sides of (3.11) with respect to K, along with (3.9), yields the similar expression

$$\sum_{j=0}^{r} 1/(j+K)^2 = (-1)[\psi^{(2)}(r+1+K) - \psi^{(2)}(K)].$$

Then by induction it can be shown that

$$\sum_{j=0}^{r} 1/(j+K)^{\gamma} = (-1)^{\gamma-1}[\psi^{(\gamma)}(r+1+K)-\psi^{(\gamma)}(K)]/(\gamma-1)!. \tag{3.12}$$

From (3.3) and (3.12) it is clear that the $T^{(\gamma)}(K)$, $\gamma = 1,2,\ldots,$ can be expressed as rational combinations of polygamma functions. Hence, from (3.1) and (3.2), the Bayes estimate $\hat{\lambda}$ and the other formulas derived in Section 2 can also be expressed using polygamma functions for integer values of the prior parameter α. Tables for di-, tri-, tetra- and pentagamma functions are given in [1]. Another source of tables of polygamma functions is [4].

4. DESIGN

The design of a surveillance test, denoted by e, is discussed in this section for a single lot and involves the choice of the time of testing t and the lot size n. Even when several lots are tested in a surveillance program, the testing is usually done successively one lot at a time. The prior information utilized at any given stage can be adequately described, at least for planning purposes, by a gamma distribution with mean and variance determined from the posterior distribution given the data on the lots already tested using equations (1.3), (1.4) and (2.6). Petrásovits [7] has shown that the actual posterior distribution with probability density function given by (2.4) and the gamma distribution with the same mean and variance have similar moments under a variety of situations. After the testing of a lot is completed, the formulas of Section 2, which were developed for any number of lots, can be applied to the data on all of the lots tested through that time.

The expected cost $C(e,\hat{\lambda})$ of a decision to be based on a surveillance test, e, may often be separated into two components, the expected sampling cost $E_{\lambda,r}[C_s(e,r)]$ and the expected loss function $E_{\lambda,r}[L(\lambda,\hat{\lambda})]$ multiplied by an appropriate scale factor, η. Thus an experiment is chosen to minimize

$$C(e,\hat{\lambda}) = E_{\lambda,r}[C_s(e,r)] + \eta E_{\lambda,r}[L(\lambda,\hat{\lambda})]$$

$$= E_{\lambda,r}[C_s(e,r)] + \eta R(\alpha,\beta) \tag{4.1}$$

where $R(\alpha,\beta)$ is the Bayes risk computed using the prior gamma distribution depending on the surveillance results already obtained in addition to the prior distribution originally specified.

The computation of $E_{\lambda,r}[C_s(e,r)]$ is usually straightforward since $C_s(e,r)$ is often independent of both r and t. In this situation, t may be specified to minimize the Bayes risk and then the relationship between the Bayes risk and the expected sampling cost may be used to select n and complete the design of the surveillance test. To help in this design process, values of t, denoted by t_0, which minimizes the Bayes risk are given in Table I for several combinations of n and coefficients of variation $\nu = (\alpha+1)^{-1/2}$ with prior mean $\mu = (\alpha+1)/\beta = 1$. The corresponding Bayes risks are displayed in Table II. If the expected cost of sampling is independent of r but depends on t as well as n, then t and n would be selected by minimizing the right side of equation (4.1) numerically utilizing equation (2.8) for the Bayes risk.

It can be shown that there exists at least one positive t_0 which minimizes the Bayes risk. Moreover, numerical computations carried out in the preparation of Tables I and II suggest that t_0 is unique, so t_0 is referred to as the optimum surveillance time. Also, it can easily be shown that in the tabling of optimum surveillance times t_0 it is only necessary to consider prior distributions with means $\mu = 1$ as in Table I. To obtain a t_0 for a prior mean $\mu \neq 1$, divide μ into the corresponding entry in Table I. Similarly $\mu = 1$ for all entries in Table II. Corresponding Bayes risks at t_0 for other prior means can be computed by multiplying the appropriate entry in Table II by μ^2.

TABLE I. Optimum surveillance time t_0 with prior mean one by prior coefficient of variation ν and sample size n

ν	n = 1	n = 5	n = 10
0.1	1.5850	1.5844	1.5838
0.2	1.5593	1.5524	1.5451
0.3	1.5179	1.4910	1.4674
0.4	1.4626	1.4019	1.3577
0.5	1.3961	1.2944	1.2320
0.6	1.3215	1.1792	1.1021
0.7	1.2418	1.0651	0.9824
0.8	1.1602	0.9572	0.8722
0.9	1.0789	0.8583	0.7737
1.0	1.0000	0.7693	0.6871
1.5	0.6717	0.4510	0.3969
2.0	0.4586	0.2921	0.2494

TABLE II. Bayes risk at the optimum surveillance time t_0 with prior mean one by prior coefficient of variation ν and sample size n

ν	n = 1	n = 5	n = 10
0.1	0.0099	0.0097	0.0094
0.2	0.0390	0.0356	0.0320
0.3	0.0854	0.0711	0.0588
0.4	0.1468	0.1110	0.0855
0.5	0.2210	0.1530	0.1115
0.6	0.3066	0.1968	0.1378
0.7	0.4026	0.2429	0.1653
0.8	0.5086	0.2920	0.1945
0.9	0.6243	0.3445	0.2261
1.0	0.7500	0.4010	0.2602
1.5	1.5338	0.7517	0.4744
2.0	2.5922	1.2273	0.7672

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